A CHARACTERISTIC SUBGROUP FOR FUSION SYSTEMS

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Abstract. As a counterpart for the prime 2 to Glauberman's ZJ-theorem, Stellmacher proves that any nontrivial 2-group S has a nontrivial characteristic subgroup $W(S)$ with the following property. For any finite Σ_4-free group G, with S a Sylow 2-subgroup of G and with $O_2(G)$ self-centralizing, the subgroup $W(S)$ is normal in G. We generalize Stellmacher's result to fusion systems. A similar construction of $W(S)$ can be done for odd primes and gives rise to a Glauberman functor.

1. Introduction

A fundamental result in the theory of finite groups is Glauberman's ZJ-theorem [6]. Let p be an odd prime, G a finite group and let S be a Sylow p-subgroup of G. Then the ZJ-theorem asserts that the center of the Thompson group $Z(J(S))$ is normal in G whenever G is $Qd(p)$-free and $C_G(O_p(G)) \leq O_p(G)$. For a finite p-group Q, the Thompson subgroup $J(Q)$ is the subgroup generated by the abelian subgroups of Q of largest order. The group $Qd(p) = (\mathbb{Z}_p \times \mathbb{Z}_p) : SL(2,p)$ is the extension of the 2-dimensional vector space over $\mathbb{F}_p$ (the field with p elements) by $SL(2,p)$ with its natural action on this vector space. A group G is H-free if no section of G is isomorphic to H; see also Section 3.

More recently, a proof of the ZJ-theorem, in the context of fusion systems, was given by Kessar and Linckelman [10]. The authors introduce the notion of $Qd(p)$-free fusion system and prove that if $\mathcal{F}$ is a $Qd(p)$-free fusion system on a finite p-group S, with p an odd prime, then $\mathcal{F}$ is controlled by $W(S)$, for any Glauberman functor W. The related notions of characteristic p-functor and Glauberman functor were initially defined in [11, Definition 1.3]; they are given below in Definition 3.4.

For $p = 2$ the ZJ-theorem does not hold anymore. In [7, Question 16.1], Glauberman asks whether there exists a subgroup which is characteristic in a Sylow 2-subgroup S of a Σ_4-free group G, with the property $C_G(O_p(G)) \leq O_p(G)$. Here Σ_4 denotes the symmetric group on four letters.

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The answer to Glauberman's question was given by Stellmacher [24], who also obtained a different proof of the ZJ-theorem [22, 23]. Stellmacher's idea was to approximate such a subgroup via subgroups of $Z(J(S))$; see [13, Section 9.4] for an overview of this approach. The main theorem in [24] (see also 6.4 in the Appendix) can be phrased as follows:

Theorem (Stellmacher): Let S be a finite nontrivial 2-group. Then there exists a nontrivial characteristic subgroup $W(S)$ of S which is normal in G, for every finite Σ_4-free group G with S a Sylow 2-subgroup and $C_G(O_2(G)) \leq O_2(G)$.

Remark that the condition (III) in [24] is not necessary. A proof of this fact uses Lemmas 6.5 and 6.6 and Remark 6.7 in the Appendix.

In this paper we generalize Stellmacher's approach to fusion systems. Our main result is a proof of Stellmacher's version of the ZJ-theorem in the context of fusion systems:

Theorem 1.1. *Let S be a finite 2-group and let $\mathcal{F}$ be a Σ_4-free fusion system over S. Then there exists a characteristic subgroup $W(S)$ of S with the property that $\mathcal{F} = N_{\mathcal{F}}(W(S))$.*

Since Stellmacher's construction of $W(S)$ gives rise to a Glauberman functor, see Section 4 for details, we can combine Theorem B in [10] with our Theorem 1.1 to obtain the more general result which is independent of the nature of the prime p:

Theorem 1.2. *Let S be a finite p-group and let $\mathcal{F}$ be a $Qd(p)$-free fusion system over S. Then there exists a characteristic subgroup $W(S)$ of S with the property that $\mathcal{F} = N_{\mathcal{F}}(W(S))$.*

Using the same construction for $W(S)$ as in the above theorem, the normal complement theorem due to Thompson [13, 9.4.7] can be phrased as:

Theorem (Thompson): Let G be a finite group, p an odd prime and S a Sylow p-subgroup of G. Then G has a normal p-complement provided $N_G(W(S))$ has such a complement.

Our third result generalizes Thompson's theorem to the class of fusion systems. This result is similar to Theorem A in [10], except that we replace the group $Z(J)$ with the group $W(S)$ defined in Section 4:

Theorem 1.3. *Let $\mathcal{F}$ be a fusion system over a finite p-group S, with p an odd prime. Then $\mathcal{F} = \mathcal{F}_S(S)$ if and only if $N_{\mathcal{F}}(W(S)) = \mathcal{F}_S(S)$.*

The paper is organized as follows. Section 2 contains background material on fusion systems. In Section 3, the notions of H-free fusion system, characteristic p-functor and Glauberman functor are defined; further properties of fusion systems are discussed. The characteristic subgroup $W(S)$ is constructed, via two different methods, in Section 4. The proofs of the theorems are given in Section 5. In the Appendix a few related results from group theory are included.

2. Background on Fusion Systems

Fusion systems were introduced by Puig in 1990 [17, 18] in an effort to axiomatize the p-local structure of a finite group and of a block of a group algebra - the work was published only recently [19] but was known to the community long before. In 2000 Broto, Levi and Oliver [5] used this axiomatic approach to solve the Martino-Priddy conjecture. The three authors gave a different definition of the fusion systems which they proved to be equivalent to Puig's definition. In this paper we use a simplified definition which we find more elegant, equivalent to the above ones [12].

We start with a more general definition, following [15].

A *category* $\mathcal{F}$ *on a p-group* S is a category whose objects are the subgroups of S and whose set of morphisms between the subgroups Q and R of S, is a set $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ of injective group homomorphisms from Q to R, with the following properties:

(1) if $Q \leq R$ then the inclusion of Q in R is a morphism in $\mathrm{Hom}_{\mathcal{F}}(Q,R)$;
(2) for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ the induced isomorphism $Q \simeq \varphi(Q)$ and its inverse are morphisms in $\mathcal{F}$;
(3) the composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

Let $\mathcal{F}_1$ be a category on S_1 and $\mathcal{F}_2$ a category on S_2. A *morphism* between $\mathcal{F}_1$ and $\mathcal{F}_2$ is a pair (α, Θ) with $\alpha \in \mathrm{Aut}(S)$ and $\Theta : \mathcal{F}_1 \to \mathcal{F}_2$ a covariant functor, such that:

(i) for any subgroup Q of S, $\alpha(Q) = \Theta(Q)$;
(ii) for any morphism φ in $\mathcal{F}_1$, $\Theta(\varphi) \circ \alpha = \alpha \circ \varphi$.

In the following we give a series of useful definitions in a category $\mathcal{F}$ on S. If there exists an isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ we say that Q and R are $\mathcal{F}$*-conjugate*.

We say that a subgroup Q of S is

(i) *fully* $\mathcal{F}$*-centralized* if $|C_S(Q)| \geq |C_S(Q')|$ for all $Q' \leq S$ which are $\mathcal{F}$-conjugate to Q.

(ii) *fully* $\mathcal{F}$*-normalized* if $|N_S(Q)| \geq |N_S(Q')|$ for all $Q' \leq S$ which are $\mathcal{F}$-conjugate to Q.

(iii) $\mathcal{F}$*-centric* if $C_S(\varphi(Q)) \subseteq \varphi(Q)$, for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$.

(iv) $\mathcal{F}$*-radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(Q)) = 1$.

