CONTROL OF FUSION IN FUSION SYSTEMS

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ABSTRACT. Let P be a p-group and $\mathcal{F}$ a fusion system on P. The aim of this paper is to give necessary and sufficient conditions on a subgroup Q of P for the normalizer $N_{\mathcal{F}}(Q)$ to be $\mathcal{F}$ itself. This generalizes a result of Gilotti and Serena on finite groups. As an application we find some classes of resistant p-groups, which are p-groups P such that the normalizer $N_{\mathcal{F}}(P)$ is equal to $\mathcal{F}$, for any fusion system $\mathcal{F}$ on P.

1. Introduction

Let p be a prime number, G a finite group and $F_p(G)$ the Frobenius category of G at p. Recall that Frobenius category of G at p is the category whose objects are the p-subgroups of G and whose morphisms are the morphisms given by conjugation by the elements of G. This category contains the p-local information of G. One can prove that Frobenius category is equivalent to its full subcategory $F_P(G)$ whose objects are the subgroups of a Sylow p-subgroup P of G.

In the '90s, Puig gave an axiomatic description for the p-local structures, generalizing the notions of Frobenius category and Brauer category. The notes in french in which Puig introduced these new concepts were revisited, and refreshed by the author in 2001 [Pu2]. The new concepts that Puig introduced are the 'full Frobenius systems' on a p-group P.

In a recent article, Broto, Levi and Oliver [BLO] identified and studied a certain class of spaces which in many ways behave like p-completed classifying spaces of finite groups. They show that these spaces occur as the "classifying spaces" of fusion systems. In fact, in the paper, they use yet another terminology which is 'saturated fusion system' and the definition they give is slightly different from that of Puig for full Frobenius systems; in the appendix of the up quoted paper they prove that the two definition are equivalent. In this paper we introduce a simplification of the definition in Broto, Levi and Oliver's paper, equivalent to the latter and, thus, also to the definition in Puig's paper. We also change the terminology form 'saturated fusion systems' to simply 'fusion systems'.

When P is a Sylow p-subgroup of a finite group G and Q a subgroup of P, Gilotti and Serena [GS] found necessary and sufficient conditions on Q for the normalizer $N_G(Q)$ to control the p-fusion in G. Recall that a subgroup H of G is said to control p-fusion in G if H contains a Sylow p-subgroup of G and the canonical inclusion of H in G induces an equivalence of categories between $F_p(H)$ and $F_p(G)$. Let $\mathcal{F}$ be a fusion system on a p-group P. As a generalization of Gilotti and Serena's result, in this paper we give necessary and sufficient conditions on a subgroup Q of P for

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the normalizer $N_{\mathcal{F}}(Q)$ to be $\mathcal{F}$ itself. This gives a new method to track resistant p-groups, which are p-groups P such that any fusion system $\mathcal{F}$ on P is equal to the normalizer $N_{\mathcal{F}}(P)$.

The paper is organized as follows. The second section is dedicated to the definition and some properties of fusion systems. For a fusion system $\mathcal{F}$ on a p-group P the set of $\mathcal{F}$-essential subgroups of P is of great interest as any morphism in $\mathcal{F}$ can be written as a composition of restrictions of $\mathcal{F}$-automorphisms of P and of its $\mathcal{F}$-essential subgroups. Section 3 contains some examples of classes of p-groups that are not realizable as $\mathcal{F}$-essential p-subgroups in any fusion system $\mathcal{F}$. Section 4 gives the generalization of Gilotti and Serena's theorem to fusion systems. The theorem in Section 4 and the results in Section 3 are used in Section 5 to find families of resistant of p-groups. The last section concentrates on the notion of normal subsystem of a fusion system $\mathcal{F}$ and gives equivalent conditions for the existence of a such normal subsystem in a particular case.

2. Fusion Systems

Fusion systems were introduced by Puig in 1990 [Pu2]. In 2000 Broto, Levi and Oliver [BLO] had a different definition of the fusion systems which they have proved to be equivalent to Puig's definition. In this paper we use a simplified definition which we find more elegant, equivalent to the above ones.

Let us start with a more general definition.

Definition 2.1. *A category $\mathcal{F}$ on a p-group P is a category whose objects are the subgroups of P and whose set of morphisms between the subgroups Q and R of P, is a set $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ of injective group homomorphisms from Q to R, with the following properties:*
(1) if $Q \leq R$ then the inclusion of Q in R is a morphism in $\mathrm{Hom}_{\mathcal{F}}(Q,R)$;
(2) for any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ the induced isomorphism $Q \simeq \phi(Q)$ and its inverse are morphisms in $\mathcal{F}$.
(3) the composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

Note that the above definition of a category on P differs from what Puig calls 'divisible Frobenius system' and what, equivalently, Broto, Levi and Oliver call 'fusion system' by the fact that we do not ask for the inner automorphisms of P to be in the category.

In a finite group G having P as a Sylow p-subgroup, every G-conjugation class of subgroups in P contains an element Q such that a Sylow p-subgroup of the G-normalizer of Q is contained in P. We give the candidates to have this property in a category $\mathcal{F}$ on P. If there exists an isomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ we say that Q and R are $\mathcal{F}$-conjugate.

Definition 2.2. *We say that a subgroup Q of P is fully $\mathcal{F}$-centralized, respectively fully $\mathcal{F}$-normalized if $|C_P(Q)| \geq |C_P(Q')|$, respectively $|N_P(Q)| \geq |N_P(Q')|$, for all $Q' \leq P$ which are $\mathcal{F}$-conjugated to Q.*

For Q, R, $T \leq P$ we denote $\mathrm{Hom}_T(Q,R) := \{u \in T \,|\, {}^uQ \leq R\}/C_T(Q)$ and $\mathrm{Aut}_T(Q) := \mathrm{Hom}_T(Q,Q)$. Other useful notations are $\mathrm{Aut}_{\mathcal{F}}(Q) := \mathrm{Hom}_{\mathcal{F}}(Q,Q)$ and $\mathrm{Out}_{\mathcal{F}}(Q) := \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$. We are now able to give the definition of a fusion system.

Definition 2.3. *A fusion system on a finite p-group P is a category $\mathcal{F}$ on P satisfying the following properties:*
FS1. $\mathrm{Hom}_P(Q,R) \subset \mathrm{Hom}_{\mathcal{F}}(Q,R)$ *for all* $Q, R \leq P$.
FS2. $\mathrm{Aut}_P(P)$ *is a Sylow p-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(P)$
FS3. Every $\phi : Q \to P$ such that $\phi(Q)$ is fully $\mathcal{F}$-normalized extends to a morphism $\bar{\phi} : N_\phi \to P$ *where* $N_\phi = \{x \in N_P(Q) \mid \exists y \in N_P(\phi(Q)),\ \phi({}^x u) = {}^y\phi(u),\ \forall u \in Q\}$.

We remark that N_ϕ is the largest subgroup of $N_P(Q)$ such that ${}^\phi(N_\phi/C_P(Q)) \leq \mathrm{Aut}_P(\phi(Q))$. Thus we have $QC_P(Q) \leq N_\phi$.

Through the rest of the section P denotes a finite p-group, Q a subgroup of P and $\mathcal{F}$ a fusion system on P. Here is an equivalent characterization of being fully $\mathcal{F}$-normalized.

Proposition 2.4 ([Li], Prop. 1.6)**.** *A subgroup Q of P is fully $\mathcal{F}$-normalized if and only if Q is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.*

In a fusion system we have analogous notions for the normalizer and the centralizer in a finite group:

Definition 2.5. *The normalizer $N_{\mathcal{F}}(Q)$ is the category on $N_P(Q)$ having as morphisms, those morphisms $\psi \in \mathcal{F}(R,T)$ satisfying that there exists a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(QR,QT)$ such that $\phi|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\phi|_R = \psi$. The centralizer $C_{\mathcal{F}}(Q)$ is the category on $C_P(Q)$ having as morphisms those morphisms $\psi \in \mathcal{F}(R,T)$ satisfying that there exists a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(QR,QT)$ such that $\phi|_Q = \mathrm{id}_Q$ and $\phi|_R = \psi$.*

In general $N_{\mathcal{F}}(Q)$ is not a fusion system on $N_P(Q)$ because Property FS2 fails to be satisfied, but it becomes one if Q is fully $\mathcal{F}$-normalized. It is the same for $C_{\mathcal{F}}(Q)$ when Q is fully $\mathcal{F}$-centralized.

Proposition 2.6 ([Pu2], Prop. 2.8)**.** *If Q is fully $\mathcal{F}$-normalized then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$. If Q is fully $\mathcal{F}$-centralized then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_P(Q)$.*

Here is some more terminology in a fusion system.

Definition 2.7. *We say that*
(i) Q is $\mathcal{F}$-centric if $C_P(\phi(Q)) \subset \phi(Q)$, for all $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$.
(ii) Q is $\mathcal{F}$-radical if $O_p(\mathrm{Out}_{\mathcal{F}}(Q)) = 1$.
(iii) Q is $\mathcal{F}$-essential if Q is $\mathcal{F}$-centric and $\mathrm{Out}_{\mathcal{F}}(Q)$ has a strongly p-embedded subgroup M (that is M contains a Sylow p-subgroup S of $\mathrm{Out}_{\mathcal{F}}(Q)$ such that ${}^\phi S \cap S = \{1\}$ for every $\phi \in \mathrm{Out}_{\mathcal{F}}(Q) \setminus M$).
(iv) Q is strongly $\mathcal{F}$-closed if for any subgroup R of Q and any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R,P)$ we have $\phi(R) \leq Q$.
(v) Q is weakly $\mathcal{F}$-closed if for any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$ we have $\phi(Q) = Q$.

An $\mathcal{F}$-centric subgroup Q of P is fully $\mathcal{F}$-centralized. Indeed, for any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$, we have

$$\phi(C_P(Q)) = \phi(Z(Q)) = Z(\phi(Q)) = C_P(\phi(Q))$$

so $|C_P(Q)| = |C_P(\phi(Q))|$ giving that all the subgroups of P in the same $\mathcal{F}$-conjugacy class as Q have the same cardinality centralizer in P. Thus they are all fully $\mathcal{F}$-centralized.

