The extension algebra of some cohomological Mackey functors

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Abstract : Let k be a field of characteristic p. We construct a new inflation functor for cohomological Mackey functors for finite groups over k. Using this inflation functor, we give an explicit presentation of the graded algebra of self extensions of the simple functor $S_{\mathbf{1}}^G$, when p is odd and G is an elementary abelian p-group.

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1. Introduction

Let k be a field and G be a finite group. The theory of *Mackey functors* and *cohomological Mackey functors* for G over k originates in the work of Green ([4]) and Dress ([3]), at the beginning of the 70's. It can be viewed as the theory of induction and restriction, when we forget the particular framework of linear representations of G over k. Many important developments have been achieved since, culminating in the comprehensive and seminal paper by Thévenaz-Webb ([5]) in 1995, where the authors introduce *the Mackey algebra* $\mu_k(G)$, and show, among many other fundamental results, that the category of Mackey functors for G over k is equivalent to the category of $\mu_k(G)$-modules. Similarly, they show that the subcategory $\mathsf{M}_k^c(G)$ of cohomological Mackey functors for G over k is equivalent to the category of $co\mu_k(G)$-modules, where $co\mu_k(G)$ is a specific quotient of $\mu_k(G)$, called *the cohomological Mackey algebra*.

The algebras $\mu_k(G)$ and $co\mu_k(G)$ share many similarities with the group algebra kG : e.g., they are finite dimensional k-vector spaces, of dimension independent on k, the Maschke theorem holds, there is a good theory of decomposition from characteristic 0 to characteristic p, etc... These resemblances raise some natural questions, whether a given theorem on kG will admit an analogue for $\mu_k(G)$ or $co\mu_k(G)$.

This was the main motivation in [2], where the question of complexity of cohomological Mackey functors was solved (in the only non-trivial case where k is a field of positive characteristic p dividing the order of G). It was also shown there how this question can be reduced to the consideration of elementary abelian p-groups E appearing as subquotients of G, and to the knowledge of enough information on the algebra $\mathcal{E} = \mathrm{Ext}^*(S_{\mathbf{1}}^E, S_{\mathbf{1}}^E)$ of self extensions of a particular simple functor $S_{\mathbf{1}}^E$ for these groups. Along the way, a presentation of this algebra was given when $p = 2$, together with a formula for the Poincaré series. In the case $p > 2$, no such presentation was given, and a conjecture was proposed for the Poincaré series of $\mathcal{E}$. This conjecture was only proved in the case $p = 3$.

This paper settles completely the case $p > 2$: a presentation of $\mathcal{E}$ is given, and, as a corollary, the forementioned conjecture is proved. The main results are the following, where $S_{1,W}^H$ denotes the simple functor for the group H defined as in 2.15. To simplify notation, when $W = k$ is the trivial module, we drop this subscript. We start by the construction of a new inflation functor for cohomological Mackey functors :

1.1. Theorem : *Let k be a field, let G be a finite group, let $N \trianglelefteq G$ and let V be a simple $k(G/N)$-module. Then there exist an exact functor $\sigma_{G/N}^G$ from $\mathsf{M}_k^c(G/N)$ to $\mathsf{M}_k^c(G)$ satisfying*

$$\sigma_{G/N}^G(S_{\mathbf{1},V}^{G/N}) = S_{\mathbf{1},\mathrm{Inf}_{G/N}^G V}^G \quad .$$

Suppose, moreover, that $G = N \rtimes H$ is the semidirect product of N by a group H.

1. *If V is a kH-module, let $\tilde{V}$ be the kG-module $\mathrm{Inf}_{G/N}^G \mathrm{Iso}_H^{G/N} V$. Then the restriction of $\tilde{V}$ to H is isomorphic to V.*
2. *The composition $\mathrm{Res}_H^G \sigma_{G/N}^G \mathrm{Iso}_H^{G/N}$ is isomorphic to the identity functor of $\mathsf{M}_k^c(H)$.*
3. *Let V and W be simple kH-modules. Then, for any $n \in \mathbb{N}$, the restriction from G to H induces a split surjection*

$$r_H^G : \mathrm{Ext}^n_{\mathsf{M}_k^c(G)}(S_{\mathbf{1},\tilde{V}}^G, S_{\mathbf{1},\tilde{W}}^G) \to \mathrm{Ext}^n_{\mathsf{M}_k^c(H)}(S_{\mathbf{1},V}^H, S_{\mathbf{1},W}^H) \quad .$$

Let now p be an odd prime, k be a field of characteristic p and G be an elementary abelian p-group of rank r. Then we can give an explicit presentation of $\mathrm{Ext}^*_{\mathsf{M}^c_k(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$, the graded algebra of self extensions of the simple functor $S_{\mathbf{1}}^G = S_{\mathbf{1},k}^G$. Here is the main result of this paper :

1.2. Theorem : *Let $\mathcal{A}$ be the graded k-algebra with generators*

$\{\hat{\tau}_\varphi \mid \varphi \in \mathrm{Hom}(G, \mathbb{F}_p^+)\}$, *in degree 1, and* $\{\hat{\gamma}_X | X \leq G, \, |X| = p\}$, *in degree 2,*

subject to the relations

(L1) $\forall \varphi, \psi \in \mathrm{Hom}(G, \mathbb{F}_p^+), \;\; \hat{\tau}_{\varphi+\psi} = \hat{\tau}_\varphi + \hat{\tau}_\psi$.

(L2) If $p \geq 5$, *then* $\forall \varphi \in \mathrm{Hom}(G, \mathbb{F}_p^+), \;\; \hat{\tau}_\varphi^2 = 0$, *and* $[\hat{\tau}_\varphi, \sum\limits_{X \nleq \mathrm{Ker}\varphi} \hat{\gamma}_X] = 0$.

If $p = 3$, *then* $\hat{\tau}_\varphi^2 = -\sum\limits_{X \nleq \mathrm{Ker}\varphi} \hat{\gamma}_X$.

(L3) $\forall \varphi \in \mathrm{Hom}(G, \mathbb{F}_p^+), \forall X, \; |X| = p, \; X \leq \mathrm{Ker}\varphi, \;\; [\hat{\tau}_\varphi, \gamma_X] = 0$.

(L4) $\forall Q \leq G, |Q| = p^2, \forall X < Q, |X| = p, \;\; [\hat{\gamma}_X, \sum\limits_{Y<Q} \hat{\gamma}_Y] = 0$.

Then there is an isomorphism of graded algebras from $\mathcal{A}$ to $\mathrm{Ext}^*_{\mathsf{M}^c_k(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$.

In view of $(L1)$, this set of generators is redundant : we choose a direct sum decomposition $G = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_{r-1} \oplus Y_r$, and for $1 \leq i \leq r$, we choose a group homomorphism $\varphi_i : G \to \mathbb{F}_p$ with kernel $\bigoplus\limits_{\substack{1 \leq j \leq r \\ j \neq i}} Y_j$.

1.3. Theorem : *Let $\mathcal{A}$ be the graded k-algebra with generators*

$\{\hat{\tau}_i | 1 \leq i \leq r\}$ *in degree 1, and* $\{\hat{\gamma}_X | X \leq G, \, |X| = p\}$ *in degree 2,*

subject to the relations

(R1) $\hat{\tau}_i\hat{\tau}_i = 0$, *if* $p \geq 5$, *or*
$\hat{\tau}_i\hat{\tau}_i = -\sum\limits_{X \nleq \mathrm{Ker}\,\varphi_i} \hat{\gamma}_X$ *if* $p = 3$.

(R2) $\hat{\tau}_i\hat{\tau}_j + \hat{\tau}_j\hat{\tau}_i = 0$ *for* $1 \leq j \leq i \leq r$, *if* $p \geq 5$, *or*
$\hat{\tau}_i\hat{\tau}_j + \hat{\tau}_j\hat{\tau}_i = \sum\limits_{X \nleq \mathrm{Ker}(\varphi_i+\varphi_j)} \hat{\gamma}_X - \sum\limits_{X \nleq \mathrm{Ker}\,\varphi_i} \hat{\gamma}_X - \sum\limits_{X \nleq \mathrm{Ker}\,\varphi_j} \hat{\gamma}_X$ *if* $p = 3$.

(R3) $[\hat{\tau}_i, \sum\limits_{X \nleq \mathrm{Ker}\,\varphi_i} \hat{\gamma}_X] = 0$*;*

(R4) $[\varphi_j(x)\hat{\tau}_i - \varphi_i(x)\hat{\tau}_j, \hat{\gamma}_{\langle x\rangle}] = 0$, *for* $1 \leq i < j \leq 1$, $x \in G$;

(R5) $[\hat{\gamma}_X, \sum\limits_{Y<Q} \hat{\gamma}_Y] = 0$, *for all* $X < Q \leq G$, $|X| = p$, $|Q| = p^2$.

Then there is an isomorphism of graded algebras from $\mathcal{A}$ *to* $\mathrm{Ext}^*_{\mathsf{M}^c_k(G)}(S^G_{\mathbf{1}}, S^G_{\mathbf{1}})$.