(v) $\mathcal{F}$*-essential* if Q is $\mathcal{F}$-centric and $\mathrm{Out}_{\mathcal{F}}(Q)$ has a strongly p-embedded proper subgroup M (that is M contains a Sylow p-subgroup P of $\mathrm{Out}_{\mathcal{F}}(Q)$ such that $P \neq {}^{\varphi}P$ and ${}^{\varphi}P \cap P = \{1\}$ for every $\varphi \in \mathrm{Out}_{\mathcal{F}}(Q) \setminus M$).

For Q, $R \leq S$ we denote $\mathrm{Hom}_S(Q,R) := \{u \in S \,|\, {}^{u}Q \leq R\}/C_S(Q)$ and $\mathrm{Aut}_S(Q) := \mathrm{Hom}_S(Q,Q)$. Other useful notations are $\mathrm{Aut}_{\mathcal{F}}(Q) := \mathrm{Hom}_{\mathcal{F}}(Q,Q)$ and $\mathrm{Out}_{\mathcal{F}}(Q) := \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$.

We are now able to give the definition of a fusion system.

A *fusion system* on a finite p-group S is a category $\mathcal{F}$ on S satisfying the following properties:

FS1. $\mathrm{Hom}_S(Q,R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q,R)$ for all $Q,R \leq S$.
FS2. $\mathrm{Aut}_S(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$.
FS3. Every $\varphi : Q \to S$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalized extends to a morphism $\widehat{\varphi} : N_\varphi \to S$ where

$$N_\varphi = \{x \in N_S(Q) \,|\, \exists y \in N_S(\varphi(Q)),\, \varphi({}^{x}u) = {}^{y}\varphi(u),\, \forall u \in Q\}\,.$$

Remark that N_φ is the largest subgroup of $N_S(Q)$ such that ${}^{\varphi}(N_\varphi/C_S(Q)) \leq \mathrm{Aut}_S(\varphi(Q))$. Thus we always have $QC_S(Q) \leq N_\varphi \leq N_S(Q)$.

If $\mathcal{F}$ is a fusion system on S and $Q \leq S$ we have the following equivalent characterization of being fully $\mathcal{F}$-normalized.

Proposition 2.1 ([14], Prop. 1.6)**.** *A subgroup Q of S is fully $\mathcal{F}$-normalized if and only if Q is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_S(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.*

In the following Lemma we recollect two useful properties involving fully $\mathcal{F}$-normalized subgroups; see also [10, Lemmas 2.2, 2.3]. For completeness we include the proofs.

Lemma 2.2. *Let $\mathcal{F}$ be a fusion system on a finite p-group S and Q a subgroup of S.*

a) *There is a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q),S)$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalized.*
b) *If Q is fully $\mathcal{F}$-normalized, then $\varphi(Q)$ is fully normalized, for any morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q),S)$.*

Proof. a) Let $\psi : Q \to S$ be a morphism with $\psi(Q)$ fully $\mathcal{F}$-normalized. By Proposition 2.1, $\mathrm{Aut}_S(\psi(Q))$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(\psi(Q))$. Since $\psi \circ \mathrm{Aut}_S(Q) \circ \psi^{-1}$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(\psi(Q))$ it follows that there exists a morphism $\tau \in \mathrm{Aut}_{\mathcal{F}}(\psi(Q))$ with $\tau\psi \circ \mathrm{Aut}_S(Q) \circ \psi^{-1}\tau^{-1} \leq \mathrm{Aut}_S(\psi(Q))$. Set $\alpha = \tau\psi$ and observe that $\alpha(Q)$ is fully $\mathcal{F}$-normalized. By the extension axiom FS3, α extends to a morphism $\varphi : N_\alpha \to S$. But since $\alpha \circ \mathrm{Aut}_S(Q) \circ \alpha^{-1} \leq \mathrm{Aut}_S(\psi(Q))$ it follows that $N_\alpha = N_S(Q)$. Henceforth there

exists a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), S)$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalized.
b) Since Q is fully $\mathcal{F}$-normalized and since φ is a morphism in $\mathcal{F}$, hence injective, we have: $\varphi(N_S(Q)) = N_S(\varphi(Q))$. □

Puig [17] gave analogous notions for the normalizer and the centralizer in a finite group:

The *normalizer of* Q *in* $\mathcal{F}$ is the category $N_{\mathcal{F}}(Q)$ on $N_S(Q)$ having as morphisms those morphisms $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, T)$, for R and T subgroups of $N_S(Q)$, satisfying that there exists a morphism $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QR, QT)$ such that $\widehat{\varphi}|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\widehat{\varphi}|_R = \varphi$. If $Q \leq S$ has the property that $\mathcal{F} = N_{\mathcal{F}}(Q)$ then we say that Q *is normal in* $\mathcal{F}$. The largest subgroup of S which is normal in $\mathcal{F}$ will be denoted $O_p(\mathcal{F})$.

The *centralizer of* Q *in* $\mathcal{F}$ is the category $C_{\mathcal{F}}(Q)$ on $C_S(Q)$ having as morphisms those morphisms $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, T)$, with R and T subgroups of $C_S(Q)$, satisfying that there exists a morphism $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QR, QT)$ such that $\widehat{\varphi}|_Q = \mathrm{id}_Q$ and $\widehat{\varphi}|_R = \varphi$.

Proposition 2.3 ([18], Prop. 2.8)**.** *If Q is fully $\mathcal{F}$-normalized then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_S(Q)$. If Q is fully $\mathcal{F}$-centralized then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_S(Q)$.*

Alperin's theorem on p-local control of fusion also holds for fusion systems. First we set up this theorem's notations and terminology. If $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ we say that φ is a maximal $\mathcal{F}$-automorphism. If $\varphi \in \mathrm{Aut}_{\mathcal{F}}(E)$, with E an $\mathcal{F}$-essential subgroup of S, we say that φ is an essential $\mathcal{F}$-automorphism. Alperin's fusion theorem asserts that the essential and maximal $\mathcal{F}$-automorphisms suffice to determine the whole fusion system $\mathcal{F}$.

Theorem 2.4 (Alperin)**.** *Any morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$ can be written as the composition of restrictions of essential $\mathcal{F}$-automorphisms, followed by the restriction of a maximal $\mathcal{F}$-automorphism. More precisely, there exists*

(a) *an integer $n \geq 0$,*
(b) *a set of $\mathcal{F}$-isomorphic subgroups of S, $Q = Q_0, Q_1, \ldots, Q_n, Q_{n+1} = \varphi(Q)$,*
(c) *a set of $\mathcal{F}$-essential, fully $\mathcal{F}$-normalized subgroups E_i of S containing Q_{i-1} and Q_i, for all $1 \leq i \leq n$,*
(d) *a set of essential automorphisms $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(E_i)$ satisfying $\psi_i(Q_{i-1}) = Q_i$, for all $1 \leq i \leq n$ and*
(e) *a maximal automorphism $\psi_{n+1} \in \mathrm{Aut}_{\mathcal{F}}(S)$ satisfying $\psi_i(Q_n) = Q_{n+1}$,*

such that we have

$$\varphi(u) = \psi_{n+1}\psi_n \ldots \psi_1(u), \text{ for all } u \in Q\,.$$

The reader can find a proof of this theorem in [21]; an alternative proof of this theorem in a different axiomatic setting was given by Puig [18, Corollary 3.9] and another in a less general form, using $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups instead of $\mathcal{F}$-essential subgroups, can be found in [5, Theorem A.10].

The classical examples of a fusion systems are the ones coming from the p-local structure of a finite group G. If S is a Sylow p-subgroup of G them we denote by $\mathcal{F}_S(G)$ the fusion system on S having as morphisms

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = N_G(P,Q)/C_G(P) = \mathrm{Hom}_G(P,Q)$$

where P, Q are subgroups of S and $N_G(P,Q) = \{g \in G \,|\, {}^gP \leq Q\}$ is the G-transporter from P to Q.