We can see if a fully $\mathcal{F}$-centralized subgroup Q of P is $\mathcal{F}$-essential by studying the Quillen complex of the outer automorphism group of Q in $\mathcal{F}$. Recall that for a finite group G the Quillen complex of G at p, denoted by $\mathcal{S}_p(G)$ is the nerve of the poset of non-trivial p-subgroups of G. Thévenaz showed [Th, Theorem 48.8] that Q is $\mathcal{F}$-essential if and only if $\mathcal{S}_p(\mathrm{Out}_{\mathcal{F}}(Q))$ is disconnected. As any non-trivial normal subgroup of $\mathrm{Out}_{\mathcal{F}}(Q)$ would connect $\mathcal{S}_p(\mathrm{Out}_{\mathcal{F}}(Q))$, we have that if Q is $\mathcal{F}$-essential then $O_p(\mathrm{Out}_{\mathcal{F}}(Q)) = 1$ giving that Q is $\mathcal{F}$-radical.

Another easy remark is that Q is strongly $\mathcal{F}$-closed if and only if $\mathrm{Hom}_{\mathcal{F}}(R, P) = \mathrm{Hom}_{\mathcal{F}}(R, Q)$, for any subgroup R of Q, and is weakly $\mathcal{F}$-closed if and only if $\mathrm{Hom}_{\mathcal{F}}(Q, P) = \mathrm{Aut}_{\mathcal{F}}(Q)$. It is clear that if Q is strongly $\mathcal{F}$-closed then Q is weakly $\mathcal{F}$-closed.

Alperin's theorem on p-local control of fusion, also holds for fusion systems. First we set up this theorem's notations and terminology. If $\phi \in \mathrm{Aut}_{\mathcal{F}}(P)$, we say that ϕ is a maximal $\mathcal{F}$-automorphism; if $\phi \in \mathrm{Aut}_{\mathcal{F}}(E)$, where E is an $\mathcal{F}$-essential subgroup of P, we say that ϕ is an essential $\mathcal{F}$-automorphism. Alperin's fusion theorem asserts in essence that the essential and maximal $\mathcal{F}$-automorphisms suffice to determine the whole fusion system $\mathcal{F}$. The principal ideas of the proof are from [Th, Theorem 48.3].

Theorem 2.8 (Alperin)**.** *Any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ can be written as the composition of restrictions of essential $\mathcal{F}$-automorphisms, followed by the restriction of a maximal $\mathcal{F}$-automorphism. More precisely, there exists*

(a) an integer $n \geq 0$,

(b) a set of $\mathcal{F}$-isomorphic subgroups of P, $Q = Q_0$, Q_1, $\ldots$, Q_n, $Q_{n+1} = \phi(Q)$,

(c) a set of $\mathcal{F}$-essential, fully $\mathcal{F}$-normalized subgroups E_i containing Q_{i-1} and Q_i, for all $1 \leq i \leq n$,

(d) a set of morphisms $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(E_i)$ satisfying $\psi_i(Q_{i-1}) = Q_i$, for all $1 \leq i \leq n$ and

(e) a morphism $\psi_{n+1} \in \mathrm{Aut}_{\mathcal{F}}(P)$ satisfying $\psi_i(Q_n) = Q_{n+1}$

such that we have

$$\phi(u) = \psi_{n+1}\psi_n \ldots \psi_1(u), \text{ for all } u \in Q\,.$$

Proof. The proof is given by induction on the index $|P : Q|$. If $|P : Q| = 1$, then $n = 0$, $P = Q$ and $\phi \in \mathrm{Aut}_{\mathcal{F}}(P)$. Suppose now that $|P : Q| > 1$. Let $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$, such that $\psi(Q)$ is fully $\mathcal{F}$-normalized.

Now, $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, \psi(Q))$ and $\psi\phi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\phi(Q), \psi(Q))$ are isomorphisms mapping to $\psi(Q)$, which is a fully $\mathcal{F}$-normalized subgroup of P. So, it suffices to find the decomposition of the theorem for an $\mathcal{F}$-isomorphism which maps to a fully $\mathcal{F}$-normalized subgroup of P, as if we find this decomposition for ψ and for $\psi\phi^{-1}$, we find one for ϕ.

So we want to decompose a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$, such that $\psi(Q)$ is fully $\mathcal{F}$-normalized. Now ${}^{\psi}\mathrm{Aut}_P(Q) \leq \mathrm{Aut}_{\mathcal{F}}(\psi(Q))$ and $\mathrm{Aut}_P(\psi(Q))$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(\psi(Q))$, by Proposition 2.4. So there is $\chi \in \mathrm{Aut}_{\mathcal{F}}(\psi(Q))$ such that ${}^{\chi\psi}\mathrm{Aut}_P(Q) \leq \mathrm{Aut}_P(\psi(Q))$. This implies that $N_{\chi\psi} = N_P(Q)$. By Property FS3 applied to $\chi\psi$, we have a morphism $\rho \in \mathrm{Hom}_{\mathcal{F}}(N_P(Q), P)$ extending $\chi\psi$. Now $|P : N_P(Q)| < |P : Q|$ so, by induction, ρ has the desired form. So, for decomposing ψ, it suffices to decompose $\chi \in \mathrm{Aut}_{\mathcal{F}}(\psi(Q))$.

In this way, by changing if necessary the notations, we have to decompose $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ for a fully $\mathcal{F}$-normalized subgroup Q of P. Applying Property FS3 to χ, there exists $\chi' \in \mathrm{Hom}_{\mathcal{F}}(N_\chi, P)$ extending χ. But N_χ contains $QC_P(Q)$. If Q is not $\mathcal{F}$-centric, then $|P : Q| > |P : QC_P(Q)|$ and we obtain the decomposition of χ' by induction. Thus we can suppose that Q is $\mathcal{F}$-centric.

If Q is $\mathcal{F}$-essential, then $\chi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and χ is of the desired form, as Q is fully $\mathcal{F}$-normalized. If not, $\mathcal{S}_p(\mathrm{Out}_{\mathcal{F}}(Q))$ is connected. As the p-subgroups of $\mathrm{Out}_{\mathcal{F}}(Q)$ are in bijection with those of $\mathrm{Aut}_{\mathcal{F}}(Q)$ containing $\mathrm{Aut}_Q(Q)$ we have also that $\mathcal{S}_p(\mathrm{Aut}_{\mathcal{F}}(Q))_{>\mathrm{Aut}_Q(Q)}$ is connected. Moreover $T := \mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$, by Proposition 2.4, as Q is fully $\mathcal{F}$-normalized. By the connectivity of $\mathcal{S}_p(\mathrm{Aut}_{\mathcal{F}}(Q))_{>\mathrm{Aut}_Q(Q)}$, there exists $T_1, \dots, T_{r-1}$ and $S_1, \dots, S_r$ p-subgroups of $\mathrm{Aut}_{\mathcal{F}}(Q)$ containing strictly $\mathrm{Aut}_Q(Q)$ such that $T \geq S_1 \leq T_1 \geq S_2 \leq T_2 \geq \dots \leq T_{r-1} \geq S_r \leq {}^{\chi}T$. Replacing, if necessary, T_i by a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ containing T_i for all $1 \leq i \leq r-1$ we suppose that $T_1, \dots, T_{r-1}$ are Sylow p-subgroups of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Now all the Sylow p-subgroups of $\mathrm{Aut}_{\mathcal{F}}(Q)$ are conjugated by elements of $\mathrm{Aut}_{\mathcal{F}}(Q)$ so there exists $\chi_i \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $T_i = {}^{\chi_i}T$ for all $1 \leq i \leq r-1$. Thus T and ${}^{\chi}T$ are connected through:

$$T \geq S_1 \leq {}^{\chi_1}T \geq S_2 \leq \dots {}^{\chi_{r-1}}T \geq S_r \leq {}^{\chi}T$$

Now, by setting $\chi_0 := \mathrm{id}$ and $\chi_r := \chi$, we have ${}^{\chi_{i-1}}S_i \leq T \geq {}^{\chi_i}S_i$ for all $1 \leq i \leq r$. So, if we denote $\theta_i := \chi_i\chi_{i-1}^{-1} \in \mathrm{Aut}_{\mathcal{F}}(Q)$, this implies that $N_{\theta_i}/Z(Q)$ contains ${}^{\chi_{i-1}}S_i$ by definition of N_{θ_i} as ${}^{\theta_i}({}^{\chi_{i-1}}S_i) = {}^{\chi_i}S_i \leq T$. Let R be the inverse image of ${}^{\chi_{i-1}}S_i$ in $N_P(Q)$. So R is contained in N_{θ_i} and contains strictly Q. As Q is fully $\mathcal{F}$-normalized, θ_i extends to $\tilde{\theta}_i \in \mathrm{Hom}_{\mathcal{F}}(N_{\theta_i}, P)$. Now the index of R in P is smaller then the index of Q in P so by induction $\tilde{\theta}_i|_R$ decomposes as in the text of the theorem. This is true for all $1 \leq i \leq r$ so, implicitly, for $\chi = \theta_r \dots \theta_1$.

The last step is to see that in this decomposition a maximal $\mathcal{F}$-automorphism commutes with an essential $\mathcal{F}$-automorphism. This gives that we can use only one maximal $\mathcal{F}$-automorphism at the left in the decomposition. Indeed, if $\nu \in \mathrm{Aut}_{\mathcal{F}}(E)$, where E is an $\mathcal{F}$-essential, fully $\mathcal{F}$-normalized subgroup and $\theta \in \mathrm{Aut}_{\mathcal{F}}(P)$, we have $\nu\theta(u) = \theta(\theta^{-1}\nu\theta)(u)$, for all $u \in \theta^{-1}(E)$ where $(\theta^{-1}\nu\theta) \in \mathrm{Hom}_{\mathcal{F}}(\theta^{-1}(E))$ and $\theta^{-1}(E)$ is an $\mathcal{F}$-essential, fully $\mathcal{F}$-normalized subgroup, as the image of E by a morphism $\mu \in \mathrm{Hom}_{\mathcal{F}}(R, P)$, for all R containing $N_P(E)$, is a fully $\mathcal{F}$-normalized subgroup of P. Indeed we have $\mu(N_P(E)) = N_P(\mu(E))$ as E is fully $\mathcal{F}$-normalized, so $|N_P(\mu(E))|$ is maximal. □

A proof of this theorem in a different axiom setting was given by Puig [Pu2, Corollary 3.9] and another in a less general form, using $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups instead of $\mathcal{F}$-essential subgroups, can be found in [BLO, Theorem A.10].