As a consequence of the above theorem, we get the following result, which was proved for $p = 3$ in ([2], Theorem 14.2) and conjectured for $p \geq 5$:

1.4. Proposition : *Let* k *be a field of odd characteristic* p, *and* $G \cong (C_p)^r$. *Then :*

1. *The algebra* $\mathcal{E} = \mathrm{Ext}^*_{\mathsf{M}^c_k(G)}(S^G_{\mathbf{1}}, S^G_{\mathbf{1}})$ *is generated by the elements* τ^G_φ *in degree 1, for* $\varphi \in \mathrm{Hom}(G, k^+)$, *and by the elements* γ^G_X *in degree 2, for* $X \leq G$ *with* $|X| = p$.
2. *The Poincaré series for* $\mathcal{E}$ *is equal to*

$$\frac{1}{(1-t)\big(1-t-(p-1)t^2\big)\big(1-t-(p^2-1)t^2\big)\ldots\big(1-t-(p^{r-1}-1)t^2\big)}\ .$$

The paper is organized as follows. In Section 2 we construct a new inflation functor for cohomological Mackey functors for a finite group G over a field k. In the case where the group is a semi-direct product, more properties of this functor are given. For completeness, Section 3 recalls a series of results on extensions of cohomological Mackey functors when G is an elementary abelian p-group G and k is a field of characteristic p. These results, taken from [2], are crucial in the second part of the paper. The use of the inflation functor constructed in Section 2 makes the proof of the exactness of the sequence in Corollary 3.10 straightforward. Section 4 deals with finding relations in the graded algebra $\mathcal{E} = \mathrm{Ext}^*_{\mathsf{M}^c_k(G)}(S^G_{\mathbf{1}}, S^G_{\mathbf{1}})$. Section 5 gives a recursive direct decomposition of this algebra. The decomposition is obtained through an involved induction on the rank of G, constructed on the technical support of Lemma 5.1. Based on this direct decomposition and the relations stated in Section 4, Section 6 builds a presentation of $\mathcal{E}$. Some arithmetics of extensions in abelian categories, needed in the paper, are presented in the Appendix.

2. Yet another inflation functor

2.1. Let k be a commutative ring with identity element, and G be a finite group. Let $\mathsf{M}^c_k(G)$ denote the category of cohomological Mackey functors for G over k. By Yoshida's Theorem, the category $\mathsf{M}^c_k(G)$ is equivalent to the category $\mathsf{Fun}_k(G)$ of k-linear contravariant functors from the category $\mathsf{perm}_k(G)$ of finitely generated permutation kG-modules, to the category k-Mod of k-modules.

2.2. When G and H are finite groups, any k-linear functor from $\mathsf{perm}_k(G)$ to $\mathsf{perm}_k(H)$ gives by precomposition a functor from $\mathsf{Fun}_k(H)$ to $\mathsf{Fun}_k(G)$. Two special cases of this situation have been considered in [2] (Section 3.12) : if U is a finite (H,G)-biset, the functor

$$\mathsf{t}_U : V \mapsto kU \otimes_{kG} V$$

from $\mathsf{perm}_k(G)$ to $\mathsf{perm}_k(H)$ yields a functor

$$\mathsf{L}_U : F \mapsto F \circ \mathsf{t}_U$$

from $\mathsf{Fun}_k(H)$ to $\mathsf{Fun}_k(G)$. Similarly, the functor

$$\mathsf{h}_U : W \mapsto \mathrm{Hom}_{kH}(kU, W)$$

from $\mathsf{perm}_k(H)$ to $\mathsf{perm}_k(G)$ yields a functor

$$\mathsf{R}_U : F \mapsto F \circ h_U$$

from $\mathsf{Fun}_k(G)$ to $\mathsf{Fun}_k(H)$, which is right adjoint to L_U.

2.3. Several special cases of this special case occur when $H = G/N$, where N is a normal subgroup of G. First, if $U = G/N$, viewed as a (G,H)-biset in the obvious way, then the functor $\mathsf{L}_U : \mathsf{Fun}_k(G/N) \to \mathsf{Fun}_k(G)$ is denoted by $\imath^G_{G/N}$, and $\mathsf{R}_U : \mathsf{Fun}_k(G) \to \mathsf{Fun}_k(G/N)$ is denoted by $\rho^G_{G/N}$.

Reversing the actions, and viewing $V = G/N$ as an (H,G)-biset, the functor $\mathsf{R}_V : \mathsf{Fun}_k(G/N) \to \mathsf{Fun}_k(G)$ is denoted by $\jmath^G_{G/N}$, and $\mathsf{L}_V : \mathsf{Fun}_k(G) \to \mathsf{Fun}_k(G/N)$ again by $\rho^G_{G/N}$ (there is no notational conflict here, as $\mathsf{L}_V \cong \mathsf{R}_U$ by [2] Proposition 3.15).

The functors $\imath^G_{G/N}$ and $\jmath^G_{G/N}$ are two inflation-like functors for cohomological Mackey functors. They are not isomorphic in general, and also both different from the usual inflation[1] $\mathrm{Inf}^G_{G/N}$ for Mackey functors (as defined in [5] Section 2). Similarly, the functor $\rho^G_{G/N}$ is a deflation functor.

2.4. When $\mathcal{C}$ is a k-linear category, recall ([1] Exemple 8.7.8 p. 97) that the

[1] Note that this inflation functor $\mathrm{Inf}^G_{G/N}$ does not preserve cohomological Mackey functors over k unless N is a p-group

karoubian envelope $\mathcal{C}^+$ of $\mathcal{C}$ is the category defined as follows : the object of $\mathcal{C}^+$ are the pairs (X,e), where X is an object of $\mathcal{C}$ and e is an idempotent in $\mathrm{End}_{\mathcal{C}}(X)$. If (X,e) and (Y,f) are objects of $\mathcal{C}^+$, then by definition

$$\mathrm{Hom}_{\mathcal{C}^+}\big((X,e),(Y,f)\big) = f\mathrm{Hom}_{\mathcal{C}}(X,Y)e \enspace ,$$

and the composition of morphism in $\mathcal{C}^+$ is induced by the composition of morphism in $\mathcal{C}$.

The category $\mathcal{C}^+$ is a k-linear category. Moreover, the correspondence

$$\left\{\begin{array}{ccc} X\in\mathcal{C} & \mapsto & (X,\mathrm{Id}_X)\in\mathcal{C}^+ \\ f\in\mathrm{Hom}_{\mathcal{C}}(X,Y) & \mapsto & f\in\mathrm{Hom}_{\mathcal{C}^+}\big((X,\mathrm{Id}_X),(Y,\mathrm{Id}_Y)\big)\end{array}\right.$$

is a fully faithful k-linear functor i from $\mathcal{C}$ to $\mathcal{C}^+$.

By composition, this functor induces a functor $\mathsf{I} : M \mapsto M\circ\mathsf{i}$ from the category $\mathcal{F}_k^+$ of k-linear functors from $\mathcal{C}^+$ to k-Mod, to the category $\mathcal{F}_k$ of k-linear functors from $\mathcal{C}$ to k-Mod, which is easily seen to be an equivalence of categories : the functor $\mathsf{J} : \mathcal{F}_k \to \mathcal{F}_k^+$ defined by

$$\mathsf{J}(L)\big(X,e\big) = L(e)\big(L(X)\big)$$

is a quasi-inverse to I.

2.5. Applying this to the category $\mathcal{C} = \mathsf{perm}_k(G)$, we observe that the category $\mathcal{C}^+$ is equivalent to the full subcategory $\mathsf{perm}_k^+(G)$ of kG-mod consisting of direct summands of finitely generated permutation kG-modules : this equivalence is induced by the functor $(X,e)\mapsto e(X)$. It follows that the category $\mathsf{M}_k^c(G)$ is equivalent to the category $\mathsf{Fun}_k^+(G)$ of contravariant k-linear functors from $\mathsf{perm}_k^+(G)$ to k-Mod.

2.6. It follows that when G and H are finite groups, any k-linear functor from $\mathsf{perm}_k^+(G)$ to $\mathsf{perm}_k^+(H)$ induces by composition a functor from $\mathsf{Fun}_k^+(H)$ to $\mathsf{Fun}_k^+(G)$, and by the above remarks, this yields a corresponding functor $\mathsf{M}_k^c(H)\to\mathsf{M}_k^c(G)$. In particular, when X is a direct summand of a finitely generated permutation (kH,kG)-bimodule, the functor $t_X = X\otimes_{kG}-$ yields a functor $L_X : \mathsf{M}_k^c(H)\to\mathsf{M}_k^c(G)$, and the functor $h_X = \mathrm{Hom}_{kH}(X,-)$ yields a functor $R_X : \mathsf{M}_k^c(G)\to\mathsf{M}_k^c(H)$.

2.7. We will consider here another special case of this kind of construction : let N be a normal subgroup of G, and let $H=G/N$. Also denote by U the set G/N, viewed as a (G,H)-biset. As for any kG-module W, there is an isomorphism of kH-modules

$$\mathrm{Hom}_{kG}(kU,W)\cong W^N \enspace ,$$

the functor $h_U : \mathsf{perm}_k^+(G) \to \mathsf{perm}_k^+(H)$ is the fixed points functor $W \mapsto W^N$.

Now $W^N \supseteq \mathrm{tr}_{\mathbf{1}}^N(W)$, and this inclusion is functorial with respect to W : the functor $W \mapsto \mathrm{tr}_{\mathbf{1}}^N(W)$ is a functor from RG-Mod to RH-Mod, which is a subfunctor of the functor $W \mapsto W^N$.

A natural question is then to know whether it can happen that the functor $W \mapsto \mathrm{tr}_{\mathbf{1}}^N(W)$ preserves direct summands of permutation modules. This is equivalent to requiring that the following condition holds :

2.8. Condition : *For any finite G-set X, the module $\mathrm{tr}_{\mathbf{1}}^N(kX)$ isomorphic to a direct summand of a permutation kH-module.*

2.9. Lemma : *Let G be a finite group, let N be a normal subgroup of G, let $H = G/N$, and let X be a finite G-set. Then there is an isomorphism of kH-modules*

$$\mathrm{tr}_{\mathbf{1}}^N(kX) \cong \bigoplus_{x \in G\backslash X} \mathrm{Ind}_{NG_x/N}^{G/N}(|N_x|k) \;\;,$$

where G_x is the stabilizer of x in G, and $N_x = N \cap G_x$.