There are examples of fusion systems that do not come from a finite group (see eg. [20]). But there are particular cases when one can construct the finite group with p-local structure equivalent to a given fusion system. The fusion system $\mathcal{F}$ is said to be *constrained* if $O_p(\mathcal{F})$ is $\mathcal{F}$-centric. Any constrained fusion system was proven to come from a finite group by Broto, Castellani, Grodal, Levi and Oliver:

Theorem 2.5. [2, 4.3] *Let $\mathcal{F}$ be a fusion system on S and suppose that there exists an $\mathcal{F}$-centric subgroup Q of S such that $N_{\mathcal{F}}(Q) = \mathcal{F}$ (in particular $\mathcal{F}$ is constrained). Then there exists a, unique up to isomorphism, finite p$'$-reduced p-constrained group $L_Q^{\mathcal{F}}$ (i.e $O_{p'}(L_Q^{\mathcal{F}}) = 1$ and $L_Q^{\mathcal{F}}/Q \leq \mathrm{Out}(Q)$), having S as a Sylow p-subgroup and such that $\mathcal{F} = \mathcal{F}_S(L_Q^{\mathcal{F}})$. Furthermore $L_Q^{\mathcal{F}} \simeq \mathrm{Aut}_{\mathcal{F}}(Q)/Z(Q)$.*

3. Further results on fusion systems

Let G be a finite group and p a prime divisor of its order. If $A \trianglelefteq B \leq G$ then B/A is a *section* of G. We say that H *is involved in* G if H is isomorphic to a section of G. If H is not involved in G then G is H*-free*.

The *fusion system $\mathcal{F}$ on S is H-free* if H is not involved in any of the groups $L_Q^{\mathcal{F}}$, for Q running over the set of $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully $\mathcal{F}$-normalized subgroups of S.

Proposition 3.1. [10, Prop. 6.3] *Let $\mathcal{F}$ be a fusion system on a finite p-group S and let Q be a fully $\mathcal{F}$-normalized subgroup of S. If $\mathcal{F}$ is H-free, then so is any fusion subsystem of $\mathcal{F}$ which lies between $N_{\mathcal{F}}(Q)$ and $N_S(Q)C_{\mathcal{F}}(Q)$. In particular, if $\mathcal{F}$ is H-free, so are $N_{\mathcal{F}}(Q)$ and $N_S(Q)C_{\mathcal{F}}(Q)$.*

Let $\mathcal{F}$ be a fusion system on S and let Q be a subgroup of S with the property $\mathcal{F} = N_{\mathcal{F}}(Q)$. The category $\mathcal{F}/Q$ on S/Q is defined as follows: for $Q \leq P, R \leq S$, a group homomorphism $\psi : P/Q \to R/Q$ is a morphism in $\mathcal{F}/Q$ if there is a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,R)$ satisfying $\psi(xQ) = \varphi(x)Q$ for all $x \in P$. The fact that $\mathcal{F}/Q$ is a fusion system on S/Q is due to Puig [17], see also [10, Proposition 2.8].

Proposition 3.2. [10, Prop. 6.4] *Let $\mathcal{F}$ be a fusion system on a finite p-group S and let Q be a normal subgroup of S such that $\mathcal{F} = N_{\mathcal{F}}(Q)$. If $\mathcal{F}$ is H-free then $\mathcal{F}/Q$ is also H-free.*

The following result generalizes the main technical step in the proof of [10, Proposition 5.2].

Proposition 3.3. *Let $\mathcal{F}$ be a fusion system on a finite p-group S and let W_i, $1 \leq i \leq n$ be subgroups of S such that*

(a) *the subgroup W_{i+1} is a characteristic subgroup of $N_S(W_i)$ for $1 \leq i \leq n-1$;*
(b) *the subgroup W_i is fully $\mathcal{F}$-normalized for $1 \leq i \leq n-1$.*

Then there exists a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(W_n), S)$ such that $\varphi(W_i)$ is fully $\mathcal{F}$-normalized for all $1 \leq i \leq n$. In particular $\varphi(N_S(W_i)) = N_S(\varphi(W_i))$. If moreover W_i is $\mathcal{F}$-centric and/or $\mathcal{F}$-radical for some $1 \leq i \leq n$, then so is $\varphi(W_i)$.

Proof. If follows from Lemma 2.2(a) that there exists a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(W_n), S)$ such that $\varphi(W_n)$ is fully $\mathcal{F}$-normalized. According to condition (a), W_{i+1} is a characteristic subgroup of $N_S(W_i)$, for $1 \leq i \leq n-1$ and therefore $N_S(W_i) \leq N_S(W_{i+1})$. In particular $N_S(W_i) \leq N_S(W_n)$ and the morphism φ is defined on $N_S(W_i)$ for all $1 \leq i \leq n$.

Next we show $\varphi(W_i)$ is fully $\mathcal{F}$-normalized for all $1 \leq i \leq n-1$. By elementary group theory $\varphi(N_S(W_i)) \leq N_S(\varphi(W_i))$. Since φ is injective it follows that $|N_S(W_i)| \leq |\varphi(N_S(W_i))|$. But, according to (b), W_i is fully $\mathcal{F}$-normalized and $|N_S(W_i)| \geq |N_S(\varphi(W_i))|$. It follows now that $|N_S(W_i)| = |N_S(\varphi(W_i))|$ which shows that $\varphi(W_i)$ is fully $\mathcal{F}$-normalized and that $\varphi(N_S(W_i)) = N_S(\varphi(W_i))$, for all $1 \leq i \leq n-1$.

Since W_i is fully $\mathcal{F}$-normalized for all $1 \leq i \leq n$, it is also fully $\mathcal{F}$-centralized. Thus $\varphi(C_S(W_i)) = C_S(\varphi(W_i))$ and if W_i is $\mathcal{F}$-centric, then so is $\varphi(W_i)$. Moreover it is a general fact that ${}^{\varphi}\mathrm{Aut}_{\mathcal{F}}(W_i) = \mathrm{Aut}_{\mathcal{F}}(\varphi(W_i))$ and ${}^{\varphi}\mathrm{Aut}_S(W_i) = \mathrm{Aut}_S(\varphi(W_i))$ so if W_i is $\mathcal{F}$-radical, then so is $\varphi(W_i)$. □

The following definition is from [10, 5.1].

Definition 3.4. A *positive characteristic functor* is a map sending any nontrivial finite p-group S to a nontrivial characteristic subgroup $W(S)$ of S such that $W(\varphi(S)) = \varphi(W(S))$ for every $\varphi \in \mathrm{Aut}(S)$. A positive characteristic functor is a *Glauberman functor* if whenever S is a Sylow p-subgroup of a $Qd(p)$-free finite group L which satisfies $C_L(O_p(L)) = Z(O_p(L))$, then $W(S)$ is normal in L.

Using the previous result we can give a different proof for Proposition 5.3 in [10].

Proposition 3.5. [10, Prop. 5.3] *Let $\mathcal{F}$ be a fusion system on a finite p-group S and let W be a positive characteristic functor. Assume that for any non-trivial proper fully $\mathcal{F}$-normalized subgroup Q of S the following holds $N_{\mathcal{F}}(Q) = N_{N_{\mathcal{F}}(Q)}(W(N_S(Q)))$. Then $\mathcal{F} = N_{\mathcal{F}}(W(S))$.*

Proof. Suppose that the conclusion does not hold. By Alperin's Fusion Theorem there exists a proper fully $\mathcal{F}$-normalized subgroup Q of S such that $\mathrm{Aut}_{N_{\mathcal{F}}(W(S))}(Q) \subset \mathrm{Aut}_{\mathcal{F}}(Q)$.