We close the section with a classical example of fusion system. Given a finite group G, a finite field k of characteristic p and a block b of kG, Alperin and Broué [AB] showed that the Brauer full subcategory under a maximal b-Brauer pair satisfies properties similar to those in Sylow's theorems. In fact the Brauer full subcategory under a maximal b-Brauer pair is a fusion system on the first component of the maximal b-Brauer pair.

Proposition 2.9 ([Li], Theorem 2.4). *Let G be a finite group, k a field of characteristic p and b a block of kG. Let (P, e) be a maximal b-Brauer pair. Then the Brauer full subcategory $\mathcal{F}_{(P,e)}(G, b)$ under (P, e_P) is a fusion system on P.*

The full subcategory $F_S(G)$ under a Sylow p-subgroup S of G of the Frobenius category $F_p(G)$ is a particular case of a Brauer category. It is well-known from the literature that $F_S(G)$ is a fusion system on S; see for instance [BLO, Prop 1.3].

3. Are there any $\mathcal{F}$-essential p-subgroups?

An important result on the group of automorphisms of a finite p-group, proved by Martin [Ma] and then, Henn and Priddy [HP] is that, for almost all p-groups, their group of automorphisms, is a p-group. On the other hand, Gorenstein and Lyons [GL] and Aschbacher [As] found the complete list short of finite groups having a strongly p-embedded subgroup. The proof is based on the classification of the simple finite groups.

We remark that there are very restrictive conditions imposed by the existence of a strongly p-embedded subgroup and also that in almost all cases, if Q is a p-group, $\mathrm{Out}(Q)$ is a p-group. Thus it is very difficult to find a candidate for $\mathrm{Out}_{\mathcal{F}}(Q)$ having a strongly p-embedded subgroup and, for most p-groups Q, $\mathrm{Out}_{\mathcal{F}}(Q)$ is forced to be a p-group for any fusion system $\mathcal{F}$ on a p-group P containing Q. This indicates that almost all p-groups Q are not realizable as $\mathcal{F}$-essential subgroups in a fusion system $\mathcal{F}$.

In this section we search families of p-groups that are not realizable as $\mathcal{F}$-essential subgroups in a fusion system $\mathcal{F}$. For $p = 2$ there are some straight forward examples: the dihedral groups of order greater than 4, the quaternion groups of order greater than 8 and the semi-dihedral groups of order greater than 16 as their groups of automorphisms are 2-groups.

Our aim is to give a sufficient condition for a p-group to not be realizable as $\mathcal{F}$-essential subgroup in a fusion system $\mathcal{F}$. For this, we need some properties of the Frattini subgroup of a p-group, which is the subgroup generated by commutators and p-th powers. We denote the Frattini subgroup of Q by $\Phi(Q)$.

The Frattini subgroup of Q is characteristic in Q. The quotient of Q by $\Phi(Q)$ is an elementary (as we quotient by the p-th powers) abelian (as we quotient by the commutators) p-group of rank n (where n is the minimal cardinal of a system of generators of Q). So we can look at $Q/\Phi(Q)$ as a vector space of dimension n over the field $\mathbf{F}_p$. Consequently, $\mathrm{Aut}(Q/\Phi(Q))$ can be seen as the group $\mathrm{GL}_n(\mathbf{F}_p)$ of the nonsingular matrices of dimension n, with coefficients in $\mathbf{F}_p$. Here is a useful result on the relation between $\mathrm{Aut}(Q)$ and $\mathrm{Aut}(Q/\Phi(Q))$.

Proposition 3.1. *Let $\phi : \mathrm{Aut}(Q) \longrightarrow \mathrm{Aut}(Q/\Phi(Q))$ be the canonical map induced by the projection $Q \longrightarrow Q/\Phi(Q)$. Then $\mathrm{Ker}(\phi)$ is a p-group.*

Proof. A result of Burnside [Gr, Theorem 5.1.4], says that any automorphism of Q of order prime to p that induces the identity on $Q/\Phi(Q)$ is, in fact, the identity on Q. So $\mathrm{Ker}(\phi)$ does not contain any nontrivial element of order prime to p, which implies that $\mathrm{Ker}(\phi)$ is a p-group. □

Lemma 3.2. *Let p be a prime number, Q a p-group, $\Phi(Q)$ its Frattini subgroup, n the rank of the elementary abelian group $Q/\Phi(Q)$ and $B \leq \mathrm{GL}_n(\mathbf{F}_p)$ the multiplicative subgroup of nonsingular lower triangular matrices. Suppose that the image of the canonical morphism $\phi : \mathrm{Aut}(Q) \longrightarrow \mathrm{Aut}(Q/\Phi(Q))$ induced by the canonical projection is isomorphic to a subgroup of B. Then Q is not realizable as an $\mathcal{F}$-essential subgroup in any fusion system $\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a fusion system on a finite p-group P, such that $Q \leq P$. We denote by $\mathrm{Im}(\phi)$ the image and by $\mathrm{Ker}(\phi)$ the kernel of the morphism ϕ. We have that $\mathrm{Im}(\phi)$ is isomorphic to a subgroup of B, so it has a unique Sylow p-subgroup. As $\mathrm{Im}(\phi) \simeq \mathrm{Aut}(Q)/\mathrm{Ker}(\phi)$ and $\mathrm{Ker}(\phi)$ is a p-group, $\mathrm{Aut}(Q)$ has a unique Sylow p-subgroup. The same is true for $\mathrm{Out}(Q)$, as it is the quotient of $\mathrm{Aut}(Q)$ by the group of the inner automorphisms of Q, which is a p-group. As $\mathrm{Out}_{\mathcal{F}}(Q)$ is a subgroup of $\mathrm{Out}(Q)$ it has also a unique Sylow p-subgroup. So $\mathcal{S}_p(\mathrm{Out}_{\mathcal{F}}(Q))$ is connected and thus Q is not $\mathcal{F}$-essential. □

This result helps us to find other families of p-groups, that are not realizable as $\mathcal{F}$-essential subgroups in any fusion system $\mathcal{F}$, for example the direct product of two cyclic p-groups of different orders or a non-abelian metacyclic p-subgroup, for p odd.

Proposition 3.3. *Let p be a prime number and $\alpha_1 \neq \alpha_2$ two positive integers. Then $Q := C_{p^{\alpha_1}} \times C_{p^{\alpha_2}}$ is not realizable as an $\mathcal{F}$-essential subgroup in any fusion system $\mathcal{F}$.*

Proof. Let x and y be two generators of Q of orders p^{α_1}, respectively p^{α_2}. The set $\{x, y\}$ is a minimal generating system for Q, so $\mathrm{Aut}(Q/\Phi(Q))$ can be seen as embedded in $\mathrm{GL}_2(\mathbf{F}_p)$. Let σ be an automorphism of Q. We have $\sigma(x) = x^a y^b$ and $\sigma(y) = x^c y^d$, where a, b, c and d are integers. Thus the image of this automorphism in $\mathrm{GL}_2(\mathbf{F}_p)$ is given by $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$.

By Lemma 3.2, a sufficient condition, for Q to not be realizable as an $\mathcal{F}$-essential subgroup in a fusion system $\mathcal{F}$, is that the image of $\mathrm{Aut}(Q)$ in $\mathrm{Aut}(Q/\Phi(Q))$ is embedded in the group of the lower triangular matrices. The latter is equivalent to showing that, for all $\sigma \in \mathrm{Aut}(Q)$ as before, p divides b, which is $\bar{b} = 0$. Another sufficient condition for Q to not be realizable as an $\mathcal{F}$-essential subgroup, is that the image of $\mathrm{Aut}(Q)$ in $\mathrm{Aut}(Q/\Phi(Q))$ to be embedded in the group of the upper triangular matrices. This is equivalent to the condition that, for all $\sigma \in \mathrm{Aut}(Q)$ as before, p divides c, which is $\bar{c} = 0$.

Suppose, without loss of generality, that $\alpha_1 > \alpha_2$. Now, the order of $\sigma(y)$ is the same as the order of y, so equal to p^{α_2}. Thus $\sigma(y)^{p^{\alpha_2}} = 1$. As $y^{p^{\alpha_2}} = 1$, we deduce that $x^{cp^{\alpha_2}} = 1$. This implies that p divides c as the order of x is greater than p^{α_2}.

Similarly, if $\alpha_1 < \alpha_2$ we prove that p divides b. □

Definition 3.4. *We say that a p-group is metacyclic if it is the extension of a cyclic p-group by another cyclic p-group. If the metacyclic p-group can be defined by a split extension we say that it is split, otherwise we say that it is non-split.*

In other words, a p-group Q is metacyclic if it has a cyclic normal subgroup such that the quotient by this subgroup is also cyclic. It is well known (see [Hu]) that a presentation by generators and relations of a metacyclic p-group is given by

$$< x, y \,|\, x^{p^m} = 1, y^{p^n} = x^{p^q}, {}^{y}x = x^{p^l+1} >$$

We say then that Q is of the type (m, n, q, l).

Using Burnside's work [Bu], Dietz [Dz] classified the presentations of non-abelian metacyclic p-subgroup for p odd, and obtained that Q is non-split if and only if

$q \neq m$ and $l < q < n$. In the following we prove that non-abelian metacyclic p-groups, for p odd, cannot be realized as $\mathcal{F}$-essential subgroups in any fusion system $\mathcal{F}$.