Proof : The module $(kX)^N$ has a k-basis consisting of the elements

$$b_x = \sum_{n \in N/N_x} nx \;\;,$$

for $x \in N\backslash X$. This basis is G/N-invariant, and the stabilizer of b_x in G/N is equal to NG_x/N. Thus

$$(kX)^N \cong \bigoplus_{x \in G\backslash X} \mathrm{Ind}_{NG_x/N}^{G/N} k \;\;.$$

The submodule $\mathrm{tr}_{\mathbf{1}}^N(kX)$ is generated by the elements $\mathrm{tr}_{\mathbf{1}}^N(x) = |N_x| b_x$, for $x \in N\backslash X$, and these elements are permuted by G/N. The lemma follows. □

In particular, as a k-module,

$$\mathrm{tr}_{\mathbf{1}}^N(kX) \cong \bigoplus_{x \in N\backslash X} (|N_x|k) \;\;.$$

Thus if $\mathrm{tr}_{\mathbf{1}}^N(kX)$ is isomorphic to a direct summand of a permutation kH-module, since any permutation kH-module is free as a k-module, it follows that $|N_x|k$ is a projective k-module. Hence, if Condition 2.8 holds, then the following condition also holds :

2.10. Condition : *For any subgroup M of N, the module $|M|k$ is a projective k-module.*

Conversely, if Condition 2.10 holds, and if X is a finite G-set, then for any $x \in X$, the k-module $|N_x|k$ is projective, hence it is isomorphic to a direct summand of k (since the map $\lambda \mapsto |N_x|\lambda$ is a surjective morphism of k-modules from k to $|N_x|k$). It follows that $|N_x|k$ is also isomorphic to a direct summand of k as $k(NG_x/N)$-module, since the action of NG_x/N on k and $|N_x|k$ is trivial. Thus $\mathrm{Ind}_{NG_x/N}^{G/N}(|N_x|k)$ is isomorphic to a direct summand of $\mathrm{Ind}_{NG_x/N}^{G/N}(k)$ as a $k(G/N)$-module. Since this is a permutation module, it follows that $\mathrm{tr}_{\mathbf{1}}^N(kX)$ is isomorphic to a direct summand of a permutation kH-module. This shows that Conditions 2.8 and 2.10 are equivalent.

2.11. Now saying that for some $m \in \mathbb{N}$, the k-module mk is projective, is equivalent to saying that there exists an idempotent e_m of k such that e_mk is equal to the annihilator $\mathrm{Ann}_k(m)$ of m in k. Moreover Condition 2.10 obviously implies the following :

2.12. Condition : *For any prime factor p of $|N|$, the k-module pk is projective. Equivalently, there exists an idempotent e_p of k such that $\mathrm{Ann}_k(p) = e_pk$.*

Conversely, if Condition 2.12 holds, let m be any integer dividing $|N|$, and denote by e_m the sum $\sum\limits_{p|m} e_p$ of the idempotents e_p corresponding to the *distinct* prime factors of m. Then e_m is an idempotent : indeed, if p and q are distinct prime numbers, then $e_pe_q = 0$, since it is both a p-torsion and a q-torsion element. Note that, more generally, if m and n are integers such that $m|n$, then $e_me_n = e_m$.

Since moreover
$$me_m = \sum_{p|m} \frac{m}{p} pe_p = 0 \ ,$$
it follows that $e_mk \subseteq \mathrm{Ann}_k(m)$.

Now write m as a product $p_1p_2\cdots p_l$ of (possibly equal) prime numbers. We prove by induction on l that $\mathrm{Ann}_k(m) = e_mk$.

If $l = 0$, then $m = 1$, and $e_1 = 0$ generates $\mathrm{Ann}_k(1) = \{0\}$. If $l > 0$, let $p = p_1$, and $n = m/p$. If $x \in \mathrm{Ann}_k(m)$, then $pnx = 0$, so there exists $y \in k$ such that $nx = e_py$. Thus $e_pnx = nx$, i.e. $n(x - e_px) = 0$. By induction hypothesis, there exists $z \in k$ such that $x = e_px + e_nz$. Thus
$$e_mx = e_me_px + e_me_nz = e_px + e_nz = x \ ,$$

since $e_m e_r = e_r$ if $r|m$. It follows that $x \in e_m k$, as was to be shown.

This shows that Conditions 2.8, 2.10, and 2.12 are equivalent.

2.13. Remark : These conditions are fulfilled in particular if the ring k is hereditary, since any ideal of k is projective in this case.

2.14. Notation : *When the normal subgroup N of G fulfills Condition 2.12, the functor $W \mapsto \mathrm{tr}_{\mathbf{1}}^N W$ from $\mathsf{perm}_k^+(G)$ to $\mathsf{perm}_k^+(G/N)$ induces an exact functor denoted by $\sigma_{G/N}^G$ from $\mathsf{M}_k^c(G/N)$ to $\mathsf{M}_k^c(G)$.*

The exactness of $\sigma_{G/N}^G$ follows from the fact that this functor is obtained by *pre-composition* with some functor.

2.15. From now on we will assume that k is a field of characteristic $p \geq 0$. Remark 2.13 shows that Condition 2.12 is fulfilled, for any normal subgroup N of G.

Recall that the simple cohomological Mackey functors for G over k are indexed by pairs (Q, V), where Q is a p-subgroup of G (which in the case $p = 0$ should be understood as $Q = \mathbf{1}$), and V is a simple $k\overline{N}_G(Q)$-module, where, as usual, $\overline{N}_G(Q) = N_G(Q)/Q$. As an object of $\mathsf{Fun}_k^+(G)$, the functor $S_{Q,V}^G$ indexed by the pair (Q, V) can be described as follows : if W is a direct summand of a permutation kG-module, then

$$S_{Q,V}^G(W) = \mathrm{tr}_{\mathbf{1}}^{\overline{N}_G(Q)} \mathrm{Hom}_k(W[Q], V) \ \ ,$$

where $W[Q]$ is the Brauer quotient of W at Q. The functorial structure with respect to W is the obvious one. In other words $S_{Q,V}^G(W)$ is the set of $k\overline{N}_G(Q)$-homomorphisms from $W[Q]$ to V which factor through a projective $k\overline{N}_G(Q)$-module.

2.16. Proposition : *Let k be a field. If $N \trianglelefteq G$, and if V is a simple $k(G/N)$-module, then*

$$\sigma_{G/N}^G(S_{\mathbf{1},V}^{G/N}) = S_{\mathbf{1},\mathrm{Inf}_{G/N}^G V}^G \ \ .$$

Proof : The value of the functor $S_{\mathbf{1},V}^{G/N}$ on a direct summand of a permutation $k(G/N)$-module W is equal to

$$S_{\mathbf{1},V}^{G/N}(W) = \mathrm{tr}_{\mathbf{1}}^{G/N} \mathrm{Hom}_k(W, V) \ \ .$$

So for $U \in \mathsf{perm}_k^+(G)$

$$\begin{aligned}\sigma_{G/N}^G(S_{\mathbf{1},V}^{G/N})(U) &= S_{\mathbf{1},V}^{G/N}\big(\mathrm{tr}_{\mathbf{1}}^N(U)\big)\\ &= \mathrm{tr}_{\mathbf{1}}^{G/N}\mathrm{Hom}_k\big(\mathrm{tr}_{\mathbf{1}}^N(U),V\big)\ \ .\end{aligned}$$

On the other hand

$$S^G_{\mathbf{1},\mathrm{Inf}_{G/N}^G V}(U) = \mathrm{tr}_{\mathbf{1}}^G\mathrm{Hom}_k(U,\mathrm{Inf}_{G/N}^G V)\ \ .$$

Let $i : \mathrm{tr}_{\mathbf{1}}^N U \hookrightarrow U$ denote the inclusion map, and let $s : U \to \mathrm{tr}_{\mathbf{1}}^N U$ be a k-linear map such that $s\circ i = \mathrm{Id}$. If $\psi \in \mathrm{Hom}_k(\mathrm{tr}_{\mathbf{1}}^N U, V)$, then $\psi = \widetilde{\psi}\circ i$, where $\widetilde{\psi} = \psi\circ s \in \mathrm{Hom}_k(U,\mathrm{Inf}_{G/N}^G V)$. Setting $\theta = \mathrm{tr}_{\mathbf{1}}^{G/N}\psi$, for any $u\in U$

$$\begin{aligned}\theta\circ\mathrm{tr}_{\mathbf{1}}^N(u) &= \sum_{g\in G/N}\sum_{n\in N} g\widetilde{\psi}(g^{-1}nu)\\ &= \sum_{g\in G/N}\sum_{n\in N} gn^{-1}\widetilde{\psi}(g^{-1}nu)\\ &= \sum_{g\in G} g\psi(g^{-1}u)\\ &= \mathrm{tr}_{\mathbf{1}}^G(\widetilde{\psi})(u)\ \ ,\end{aligned}$$

thus $\theta\circ\mathrm{tr}_{\mathbf{1}}^N = \mathrm{tr}_{\mathbf{1}}^G(\widetilde{\psi})$. This shows that the map

$$I : \mathrm{Hom}_k(\mathrm{tr}_{\mathbf{1}}^N U, V) \to \mathrm{Hom}_k(U,\mathrm{Inf}_{G/N}^G V)$$

sending θ to $\theta\circ\mathrm{tr}_{\mathbf{1}}^N$ maps $\mathrm{tr}_{\mathbf{1}}^{G/N}\mathrm{Hom}_k(\mathrm{tr}_{\mathbf{1}}^N U,V)$ into $\mathrm{tr}_{\mathbf{1}}^G\mathrm{Hom}_k(U,\mathrm{Inf}_{G/N}^G V)$. The map I is obviously injective, as $\mathrm{tr}_{\mathbf{1}}^N : U\to\mathrm{tr}_{\mathbf{1}}^N U$ is surjective.