Set $W_1 = Q$ and define recursively $W_{i+1} := W(N_S(W_i))$. So W_{i+1} is characteristic in $N_S(W_i)$ implying that we get the following inclusions $N_S(W_i) < N_S(N_S(W_i)) \leq N_S(W_{i+1})$ this in its turn implies there exists $n \geq 1$ such that the sequence of $N_S(W_i)$ for $1 \leq i \leq n$ is strictly increasing and $N_S(W_n) = S$. Observe that if $N_S(W_i) = N_S(W_{i+1})$ then $N_S(W_i) = N_S(N_S(W_i)) \leq S$ and by an elementary property of p-groups it follows that $N_S(W_i) = S$, thus indeed the sequence eventually reaches S.

Moreover the sequence $\{W_i,\, 1 \leq i \leq n+1\}$ can be chosen so that all its terms are fully $\mathcal{F}$-normalized. This can be done recursively by applying Proposition 3.3, for all $2 \leq k \leq n+1$ to the partial subsequences $\{W_i,\, 1 \leq i \leq k\}$. The W_i's are successively modified by replacing them with their images through the morphism φ given by Proposition 3.3.

Consider the sequence of the normalizers in $\mathcal{F}$ of the W_i's for $1 \leq i \leq n+1$. Given that W_i, $1 \leq i \leq n$ are fully $\mathcal{F}$-normalized, we have that $N_{\mathcal{F}}(W_i)$ is a fusion system on $N_S(W_i)$. It follows from our assumption that $N_{\mathcal{F}}(W_i) \subseteq N_{\mathcal{F}}(W_{i+1})$ for all $1 \leq i \leq n-1$. But then $N_{\mathcal{F}}(Q) = N_{\mathcal{F}}(W_1) \subseteq N_{\mathcal{F}}(W_{n+1}) = N_{\mathcal{F}}(W(S))$. At the level of morphisms on Q this gives $\mathrm{Aut}_{\mathcal{F}} Q \subseteq \mathrm{Aut}_{N_{\mathcal{F}}(W(S))}(Q)$ which is a contradiction with the initial supposition on Q. □

We denote by $SC_{\mathcal{F}}(Q)$ the category on S having as morphisms all group homomorphisms $\varphi : P \to R$, for P and R subgroups of S, for which there exists a morphism $\psi : QP \to QR$ and $x \in S$ such that $\psi|_Q = c_x$ (the morphism induced by conjugation by x) and $\psi|_P = \varphi$.

The previous proposition is used in [10] to prove the next important result.

Proposition 3.6. [10, Prop. 3.4] *Let S be a finite p-group and let Q be a normal subgroup of S. Let $\mathcal{F}$ and $\mathcal{G}$ be fusion systems on S such that $\mathcal{F} = SC_{\mathcal{F}}(Q)$ and such that $\mathcal{G} \subseteq \mathcal{F}$. Let P be a normal subgroup of S containing Q. We have $\mathcal{G} = N_{\mathcal{F}}(P)$ if and only if $\mathcal{G}/Q = N_{\mathcal{F}/Q}(P/Q)$.*

At the end of this section we give an application of the Frattini argument to fusion systems. The group theoretic result states that, if G is a finite group, then the following factorization holds: $G = C_G(Q)N_G(R)$ with $Q = O_p(G)$ and $R = C_G(QC_S(Q))$. This is easily seen to be true as $C_G(Q) \trianglelefteq G$ and $N_G(C_S(Q)) \leq N_G(R)$, then an application of the Frattini argument gives the result.

Lemma 3.7. *Let $\mathcal{F}$ be a fusion system on S, $Q = O_p(\mathcal{F})$ and $R = QC_S(Q)$. Set $\mathcal{F}_1 = SC_{\mathcal{F}}(Q)$ and $\mathcal{F}_2 = N_{\mathcal{F}}(R)$. Then $\mathcal{F} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle$.*

Proof. First remark that $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems on S with $\mathcal{F}$ containing the other two. By Alperin's fusion theorem (see Theorem 2.4), it is enough to prove that for every $\mathcal{F}$-centric, $\mathcal{F}$-radical, fully $\mathcal{F}$-normalized subgroup U of S we have $\mathrm{Aut}_{\mathcal{F}}(U) = \mathrm{Aut}_{\mathcal{G}}(U)$ with $\mathcal{G} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle$.

We shall prove that every $\varphi \in \mathrm{Aut}_{\mathcal{F}}(U)$ can be written as composition of morphisms in $\mathcal{F}_1$ and $\mathcal{F}_2$ and thus will be contained in $\mathrm{Aut}_{\mathcal{G}}(U)$. This will finish our proof as the opposite inclusion is clearly satisfied.

Given that U is $\mathcal{F}$-centric and $\mathcal{F}$-radical we have by [16, Proposition 5.6] that $Q \leq U$. Hence φ restricts to an automorphism $\theta \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Now we have that N_θ contains U and R so it contains UR. Given that Q is fully $\mathcal{F}$-normalized θ extends to $\chi \in \mathrm{Hom}_{\mathcal{F}}(UR, S)$. Moreover $\chi(R) = \chi(Q)C_S(\chi(Q)) = R$ so in fact $\chi \in \mathrm{Hom}_{\mathcal{F}_2}(UR, S)$.

Denote by ψ the restriction to U of χ; then $\psi \in \mathrm{Hom}_{\mathcal{F}_2}(U, \psi(U))$. Both φ and ψ restrict as θ on Q so $\varphi \circ \psi^{-1}$ belongs to $\mathrm{Hom}_{\mathcal{F}_1}(\psi(U), U)$. The conclusion in the lemma follows as $\varphi = \varphi \circ \psi^{-1} \circ \psi \in \mathrm{Aut}_{\mathcal{G}}(Q)$. □

Next, we give a straightforward result on fusion control in fusion systems.

Lemma 3.8. *Let W be a fully $\mathcal{F}$-normalized subgroup of S and suppose that there are two fusion subsystems $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ such that $\mathcal{F} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle$. If moreover $\mathcal{F}_1 = N_{\mathcal{F}_1}(W)$ and $\mathcal{F}_2 = N_{\mathcal{F}_2}(W)$. Then $\mathcal{F} = N_{\mathcal{F}}(W)$.*

Proof. We have that $\langle N_{\mathcal{F}_1}(W), N_{\mathcal{F}_2}(W) \rangle \subseteq N_{\mathcal{F}}(W) \subseteq \mathcal{F}$. The result follows. □

4. A characteristic subgroup of S

Let S be a finite p-group. In this section we construct a subgroup $W(S)$ which is characteristic in S and such that $\Omega(Z(S)) \leq W(S) \leq \Omega(Z(J(S)))$, and with the property that $W(S) \trianglelefteq \mathcal{E}$ for all $(\varphi, \mathcal{E}) \in \mathcal{U}_J$, with $\mathcal{U}_J$ a class of embeddings defined below. The notation $\Omega(H)$ stands for the subgroup of H generated by all the elements of order p, while $J(S)$ denotes the Thompson subgroup of S defined in the Introduction. We shall give below two different, although equivalent, constructions of this characteristic subgroup of S which we shall denote $W(S)$ and W. The first construction follows the approach developed by Stellmacher [22, 24] for finite groups, in which such a subgroup is approximated from various subgroups of $Z(J(S))$. The second construction uses basic properties of fusion systems.