Proposition 3.5. *Let p be an odd prime and $Q =< x, y >$ be a non-abelian metacyclic p-group. Then Q is not realizable as $\mathcal{F}$-essential subgroup in any fusion system $\mathcal{F}$.*

Proof. Suppose that Q is of type (m, n, q, l) and chose x and y such that $q = m$ if Q is split. The set $\{x, y\}$ is a minimal generator system for Q, so $\mathrm{Aut}(Q/\Phi(Q))$ can be seen as embedded in $\mathrm{GL}_2(\mathbf{F}_p)$. Let σ be an automorphism of Q. We have $\sigma(x) = x^a y^b$ and $\sigma(y) = x^c y^d$, where a, b, c and d are integers. As in the previous proof, for Q to not be realizable as $\mathcal{F}$-essential subgroup in a fusion system $\mathcal{F}$, it suffices to show that p divides b for all $\sigma \in \mathrm{Aut}(Q)$ or that p divides c for all $\sigma \in \mathrm{Aut}(Q)$. The relations in Q give that

$$\begin{aligned} &(1) \qquad (\sigma(x))^{p^m} = 1\,, \\ &(2) \qquad (\sigma(y))^{p^n} = (\sigma(x))^{p^q}\,, \\ &(3) \qquad {}^{\sigma(y)}\sigma(x) = (\sigma(x))^{p^l+1}\,. \end{aligned}$$

We compute now, using the generators, the powers of $\sigma(x)$ and of $\sigma(y)$, and ${}^{\sigma(y)}\sigma(x)$, by putting $\sigma(x) = x^a y^b$ and $\sigma(y) = x^c y^d$. These computations will be useful later in the proof. Let r and s be two positive integers.

$$\begin{aligned} (\sigma(x))^s &= (x^a y^b)^s = x^{a\sum_{i=0}^{s-1}(p^l+1)^{ib}} y^{bs}\,, \\ (\sigma(y))^r &= (x^c y^d)^r = x^{c\sum_{i=0}^{r-1}(p^l+1)^{id}} y^{dr}\,, \\ {}^{\sigma(y)}\sigma(x) &= (x^c y^d)(x^a y^b)(x^c y^d)^{-1} = x^c y^d x^a y^b y^{-d} x^{-c} \\ &= x^{c+a(p^l+1)^d - c(p^l+1)^b} y^b\,. \end{aligned}$$

We distinguish two cases.

If $l \geq n$, then Q is necessarily split. Combining the relation (2) with the identities computed above and using the fact that $q = m$, by the choice we made before, we obtain

$$x^{c\sum_{i=0}^{p^n-1}(p^l+1)^{id}} y^{dp^n} = (\sigma(x))^{p^m}\,.$$

But $(\sigma(x))^{p^m} = 1$, so $y^{dp^n} = 1$ and $x^{c\sum_{i=0}^{p^n-1}(p^l+1)^{id}} = 1$. The second equality is equivalent to $c\sum_{i=0}^{p^n-1}(p^l+1)^{id} \equiv 0 \pmod{p^m}$. As $(p^l+1)^d \equiv 1 \pmod{p}$, the sum $\sum_{i=0}^{p^n-1}(p^l+1)^{id}$ is exactly divisible by p^n (i.e. divisible by p^n and non-divisible by p^{n+1}). Now, Q is non-abelian, so $l < m$, which implies, as $l \geq n$, that $n < m$. This forces c to be a multiple of p.

If $l < n$, combine the relation (3) with the identities computed above and obtain

$$x^{c+a(p^l+1)^d - c(p^l+1)^b} y^b = x^{a\sum_{i=0}^{p^l}(p^l+1)^{ib}} y^{b(p^l+1)}\,.$$

This implies that $y^{b(p^l+1)} y^{-b} \in < x >$, which is equivalent to $bp^l \equiv 0 \pmod{p^n}$. Thus b is a multiple of p. □

4. Main result

As in Section 2, let $\mathcal{F}$ be a fusion system on a p-group P. If $\mathcal{F} = F_P(G)$ for a finite group G and H is a subgroup of G, containing P, then H controls the p-fusion in G if and only if $F_P(H) = F_P(G)$. Let us define, in general, the notion of subsystem of a fusion system and study when a subsystem can be equal to the whole fusion system.

Definition 4.1. *Let $\mathcal{F}$ be a fusion system on P. A subsystem $\mathcal{F}'$ of $\mathcal{F}$ is a subcategory of $\mathcal{F}$, which is itself a fusion system on a subgroup Q of P.*

In what follows, we give necessary and sufficient conditions on a fully $\mathcal{F}$-normalized subgroup Q of P, for the subsystem $N_{\mathcal{F}}(Q)$ to be equal to $\mathcal{F}$. Remark that if $\mathcal{F} = F_P(G)$, where P is a Sylow p-subgroup of G and Q is a fully $\mathcal{F}$-normalized subgroup of P, then $N_{\mathcal{F}}(Q) = F_{N_P(Q)}(N_G(Q))$. Indeed, $N_P(Q)$ is a Sylow p-subgroup of $N_G(Q)$. Moreover, any morphism in $F_{N_P(Q)}(N_G(Q))$ between two subgroups R and T of $N_P(Q)$ is given by the conjugation by an element h of $N_G(Q)$. As ${}^hQ = Q$ this can be regarded as a morphism by conjugation by h between $Q \cdot R$ and $Q \cdot T$. Thus $\mathrm{Hom}_{F_{N_P(Q)}(N_G(Q))}(R, T) \subset \mathrm{Hom}_{N_{\mathcal{F}}(Q)}(R, T)$. The inclusion in the other sense is obvious.

Let us recall what is a central series for a finite group.

Definition 4.2. *Let Q be a p-group. A central series for Q is a series $1 = Q_0 \leq Q_1 \leq Q_2 \leq \ldots \leq Q_n = Q$ such that Q_i is normal in Q and $Q_{i+1}/Q_i \leq Z(Q/Q_i)$ for all $1 \leq i \leq n$. If we have equality in the last relation for all $1 \leq i \leq n$ then the series is called the upper central series.*

For a fusion system $\mathcal{F}$ on P we can define the quotient $\overline{\mathcal{F}}$ of $\mathcal{F}$ by a strongly $\mathcal{F}$-closed subgroup Q of P.

Definition 4.3. *Let $\mathcal{F}$ be a fusion system on P and Q a strongly $\mathcal{F}$-closed subgroup of P. We define $\overline{\mathcal{F}} := \mathcal{F}/Q$ as the category on P/Q whose objects are the subgroups of P/Q and whose morphisms are those induced by $\mathcal{F}$.*

In fact if G is a finite group having P as Sylow p-subgroup, and Q is a subgroup of P normal in G, then Q is strongly $F_P(G)$-closed and, moreover $F_P(G)/Q = F_{P/Q}(G/Q)$. The fact that Q is strongly $F_P(G)$-closed is straight forward, as any morphism in $F_P(G)$ given by conjugation by an element g of G mapping from a subgroup R of Q extends in a morphism by conjugation by g which maps from Q and we have ${}^gR \leq {}^gQ = Q$ as G normalizes Q. The equality $F_P(G)/Q = F_{P/Q}(G/Q)$ comes simply from the fact that the objects and the morphisms of the two categories are trivially the same.

We prove now that $\mathcal{F}/Q$ is a fusion system. This result is due to Puig, but we prefer to give here the proof as our formulation is different from Puig's one. For any subgroup R of P denote by $\overline{R}$ its canonical projection in $\overline{P} := P/Q$.

Proposition 4.4 ([Pu2], Prop. 2.15)**.** *Let $\mathcal{F}$ be a fusion system on P and Q a strongly $\mathcal{F}$-closed subgroup of P. Then $\overline{\mathcal{F}} := \mathcal{F}/Q$ is a fusion system on P/Q.*

Proof. We verify the three properties for $\overline{\mathcal{F}}$. Property FS1 is trivially satisfied.

By construction of $\overline{\mathcal{F}}$, the canonical morphism $\mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}_{\overline{\mathcal{F}}}(\overline{P})$ is surjective and the image of $\mathrm{Aut}_P(P)$ by this morphism is $\mathrm{Aut}_{\overline{P}}(\overline{P})$. As $\mathcal{F}$ is a fusion system, the group $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$, so $\mathrm{Aut}_{\overline{P}}(\overline{P})$ is a Sylow p-subgroup of $\mathrm{Aut}_{\overline{\mathcal{F}}}(\overline{P})$, thus Property FS2 is satisfied in $\overline{\mathcal{F}}$.

We verify now Property FS3. Consider $\overline{\phi} \in \mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{R}, \overline{P})$, R the inverse image of $\overline{R}$ in P and $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ a representative of $\overline{\phi}$ in $\mathcal{F}$. Denote $T := \phi(R)$. Suppose that $\overline{T}$ is fully $\overline{\mathcal{F}}$-normalized. We have $Q \leq R$ so $Q \leq N_P(R)$ giving that $N_P(R)$ is the inverse image of $N_{\overline{P}}(\overline{R})$. The same is true for T as $Q = \phi(Q) \leq T$, and also for any subgroup R' of P, $\mathcal{F}$-conjugated to T. Thus, using that $\overline{T}$ is fully $\overline{\mathcal{F}}$-normalized we have $|N_P(R)'| = |N_{\overline{P}}(\overline{R}')| \cdot |Q| \leq |N_{\overline{P}}(\overline{T})| \cdot |Q| = |N_P(T)|$ so T is fully $\mathcal{F}$-normalized.

By Property FS3 applied to ϕ, there exists $\rho \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, P)$ extending ϕ. As Q is strongly $\mathcal{F}$-closed, we obtain $\rho(Q) = Q$, so $\overline{\rho}$ is a morphism in $\mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{N_\phi}, \overline{P})$. Moreover $\overline{N_\phi} = N_{\overline{\phi}}$. The inclusion of $\overline{N_\phi}$ in $N_{\overline{\phi}}$ is obvious. For the other inclusion denote by N the inverse image of $N_{\overline{\phi}}$ in P. The canonical projection on P/Q induces a morphism between $\mathrm{Aut}_P(R)$ and $\mathrm{Aut}_{\overline{P}}(\overline{R})$. We have that $\overline{{}^{\phi}(\mathrm{Aut}_N(R))} = {}^{\overline{\phi}}\mathrm{Aut}_{N_{\overline{\phi}}}(\overline{R}) \leq \mathrm{Aut}_{\overline{P}}(\overline{T})$ thus ${}^{\phi}(\mathrm{Aut}_N(R)) \leq \mathrm{Aut}_P(T)$ implying that $N \leq N_\phi$. So $\overline{\rho}$ extends $\overline{\phi}$ to $N_{\overline{\phi}}$ and Property FS3 is satisfied for $\overline{\mathcal{F}}$. □

Remark 4.5. Let $\mathcal{F}$ be a fusion system on P and Q a normal subgroup of P. Then $N_P(Q) = P$, so Q is fully $\mathcal{F}$-normalized. Even if Q is not necessarily strongly $\mathcal{F}$-closed, we have that, Q is strongly $\mathcal{N}$-closed where $\mathcal{N} = N_{\mathcal{F}}(Q)$. Indeed, for any subgroup $R \leq Q$ and any morphism $\phi \in \mathrm{Hom}_{\mathcal{N}}(R, N_P(Q))$ there exists a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q \cdot R, P) = \mathrm{Hom}_{\mathcal{F}}(Q, P)$ such that $\psi(u) = \phi(u)$, for all $u \in R$ and $\psi(Q) = Q$. As a consequence, we have $\phi(R) = \psi(R) \leq \psi(Q) = Q$ and as this is true for all the subgroups of Q, the latter is strongly $\mathcal{N}$-closed.

