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CONJECTURES ON FINITE AND p-LOCAL GROUPS

Algebraic topology and group theory are strongly interacting mathematical fields and in the past decade new connections between the p-completion of the classifying space of a group and its p-local structure were discovered. A seminal theorem relating the p-local structure with the mod p cohomology of a finite group is the following [Mi].

Theorem (Mislin 1990). *A subgroup H of a finite group G controls the p-fusion in G if and only if the canonical inclusion of H in G induces an isomorphism in all degrees between the mod p cohomology rings of G and H.*

In other words G and H have the same p-local structure if and only if they have the same mod p cohomology.

The conjectures and questions below continue in this vein. In the context of group representations and local group theory, a fundamental conjecture is Alperin's Weight Conjecture [Al]. Here is a formulation with a topological flavor.

Definition 1. Let G be a finite group. A p-chain C of G is a strictly increasing chain

$$C : P_0 < P_1 < \cdots < P_n$$

of p-subgroups of G. We let C_i denote the initial subchain terminating at P_i. The chain C is radical if $P_0 = O_p(G)$ and, for each i, $P_i = O_p(N_G(C_i))$.

We denote by $\mathcal{R}(G)$ the set of all radical p-chains of G, and by $\mathcal{R}(G)/G$ its orbit space.

Definition 2. Fix a prime p. Let ϕ be an ordinary irreducible character of the finite group H. The defect $d(\phi)$ of ϕ is the largest non-negative integer d such that p^d divides $\dfrac{|H|}{\phi(1)}$.

Definition 3. Let G be a finite group and B a p-block of G. For any p-chain C of G we denote by $k(C, B, d)$ the number of characters $\phi \in Irr(N_G(C))$ having defect $d(\phi) = d$ and belonging to a p-block $B(\phi)$ of $N_G(C)$ such that the induced p-block $B(\phi)^G$ is B.

Conjecture 4. *Let p be a prime, G a finite group with $O_p(G) = 1$, and B a p-block of G which is not of defect 0. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} k(C, B, d) = 0.$$

Next, here is a statement about finite groups. In some sense it is not a conjecture because the first author is fairly certain it can be proved easily as a corollary of the Classification of the Finite Simple Groups. The conjecture is that there is an elementary proof, perhaps following from further advances in the p-modular representation theory of finite groups. (The case $p = 2$ is a theorem of Goldschmidt [Go].)

Definition 5. Let p be a prime and P a p-group. Then $\Omega_1(P)$ is the subgroup of P generated by all elements of order p.

Conjecture 6. *Let p be a prime and let G be a finite group having an abelian Sylow p-subgroup A. Suppose that B is a strongly closed subgroup of A with respect to G, i.e. if $x \in B$ and $g \in G$ with $x^g \in A$, then $x^g \in B$. Then there exists a normal subgroup N of G having a Sylow p-subgroup B^* such that $\Omega_1(B) = \Omega_1(B^*)$. In particular, if $A = \Omega_1(A)$ and B is strongly closed in A with respect to G, then $B \in \mathrm{Syl}_p(N)$ for some normal subgroup N of G.*

The following conjectures are related to the recent work of Broto, Levi, and Oliver [BLO] on the theory of so-called p-local groups. The underlying structure of a p-local group is a fusion system on a finite p-group P. Fusion systems were introduced by Puig in 1990 [Pu]. Here is the definition of a fusion system on P. First let us start with a more general definition

Definition 7. A *category $\mathcal{F}$ on a finite p-group P* is a category whose objects are the subgroups of P and whose set of morphisms between the subgroups Q and R of P, is the set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ of injective group homomorphisms from Q to R, with the following properties:
(a) if $Q \leq R$ then the inclusion of Q in R is a morphism in $\mathrm{Hom}_{\mathcal{F}}(Q, R)$.
(b) for any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ the induced isomorphism $Q \simeq \phi(Q)$ and its inverse are morphisms in $\mathcal{F}$.
(c) composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

And now the definition of a fusion system:

Definition 8. A *fusion system* $\mathcal{F}$ on a finite p-group P is a category on P satisfying the following properties:
(1) $\mathrm{Hom}_P(Q, R) \subset \mathrm{Hom}_{\mathcal{F}}(Q, R)$ for all $Q, R \leq P$.
(2) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.
(3) Every $\phi : Q \to P$ such that $|N_P(\phi(Q))|$ is maximal in the $\mathcal{F}$-isomorphism class of Q, extends to $\bar{\phi} : N_\phi \to P$ where

$$N_\phi = \{x \in N_P(Q) \,|\, \exists\, y \in N_P(\phi(Q)),\ \phi({}^x u) = {}^y\phi(u)\, \forall\, u \in Q\}\,.$$

Axiom (2) is saying that P is a 'Sylow p-subgroup' of $\mathcal{F}$ and the extension in Axiom (3) is equivalent to saying that any p-subgroup can be embedded by conjugation in a Sylow p-subgroup. In particular, if G is a finite group, p is a prime divisor of $|G|$, and $P \in \mathrm{Syl}_p(G)$, then the morphisms given by conjugation by elements of G between the subgroups of P determine a fusion system $\mathcal{F}_G(P)$ on P.