Conversely, let $\varphi\in\mathrm{Hom}_k(U,\mathrm{Inf}_{G/N}^G V)$, and set $\Xi = \mathrm{tr}_{\mathbf{1}}^G\varphi$. Then for $u\in U$

$$\begin{aligned}\Xi(u) &= \sum_{g\in G} g\varphi(g^{-1}u)\\ &= \sum_{g\in G/N}\sum_{n\in N} gn\varphi(n^{-1}g^{-1}u)\\ &= \sum_{g\in G/N}\sum_{n\in N} g\varphi(g^{-1}n^{-1}u)\\ &\quad (\text{since } n^{-1}g^{-1}w = g^{-1}{\cdot}gn^{-1}g^{-1}\cdot u)\\ &= \big(\mathrm{tr}_{\mathbf{1}}^{G/N}(\varphi\circ i)\big)(\mathrm{tr}_{\mathbf{1}}^N u)\ \ ,\end{aligned}$$

so $\Xi = I\big(\mathrm{tr}_{\mathbf{1}}^{G/N}(\varphi\circ i)\big)$, showing that I induces an isomorphism

$$\mathrm{tr}_{\mathbf{1}}^{G/N}\mathrm{Hom}_k(\mathrm{tr}_{\mathbf{1}}^N U,V) \to \mathrm{tr}_{\mathbf{1}}^G\mathrm{Hom}_k(U,\mathrm{Inf}_{G/N}^G V)\ \ ,$$

which is obviously functorial in U. This completes the proof. $\square$

2.17. Proposition : *Let $G = N \rtimes H$ be the semidirect product of N by a group H.*

1. *If V is a kH-module, let $\tilde{V}$ be the kG-module $\mathrm{Inf}_{G/N}^G \mathrm{Iso}_H^{G/N} V$. Then the restriction of $\tilde{V}$ to H is isomorphic to V.*
2. *The composition $\mathrm{Res}_H^G \sigma_{G/N}^G \mathrm{Iso}_H^{G/N}$ is isomorphic to the identity functor of $\mathsf{M}_k^c(H)$.*
3. *Let V and W be simple kH-modules. Then, for any $n \in \mathbb{N}$, the restriction from G to H induces a split surjection*

$$r_H^G : \mathrm{Ext}^n_{\mathsf{M}_k^c(G)}(S^G_{\mathbf{1},\tilde{V}}, S^G_{\mathbf{1},\tilde{W}}) \to \mathrm{Ext}^n_{\mathsf{M}_k^c(H)}(S^H_{\mathbf{1},V}, S^H_{\mathbf{1},W}) \ \ .$$

Proof : Assertion 1 is obvious. For Assertion 2, let W be a direct summand of a permutation kH-module, and let W' be the kH-module defined by

$$W' = \mathrm{Iso}_{G/N}^H \mathrm{tr}_{\mathbf{1}}^N \mathrm{Ind}_H^G W \ \ .$$

The set N is a set of representatives of G/H, so

$$\mathrm{Ind}_H^G W \cong \bigoplus_{n \in N} n \otimes W \ \ .$$

Moreover, for any $n \in N$ and $w \in W$,

$$\mathrm{tr}_{\mathbf{1}}^N (n \otimes w) = \sum_{x \in N} xn \otimes w = \sum_{x \in N} x \otimes w \ \ .$$

This shows that the map $\theta : w \in W \mapsto \sum_{x \in N} x \otimes w \in W'$ is a k-linear isomorphism from W to W'. Moreover, for any $h \in H$

$$\begin{aligned} h(\sum_{x\in N} x \otimes w) &= \sum_{x \in N} hx \otimes w \\ &= \sum_{x\in N} hxh^{-1} \cdot h \otimes w = \sum_{x \in N} xh \otimes w \\ &= \sum_{x \in N} x \otimes hw \ \ . \end{aligned}$$

Thus θ is actually an isomorphism of kH-modules $W \to W'$, which is obviously functorial with respect to W. So the functor $F = \mathrm{Iso}_{G/N}^H \mathrm{tr}_{\mathbf{1}}^N \mathrm{Ind}_H^G$ is isomorphic to the identity functor of $\mathsf{perm}_k^+(H)$. Assertion 2 follows, since the functor $\mathrm{Res}_H^G\sigma_{G/N}^G\mathrm{Iso}_H^{G/N}$ is the endofunctor of $\mathsf{M}_k^c(H)$ obtained by composition with F.

Assertion 1 implies that $\mathrm{Res}_H^G S_{\mathbf{1},\tilde{V}}^G \cong S_{\mathbf{1},V}^H$, so the restriction functor Res_H^G induces a k-linear map

$$r_H^G : \mathrm{Ext}^n_{\mathsf{M}_k^c(G)}(S_{\mathbf{1},\tilde{V}}^G, S_{\mathbf{1},\tilde{W}}^G) \to \mathrm{Ext}^n_{\mathsf{M}_k^c(H)}(S_{\mathbf{1},V}^H, S_{\mathbf{1},W}^H) \ .$$

Conversely, the functor $\sigma_{G/N}^G\mathrm{Iso}_H^{G/N}$ is an exact functor from $\mathsf{M}_k^c(H)$ to $\mathsf{M}_k^c(G)$, which sends $S_{\mathbf{1},V}^H$ to $S_{\mathbf{1},\tilde{V}}^G$, by Proposition 2.16. This yields a k-linear map

$$s_H^G : \mathrm{Ext}^n_{\mathsf{M}_k^c(H)}(S_{\mathbf{1},V}^H, S_{\mathbf{1},W}^H) \to \mathrm{Ext}^n_{\mathsf{M}_k^c(G)}(S_{\mathbf{1},\tilde{V}}^G, S_{\mathbf{1},\tilde{W}}^G) \ .$$

Since by Assertion 2, the functor $\mathrm{Res}_H^G\sigma_{G/N}^G\mathrm{Iso}_H^{G/N}$ is isomorphic to the identity functor of $\mathsf{M}_k^c(H)$, the composition $r_H^G \circ s_H^G$ is an isomorphism, so r_H^G is split surjective. □

3. Extensions of cohomological Mackey functors

3.1. Hypothesis : *From now on, we assume that k is a field of positive characteristic p and that G is a finite p-group.*

In this section we recall some notation and results from [2] about extensions of cohomological Mackey functors for elementary abelian p-groups. When H is a subgroup of G, we denote by $K_G(H)$ the set of complements of H in G, i.e. the set of subgroups T of G such that $H \oplus T = G$.

3.2. ([2] Corollary 8.3) When X is a subgroup of order p of G, there exist a unique cohomological Mackey functor $\begin{pmatrix} S_X^G \\ S_{\mathbf{1}}^G \end{pmatrix}$, up to isomorphism, which fits into a non split exact sequence in $\mathsf{M}_k^c(G)$ of the form

$$0 \to S_{\mathbf{1}}^G \to \begin{pmatrix} S_X^G \\ S_{\mathbf{1}}^G \end{pmatrix} \to S_X^G \to 0 \ . \tag{3.3}$$

There is also a unique cohomological Mackey functor $\begin{pmatrix} S_{\mathbf{1}}^G \\ S_X^G \end{pmatrix}$, up to isomorphism, which fits into a non split exact sequence in $\mathsf{M}_k^c(G)$ of the form

$$0 \to S_X^G \to \begin{pmatrix} S_{\mathbf{1}}^G \\ S_X^G \end{pmatrix} \to S_{\mathbf{1}}^G \to 0 \ , \tag{3.4}$$

and $\begin{pmatrix} S_{\mathbf{1}}^G \\ S_X^G \end{pmatrix}$ is isomorphic to the dual $\begin{pmatrix} S_X^G \\ S_{\mathbf{1}}^G \end{pmatrix}^*$ of $\begin{pmatrix} S_X^G \\ S_{\mathbf{1}}^G \end{pmatrix}$.

We denote by $\gamma_X^G \in \mathrm{Ext}^2_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$ the class of the exact sequence

$$\Gamma_X^G : 0 \to S_{\mathbf{1}}^G \to \begin{pmatrix} S_X^G \\ S_{\mathbf{1}}^G \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}}^G \\ S_X^G \end{pmatrix} \to S_{\mathbf{1}}^G \to 0$$

obtained by splicing the two previous short exact sequences ([2] Notation 7.4).

3.5. ([2] Section 14) When $p > 2$, and $\varphi \in \mathrm{Hom}(G, k^+)$, let U_φ^G denote the kG-module $k \oplus k$, on which G acts by

$$\forall g \in G,\ \forall (x, y) \in k^2,\ \ g(x, y) = \big(x + \varphi(g)y, y\big)\ \ .$$

Let T_φ^G denote the unique Mackey functor for G over k such that $T_\varphi^G(H)$ is zero if H is a non trivial subgroup of G, and $T_\varphi^G(\mathbf{1}) \cong U_\varphi^G$. The functor T_φ^G is cohomological, and fits in an exact sequence

$$0 \to S_{\mathbf{1}}^G \to T_\varphi^G \to S_{\mathbf{1}}^G \to 0\ \ ,$$

in $\mathsf{M}_k^c(G)$, which is non split if $\varphi \neq 0$. We denote by τ_φ^G the class of this extension in $\mathrm{Ext}^1_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$. When $\varphi \in \mathrm{Hom}(G, \mathbb{F}_p^+)$, we denote by the same symbol the composition of φ with the inclusion $\mathbb{F}_p^+ \hookrightarrow k^+$, and by τ_φ the corresponding element of $\mathrm{Ext}^1_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$.

3.6. The following conjecture was proposed in [2], and proved there for $p = 3$ ([2] Theorem 14.2) :

3.7. Conjecture : *Let k be a field of odd characteristic p, and $G \cong (C_p)^r$. Then :*

1. *The algebra $\mathcal{E} = \mathrm{Ext}^*_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$ is generated by the elements τ_φ^G in degree 1, for $\varphi \in \mathrm{Hom}(G, k^+)$, and by the elements γ_X^G in degree 2, for $X \leq G$ with $|X| = p$.*
2. *The Poincaré series for $\mathcal{E}$ is equal to*

$$\frac{1}{(1-t)\big(1-t-(p-1)t^2\big)\big(1-t-(p^2-1)t^2\big)\ldots\big(1-t-(p^{r-1}-1)t^2\big)}\ \ .$$

3.8. Proposition : [[2] Proposition 8.7 and Proposition 10.1] *Let k be a field of characteristic $p > 0$, let G be an elementary abelian p-group, and let*

H be a subgroup of index p in G. Set $I = \mathrm{Ind}_H^G S_{\mathbf{1}}^H$.