We continue with two technical lemmas.

Lemma 4.6. *Let Q a p-group, T a normal subgroup of Q and for any subgroup R of Q denote by $\overline{R}$ its canonical projection in $\overline{Q} := Q/T$. If the series given by $Q = Q_n \geq Q_{n-1} \geq \ldots \geq Q_1 \geq 1$ is a central series of Q and $T \leq Q_1$ then $\overline{Q} = \overline{Q_n} \geq \overline{Q_{n-1}} \geq \ldots \geq \overline{Q_1} \geq 1$ is a central series of $\overline{Q}$. If, moreover, $Q_1 = T$ and T is central in Q then the converse is also true.*

Proof. For the first part of the lemma we use the property that $\overline{Z(Q)} \leq Z(\overline{Q})$ and $\overline{Z(Q/Q_i)} \leq Z(\overline{Q}/\overline{Q_i})$. So the relations $\overline{Q_1} \leq Z(\overline{Q})$ and $\overline{Q_{i+1}}/\overline{Q_i} \leq Z(\overline{Q}/\overline{Q_i})$ are always satisfied, given that $\overline{Q_1} \leq \overline{Z(Q)}$ and $\overline{Q_{i+1}}/\overline{Q_i} = \overline{Q_{i+1}/Q_i} \leq \overline{Z(Q/Q_i)}$. We deduce that $\overline{Q} = \overline{Q_n} \geq \overline{Q_{n-1}} \geq \ldots \geq \overline{Q_1} \geq 1$ is a central series of $\overline{Q}$. For the converse, T is central in Q, so $Q_1 = T \leq Z(Q)$. The others verifications are straight forward, using the fact that $Q/Q_i \simeq \overline{Q}/\overline{Q_i}$ and $Q_{i+1}/Q_i \simeq \overline{Q_{i+1}}/\overline{Q_i}$, for all $i = 1, \ldots, n-1$. □

Lemma 4.7. *Let P be a p-group, $\mathcal{F}$ a fusion system on P and T a strongly $\mathcal{F}$-closed subgroup of P. Let $\overline{\mathcal{F}} := \mathcal{F}/T$, $Q \leq P$ and denote by $\overline{} : P \to P/T$ the canonical projection. If Q is strongly, respectively weakly $\mathcal{F}$-closed, then $\overline{Q}$ is strongly, respectively weakly $\overline{\mathcal{F}}$-closed. If Q contains T and $\overline{Q}$ is weakly $\overline{\mathcal{F}}$-closed, then Q is weakly $\mathcal{F}$-closed.*

Proof. By definition, Q is strongly $\mathcal{F}$-closed if and only if for all $R \leq Q$, we have $\mathrm{Hom}_{\mathcal{F}}(R, P) = \mathrm{Hom}_{\mathcal{F}}(R, Q)$. This implies that $\mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{R}, \overline{P}) = \mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{R}, \overline{Q})$, as the morphisms in $\overline{\mathcal{F}}$ are induced by the morphisms in $\mathcal{F}$. As any subgroup of $\overline{Q}$ has a preimage in Q, we have the equivalence with the fact that $\overline{Q}$ is strongly $\overline{\mathcal{F}}$-closed.

The fact that Q is weakly $\mathcal{F}$-closed, implies by the same type of argument that $\overline{Q}$ is weakly $\overline{\mathcal{F}}$-closed.

Suppose now that $T \leq Q$ and $\overline{Q}$ is weakly $\overline{\mathcal{F}}$-closed. Let $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$. It induces a morphism $\overline{\phi} \in \mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{Q},\overline{P})$. As $\overline{Q}$ is weakly closed in $\overline{\mathcal{F}}$ it follows that $\overline{\phi} \in \mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{Q})$. This makes that $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q)$, so Q is weakly $\mathcal{F}$-closed. □

In the middle of the '80s, Gilotti and Serena [GS] gave necessary and sufficient conditions for the normalizer of a p-subgroup of a finite group G to control the p-fusion in G. In the last part of this section we generalize this result to fusion systems. This is the main result of our paper.

Theorem 4.8. *Let P be a finite p-group, Q a subgroup of P and $\mathcal{F}$ a fusion system on P. Then $N_{\mathcal{F}}(Q) = \mathcal{F}$ if and only if Q is strongly $\mathcal{F}$-closed and admits a central series $Q = Q_n \geq Q_{n-1} \geq \ldots \geq Q_1 \geq 1$ with Q_i weakly $\mathcal{F}$-closed, for all $1 \leq i \leq n-1$.*

Proof. Put $\mathcal{N} := N_{\mathcal{F}}(Q)$.

$\Longrightarrow$ Suppose that $\mathcal{N} = \mathcal{F}$. Let R be a subgroup of Q and $\phi \in \mathrm{Hom}_{\mathcal{F}}(R,P)$. By hypothesis $\phi \in \mathrm{Hom}_{\mathcal{N}}(R,P)$, so the morphism ϕ extends to a morphism $\psi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Thus, $\phi(R) = \psi(R) \leq \psi(Q) = Q$, which gives that Q is strongly $\mathcal{F}$-closed. Consider the upper central series $Q = Q_n \geq Q_{n-1} \geq \ldots \geq Q_1 \geq 1$. Fix $1 \leq i \leq n-1$ and let $\phi_i \in \mathrm{Hom}_{\mathcal{F}}(Q_i,P)$. As before, we have $\phi_i \in \mathrm{Hom}_{\mathcal{N}}(Q_i,P)$. But Q_i is characteristic in Q, so $Q_i = \phi_i(Q_i)$, as ϕ lifts to a morphism in $\mathrm{Aut}_{\mathcal{F}}(Q)$. Thus Q_i is weakly $\mathcal{F}$-closed, for all $1 \leq i \leq n-1$.

$\Longleftarrow$ Suppose that the group Q is strongly $\mathcal{F}$-closed and admits a central series $Q = Q_n \geq Q_{n-1} \geq \ldots \geq Q_1 > 1$ where Q_i is weakly $\mathcal{F}$-closed for all $1 \leq i \leq n-1$. Given that Q_i is weakly $\mathcal{F}$-closed we have $Q_i \lhd P$, for all $1 \leq i \leq n$. In particular, $N_P(Q) = P$ so Q is fully $\mathcal{F}$-normalized and $\mathcal{N}$ is a fusion system on P by Proposition 2.6. Denote by $T := Q_1 \cap Z(P)$. As Q_1 is normal in P, we have that Q intersects nontrivialy the centre $Z(P)$, so T is not trivial.

We first prove that $\mathrm{Hom}_{\mathcal{F}}(T,P) = \mathrm{Hom}_{\mathcal{N}}(T,P)$. Let $\phi \in \mathrm{Hom}_{\mathcal{F}}(T,P)$. By the fact that Q is strongly $\mathcal{F}$-closed we have $\phi(T) \leq Q$. As $Q_1 \leq Z(Q)$, we obtain that T and $\phi(T)$ are subgroups of $C_P(Q_1)$ and, consequently $Q_1 \leq C_P(\phi(T))$. But T is fully $\mathcal{F}$-normalized, as $N_P(T) = P$. So ϕ^{-1} lifts to $\chi \in \mathrm{Hom}_{\mathcal{F}}(C_P(\phi(T)),P)$. Thus $\chi|_{Q_1} \in \mathrm{Hom}_{\mathcal{F}}(Q_1,P)$ and, by the weak $\mathcal{F}$-closure of Q_1, we have $\chi|_{Q_1} \in \mathrm{Aut}_{\mathcal{F}}(Q_1)$. As Q_1 is unique in its $\mathcal{F}$-conjugacy class in P, we deduce that Q_1 is fully $\mathcal{F}$-normalized. So $\chi|_{Q_1}$ extends, by Property FS3, to $\widetilde{\chi} \in \mathrm{Hom}_{\mathcal{F}}(C_P(Q_1),P)$. Now, $Q \leq C_P(Q_1)$, as $Q_1 \leq Z(Q)$. We obtain, by restricting χ to Q, that $\widetilde{\chi}|_Q \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$, so, by the strong $\mathcal{F}$-closure of Q, we have $\widetilde{\chi}|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$. This implies that $\widetilde{\chi}|_T$ is a morphism in $\mathrm{Hom}_{\mathcal{N}}(\phi(T),P)$. So $\phi = \chi^{-1}|_T = \widetilde{\chi}^{-1}|_T$ is in $\mathrm{Hom}_{\mathcal{N}}(T,P)$.

Back to the general case. By Alperin's fusion theorem $\mathcal{F} = \mathcal{N}$ if $\mathrm{Aut}_{\mathcal{F}}(U) = \mathrm{Aut}_{\mathcal{N}}(U)$, for all $\mathcal{F}$-centric, fully $\mathcal{F}$-normalized U. The proof is by induction on the number of morphisms in $\mathcal{F}$. If $\mathcal{F}$ is the trivial fusion system on P, then $\mathrm{Aut}_{\mathcal{F}}(P) = \{id\}$. As $\mathrm{Aut}_{\mathcal{F}}(P)$ contains the inner automorphisms of P we have $\mathrm{Aut}_P(P) = \{id\}$. Thus P is necessarily abelian and the theorem is straight forward.

Denote by $\mathcal{C} := C_{\mathcal{F}}(T)$, let U be $\mathcal{F}$-centric, fully $\mathcal{F}$-normalized and let $\phi \in \mathrm{Aut}_{\mathcal{F}}(U)$. As $C_P(T) = P$, the group T is fully $\mathcal{F}$-centralized, so $\mathcal{C}$ is a fusion system on P by Proposition 2.6. Denote by $\mathcal{N}' := N_{\mathcal{C}}(Q)$. As Q is fully $\mathcal{C}$-normalized, it follows that $\mathcal{N}'$ is a fusion system on P. We have clearly $\mathrm{Hom}_{\mathcal{N}'}(R,P) = \mathrm{Hom}_{\mathcal{N}}(R,P) \cap \mathrm{Hom}_{\mathcal{C}}(R,P)$ for any subgroup R of P. We distinguish two cases.