4.1. **The group W(S).** An *embedding* is a pair $(\varphi, \mathcal{E})$ where $\varphi \in \mathrm{Aut}(S)$ and $\mathcal{E}$ is a category on $\varphi(S) = S$. Let $\mathcal{C}$ denote the family of all embeddings of S. A nonempty subclass $\mathcal{U}$ of $\mathcal{C}$ is *characteristically closed* if $(\varphi\alpha, \mathcal{E}) \in \mathcal{U}$ whenever $(\varphi, \mathcal{E}) \in \mathcal{U}$ and $\alpha \in \mathrm{Aut}(S)$.

An *equivalence* between two embeddings $(\varphi_1, \mathcal{E}_1)$ and $(\varphi_2, \mathcal{E}_2)$ is a morphism $(\alpha, \Theta) : \mathcal{E}_1 \to \mathcal{E}_2$ with $\alpha\varphi_1 = \varphi_2$ and $\mathrm{Hom}_{\mathcal{E}_2}(\varphi_2(Q), \varphi_2(R)) = \alpha \circ \mathrm{Hom}_{\mathcal{E}_1}(\varphi_1(Q), \varphi_1(R)) \circ \alpha^{-1}{}_{|\varphi_1(Q)}$. The equivalence of embeddings defines an equivalence relation on $\mathcal{U}$. Since S is a finite group, the collection of equivalence classes $[\mathcal{U}]$ is a finite set.

Let $O_S(\mathcal{U})$ denote the largest subgroup of S which satisfies the property that $\varphi(O_S(\mathcal{U}))$ is normal in $\mathcal{E}$ for every embedding $(\varphi, \mathcal{E})$ in $\mathcal{U}$.

Lemma 4.1. *Let $\mathcal{U}$ be a characteristically closed subclass of $\mathcal{C}$ and let $\alpha \in \mathrm{Aut}(S)$. Let Q be a subgroup of S with the property that $\varphi(Q) \trianglelefteq \mathcal{E}$ for every $(\varphi, \mathcal{E}) \in \mathcal{U}$. Then $\varphi(\alpha(Q)) \trianglelefteq \mathcal{E}$ for every $(\varphi, \mathcal{E})$. In particular $O_S(\mathcal{U})$ is a characteristic subgroup of S.*

Proof. Observe that since $\mathcal{U}$ is characteristically closed and $(\varphi, \mathcal{E}) \in \mathcal{U}$ then $(\varphi\alpha, \mathcal{E}) \in \mathcal{U}$ and thus $\varphi(\alpha(Q)) \trianglelefteq \mathcal{E}$ for every $(\varphi, \mathcal{E}) \in \mathcal{U}$. The fact that $O_S(\mathcal{U})$ is $\mathrm{Aut}(S)$-invariant follows from its definition. □

Let $\mathcal{U}_J$ denote the class of embeddings $(\varphi, \mathcal{E})$ which satisfy the following conditions:

- $(U1)$ $\mathcal{U}_J$ is characteristically closed.
- $(U2)$ $J(\varphi(S)) = J(S)$ is normal in $\mathcal{E}$ for all $(\varphi, \mathcal{E}) \in \mathcal{U}_J$.
- $(U3)$ $\mathcal{E}$ is a $Qd(p)$-free fusion system.

For a p-group P we set $A(P) = \Omega(Z(P))$ and $B(P) = \Omega(Z(J(P)))$. Remark that $A(P) \leq B(P)$ as $Z(P) \leq J(P)$. Note that $\alpha(A(P)) = A(\alpha(P))$ and $\alpha(B(P)) = B(\alpha(P))$ for all $\alpha \in \mathrm{Aut}(P)$, as $A(P)$ and $B(P)$ are characteristic subgroups of P.

Define recursively a subgroup $W(S) \leq B(S)$ as follows. Let

$$W_0 := A(S) = \Omega(Z(S)) \leq B(S)$$

and assume that for $i \geq 1$ the subgroups $W_0, W_1, \ldots W_{i-1}$ with

$$W_0 < W_1 < \ldots W_{i-1} \leq B(S)$$

are defined. If $\varphi(W_{i-1}) \trianglelefteq \mathcal{E}$ for all $(\varphi, \mathcal{E}) \in \mathcal{U}_J$ then set $W(S) := W_{i-1}$. Otherwise, choose $(\varphi_i, \mathcal{E}_i) \in \mathcal{U}_J$ to be such that $\varphi_i(W_{i-1})$ is not normal in $\mathcal{E}_i$ and define

$$W_i := \varphi_i^{-1}\langle \varphi_i(W_{i-1})^{\mathcal{E}_i} \rangle = \varphi_i^{-1}\langle \psi(\varphi_i(W_{i-1})) : \psi \in \mathrm{Hom}_{\mathcal{E}_i}(\varphi_i(W_{i-1}), \varphi_i(S)) \rangle$$

to be the preimage in S of the group generated by the $\mathcal{E}_i$-orbit of $\varphi_i(W_{i-1})$.

Since $B(\varphi_i(S))$ is a characteristic subgroup of $J(\varphi_i(S))$, which in its turn is normal in $\mathcal{E}_i$, it follows that $B(\varphi_i(S))$ is also normal in $\mathcal{E}_i$. Clearly $\varphi_i(W_i) \leq B(\varphi_i(S))$ since $\varphi_i(W_{i-1}) \leq B(\varphi_i(S))$ by construction, and since $\varphi_i(W_i)$ is generated by various conjugates of $\varphi_i(W_{i-1})$. Thus we have:

$$A(\varphi_i(S)) \leq \varphi_i(W_{i-1}) < \varphi_i(W_i) \leq B(\varphi_i(S)) \trianglelefteq \mathcal{E}_i$$

as $A(\varphi_i(S)) = \varphi_i(A(S))$ and $B(\varphi_i(S)) = \varphi_i(B(S))$ for $\varphi_i \in \operatorname{Aut}(S)$. Then it follows:

$$A(S) \leq W_{i-1} < W_i \leq B(S)$$

As S is finite, this recursive definition terminates after a finite number n of steps and $W(S) := W_n$. Therefore we obtain a chain of subgroups of $Z(J(S))$:

$$A(S) = W_0 < W_1 \ldots < W_i < \ldots < W_n = W(S) \leq B(S)$$

and $\varphi(W(S)) \trianglelefteq \mathcal{E}$ for all $(\varphi, \mathcal{E}) \in \mathcal{U}_J$.

The group $W(S)$ depends on S only and it is independent of the pairs $(\varphi_i, \mathcal{E}_i)$. To see this assume that we defined in an analogous way

$$W_0 = \overline{W}_0 < \overline{W}_1 < \ldots < \overline{W}_{\tilde{n}} =: \overline{W}(S)$$

for suitable pairs $(\overline{\varphi}_j, \overline{\mathcal{E}}_j)$ in $\mathcal{U}_J$ and for $j = 1, \ldots \overline{n}$. First note that $\overline{W}_0 = W_0 \leq W(S) \cap \overline{W}(S)$. Thus $\overline{\varphi}_1(\overline{W}_0) = \overline{\varphi}_1(W_0) \leq \overline{\varphi}_1(W(S)) = W(S)$ as $W(S) \trianglelefteq \overline{\mathcal{E}}_1$, and therefore $\overline{\varphi}_1(\overline{W}_1) = \langle \overline{\varphi}_1(\overline{W}_0)^{\overline{\mathcal{E}}_1} \rangle \leq W(S)$ which implies $\overline{W}_1 \leq W(S)$. Proceed by induction on j; a similar argument shows that since $\overline{W}_{j-1} \leq W(S)$ then $\overline{\varphi}_j(\overline{W}_j) \leq W(S)$ and $\overline{W}_j \leq W(S)$. Therefore $\overline{W}(S) \leq W(S)$. Similarly $W(S) \leq \overline{W}(S)$ and thus $W(S) = \overline{W}(S)$.