1. *Let R and S denote respectively the radical and the socle of I, as an object of $\mathsf{M}_k^c(G)$. Then $I \supset R \supseteq S \supset \{0\}$, and $I/R \cong S \cong S_{\mathbf{1}}^G$. Moreover*
$$R/S \cong L \oplus \bigoplus_{X \in K_G(H)} S_X^G \ ,$$
where L is a functor all of whose composition factors are isomorphic to $S_{\mathbf{1}}^G$, with multiplicity $p-2$.

2. *Let $Y \in K_G(H)$. The functor R has a subfunctor J isomorphic to $\iota_{G/Y}^G(S_{\mathbf{1}}^{G/Y})$, and there is an isomorphism*
$$R/J \cong L \oplus \bigoplus_{X \in K_G(H)-\{Y\}} S_X^G \ ,$$

3.9. Corollary : [[2] Corollary 10.3] *With the same notation, there is a long exact sequence of extension groups*

$$\begin{tikzcd}
\cdots \arrow[r] & L(n-1) \oplus \bigoplus_{X\in\mathcal{X}} E_G(n-2) \arrow[r] & E_G(n) \arrow[r] & E_H(n) \arrow[dll, rounded corners] \\
 & L(n) \oplus \bigoplus_{X\in\mathcal{X}} E_G(n-1) \arrow[r] & E_G(n+1) \arrow[r] & E_H(n+1) \cdots
\end{tikzcd} \ ,$$

where $E_G(n) = \mathrm{Ext}^n_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$, $E_H(n) = \mathrm{Ext}^n_{\mathsf{M}_k^c(H)}(S_{\mathbf{1}}^H, S_{\mathbf{1}}^H)$, $L(n) = \mathrm{Ext}^n_{\mathsf{M}_k^c(G)}(L, S_{\mathbf{1}}^G)$, and $\mathcal{X} = K_G(H) - \{Y\}$.

Moreover, it is easy to check that the map $E_G(n) \to E_H(n)$ in this corollary is induced by the restriction functor Res_H^G, since $\mathrm{Res}_H^G S_{\mathbf{1}}^G \cong S_{\mathbf{1}}^H$. As k is a field by Hypothesis 3.1, the conclusion of Proposition 2.17 holds and this map is (split) surjective. Thus :

3.10. Corollary : *With the same notation, for any $n \in \mathbb{N}$, there is a short exact sequence of extension groups*

$$0 \longrightarrow L(n-1) \oplus \bigoplus_{X\in\mathcal{X}} E_G(n-2) \to E_G(n) \to E_H(n) \to 0 \ .$$

4. Mackey functors concentrated at 1 and relations in $\mathcal{E}$

We first consider some generalizations of the functor L of Proposition 3.8 :

4.1. Definition : *Let k be a field of characteristic $p > 0$, and G be a finite p-group.*

- *A Mackey functor M for G over k is said to be* concentrated at $\mathbf{1}$ *if $M(H) = \{0\}$ for any non trivial subgroup H of G.*
- *A kG-module V is said to* have zero traces *if*

$$\mathrm{tr}_{\mathbf{1}}^X(V) = \{0\}$$

for any non trivial subgroup X of G.

4.2. Remark : 0) A Mackey functor concentrated at $\mathbf{1}$ is cohomological.

1) A (finitely generated) Mackey functor for G over k is concentrated at $\mathbf{1}$ if and only if all its composition factors are isomorphic to $S_{\mathbf{1}}^G$.

2) By transitivity of traces, a kG-module V has zero traces if and only if $\mathrm{tr}_{\mathbf{1}}^X(V) = \{0\}$ for any subgroup X *of order* p of G.

4.3. Proposition : *Let k be a field of characteristic p, and G be a finite p-group.*

1. *Let M be a Mackey functor for G over k. If M is concentrated at $\mathbf{1}$, then M is cohomological, and the kG-module $V = M(\mathbf{1})$ has zero traces.*
2. *If V is a kG-module having zero traces, then there is a unique Mackey functor $\widehat{V}$ for G over k such that $\widehat{V}$ is concentrated at $\mathbf{1}$ and $\widehat{V}(\mathbf{1}) \cong V$ as kG-modules.*
3. *If M is an object of $\mathsf{M}_k^c(G)$, let M^0 denote the kG-submodule*

$$M^0 = \bigcap_{1 < X \leq G} \mathrm{Ker}\ t_{\mathbf{1}}^X$$

of $M(1)$. Then M^0 has zero traces, and $\widehat{M^0}$ is the largest subfunctor of M concentrated at $\mathbf{1}$.

4. *The correspondences $M \mapsto M(\mathbf{1})$ and $V \mapsto \widehat{V}$ are mutual inverse equivalences of categories between the full subcategory of $\mathsf{M}_k^c(G)$ whose ob-*

jects are concentrated at $\mathbf{1}$*, and the full subcategory of* kG-Mod *whose objects are modules with zero traces.*

Proof : For Assertion 1, observe that for any subgroup X of G, and any $v \in V = M(\mathbf{1})$

$$r_{\mathbf{1}}^X t_{\mathbf{1}}^X v = \sum_{x \in X} x \cdot v = \mathrm{tr}_{\mathbf{1}}^X(v) \enspace .$$

This has to be zero if X is non-trivial, since $M(X) = \{0\}$ by assumption. Moreover M is obviously cohomological.

For Assertion 2, it is straightforward to check that if V has zero traces, then the assignments $M(\mathbf{1}) = V$ and $M(H) = \{0\}$ for $\mathbf{1} < H \leq G$, define a Mackey functor M for G over k. This proves the existence part of Assertion 2. Uniqueness is straightforward.

Assertion 3 is also straightforward, and Assertion 4 follows easily. □

4.4. Example : Let H be a subgroup of index p of G, and let $W = \mathrm{Ind}_H^G k$. Let $\varepsilon : W \to k$ denote the augmentation map, and set $V = \mathrm{Ker}\,\varepsilon$. Then the kG-module V has zero traces : indeed, if X is a subgroup of order p of G, then either $X \leq H$, and then $\mathrm{tr}_{\mathbf{1}}^X(W) = \{0\}$, since X acts trivially on W. Or $X \not\leq H$, and then $\mathrm{Res}_X^G W \cong \mathrm{Ind}_{\mathbf{1}}^X k \cong kX$, so there is an exact sequence

$$0 \to \mathrm{Res}_X^G V \to kX \to k \to 0 \enspace ,$$

showing that $\mathrm{tr}_{\mathbf{1}}^X(V) = \{0\}$ also in this case.

The module $\mathrm{Ind}_H^G k$ is inflated from the free module of rank 1 for the cyclic group G/H. Hence, it is uniserial, and all its subquotients are indecomposable, and characterized by their dimension, up to isomorphism. We denote by U_a the subquotient of dimension a, for $a \in \{1, \ldots p\}$. If $a \leq p-1$, the module U_a is isomorphic to a submodule of V, hence it has zero traces, and we denote by T_a the functor $\widehat{U_a}$.

Let $\varphi \in \mathrm{Hom}(G, k^+)$ with kernel H. Then the functor T_φ^G introduced in Paragraph 3.5 is isomorphic to T_2. Similarly, the functor L of Proposition 3.8 is isomorphic to T_{p-2}, since $L(1) \cong U_{p-2}$. We also denote by M the functor $T_{p-1} = \widehat{V}$. The two short exact sequences

$$(\mathbf{4.5}) \qquad 0 \to k \to U_{a+1} \to U_a \to 0 \enspace , \enspace \text{and} \enspace 0 \to U_a \to U_{a+1} \to k \to 0$$

of kG-modules yield corresponding exact sequences

$$(\mathbf{4.6}) \qquad 0 \to S_{\mathbf{1}}^G \to T_{a+1} \to T_a \to 0 \enspace , \enspace \text{and} \enspace 0 \to T_a \to T_{a+1} \to S_{\mathbf{1}}^G \to 0$$

of cohomological Mackey functors for G over k. In particular,

$$(\mathbf{4.7}) \qquad 0 \to k \to V \to U \to 0 \enspace , \enspace \text{and} \enspace 0 \to U \to V \to k \to 0$$

yield

$$(\mathbf{4.8}) \qquad 0 \to S_{\mathbf{1}}^G \to M \to L \to 0 \ , \quad \text{and} \ \ 0 \to L \to M \to S_{\mathbf{1}}^G \to 0\,.$$

We will denote by $\tau_{\varphi,L}$ the element of $\mathrm{Ext}^1_{\mathsf{M}_k^c(G)}(L, S_{\mathbf{1}}^G)$, respectively $\tau_{L,\varphi}$ the element of $\mathrm{Ext}^1_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, L)$ corresponding to this last two short exact sequences.

Assertion 2 of Proposition 3.8 can now be rephrased as follows : there exists a subfunctor Σ of R, containing S, such that

$$R = M + \Sigma, \quad S = M \cap \Sigma, \quad \Sigma/S \cong \mathop{\oplus}_{X \in K_G(H)} S_X^G \ .$$

Setting $K = L + \Sigma$, this gives the following diagram of subfunctors of I :

$$\begin{tikzcd}[column sep=small]
 & & I \arrow[d, dash] & & \\
 & & R \arrow[ddll, dash] \arrow[dr, dash] & & \\
 & & & K \arrow[ddll, dash] \arrow[dr, dash] & \\
M \arrow[dr, dash] & & & & \Sigma \arrow[ddll, dash] \\
 & L \arrow[dr, dash] & & & \\
 & & S \arrow[d, dash] & & \\
 & & \{0\}\,. & &
\end{tikzcd}$$

We give now a series of relations between elements in $\mathrm{Ext}^1_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$ and $\mathrm{Ext}^2_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$. For simplicity, we will drop the exponent G, and write τ_φ, γ_X, $S_{\mathbf{1}}$, T_φ, U_φ instead of τ_φ^G, γ_X^G, $S_{\mathbf{1}}^G$, T_φ^G, U_φ^G respectively. We start with a linearity relation between the τ's

4.9. Lemma : *Let φ and ψ be two morphisms in* $\mathrm{Hom}(G, k^+)$. *Then* $\tau_\varphi + \tau_\psi = \tau_{\varphi+\psi}$ *in* $\mathrm{Ext}^1_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}, S_{\mathbf{1}})$.