Case 1: $\mathcal{F} \neq \mathcal{C}$. As U is $\mathcal{F}$-centric, it contains $Z(P)$, so, a fortiori, T. Thus, we have $\phi|_T \in \mathrm{Hom}_{\mathcal{F}}(T,P)$. By Property FS3 using the fact that T is fully $\mathcal{F}$-normalized as $N_P(T) = P$, the morphism $\psi := \phi^{-1}|_{\phi(T)}$ lifts to $\widetilde{\psi} \in \mathrm{Hom}_{\mathcal{F}}(N_\psi, P)$. In order to prove that $\phi \in \mathrm{Aut}_{\mathcal{N}}(U)$ it is sufficient to show that $UQ \leq N_\psi$. In fact we prove that $UQ \leq C_P(\phi(T))$.

As ϕ is an automorphism of U and $Z(U)$ is characteristic in U we have that $\phi(Z(U)) \leq Z(U)$. But $T \leq Z(P) \leq Z(U)$ so $\phi(T) \leq \phi(Z(U)) = Z(U)$ which gives that $U \leq C_P(\phi(T))$ (*).

By the fact that Q is strongly $\mathcal{F}$-closed we have $\phi(T) \leq Q$. As $Q_1 \leq Z(Q)$, we obtain that $T, \phi(T) \leq C_P(Q_1)$ and so that $Q_1 \leq C_P(\phi(T))$. But T is fully $\mathcal{F}$-normalized, as $N_P(T) = P$. So ψ lifts to $\widetilde{\psi} \in \mathrm{Hom}_{\mathcal{F}}(C_P(\phi(T)), P)$. Thus $\widetilde{\psi}|_{Q_1} \in \mathrm{Hom}_{\mathcal{F}}(Q_1, P)$ and, by the weak $\mathcal{F}$-closure of Q_1, we have $\widetilde{\psi}|_{Q_1} \in \mathrm{Aut}_{\mathcal{F}}(Q_1)$. Now, $Q \leq C_P(Q_1)$, as $Q_1 \leq Z(Q)$. Moreover $\phi(T) \leq Q_1$ so $Q \leq C_P(\phi(T))$ (**).

By (*) and (**) $QU \leq C_P(\phi(T))$ so $\widetilde{\psi}$ is defined also on QU. Thus $\widetilde{\psi}\phi \in \mathrm{Hom}_{\mathcal{F}}(U,P) \cap \mathcal{C}(T)$. By the induction hypothesis applied to $\mathcal{C}$, we have $\widetilde{\psi}\phi \in \mathrm{Hom}_{\mathcal{N}'}(U,P)$. As $\mathrm{Hom}_{\mathcal{N}'}(U,P) \leq \mathrm{Hom}_{\mathcal{N}}(U,P)$ we obtain that $\widetilde{\psi}\phi \in \mathrm{Hom}_{\mathcal{N}}(U,P)$. But, $\widetilde{\psi}|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$ so $\widetilde{\psi} \in \mathrm{Hom}_{\mathcal{N}}(U,P)$, which implies that $\phi \in \mathrm{Hom}_{\mathcal{N}}(U,P)$ and finishes the proof in this case.

Case 2: $\mathcal{F} = \mathcal{C}$. In this case T is strongly $\mathcal{F}$-closed. Indeed let $R \leq T$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(R,P)$. As $\mathcal{F} = \mathcal{C}$ the morphism ϕ extends to $\psi \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $\psi = \mathrm{id}_T$ and so $\phi(R) = R \leq T$.

Denote by $\overline{\mathcal{F}} := \mathcal{F}/T$. Using Proposition 2.6, $\overline{\mathcal{F}}$ is a fusion system on $\overline{P}$. Moreover, by Lemma 4.6, $\overline{Q} = \overline{Q_n} \geq \overline{Q_{n-1}} \geq \ldots \geq \overline{Q_1} \geq 1$ is a central series of $\overline{Q}$ and, by Lemma 4.7, $\overline{Q}$ is strongly $\overline{\mathcal{F}}$-closed and $\overline{Q}_i$ is weakly $\overline{\mathcal{F}}$-closed, for all $1 \leq i \leq n-1$. Since T is not trivial, we have less morphisms in $\overline{\mathcal{F}}$ then in $\mathcal{F}$ and by induction hypothesis, $\overline{\mathcal{F}} = N_{\overline{\mathcal{F}}}(\overline{Q}) =: \overline{\mathcal{N}}$, so $\overline{\phi} \in \mathrm{Aut}_{\overline{\mathcal{N}}}(\overline{U})$. We lift $\overline{\phi}$ in $\psi \in \mathrm{Aut}_{\mathcal{N}}(U)$ as the canonical projection of $\mathrm{Aut}_{\mathcal{N}}(U)$ on $\mathrm{Aut}_{\overline{\mathcal{N}}}(\overline{U})$ is surjective. Moreover, for all $x \in U$ we have $\psi\phi^{-1}(x)x^{-1} \in T$. Denote by $K := \{\theta \in \mathrm{Aut}_{\mathcal{F}}(U) | \theta(x)x^{-1} \in T, \text{ for all } x \in U\}$. Now, K is a subgroup of $\mathrm{Aut}(U)$ which centralizes U/T by construction and T by the fact that $\mathcal{F} = \mathcal{C}$. Thus K centralizes the quotients of the normal series $1 \triangleleft T \triangleleft U$. This implies that K is a p-group. Moreover, K is a normal subgroup of $\mathrm{Aut}_{\mathcal{F}}(U)$, as, for all $\chi \in \mathrm{Aut}_{\mathcal{F}}(U)$ and $\theta \in K$, we have

$$({}^{\chi}\theta)(x)x^{-1} = \chi\big(\theta(y)y^{-1}\big) \in \chi(T) = T \quad \text{ where } y = \chi^{-1}(x), \text{ for all } x \in U\,.$$

As U is fully $\mathcal{F}$-normalized, $\mathrm{Aut}_P(U)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(U)$ by Proposition 2.4. Moreover, K is a normal subgroup of $\mathrm{Aut}_{\mathcal{F}}(U)$, so $K \leq \mathrm{Aut}_P(U)$. As $\psi\phi^{-1} \in K$, there exists $u \in P$ such that $\psi\phi^{-1}(x) = {}^{u}x$, for all $x \in U$. Given that Q is normal in P, we obtain $\mathrm{conj}(u)(Q) = Q$ and thus $\mathrm{conj}(u) \in \mathrm{Aut}_{\mathcal{N}}(U)$. Finally $\phi = \mathrm{conj}(u^{-1})\psi \in \mathrm{Aut}_{\mathcal{N}}(U)$. □

Remark 4.9. In the particular case when $Q = P$ is a Sylow p-subgroup of a finite group G, Martino and Priddy [MP, Theorem 4.1] give an equivalent condition for the normalizer of P to control p-fusion in G. The condition is that the subgroups $\Omega_k(P)$ of P, generated by the elements of order p^k or less, form a central series for

P. As the subgroups $\Omega_k(P)$ are unique in P, they are a fortiori weakly $F_P(G)$-closed. Thus Martino and Priddy's result can be obtained as a corollary of Theorem 4.8.

5. Resistant groups

The theorem in Section 4 gives in particular necessary and sufficient conditions on a finite p-group P to have $\mathcal{F} = N_{\mathcal{F}}(P)$ for a fusion system $\mathcal{F}$ on P. When the last equality is true for all fusion systems on P we say that P is resistant. This is a generalization of the notion of 'resitant groups' we introduced in the special case of fusion systems of finite groups [St, Definition 2.3]. By a theorem of Mislin [Mi] the latter notion is equivalent to what Martino and Priddy [MP] call 'Swan groups'. Recall the P is a Swan group if for any finite group G having P as a Sylow p-subgroup, the restriction gives an isomorphism between the mod-p cohomology rings $H^*(G)$ and $H^*(N_G(P))$.

Definition 5.1. *We say that a p-group P is resistant if for any fusion system $\mathcal{F}$ on P, we have $\mathcal{F} = N_{\mathcal{F}}(P)$.*

By Alperin's theorem a p-group P is resistant if all its subgroups are not realizable as $\mathcal{F}$-essential subgroups for any fusion system $\mathcal{F}$ on P. One can prove [Pu1, Ch. IV, Proposition 2] that the latter condition is also necessary. Theorem 4.8 gives a new criteria for P to be resistant; that is if and only if P possesses a central series composed of weakly $\mathcal{F}$-closed subgroups, for any fusion system $\mathcal{F}$ on P. The condition that P is strongly $\mathcal{F}$-closed is trivially satisfied. Abelian p-groups are resistant as the upper central series is reduced to P. The following lemma is an important ingredient in finding families of resistant groups.

Lemma 5.2. *Let p be an odd prime, Q an elementary abelian p-group of order p^2 and H a subgroup of the group of automorphisms of Q. Suppose that H has a nontrivial Sylow p-subgroup R and let P be a p-group such that the short sequence $1 \longrightarrow Q \longrightarrow P \longrightarrow R \longrightarrow 1$ is exact.*

a) Suppose that for any morphism $\phi \in N_H(R)$, there exists an automorphism ψ of P, such that $\psi(u) = \phi(u)$, for all $u \in Q$ and $\psi\rho\psi^{-1} = \phi\rho\phi^{-1}$, for all $\rho \in R$. Then the short exact sequence $1 \longrightarrow Q \longrightarrow P \longrightarrow R \longrightarrow 1$ extends to a short exact sequence $1 \longrightarrow Q \longrightarrow L \longrightarrow H \longrightarrow 1$.

b) Suppose that H has at least two distinct Sylow p-subgroups and the short exact sequence $1 \longrightarrow Q \longrightarrow P \longrightarrow R \longrightarrow 1$ extends to a short exact sequence $1 \longrightarrow Q \longrightarrow L \longrightarrow H \longrightarrow 1$. Then P is not isomorphic to $C_{p^2} \rtimes C_p$.