Lemma 4.2. *Let $\alpha \in \operatorname{Aut}(S)$. Then $W(\alpha(S)) = \alpha(W(S))$. In particular, $W(S)$ is a characteristic subgroup of S, nontrivial if S is nontrivial.*

Proof. The mapping $(\varphi, \mathcal{E}) \to (\varphi\alpha, \mathcal{E})$ defines a bijection on $\mathcal{U}_J$. Under this map, the chain of subgroups:

$$A(S) = W_0 < \ldots < W_i < \ldots < W_n = W(S) \leq B(S)$$

is taken to the following chain:

$$A(S) = \alpha(W_0) < \alpha(W_1) < \ldots < \alpha(W_i) < \ldots < \alpha(W_n) = \alpha(W(S)) \leq B(S).$$

Therefore $W(\alpha(S)) = \alpha(W(S))$. The last statement follows from the fact that $\Omega(Z(S)) \leq W(S)$ and $Z(S) \neq 1$ if $S \neq 1$. □

4.2. **The group W.** Denote by $\mathcal{C}_J$ the class of categories $\mathcal{F}$ on S which satisfy the following conditions:

$(C1)$ $J(S)$ is normal in $\mathcal{F}$ for all $\mathcal{F} \in \mathcal{C}_J$.
$(C2)$ $\mathcal{F}$ is a $Qd(p)$-free fusion system.

Proposition 4.3. *Let $W_0 = \Omega(Z(S))$ and define*

$$W := \langle \psi(W_0) \mid \psi \in \mathrm{Hom}_{\mathcal{F}}(J(S), S) \text{ for } \mathcal{F} \in \mathcal{C}_J \rangle.$$

The subgroup W is a nontrivial characteristic subgroup of S.

Proof. For all $\alpha \in \mathrm{Aut}(S)$, we will show that $\alpha(W) = W$. Let $\mathcal{F}$ be a category on S. Denote by $\mathcal{F}^\alpha$ the category on S having as sets of morphisms

$$\mathrm{Hom}_{\mathcal{F}^\alpha}(Q, R) = \alpha^{-1} \circ \mathrm{Hom}_{\mathcal{F}}(\alpha(Q), \alpha(R)) \circ \alpha\,.$$

Note that if $\mathcal{F} \in \mathcal{C}_J$ then $\mathcal{F}^\alpha \in \mathcal{C}_J$, and if $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ then $\alpha\psi\alpha^{-1} \in \mathrm{Hom}_{\mathcal{F}^{\alpha^{-1}}}(\alpha(Q), \alpha(R))$. Thus we have:

$$\begin{aligned}\alpha(W) &:= \langle \alpha\psi(W_0) \mid \psi \in \mathrm{Hom}_{\mathcal{F}}(J(S), S) \text{ for } \mathcal{F} \in \mathcal{C}_J \rangle = \\ &= \langle \alpha\psi\alpha^{-1}(\alpha(W_0)) \mid \alpha\psi\alpha^{-1} \in \mathcal{F}^{\alpha^{-1}}(\alpha(J(S)), \alpha(S)) \text{ for } \mathcal{F} \in \mathcal{C}_J \rangle = \\ &= \langle \widetilde{\psi}(W_0) \mid \widetilde{\psi} \in \mathcal{F}^{\alpha^{-1}}(J(S), S) \text{ for } \mathcal{F} \in \mathcal{C}_J \rangle \\ &\leq W\end{aligned}$$

where in the last equality we use that W_0 and $J(S)$ are characteristic subgroups of S. But since α is injective, it follows that $|W| = |\alpha(W)|$ and therefore $\alpha(W) = W$, proving that W is characteristic in S. □

4.3. **Stellmacher functor.** Given that $(\varphi, \mathcal{F})$ and $(\alpha\varphi, \mathcal{F}^{\alpha^{-1}})$ are equivalent as embeddings and $\mathcal{F} \in \mathcal{C}_J$ if and only if $\mathcal{F}^\alpha \in \mathcal{C}_J$ for any $\varphi, \alpha \in \mathrm{Aut}(S)$ and $\mathcal{F}$ a category on S, the two definitions $W(S)$ and W represent the same subgroup of S, that is $W = W(S)$. It follows from Lemma 4.2 that the functor $S \to W(S)$, for S a finite p-group, is a positive characteristic functor in the sense of Definition 3.4. We shall call the functor $S \to W(S)$, with $W(S)$ constructed via one of the methods from this Section, a *Stellmacher functor*.

The Thompson subgroup of S, $J(S)$ is a characteristic, centric subgroup. Thus using $(C1)$, any $\mathcal{F} \in \mathcal{C}_J$ is a constrained fusion system on S and by Theorem 2.5, there exists a p-constrained finite group L with $\mathcal{F} = \mathcal{F}_L(S)$ and satisfying the following conditions: S is a Sylow p-subgroup of L; $C_L(O_p(L)) \leq O_p(L)$ and L is $Qd(p)$-free. It follows from our construction that $W(S)$ is a characteristic subgroup of S which is also normal in L. The construction of $W(S)$ depends on S only, and the subgroup $W(S)$ is constructed in the same way as Stellmacher does in the context of finite groups so it is the same subgroup of S. Finally, notice that $S \to W(S)$ is also a Glauberman functor, see Definition 3.4.

5. Proofs of the Theorems

Proof of Theorem 1.1. Let S be a finite 2-group and let $\mathcal{F}$ be a Σ_4-free fusion system on S. Also $W(S)$ is the characteristic subgroup of S defined in Section 4.

The statement of Theorem 1.1 is true for the smallest fusion system on S, which is $\mathcal{F}_S(S)$. Suppose now by induction that all proper Σ_4-free subsystems and all Σ_4-free quotient systems $\mathcal{F}/Q$, with Q a nontrivial normal subgroup of $\mathcal{F}$ satisfy Theorem 1.1.

If $O_2(\mathcal{F}) = 1$ then for every non-trivial fully $\mathcal{F}$-normalized subgroup P of S we have that $N_{\mathcal{F}}(P)$ is a proper subsystem of $\mathcal{F}$ (otherwise $P \leq O_2(\mathcal{F})$). But then $N_{\mathcal{F}}(P)$ satisfies Theorem 1.1 by induction as it is Σ_4-free by Proposition 3.1. Hence $N_{\mathcal{F}}(P) = N_{N_{\mathcal{F}}(P)}(W(N_S(P)))$ for every non-trivial fully $\mathcal{F}$-normalized subgroup P. An application of Proposition 3.5 gives now that $\mathcal{F} = N_{\mathcal{F}}(W(S))$. So we can suppose $(H1)$: $O_2(\mathcal{F}) \neq 1$.

Set $Q := O_2(\mathcal{F})$ and $R := QC_S(Q)$. If $Q = R$ then Q is $\mathcal{F}$-centric. Consequently $\mathcal{F}$ is a constrained fusion system. According to [3, Proposition 4.3] there exists a $2'$-reduced 2-constrained finite group L_Q, which is an extension of $\mathrm{Aut}_{\mathcal{F}}(Q)$ by $Z(Q)$ and having S as a Sylow 2-subgroup. Thus $\mathcal{F} = \mathcal{F}_S(L_Q)$. Since $\mathcal{F}$ is Σ_4-free, the group L_Q is Σ_4-free, by definition and given the L_Q is 2-constrained we have $C_{L_Q}(O_2(L_Q)) \leq O_2(L_Q)$. Then, according to Stellmacher's main theorem in [24], see also the Introduction, the group $W(S)$ is normal in L_Q. This in its turn implies that $W(S) \trianglelefteq \mathcal{F}_S(L_Q)$ and therefore $\mathcal{F} = N_{\mathcal{F}}(W(S))$. Thus we can also make the assumption $(H2)$: $Q \neq R$ implying moreover that $N_{\mathcal{F}}(R)$ is a proper subsystem of $\mathcal{F}$.