Proof : Let $0 \to S_{\mathbf{1}} \to T_\varphi \to S_{\mathbf{1}} \to 0$ be the representative for τ_φ and $0 \to S_{\mathbf{1}} \to T_\psi \to S_{\mathbf{1}} \to 0$ be the representative for τ_ψ described in Paragraph 3.5. Given that both τ_φ and τ_ψ are concentrated at the trivial subgroup, this will also be the case for $\tau_\varphi + \tau_\psi$.

We construct a representative $0 \to S_{\mathbf{1}} \to T \to S_{\mathbf{1}} \to 0$ for the sum as in Lemma A.2 working with the G-modules U_φ and U_ψ that give the functors T_φ, respectively T_ψ. Then $T(1)$ is given by the following sequence of pushout and pullback

$$\begin{tikzcd} k \oplus k \arrow[r, "i \oplus j"] \arrow[d, "\pi_1+\pi_2"'] & U_\varphi \oplus U_\psi \arrow[d] \\ k \arrow[r] & U' \end{tikzcd} , \qquad \begin{tikzcd} U' \arrow[r, "\widetilde{s\oplus t}"] & k \oplus k \\ T(1) \arrow[u] \arrow[r] & k \arrow[u, "\Delta"'] \end{tikzcd} .$$

The same computation as in Lemma A.2 gives

$$T(1) \simeq \{(a_1, a_2, b_1, b_2, c) \in k^5 | a_2 = b_2\}/\{(0, d_1, 0, d_2, -d_1 - d_2)| d_1, d_2 \in k\} .$$

This module has dimension 2, the action of G on the class of (a_1, a_2, b_1, b_2, c) is induced by the action on U_φ for the first two terms, the action on U_ψ for the next two terms and is trivial on the last. More precisely,

$$g[(a_1, a_2, b_1, b_2, c)] = [(a_1 + \varphi(g)a_2, a_2, b_1 + \psi(g)b_2, b_2, c)] .$$

Thus $T(1)$ is a G-module with k-basis $\{[(0, 1, 0, 1, 0)], [(0, 0, 0, 0, 1)]\}$ and action given by

$$g[(0, 0, 0, 0, 1)] = [(0, 0, 0, 0, 1)]$$

and

$$g[(0, 1, 0, 1, 0)] = [(\varphi(g)1, 1, \psi(g), 1, 0)] = [(0, 1, 0, 1, 0)] + [(0, 0, 0, 0, \varphi(g) + \psi(g))] .$$

This means that $T(1) \cong U_{\varphi+\psi}$ and that $T \cong T_{\varphi+\psi}$. □

The following lemmas give relations involving $(\tau_\varphi)^2$. We obtain different relations in the cases $p = 3$ and $p \geq 5$. This is tidily related to the existence of the functor T_3 only when $p \geq 5$. This functor is constructed in Example 4.4 as an application of Proposition 4.3 . The main ingredient used in both lemmas is Lemma 12.2 of [2], that we use to detect zero elements in $\mathrm{Ext}^2_{\mathsf{M}^c_k(G)}(S_{\mathbf{1}}, S_{\mathbf{1}})$.

4.10. Lemma : *Let k be a field of characteristic $p \geq 5$ and let $\varphi \in \mathrm{Hom}(G, k^+)$. Then $(\tau_\varphi)^2 = 0$ in $\mathrm{Ext}^2_{\mathsf{M}^c_k(G)}(S_{\mathbf{1}}, S_{\mathbf{1}})$.*

Proof : Recall from Example 4.4 that $T_\varphi = \begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \end{pmatrix} \cong T_2$, when $H = \mathrm{Ker}\,\varphi$. Since $p \geq 5$, we have $p-1 \geq 3$, so the module U_3 defined in Example 4.4 has zero traces. This module has a filtration

$$\{0\} \subset k \subset U_2 \subset U_3 \ ,$$

such that $U_3/k \cong U_2$ and $U_3/U_2 \cong k$. It follows that T_3 has a filtration

$$\{0\} \subset S_{\mathbf{1}} \subset T_2 \subset T_3$$

such that $T_3/S_{\mathbf{1}} \cong T_2$ and $T_3/T_2 \cong S_{\mathbf{1}}$. Applying Lemma 12.2 of [2] to this filtration shows that the sequence

$$0 \to S_{\mathbf{1}} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \end{pmatrix} \to S_{\mathbf{1}} \to 0$$

represents the zero element of $\mathrm{Ext}^2_{\mathsf{M}^c_k(G)}(S_{\mathbf{1}}, S_{\mathbf{1}})$. But this represents precisely $(\tau_\varphi)^2$. □

In the case $p = 3$, using the notation of Example 4.4, we have $M = T_2$ and $L = S_{\mathbf{1}}$. Hence, the construction in the proof of previous lemma cannot be used. We use instead the decomposition of the functor I from Proposition 3.8.

4.11. Hypothesis : *We assume from now on that $G \cong (C_p)^r$ is an elementary abelian p-group of rank r.*

4.12. Lemma : *Let k be a field of characteristic $p = 3$. Let G be an elementary abelian p-group, and $\varphi \in \mathrm{Hom}(G, \mathbb{F}_p^+)$. Then*

$$(\tau_\varphi)^2 = - \sum_{X \nleq \mathrm{Ker}\,\varphi} \hat{\gamma}_X$$

in $\mathrm{Ext}^2_{\mathsf{M}^c_k(G)}(S_{\mathbf{1}}, S_{\mathbf{1}})$.

Proof : If $\varphi = 0$, there is nothing to prove, since $\tau_\varphi = 0$ and the summation in the right hand side is empty. So we assume $\varphi \neq 0$, and set $H = \mathrm{Ker}\varphi$. With the notation of Proposition 3.8, let $I \supset R \supset S_{\mathbf{1}}$ be a filtration of the functor $I = \mathrm{Ind}_H^G S_{\mathbf{1}}^H$. Then we have $I/R \simeq S_{\mathbf{1}}$ and, using that $R/S_{\mathbf{1}} \cong S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X$, we obtain $I/S_{\mathbf{1}} \simeq \begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X \end{pmatrix}$. Moreover, we have that $R \cong \begin{pmatrix} S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X \\ S_{\mathbf{1}} \end{pmatrix}$. Lemma 12.2 of [2] applied to this filtration gives that the exact sequence

$$0 \to S_{\mathbf{1}} \to \begin{pmatrix} S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X \\ S_{\mathbf{1}} \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X \end{pmatrix} \to S_{\mathbf{1}} \to 0$$

represents the zero element of $\mathrm{Ext}^2_{\mathsf{M}^c_k(G)}(S_{\mathbf{1}},S_{\mathbf{1}})$. The result follows as this sequence also represents $(\tau_\varphi)^2+\sum\limits_{X\in K_G(H)}\hat{\gamma}_X$. □

We prove now two commutation relations satisfied by the elements τ_φ and γ_X in $\mathrm{Ext}^3_{\mathsf{M}^c_k(G)}(S_{\mathbf{1}},S_{\mathbf{1}})$:

4.13. Lemma : *Let k be a field of characteristic $p\geq 3$ and G be an elementary abelian p-group. Then*

$$\gamma_X\tau_\varphi=\tau_\varphi\gamma_X$$

for all $\varphi:G\to k^+$ and $X\leq \mathrm{Ker}\,\varphi$, $|X|=p$.

Proof : We fix $\varphi:G\to k^+$ and $X\leq\mathrm{Ker}\,\varphi$, $|X|=p$. Let M be the functor defined by the following data. First set $M(1)=U_\varphi$. Recall that this means $M(1)\cong k\oplus k$ as k-vector spaces and the action by $g\in G$ is given by $g(x,y)=(x+\varphi(g)y,y)$. Then set $M(X)=k\oplus k=\{(s,t)|s,t\in k\}$ with trivial G-action and $M(H)=0$ for all subgroups H of G different from $\mathbf{1}$ and X. Also set $t^X_{\mathbf{1}}(1,0)=(0,0)$, $t^X_{\mathbf{1}}(0,1)=(1,0)$, $r^X_{\mathbf{1}}(1,0)=(0,0)$ and $r^X_{\mathbf{1}}(0,1)=(1,0)$. It is easy to check that M is a cohomological Mackey functor. The fact that $X\leq\mathrm{Ker}\,\varphi$ gives that the action of $u\in X$ is trivial on U_φ and, thus $\sum\limits_{u\in X}u\cdot(x,y)=p\cdot(x,y)=0=r^X_{\mathbf{1}}t^X_{\mathbf{1}}(x,y)$.