Proof. a) To show that the short exact sequence $1 \longrightarrow Q \longrightarrow P \longrightarrow R \longrightarrow 1$ extends to the short exact sequence $1 \longrightarrow Q \longrightarrow L \longrightarrow H \longrightarrow 1$, it suffices to prove [Br, pp. 84-85] that the cohomology class α, determined by P in $H^2(R, Q)$, is H-stable, i.e. for all $\phi \in H$ we have

$$\mathrm{Res}^{R}_{R\cap {}^{\phi}R}\, \alpha = \mathrm{Res}^{{}^{\phi}R}_{R\cap {}^{\phi}R}\, \mathrm{c}^*_{\phi}(\alpha) \quad (\bullet)\,.$$

Here Res is the restriction in cohomology and c^*_{ϕ} is the morphism induced in cohomology by the conjugation by ϕ. If $R \neq {}^{\phi}R$ then $R \cap {}^{\phi}R = 1$, as R is a p-group of order p, and the relation $(\bullet)$ is clearly is satisfied. Suppose that $R = {}^{\phi}R$. By the hypothesis, there exists an automorphism ψ of P such that $\psi(u) = \phi(u)$, for all $u \in Q$ and $\psi\rho\psi^{-1} = \phi\rho\phi^{-1}$ for all $\rho \in R$. This implies that $\alpha = \mathrm{c}^*_{\phi}(\alpha)$ and

($\bullet$) is once again satisfied. Indeed, let $1 \longrightarrow Q \longrightarrow P' \longrightarrow R \longrightarrow 1$ be an exact short sequence representing the class $\mathrm{c}_\phi^*(\alpha) \in H^2(R,Q)$. By the definition of c_ϕ^* (see [Br, p. 80]), there exists an isomorphism $f : P \longrightarrow P'$ such that the next diagram commutes

$$\begin{array}{cccccccccc} (\alpha) & 1 & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & R & \longrightarrow & 1 \\ & & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{conj}(\phi)} & & \\ (\mathrm{c}_\phi^*(\alpha)) & 1 & \longrightarrow & Q & \longrightarrow & P' & \longrightarrow & R & \longrightarrow & 1 \end{array}$$

On the other hand, the condition on the existence of ψ gives the next commutative diagram.

$$\begin{array}{cccccccccc} (\alpha) & 1 & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & R & \longrightarrow & 1 \\ & & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{conj}(\phi)} & & \\ (\alpha) & 1 & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & R & \longrightarrow & 1 \end{array}$$

Taking $g := f\psi^{-1} : P \longrightarrow P'$ we obtain

$$\begin{array}{cccccccccc} (\alpha) & 1 & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & R & \longrightarrow & 1 \\ & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f\psi^{-1}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ (\mathrm{c}_\phi^*(\alpha)) & 1 & \longrightarrow & Q & \longrightarrow & P' & \longrightarrow & R & \longrightarrow & 1 \end{array}$$

which gives that $\alpha = \mathrm{c}_\phi^*(\alpha)$.

b) The proof of the second part of the lemma is due to Jacques Thévenaz. It is based on the fact that $H^2(\mathrm{SL}_2(\mathbf{F}_p), Q) = 0$, as we already showed in [St]. Now, by the first part of the proof, α extends to a cohomology class in $H^2(H,Q)$. Moreover H contains $\mathrm{SL}_2(\mathbf{F}_p)$, given that H has at least two Sylow p-subgroups, and any two Sylow p-subgroups of $\mathrm{GL}_2(\mathbf{F}_p)$ generate $\mathrm{SL}_2(\mathbf{F}_p)$. So, α extends to a cohomology class in $H^2(\mathrm{SL}_2(\mathbf{F}_p), Q)$, which implies that α is trivial. We obtain that P is not isomorphic to $C_{p^2} \rtimes C_p$ as the cohomology class induced by $C_{p^2} \rtimes C_p$ in $H^2(Q,R)$ is not trivial (in other words, $C_{p^2} \rtimes C_p$ is not a semi-direct product of $C_p \times C_p$ by C_p). $\square$

First we prove a similar result to those obtained in [St], for the generalized extraspecial p-groups. We recall that a generalized extraspecial p-group S is a p-group satisfying that $\Phi(S) = [S,S] \simeq C_p$.

Theorem 5.3. *Let P be an extraspecial generalized p-group. Then P is resistant, except for the case where $P = E \times A$ with A elementary abelian and E dihedral of order 8 (when $p = 2$) or of order p^3 and exponent p (when p is odd).*

Proof. If P is such a p-group, the upper central series is simply $1 \lhd \Phi(P) \lhd P$. By Theorem 4.8 P is resistant if and only if $\Phi(P)$ is weakly $\mathcal{F}$-closed, for any fusion system $\mathcal{F}$ on P. By Alperin's theorem, any morphism in $\mathrm{Hom}_{\mathcal{F}}(\Phi(P), P)$ is decomposable in restrictions of maximal and essential $\mathcal{F}$-automorphisms. As the maximal $\mathcal{F}$-automorphisms stabilize $\Phi(P)$, we need only to prove the same for the essential $\mathcal{F}$-automorphisms.

Let $\phi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, where Q is an $\mathcal{F}$-essential, fully $\mathcal{F}$-normalized subgroup. As Q is $\mathcal{F}$-centric, it contains $Z(P)$ and $Q/Z(P)$ contains a maximal isotropic subspace of $P/Z(P)$ for the symplectic form $\beta : P/Z(P) \times P/Z(P) \to \Phi(P)$, defined by $\beta(x, y) = [x, y]$ (see [St] for more details). Moreover, if Q is not elementary abelian, then $\Phi(Q) = \Phi(P)$. Any automorphism of Q preserves $\Phi(Q)$ and so ϕ stabilizes $\Phi(P)$. We have the same property for ϕ if $|P/Z(P)| > p^2$. Indeed, $|P : C_P(\phi(\Phi(P)))| \leq p$, as the centralizer of a subgroup of order p of P is of index at most p in P, and we deduce that $C_P(\phi(\Phi(P)))/Z(P)$ is not contained in a maximal isotropic subspace of $P/Z(P)$ with respect to β. Thus $C := C_P(\phi(\Phi(P)))$ is not elementary abelian. So $\Phi(C)$ is not trivial and we have $\Phi(C) = \Phi(P)$. Now, as $N_P(\Phi(P)) = P$, it follows that $\Phi(P)$ is fully $\mathcal{F}$-normalized. Thus ϕ^{-1} extends to $\psi \in \mathrm{Hom}_{\mathcal{F}}(C, P)$ by Property FS3. We deduce that $\phi(\Phi(P))$ is the Frattini subgroup of C, so $\phi(\Phi(P)) = \Phi(P)$.

So we can restrict our research to the case where Q is elementary abelian and $|P : Z(P)| = p^2$. In this case Q is of index p in P and we have $P \simeq E \times A$ where E is an extraspecial p-group of order p^3 and A a p-group elementary abelian. Moreover $E \simeq P/A$ contains a subgroup isomorphic to $Q/A \simeq C_p \times C_p$, as Q is elementary abelian, so E is not isomorphic to Q_8. The only case that remains to exclude is the case $E \simeq C_{p^2} \rtimes C_p$, for p odd. Denote by $K := (P/Q)$ regarded as subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q/A)$.

As Q is a normal subgroup of P, it is fully $\mathcal{F}$-normalized. If $K = {}^{\phi}K$, so ϕ extends to $\psi \in \mathrm{Aut}_{\mathcal{F}}(P)$ by Property FS3. Now, ψ fixes $\Phi(P)$, and so does ϕ.

It remains to verify the case where $K \neq {}^{\phi}K$. We consider K and ${}^{\phi}K$ as p-subgroups of $\mathrm{Aut}(Q/A) \simeq \mathrm{GL}_2(\mathbf{F}_p)$. In this context K and ${}^{\phi}K$ generate $\mathrm{SL}_2(\mathbf{F}_p)$.

We show now that the extension $1 \longrightarrow Q/A \longrightarrow E \longrightarrow K \longrightarrow 1$ satisfies the hypothesis of Lemma 5.2. Indeed, for any morphism $\phi' \in \mathrm{Hom}_{\mathcal{F}}(Q)$ fixing K, we have $K = {}^{\phi'}K$, so $N_{\phi'}$ contains the preimage of K in P, which is P itself. So $N_{\phi'} = P$ and as Q is fully $\mathcal{F}$-normalized, ϕ' lifts in $\mathrm{Aut}_{\mathcal{F}}(P)$ by Property FS3.

Thus, the extension $1 \longrightarrow Q/A \longrightarrow E \longrightarrow K \longrightarrow 1$ satisfies the hypothesis of Lemma 5.2 so it lifts to $1 \longrightarrow Q/A \longrightarrow L \longrightarrow < K, {}^{\phi}K > \longrightarrow 1$. Moreover, K and ${}^{\phi}K$ are two distinct Sylow p-subgroups of $< K, {}^{\phi}K >$ so, by the same Lemma 5.2, E is not isomorphic to $C_{p^2} \rtimes C_p$.

The remaining cases are really exceptions to the theorem, as proved in [St]. □

The second class of resistant p-groups is those of metacyclic p-groups, for p odd. In the special case of the Frobenius category of a finite group this is a result by Dietz [Dz] using some cohomological methods. The next result generalizes the latter and the proof uses a different approach.

Proposition 5.4. *Let P a metacyclic p-group, with p odd. Then P is resistant.*

Proof. Consider that P is given by generators and relations

$$P := < u, v | u^{p^m} = 1, v^{p^n} = u^{p^q}, {}^{v}u = u^{p^l+1} > \ .$$

In other words, P is of the type (m, n, q, l) and is defined by an extension:

$$1 \longrightarrow C_{p^m} \hookrightarrow P \longrightarrow C_{p^n} \longrightarrow 1 \ .$$

If P is abelian then it is trivially resistant. If not, P is necessarily non-cyclic. In the case where P is a non-cyclic metacyclic p-group, for p odd, Nadia Mazza [Mz] proved that P posses a unique subgroup Q isomorphic to $C_p \times C_p$. This implies that if P posses a subgroup isomorphic to $C_p^\alpha \times C_p^\alpha$, for a positive integer number α, this subgroup is unique.

The aim is now to find a central series for P composed of subgroups weakly $\mathcal{F}$-closed. We procede by induction on the order of P. The case where $|P| \leq p^2$ is trivial as P is abelian.