Next we will see that we also have $(H3)$: $SC_{\mathcal{F}}(Q) \neq \mathcal{F}$. Indeed suppose that $SC_{\mathcal{F}}(Q) = \mathcal{F}$. Then $\mathcal{F}/Q$ is a proper quotient system of $\mathcal{F}$ which is Σ_4-free by Proposition 3.2. The induction hypothesis gives now $\mathcal{F}/Q = N_{\mathcal{F}/Q}(W(S/Q))$. Next, Proposition 3.6 gives that $\mathcal{F} = N_{\mathcal{F}}(U)$ where U is the preimage in S of $W(S/Q)$. As $U \trianglelefteq \mathcal{F}$ it follows that $U \leq Q$, but this leads to a contradiction given that $W(S/Q) \neq 1$.

According to Lemma 3.7, we have $\mathcal{F} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ with $\mathcal{F}_1 = SC_{\mathcal{F}}(Q)$ and $\mathcal{F}_2 = N_{\mathcal{F}}(R)$. By $(H2)$ and $(H3)$ both $\mathcal{F}_1$ and $\mathcal{F}_2$ are proper subsystems of $\mathcal{F}$, the induction hypothesis gives that $W(S) \trianglelefteq \mathcal{F}_1$ and that $W(S) \trianglelefteq \mathcal{F}_2$.

Notice that $W(S)$ is fully $\mathcal{F}$-normalized. By Lemma 2.2, there exists a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(W(S)), S)$ with $\varphi(W(S))$ fully $\mathcal{F}$-normalized. As $N_S(W(S)) = S$ and since $W(S)$ is characteristic in S it follows that $W(S) = \varphi(W(S))$ is fully $\mathcal{F}$-normalized.

Finally, an application of Lemma 3.8 gives the result: $\mathcal{F} = N_{\mathcal{F}}(W(S))$.

Proof of Theorem 1.2. Let S be a finite p-group. Recall that the construction $W(S)$ described in Section 4 and which associates to S a nontrivial characteristic subgroup $W(S)$ gives rise to a Glauberman functor.

Assume now that $\mathcal{F}$ is a $Qd(p)$-free fusion system on S. If p is an odd prime, it follows from [10, Theorem B] that $\mathcal{F} = N_{\mathcal{F}}(W(S))$. If $p = 2$ then $Qd(2) = \Sigma_4$ and the result is given by Theorem 1.1.

Proof of Theorem 1.3. Let p be an odd prime. Let $\mathcal{F}$ be a fusion system over a finite p-group S. Let $W(S)$ be the characteristic subgroup of S given by the Stellmacher functor. Since $\mathcal{F}_S(S) \subseteq N_{\mathcal{F}}(W(S)) \subseteq \mathcal{F}$ it is enough to show that if $N_{\mathcal{F}}(W(S)) = \mathcal{F}_S(S)$ then $\mathcal{F} = \mathcal{F}_S(S)$. The proof is similar to that of Theorem A in [10]; for the sake of completeness we will provide the details.

Let $\mathcal{F}$ be a minimal counterexample to Theorem 1.3; thus $N_{\mathcal{F}}(W(S)) = \mathcal{F}_S(S)$ but $\mathcal{F} \neq \mathcal{F}_S(S)$, and all the proper subsystems and quotient systems of $\mathcal{F}$ satisfy Theorem 1.3. Under this assumption we show that $\mathcal{F}$ is a constrained fusion system by proving that $Q := O_p(\mathcal{F})$ is a nontrivial $\mathcal{F}$-centric proper subgroup of S. This is attained in the following six steps.

Step 1 : Any fusion system $\mathcal{G}$ on S which is properly contained in $\mathcal{F}$ is equal to $\mathcal{F}_S(S)$.

As $\mathcal{G} \subset \mathcal{F}$ it follows that $N_{\mathcal{G}}(W(S)) \subseteq N_{\mathcal{F}}(W(S)) = \mathcal{F}_S(S)$. But $W(S)$ is a characteristic subgroup of S and therefore $\mathcal{F}_S(S) \subseteq N_{\mathcal{G}}(W(S))$. Thus $N_{\mathcal{G}}(W(S)) = \mathcal{F}_S(S)$ and the minimality assumption on $\mathcal{F}$ implies that $\mathcal{G} = \mathcal{F}_S(S)$.

Step 2 : Let P be a fully $\mathcal{F}$-normalized subgroup of S and set $A = N_S(P)$. Then there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(A, S)$ such that both $\varphi(P)$ and $\varphi(W(A))$ are fully $\mathcal{F}$-normalized.

By Lemma 2.2(a), there is a morphism $\varphi : N_S(W(A)) \to S$ such that $\varphi(W(A))$ is fully $\mathcal{F}$-normalized. Since $W(A)$ is a characteristic subgroup of A, we have $N_S(P) = A \leq N_S(A) \leq N_S(W(A))$ and the morphism φ can be restricted to $\varphi : A \to S$. According to Lemma 2.2(b), the group $\varphi(P)$ is also fully $\mathcal{F}$-normalized.

Step 3 : The subgroup $Q = O_p(\mathcal{F})$ is nontrivial.

Recall that it is assumed that $\mathcal{F}_S(S) \subset \mathcal{F}$. Alperin's fusion theorem implies that there is a fully $\mathcal{F}$-normalized subgroup P of S with $\mathcal{F}_A(A) \subset N_{\mathcal{F}}(P)$, for $A = N_S(P)$. Choose the subgroup P such that:

a). $W(A)$ is fully $\mathcal{F}$-normalized;

b). $N_S(P) = A$ has maximal order among subgroups T with $\mathcal{F}_{N_S(T)}(N_S(T)) \subset N_{\mathcal{F}}(T)$.

The choice of P and the fact that A is a proper subgroup of $N_S(W(A))$ implies that $N_{\mathcal{F}}(W(A)) = \mathcal{F}_{N_S(W(A))}(N_S(W(A)))$. Therefore $N_{N_{\mathcal{F}}(P)}(W(A)) = \mathcal{F}_A(A)$.

If $N_{\mathcal{F}}(P) \subset \mathcal{F}$ then the minimality assumption on $\mathcal{F}$ implies that $N_{\mathcal{F}}(P) = \mathcal{F}_R(R)$, which contradicts our choice of P. Thus we have $N_{\mathcal{F}}(P) = \mathcal{F}$ and $1 \neq P \trianglelefteq \mathcal{F}$. Hence $1 \neq P \leq Q$ which proves that $Q \neq 1$.

Step 4 : Q is a proper subgroup of S.

If $Q = S$ then $\mathcal{F} = N_{\mathcal{F}}(S) = N_{\mathcal{F}}(W(S)) = \mathcal{F}_S(S)$ contradicting our assumption on $\mathcal{F}$.

Step 5 : $SC_{\mathcal{F}}(Q) = \mathcal{F}_S(S)$ when $Q = O_p(\mathcal{F})$.