The functor M has a subfunctor isomorphic to $\begin{pmatrix}S_X\\S_{\mathbf{1}}\end{pmatrix}$ and the quotient by this subfunctor is isomorphic to $\begin{pmatrix}S_{\mathbf{1}}\\S_X\end{pmatrix}$. Define the Mackey functor M' by $M'(1)=U_\varphi$, $M'(X)=k$, $r^X_{\mathbf{1}}(1)=(1,0)$ and $t^X_{\mathbf{1}}(x,y)=0$. It is straight forward that M' is a cohomological Mackey functor, that T_φ is a subfunctor of M' and that $M'=\begin{pmatrix}\begin{pmatrix}S_X\\S_{\mathbf{1}}\end{pmatrix}\\S_{\mathbf{1}}\end{pmatrix}$. Also, define the Mackey functor M'' by $M''(1)=U_\varphi$, $M''(X)=k$, $r^X_{\mathbf{1}}(1)=(0,0)$ and $t^X_{\mathbf{1}}(x,y)=y$. Again, it is straight forward that M'' is a cohomological Mackey functor, that T_φ is a factor of M'' when factoring out the socle S_X and that $M''=\begin{pmatrix}S_{\mathbf{1}}\\\begin{pmatrix}S_{\mathbf{1}}\\S_X\end{pmatrix}\end{pmatrix}$. Moreover, we get an exact sequence

$$0\to S_{\mathbf{1}}\to M'\to M\to M''\to S_{\mathbf{1}}\to 0\,.$$

It is now straightforward to show that this sequence is Yoneda equivalent to

$$0 \to S_{\mathbf{1}} \to \begin{pmatrix} S_X \\ S_{\mathbf{1}} \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_X \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \end{pmatrix} \to S_{\mathbf{1}} \to 0$$

via the canonical projections $M' \to \begin{pmatrix} S_X \\ S_{\mathbf{1}} \end{pmatrix}$, $M \to \begin{pmatrix} S_{\mathbf{1}} \\ S_X \end{pmatrix}$ and, respectively, $M'' \to \begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \end{pmatrix}$ and also equivalent to

$$0 \to S_{\mathbf{1}} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \end{pmatrix} \to \begin{pmatrix} S_X \\ S_{\mathbf{1}} \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_X \end{pmatrix} \to S_{\mathbf{1}} \to 0$$

via the canonical injections $\begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \end{pmatrix} \to M'$, $\begin{pmatrix} S_X \\ S_{\mathbf{1}} \end{pmatrix} \to M$ and, respectively, $\begin{pmatrix} S_{\mathbf{1}} \\ S_X \end{pmatrix} \to M''$. Given that the last two exact sequences represent $\gamma_X\tau_\varphi$, respectively, $\tau_\varphi\gamma_X$, the result follows. □

4.14. Lemma : *Let k be a field of characteristic $p \geq 3$ and G be an elementary abelian p-group. Then*

$$(\sum_{X \not\leq \operatorname{Ker}\varphi} \gamma_X)\tau_\varphi = \tau_\varphi(\sum_{X \not\leq \operatorname{Ker}\varphi} \gamma_X) \quad .$$

Proof : Again if $\varphi = 0$, there is nothing to prove, so we assume $\varphi \neq 0$, and set $H = \operatorname{Ker}\varphi$. Let $T = T_\varphi^G$ be the functor described in Paragraph 3.5 and let L, respectively M, be the functors described in Example 4.4.

- **First step**.

$$(*) \quad \tau_{L,\varphi}\tau_\varphi + \sum_{X \in K_G(H)} \gamma_{L,X} = 0$$

where the notation is as follows : the functor L was defined in Proposition 3.8, and the element $\tau_{L,\varphi}$ in Example 4.4.

Let $i : S_{\mathbf{1}} \to L$ be the inclusion map corresponding to the isomorphism from $S_{\mathbf{1}}$ to the socle of L. Taking the image under i of Extension 3.3 yields the following diagram

$$\begin{tikzcd}
0 \arrow[r] & S_{\mathbf{1}} \arrow[r] \arrow[d, "i"] & \begin{pmatrix} S_X \\ S_{\mathbf{1}} \end{pmatrix} \arrow[r] \arrow[d] & S_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & \begin{pmatrix} S_X \\ L \end{pmatrix} \arrow[r] & S_X \arrow[r] & 0
\end{tikzcd}$$

whose bottom line can be spliced with Extension 3.4 to give the exact sequence

$$0 \to L \to \begin{pmatrix} S_X \\ L \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_X \end{pmatrix} \to S_{\mathbf{1}} \to 0 \quad ,$$

which defines the element $\gamma_{L,X} \in \mathrm{Ext}^2_{\mathsf{M}^c_k(G)}(S_{\mathbf{1}}, L)$.

We apply Lemma 12.2 in [2] to the sequence of functors

$$\{0\} \subset L \subset R \subset I \quad ,$$

where R and I are defined in Proposition 3.8. The quotient R/L is isomorphic to $S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X$, so R can be represented by $\begin{pmatrix} S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X \\ L \end{pmatrix}$.

The quotient I/R is isomorphic to $S_{\mathbf{1}}$. The quotient I/L is isomorphic to $\begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X \end{pmatrix}$. Thus one gets a four terms exact sequence

$$0 \to L \to \begin{pmatrix} S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X \\ L \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_{\mathbf{1}} \oplus \bigoplus_{X \in K_G(H)} S_X \end{pmatrix} \to S_{\mathbf{1}} \to 0$$

representing a zero extension in $\mathrm{Ext}^2(S_{\mathbf{1}}, L)$, which is in fact (*).

- **Second step**.

$$(**) \quad \tau_{\varphi,L}\gamma_{L,X} = \tau_\varphi \gamma_X$$

where $\tau_{\varphi,L} = (0 \to S_{\mathbf{1}} \to M \to L \to 0)$

The left hand side of (**) is represented by

$$0 \to S_{\mathbf{1}} \to M \to \begin{pmatrix} S_X \\ L \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_X \end{pmatrix} \to S_{\mathbf{1}} \to 0 \quad ,$$

and the right hand side is represented by

$$0 \to S_{\mathbf{1}} \to T \to \begin{pmatrix} S_X \\ S_{\mathbf{1}} \end{pmatrix} \to \begin{pmatrix} S_{\mathbf{1}} \\ S_X \end{pmatrix} \to S_{\mathbf{1}} \to 0 \quad .$$

One constructs an equivalence of these extensions by taking inclusions maps $T \hookrightarrow M$, respectively $\begin{pmatrix} S_X \\ S_{\mathbf{1}} \end{pmatrix} \hookrightarrow \begin{pmatrix} S_X \\ L \end{pmatrix}$, and the identity elsewhere. The equality in (**) follows.

- **Third step**. Recall that we have

$$(*) \quad \tau_{L,\varphi}\tau_{\varphi} + \sum_{X \in K_G(H)} \gamma_{L,X} = 0$$

$$(**) \quad \tau_{\varphi,L}\gamma_{L,X} = \tau_{\varphi}\gamma_X$$

$$(***) \quad \tau_{\varphi,L}\tau_{L,\varphi} + \sum_{X \in K_G(H)} \gamma_X = 0$$

Left-multiplying (*) by $\tau_{\varphi,L}$ and right-multiplying (***) by τ_{φ} we get:

$$\Big(\sum_{X \in K_G(H)} \gamma_X\Big)\tau_{\varphi} = \tau_{\varphi,L}\Big(\sum_{X \in K_G(H)} \gamma_{L,X}\Big)$$

Then by (**) we replace the right term to get

$$\Big(\sum_{X \in K_G(H)} \gamma_X\Big)\tau_{\varphi} = \tau_{\varphi}\Big(\sum_{X \in K_G(H)} \gamma_X\Big)$$

hence $(\sum_{X \in K_G(H)} \gamma_X)$ and τ_{φ} commute. If G is cyclic, the sum has only one term γ_X commuting with (the unique) τ_{φ}. $\square$

5. Inductive construction of a basis of $\mathcal{E}$

Recall that k be a field of characteristic $p \geq 3$ and G is an elementary abelian p-group of rank r. Our aim is to construct a basis of the algebra $\mathrm{Ext}^*_{\mathsf{M}^c_k(G)}(S^G_{\mathbf{1}}, S^G_{\mathbf{1}})$. We do this by induction. The technical structure of the induction is build on the following general result on commutative diagrams :

5.1. Lemma : *Suppose we have the following diagram of finite dimensional k-vector spaces, where the four exterior triangles and the triangle (E_0, F_2, F_1)*

are commutative.

$$
\begin{tikzcd}
 & & E_1 & & \\
 & F_2 \arrow[ur] \arrow[rr] & & F_1 \arrow[ul] \arrow[dr] & \\
E_1 \arrow[ur] \arrow[dr] & & & & E_2 \\
 & F_1 \arrow[u] \arrow[rr] & & F_0 \arrow[u] \arrow[ur] & \\
 & & E_0 \arrow[ul] \arrow[ur] \arrow[uuul] \arrow[uuur] & &
\end{tikzcd}
$$

Suppose that, in the above diagram, the two maps $E_0 \to F_1$ are the same and that all the sequences

$$E_r \longrightarrow F_i \xrightarrow{\alpha} F_j \longrightarrow E_s\,,$$

are exact, where α is any of the horizontal or vertical maps. Suppose moreover that

(H1) $\mathrm{Ker}(E_0 \to F_0) = \mathrm{Ker}(E_0 \to F_1)$*,*

(H2) $\mathrm{Ker}(E_1 \to F_1) = \mathrm{Ker}(E_1 \to F_2)$*,*

(H3) $E_0 \to F_0 \to E_2$ is exact,

(H4) $\mathrm{Im}(E_1 \to F_1) = \mathrm{Im}(F_0 \to F_1)$*.*

Then we have

(C1) $\mathrm{Im}(F_1 \to E_1) = \mathrm{Im}(F_2 \to E_1)$

(C2) $\mathrm{Im}(F_0 \to E_2) = \mathrm{Im}(F_1 \to E_2)$

(C3) $E_0 \to F_1 \to E_2$ is exact

(C4) $\mathrm{Im}(E_1 \to F_2) = \mathrm{Im}(F_1 \to F_2)$ *and $F_0 \cong F_1 \cong F_2$*

(C5) $E_1 \to F_2 \to E_1$ is exact.