If we manage to find $1 \neq P_1 \leq Z(P)$ weakly $\mathcal{F}$-closed, then we can apply the induction hypothesis to $\overline{\mathcal{F}} := N_{\mathcal{F}}(P_1)/P_1$, which is a fusion system on P/P_1, by Proposition 4.4, given that P_1 is strongly $N_{\mathcal{F}}(P_1)$-closed. Indeed, P/P_1 is a metacyclic p-group of order strictly inferior to P. By induction hypothesis P/P_1 admits a central series $1 = \overline{P}_1 \lhd \overline{P}_2 \lhd \ldots \lhd \overline{P}_n = P/P_1$ with $\overline{P}_i$ weakly $\overline{\mathcal{F}}$-closed, for $1 \leq i \leq n$. Let $\pi : P \longrightarrow P/P_1$ be the canonical projection. Consider the subnormal series for P, given by $1 = P_0 \lhd P_1 \lhd P_2 \lhd \ldots \lhd P_n = P$ with $P_i = \pi^{-1}(\overline{P}_i)$ for $1 \leq i \leq n$. By the second part of Lemma 4.6, this is a central series for P. Moreover, by Lemma 4.7, P_i is weakly $\mathcal{N}$-closed where $\mathcal{N} := N_{\mathcal{F}}(P_1)$, for all $2 \leq i \leq n$. Now, we prove that P_i is also weakly $\mathcal{F}$-closed, for all $2 \leq i \leq n$. Indeed, let $\phi \in \mathrm{Hom}_{\mathcal{F}}(P_i, P)$. We have $\phi|_{P_1} \in \mathrm{Hom}_{\mathcal{F}}(P_1, P)$. But P_1 is weakly $\mathcal{F}$-closed, which gives that $\phi(P_1) = P_1$ and, in consequence that $\phi \in \mathrm{Hom}_{\mathcal{N}}(P_i, P)$. Now, P_i is weakly $\mathcal{N}$-closed, so $\phi(P_i) = P_i$ and we obtain that P_i is weakly $\mathcal{F}$-closed.

We search now a nontrivial central subgroup P_1 of P which is weakly $\mathcal{F}$-closed. Recall that Q is the unique subgroup of P isomorphic to $C_p \times C_p$.

If $Q \leq Z(P)$ then we take $P_1 := Q$ which is weakly $\mathcal{F}$-closed, as it is unique in its $\mathcal{F}$-conjugacy class. If not we have necessarily $< u^{p^{m-1}} > \leq Z(P)$ and $Z(P) \simeq C_{p^l}$. We want to see that $< u^{p^{m-1}} >$ is normalized by the maximal and essential $\mathcal{F}$-automorphisms of P. The maximal $\mathcal{F}$-automorphisms normalize $Z(P)$ and, a fortiori, $< u^{p^{m-1}} >$ as it is the unique subgroup of order p of $Z(P)$.

Let E be an $\mathcal{F}$-centric, fully $\mathcal{F}$-normalized subgroup of P, candidate to be $\mathcal{F}$-essential. By elimination of metacyclic p-groups that are not realizable as $\mathcal{F}$-essential subgroups (see Proposition 3.3 and Proposition 3.5), we deduce that E is isomorphic to $C_{p^\alpha} \times C_{p^\alpha}$, for a positive integer α.

Let $\Phi(E)$ be the Frattini subgroup of E. The kernel of the canonical application $\phi : \mathrm{Aut}_{\mathcal{F}}(E) \longrightarrow \mathrm{Aut}(E/\Phi(E))$ is a p-group. Thus, as E is an $\mathcal{F}$-essential subgroup and $\mathrm{Aut}_E(E) = 1$, the complex $\mathcal{S}_p(\mathrm{Aut}_{\mathcal{F}}(E))$ is disconnected. So, $O_p(\mathrm{Aut}_{\mathcal{F}}(E))$ is trivial, which gives that the kernel of ϕ is also trivial. We deduce that a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(E)$ is of order p, as $\mathrm{Aut}(E/\Phi(E)) \simeq \mathrm{GL}_2(\mathbf{F}_p)$.

On the other hand, E is the unique subgroup of P isomorphic to $C_{p^\alpha} \times C_{p^\alpha}$, so it is characteristic in P. In this way $N_P(E) = P$ and E is fully $\mathcal{F}$-normalized. As, moreover, E is $\mathcal{F}$-centric, P/E is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(E)$. We have seen that a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(E)$ is of order p, which implies that $|P : E| = p$.

So P is a metacyclic subgroup, of order $p^{2\alpha+1}$, having a subgroup E isomorphic to $C_{p^\alpha} \times C_{p^\alpha}$ and a centre $Z(P)$ cyclic. This is equivalent to the fact that u does not commute with $v^{p^{n-1}}$ and that $u^{p^{m-\alpha}}$ commutes with $v^{p^{n-\alpha}}$.

But $v^{p^{n-1}} u v^{-p^{n-1}} = u^{(p^l+1)^{p^{n-1}}}$ and if we want that $u^{(p^l+1)^{p^{n-1}}} \neq u$ then necessarily $n + l - 1 < m$. On the other hand,

$$u^{p^{m-\alpha}} = v^{p^{n-\alpha}} u^{p^{m-\alpha}} v^{-p^{n-\alpha}} = u^{p^{m-\alpha}(p^l+1)^{p^{n-\alpha}}}$$

and, using the fact that $(p^l+1)^{p^{n-\alpha}}-1$ is exactly divisible by $p^{l+n-\alpha}$, it follows that $(m-\alpha)+(l+n-\alpha)\geq m$. As $m+n=2\alpha+1$, we obtain that $1+l\geq m$. In fact, we have $m=l+1$, as $l<m$. We come back now to the inequality $n+l-1<m$ in which we replace $m=l+1$ in such way that we obtain $n<2$. So $n=1$ which gives that $\alpha=1$ and $m=2$. We reach the conclusion that P have to be a p-group extraspecial of order p^3 and exponent p^2, case where we have seen, in the last theorem that any morphism in $\mathrm{Aut}_{\mathcal{F}}(E)$ fixes the centre of P.

So, if the centre of P is cyclic, $< u^{p^{m-1}} >$ is weakly $\mathcal{F}$-closed and we can take $P_1 =< u^{p^{m-1}} >$. □

Another example we are interested in is the Sylow 2-subgroup of the Suzuki group $Sz(8)$. In [Bru1] Broué conjectured that the equivalence class of the derived category of a block algebra only depends on the equivalence class of the underlying fusion system. Only short time thereafter, he realized [Bru2] that the conjecture is false in this generality, by giving as example the 2-local structure of $Sz(8)$ and of the normalizer of its Sylow 2-subgroup. The normalizer controls the 2-fusion in $Sz(8)$, giving that their 2-local structures are the same. But the derived categories associated to the principal blocks are not equivalent.

As an application of our theorem we give a short proof of the known fact that the normalizer of the Sylow 2-subgroup controls the 2-fusion in $Sz(8)$. Denote by P the Sylow 2-subgroup of $Sz(8)$. We show that P is resistant which implies the above assertion. This is easy to prove by looking at the structure of P. Indeed the upper central series for P is $1\lhd Z(P)\lhd P$, where $Z(P)$ is the centre of P. Moreover $Z(P)$ is the unique elementary abelian subgroup of rank 3 of P. This implies that $Z(P)$ is weakly $\mathcal{F}$-closed for any fusion system $\mathcal{F}$ on P. Applying our main theorem we obtain that P is resistant.

6. Normal Subsystems

Recently, Linckelman [Li], motivated by the reduction of some problems on fusion systems introduced the notion of *normal fusion system*. Here is his approach:

Definition 6.1. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and $\mathcal{F}'$ a fusion subsystem of $\mathcal{F}$ on a subgroup P' of P. We say that $\mathcal{F}'$ is normal in $\mathcal{F}$ if P' is strongly $\mathcal{F}$-closed and if for every isomorphism $\phi : Q \to Q'$ in $\mathcal{F}$ and any two subgroups R, R' of $Q\cap P'$ we have*

$$\phi\circ \mathrm{Hom}_{\mathcal{F}'}(R,R')\circ\phi^{-1}\subseteq \mathrm{Hom}_{\mathcal{F}'}(\phi(R),\phi(R'))\,.$$

As a corollary to our main theorem we prove some properties on normal fusion subsystems.

Proposition 6.2. *Let P be a finite p-group, $\mathcal{F}$ a fusion system on P and Q a strongly $\mathcal{F}$-closed subgroup of P. Then $\mathcal{G}:=F_Q(Q)$ is normal in $\mathcal{F}$ if and only if $N_{\mathcal{F}}(Q)=\mathcal{F}$.*

Proof. If $N_{\mathcal{F}}(Q)=\mathcal{F}$ then any morphism $\phi\in \mathrm{Hom}_{\mathcal{F}}(R,P)$ where R is a subgroup of Q, lifts to a morphism $\phi\in\mathrm{Aut}_{\mathcal{F}}(Q)$ as Q is strongly $\mathcal{F}$-closed. To prove that $\mathcal{G}$ is normal in $\mathcal{F}$ is is sufficient to prove that for any $u\in Q$ and any morphism $\phi\in\mathrm{Hom}_{\mathcal{F}}(< R,\phi(R) >,P)$, the morphism $\psi:=\phi\mathrm{conj}_u\phi^{-1}\in\mathrm{Hom}_{\mathcal{F}}(\phi(R),\phi({}^uR))$ is also in $\mathrm{Hom}_{\mathcal{G}}(\phi(R),\phi({}^uR))$. But this is straight forward as ϕ lifts to $\tilde{\phi}$ so $\psi=\mathrm{conj}_{\tilde{\phi}(u)}$ which is a morphism in $\mathrm{Hom}_{\mathcal{G}}(\phi(R),\phi({}^uR))$.

If $\mathcal{G}$ is a normal subsystem in $\mathcal{F}$ we apply the main result to prove that $N_{\mathcal{F}}(Q) = \mathcal{F}$. As Q is strongly $\mathcal{F}$-closed it suffices to prove that the components of the upper central series of Q are all weakly $\mathcal{F}$-closed. We prove this property in general for any characteristic subgroup R of Q. Let $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, \phi(R))$ with $\phi(R)$ fully $\mathcal{F}$-normalized. As $\mathcal{G}$ is normal in $\mathcal{F}$ we have that for any $u \in Q$, $\psi := \phi \mathrm{conj}_u \phi^{-1}$ is a morphism in $\mathrm{Aut}_{\mathcal{G}}(\phi(R))$ so equal to conj_v for a $v \in Q$. So $u \in N_\phi$ for all $u \in Q$. By Property FS3 the morphism ϕ extends to $\tilde{\phi} \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. But Q is strongly $\mathcal{F}$-closed, so $\tilde{\phi} \in \mathrm{Aut}_{\mathcal{F}}(Q)$. We obtain that ϕ is the restriction to R of an automorphism of Q. As R is characteristic in Q we have that $\phi(R) = R$, so R is weakly $\mathcal{F}$-closed.

□

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