We say that a fusion system is *exotic* if it does not arise in this way. There is some dispute about the correct definition of a normal subobject in this theory. Here is Markus Linckelmann's definition of a normal subsystem [Li]. First let's introduce the notion of *strongly $\mathcal{F}$-closed* subgroups.

Definition 9. Let $\mathcal{F}$ be a fusion system on a finite p-group P and Q a subgroup of P. We say that Q is *strongly $\mathcal{F}$-closed* if for any subgroup R of Q and any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ we have $\phi(R) \leq Q$.

And now the notion of normal fusion subsystem.

Definition 10. Let $\mathcal{F}$ be a fusion system on a finite p-group P and $\mathcal{F}'$ a fusion subsystem of $\mathcal{F}$ on a subgroup P' of P. We say that $\mathcal{F}'$ is *normal* in $\mathcal{F}$ if P' is strongly $\mathcal{F}$-closed and if for every isomorphism $\phi : Q \to Q'$ in $\mathcal{F}$ and any two subgroups R, R' of $Q \cap P'$ we have

$$\phi \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ \phi^{-1} \subseteq \mathrm{Hom}_{\mathcal{F}'}(\phi(R), \phi(R')) .$$

Here is a conjecture on normal fusion systems:

Conjecture 11. *Let $\mathcal{F}$ be a fusion system on P and P' a strongly $\mathcal{F}$-closed subgroup of P. Then there exists a normal fusion subsystem $\mathcal{F}'$ of $\mathcal{F}$ on P'.*

A *simple fusion* system is a fusion system that has no non-trivial normal subsystems. In particular, if G is a finite simple group which does not have a proper strongly p-embedded subgroup, then $\mathcal{F}_G(P)$ is a simple fusion system.

When p is odd, it seems to be fairly easy to construct examples of exotic simple p-local groups. On the other hand when $p = 2$, the only known exotic examples live in a single infinite family, $\mathcal{F}_{Sol}(q)$, which may be regarded as the analogue of finite Chevalley groups with respect to the exotic 2-compact group of Dwyer and Wilkerson.

Conjecture 12. *The family $\mathcal{F}_{Sol}(q)$ contains all of the exotic simple 2-local groups.*

This conjecture is hard to believe. On the other hand, it has thus far been impossible to dream up other examples. It is conceivable that it could be proved by a lengthy and elaborate analysis in the vein of the traditional 2-local analysis of finite simple groups used in the proof of the Classification Theorem. It would be much more interesting if homotopy-theoretic tools could be brought to bear to prove this result. That would really give hope for new and exciting applications of topology to finite group theory.

Simple Lie groups are of course linked to real reflection groups. Similarly, p-compact groups are linked to p-adic reflection groups. This suggests the following question.

Question 13. *Is there a family of interesting topological objects connected to quaternionic reflection groups?* [Co]

As the Dwyer-Wilkerson 2-compact group [DW] is, in some sense, a 45-dimensional object, it is a plausible conjecture that it might have some connection to the 45-dimensional algebra $SH(3, \mathbf{O})$ of 3×3 skew hermitian matrices over the octonions (with bracket multiplication).

Question 14. *What is the automorphism group of this algebra and of related algebras, e.g. over rings of integral octonions?*

Finally, we note that another source of examples for fusion systems are the fusion systems coming from p-blocks of group algebras, given by the conjugations between the Brauer pairs in a maximal Brauer pair of the p-block. Such examples are called Brauer categories [AB]. Here's a natural question one can ask.

Question 15. *Are there Brauer categories which are exotic fusion systems?*

It is pretty hard to check that a given fusion system is not a Brauer category. Up to now, the only known way to do it is by reduction to Brauer categories of quasisimple groups and then by using the classification of finite simple groups [Ke], [KS]. There are examples of embedded fusion systems where the minimal and the maximal ones come from finite groups and the intermediate ones are exotic. It is not yet known whether these exotic fusion systems are Brauer categories.

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