We have $SC_{\mathcal{F}}(Q) \subseteq \mathcal{F}$. If $SC_{\mathcal{F}}(Q) \subset \mathcal{F}$ then *Step 1* implies that $SC_{\mathcal{F}}(Q) = \mathcal{F}_S(S)$ and we are done. Assume now that $SC_{\mathcal{F}}(Q) = \mathcal{F}$ and recall that $\mathcal{F} \neq \mathcal{F}_S(S)$. An application of Proposition 3.6, with $\mathcal{G} = \mathcal{F}_S(S)$ and $\mathcal{F} = N_{\mathcal{F}}(Q)$, gives that $\mathcal{F}/Q \neq \mathcal{F}_S(S)/Q = \mathcal{F}_{S/Q}(S/Q)$. By *Step 3* the subgroup Q is nontrivial and the minimality assumption on $\mathcal{F}$ implies that $N_{\mathcal{F}/Q}(W(S/Q)) \neq \mathcal{F}_{S/Q}(S/Q)$. Let P be the inverse image of $W(S/Q)$ in S. Notice that $W(S/Q) \neq 1$, by the definition of $W(S)$, and thus P properly contains Q. Also $P \trianglelefteq S$ and $N_{\mathcal{F}/Q}(W(S/Q)) = N_{\mathcal{F}/Q}(P/Q)$. Another application of Proposition 3.6 gives that $N_{\mathcal{F}}(P) \neq \mathcal{F}_S(S)$. Since $N_{\mathcal{F}}(P) \subseteq \mathcal{F}$, *Step 1* implies that $N_{\mathcal{F}}(P) = \mathcal{F}$ which is a contradiction with the fact that P contains Q properly.

Step 6 : The subgroup Q is $\mathcal{F}$-centric.

If $Q = R = SC_S(Q)$ then Q is $\mathcal{F}$-centric and we are done. So let us assume that $Q < R$. Notice that $R \trianglelefteq S$. Then $N_{\mathcal{F}}(R)$ is a proper subsystem of $\mathcal{F}$ and an application of *Step 1* gives that $N_{\mathcal{F}}(R) = \mathcal{F}_S(S)$. Recall also that by the previous step, $SC_{\mathcal{F}}(Q) = \mathcal{F}_S(S)$. Therefore, Lemma 3.7 implies that $\mathcal{F} = \mathcal{F}_S(S)$, which is a contradiction to our choice of $\mathcal{F}$. Thus we must have $Q = R$.

Since $Q = O_p(\mathcal{F})$ is a nontrivial normal centric subgroup of $\mathcal{F}$, the fusion system $\mathcal{F}$ is constrained. But this means by [2, 4.3], that there is a p'-reduced p-constrained finite group L with S as a Sylow p-subgroup and such that $Q = O_p(L)$. Furthermore $\mathcal{F} = \mathcal{F}_S(L)$ and therefore $N_{\mathcal{F}}(W(S)) = \mathcal{F}_S(N_L(W(S))$.

Since $N_{\mathcal{F}}(W(S)) = \mathcal{F}_S(S)$ it follows that $N_L(W(S))$ has a p-complement; see Remark 6.10 in Appendix. According to the normal p-complement theorem of Thompson, 6.11 below, it follows that L has a p-complement. Therefore $\mathcal{F}_S(S) = \mathcal{F}_S(L)$ and we reached a contradiction with our assumption on $\mathcal{F}$. This concludes the proof of the Theorem 1.3.

6. Appendix

Let p be an odd prime, G a finite group and S a Sylow p-subgroup of G. We say that G is *p-stable* if and only if for every p-subgroup Q of G and every element x of $N_G(Q)$ such that $[Q, x, x] = 1$, we have that $xC_G(Q) \in O_p(N_G(Q)/C_G(Q))$.

A classic result of special significance to the theory of finite groups is Glauberman's ZJ-theorem [6]:

Theorem 6.1 (Glauberman). Let p be an odd prime. Let G be a finite, p-stable group such that $C_G(O_p(G)) \leq O_p(G)$. Then $Z(J(S))$ is a normal subgroup of G.

Using the following:

Proposition 6.2 (14.7, [7]). Assume that p is odd and that G is a finite group. Then the following conditions on G are equivalent:
(a) the group $Qd(p)$ is not involved in G;
(b) every section of G is p-stable.

the ZJ-theorem can be reformulated as follows:

Theorem 6.3 (Glauberman). Let p be an odd prime and let G be a $Qd(p)$-free finite group with $C_G(O_p(G)) \leq O_p(G)$. Then $Z(J(S))$ is a normal subgroup of G.

For $p = 2$ the ZJ-theorem does not hold anymore; see [7, Section 11]. As noted by Glauberman [7] a necessary and sufficient condition for every section of G to be 2-stable is that G have a normal 2-complement, which is too strong to be useful.

In a couple of papers [22, 24], Stellmacher proved an analogous version of Glauberman's ZJ-theorem, by constructing a characteristic subgroup $W(S)$ of S and extending the result for $p = 2$. An overview of his method, including a sketch of the proof for the odd prime case can be found in [13, Section 9.4]. The main theorem in [24] reads as follows:

Theorem 6.4 (Stellmacher). Let S be a nontrivial finite 2-group. Suppose that G is a finite group satisfying the following:
(I) G is Σ_4-free,
(II) $S \in \mathrm{Syl}_2(G)$ and $C_G(O_2(G)) \leq O_2(G)$,
(III) Every non-abelian simple section of G is isomorphic to $Sz(2^{2n+1})$ or $PSL_2(3^{2n+1})$.
Then there exists a nontrivial characteristic subgroup $W(S)$ of S which is normal in G.

Next, consider a couple of useful lemmas:

Lemma 6.5 (Chp. II, Lemma 2.3, [8]). The following conditions are equivalent:
(a) Σ_4 is involved in G;
(b) There exists a 2-subgroup Q of G such that Σ_3 is involved in $N_G(Q)/C_G(Q)$.

Lemma 6.6 (Chp. II, Corollary 7.3, [8]). Let G be a non-abelian simple group. The following are equivalent:

(a) G is Σ_3-free;
(b) G is isomorphic to $Sz(2^{2n+1})$ or $PSL(2, 3^{2n+1})$.

Remark 6.7. Note that if G is Σ_3-free then G is Σ_4-free. A finite group G with $C_G(O_2(G)) \leq O_2(G)$ is Σ_4-free if and only if $G/O_2(G)$ is Σ_3-free [24].

Using the previous two lemmas and remark, we can rephrase Stellmacher's Theorem 6.4 as follows:

Theorem 6.8 (Stellmacher). Let S be a finite nontrivial 2-group. Then there exists a nontrivial characteristic subgroup $W(S)$ of S which is normal in G, for every finite Σ_4-free group G with S a Sylow 2-subgroup and $C_G(O_2(G)) \leq O_2(G)$.

If $G = SO_{p'}(G)$, with S a Sylow p-subgroup of G, we say that G *has a normal p-complement*. A standard result due to Frobenius (see [7, 8.6] for example) is given below:

Theorem 6.9 (Frobenius). The following conditions are equivalent for a finite group G with Sylow p-subgroup S:
(a) G has a normal p-complement;
(b) if Q is a non-identity subgroup of G then $N_G(Q)/C_G(Q)$ is a p-group;
(c) if Q is a non-identity p-subgroup of G then $N_G(Q)$ has a normal p-complement;
(d) if two elements of S are conjugate in G, they are conjugate in S.

Remark 6.10. The equivalence (a) $\Leftrightarrow$ (d) in the above theorem, states that G has a normal p-complement if and only if S controls fusion in G. In the language of fusion systems, S controls G fusion if and only if $\mathcal{F}_S(S) = \mathcal{F}_S(G)$.

For odd primes Frobenius' result was improved by a result of Thompson. We give below a version of Thompson's p-complement theorem which uses Stellmacher's characteristic subgroup $W(S)$:

Theorem 6.11. [13, 9.4.7] Let G be a group, p an odd prime, and $S \in \mathrm{Syl}_p(G)$. Then G has a normal p-complement if and only if $N_G(W(S))$ has a normal p-complement.

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