Proof : Choose a direct decomposition $F_0 = U \oplus V$ such that

$$E_0 \cong \mathrm{Ker}(E_0 \to F_0) \oplus U \quad .$$

Given (H1) and the fact that the triangle (E_0, F_1, F_0) is commutative, the horizontal map in this triangle induces an isomorphism between $\mathrm{Im}(E_0 \to F_1)$

and $\mathrm{Im}(E_0 \to F_0)$. Similarly, (H2) and the fact that the triangle (E_1, F_1, F_2) is commutative, implies that the vertical map in this triangle induces an isomorphism between $\mathrm{Im}(E_1 \to F_1)$ and $\mathrm{Im}(E_1 \to F_2)$.

Moreover, one has $\mathrm{Im}(E_0 \to F_0) \cong U$ and, using (H4) and the exactness of the sequence $E_0 \to F_0 \to F_1$, that $\mathrm{Im}(E_1 \to F_1) \cong V$.

Moreover $\mathrm{Im}(F_1 \to F_0) = \mathrm{Ker}(F_0 \to E_2) = \mathrm{Im}(E_0 \to F_0) \cong U$. Thus we have $F_1 \cong \mathrm{Im}(E_1 \to F_1) \oplus \mathrm{Im}(F_1 \to F_0) \cong V \oplus U$, and it follows that $\mathrm{Im}(F_1 \to F_2) \cong F_1/\mathrm{Im}(E_0 \to F_1) \cong V$ and

$$\mathrm{Im}(F_1 \to E_1) \cong F_1/\mathrm{Im}(F_0 \to F_1) \cong (U \oplus V)/V \cong U \ .$$

Since the sequence $F_1 \to F_2 \to E_1$ is exact, one gets

$$F_2 \cong V \oplus \mathrm{Im}(F_2 \to E_1) \ .$$

By the commutativity of the triangle (E_1, F_2, F_1),

$$\mathrm{Im}(F_2 \to E_1) \subseteq \mathrm{Im}(F_1 \to E_1) \cong U \ .$$

Since the sequence $E_1 \to F_2 \to F_1$ is exact, we get that

$$F_2 \cong V \oplus \mathrm{Im}(F_2 \to F_1) \ .$$

By the commutativity of the triangle (E_0, F_2, F_1), we have

$$\mathrm{Im}(F_2 \to F_1) \supseteq \mathrm{Im}(E_0 \to F_1) \cong U \ .$$

Hence F_2 is contained in and, respectively contains a k-vector space isomorphic to $U \oplus V$ implying that $F_2 \cong U \oplus V$ and that the previous inclusions are equalities.

In particular $\mathrm{Im}(F_2 \to E_1) \cong U$, hence (C1). Moreover, $\mathrm{Im}(E_1 \to F_2) = \mathrm{Im}(F_1 \to F_2) \cong V$, and (C4) follows. Also

$$\mathrm{Im}(F_1 \to E_2) \cong F_1/\mathrm{Im}(F_2 \to F_1) \cong (U \oplus V)/U \cong V \ ,$$

and, using (H3), we have $\mathrm{Im}(F_0 \to E_2) \cong F_0/\mathrm{Im}(E_0 \to F_0) \cong V$. The commutativity of the triangle (E_2, F_0, F_1) implies then (C2). In the triangle (E_0, F_1, F_2), we have $\mathrm{Im}(E_0 \to F_1) = \mathrm{Im}(F_2 \to F_1)$ and, from the exactness of the sequence $F_2 \to F_1 \to E_2$, we have $\mathrm{Im}(F_2 \to F_1) = \mathrm{Ker}(F_1 \to E_2)$. Now (C3) follows. Lastly, $\mathrm{Im}(E_1 \to F_2) = \mathrm{Im}(F_1 \to F_2) = \mathrm{Ker}(F_2 \to E_1)$ and (C5) follows. □

5.2. First we fix some notation : we set $F_a(n) := \mathrm{Ext}^n_{\mathsf{M}^c_k(G)}(T_a, S^G_{\mathbf{1}})$. In particular, $E_G(n) = F_1(n)$ and $L(n) = F_{p-2}(n)$. Applying $\mathrm{Hom}_{\mathsf{M}_k^c(G)}(-, S_{\mathbf{1}}^G)$ to the short exact sequences of Mackey functors in 4.4, yields two long exact sequences

$$0 \to E_G(0) \to F_{a+1}(0) \to F_a(0) \to E_G(1) \to \dots$$

and

$$0 \to F_a(0) \to F_{a+1}(0) \to E_G(0) \to F_a(1) \to \dots$$

In what follows $E_G(n-1) \to F_a(n)$, $E_G(n) \to F_a(n)$, $F_a(n) \to E_G(n)$, $F_a(n) \to E_G(n+1)$, $F_a(n) \to F_{a+1}(n)$ or $F_{a+1}(n) \to F_a(n)$ are the maps in the long exact sequences above.

5.3. Proposition : *Let $p > 5$. Then, for all n, the sequence*

$$E_G(n) \to L(n) \to E_G(n)$$

given as above is exact and $\dim L(n) = \dim E_G(n)$*. Moreover, the sequence*

$$E_G(n) \to E_G(n+1) \to E_G(n+2)$$

given by multiplication from the right with τ_φ, is exact, for any non trivial homomorphism $\varphi : G \to k^+$ and we have a direct decomposition of $E_G(n)$ inductively given by

$$\begin{array}{rclcl}
E_G(n) & \cong & E_H^G(n) & \oplus & E_H^G(n-1)\tau_\varphi \\
& \oplus & E_H^G(n-2)s_\varphi & \oplus & E_H^G(n-3)\tau_\varphi s_\varphi \\
& \oplus & E_H^G(n-4)s_\varphi^2 & \oplus & E_H^G(n-5)\tau_\varphi s_\varphi^2 \\
& \oplus & \dots & & \\
& \oplus & \bigoplus\limits_{X\in\mathcal{X}} E_G(n-2)\gamma_X & \oplus & \bigoplus\limits_{X\in\mathcal{X}} E_G(n-3)\gamma_X\tau_\varphi \\
& \oplus & \bigoplus\limits_{X\in\mathcal{X}} E_G(n-4)s_\varphi\gamma_X & \oplus & \bigoplus\limits_{X\in\mathcal{X}} E_G(n-5)s_\varphi\gamma_X\tau_\varphi \\
& \oplus & \bigoplus\limits_{X\in\mathcal{X}} E_G(n-6)s_\varphi^2\gamma_X & \oplus & \bigoplus\limits_{X\in\mathcal{X}} E_G(n-7)s_\varphi^2\gamma_X\tau_\varphi \\
& \oplus & \dots & &
\end{array}$$

where $E_H^G(m) := \sigma_{G/Y}^G \mathrm{Iso}_H^{G/Y} E_H(m)$ *and* $s_\varphi := \sum\limits_{X\in K_G(H)} \gamma_X$.

5.4. We do the proof by simultaneous induction on n and $|G|$. The cases $n = 0$ or $G = \mathbf{1}$ are trivial so we have the starting point for the induction.

We suppose the proposition true for all the proper subgroups K of G and all m, and, respectively, for $K = G$ and $m \leq n$, and we prove it for G and $n+1$.

5.5. By induction, we have that the sequence

$$E_K(m-1) \to E_K(m) \to E_K(m+1)$$

given by multiplication from the right with τ_ψ, is exact, for K and m as in the previous paragraph, and for any non trivial homomorphism $\psi : K \to k^+$. To ease notation, we set $E(n) := E_G(n)$, while still writing $E_K(n)$ when K is different from G.

5.6. The four exterior triangles and the triangle $\big(E(n-1), F_{d+2}(n), F_{d+1}(n)\big)$ in the following diagram

$$\begin{tikzcd}
 & & E(n) & & \\
 & F_{d+2}(n) \arrow[ur] \arrow[rr] & & F_{d+1}(n) \arrow[ul] \arrow[dr] & \\
E(n) \arrow[ur] \arrow[dr] & & & & E(n+1) \\
 & F_{d+1}(n) \arrow[uu] \arrow[rr] & & F_d(n) \arrow[uu] \arrow[ur] & \\
 & & E(n-1) \arrow[ul] \arrow[ur] \arrow[uuul] \arrow[uuur] & &
\end{tikzcd}$$

are commutative for all n and $1 \leq d \leq p-4$. Setting $E_i = E(n+i-1)$ and $F_i = F_{d+i}(n)$ for $i \in \{0,1,2\}$, for $d = 1$ the corresponding hypothesis (H3) in Lemma 5.1 is the fact that $E(n-1) \to E(n) \to E(n+1)$ is exact and the hypotheses (H1), (H2), (H4) are easy to check. This starts the induction on d. Then, (H1) to (H4) are satisfied for all $d \leq p-4$. Indeed, the conclusions (C3) and (C4) for $F_i = F_{d+i}(n)$ in the same Lemma 5.1 are the hypotheses (H3) and (H4) for $F_i = F_{d+i+1}(n)$. Also, we have

$$\mathrm{Ker}\big(E(n) \to F_a(n+1)\big) = \mathrm{Ker}\big(E(n) \to F_b(n+1)\big),\ \forall a, b \in \{1, \dots, p-3\} \text{ and}$$

$$\mathrm{Ker}\big(E(n+1) \to F_a(n+1)\big) = \mathrm{Ker}\big(E(n+1) \to F_b(n+1)\big),\ \forall a, b \in \{2, \dots, p-2\}.$$

Hence (H1)-(H2) for $F_i = F_{i+d+1}(n)$ are obtained from (C1) and (C2) for $F_i = F_{d+i}(n)$ and the long exact sequences in 5.2:

$$\mathrm{Im}\big(F_a(n) \to E(n)\big) = \mathrm{Ker}\big(E(n) \to F_{a-1}(n+1)\big),\ \forall a \in \{2, \dots, p-2\} \text{ and}$$

$$\mathrm{Im}\big(F_a(n) \to E(n+1)\big) = \mathrm{Ker}\big(E(n+1) \to F_{a+1}(n+1)\big), \forall a \in \{1, \dots, p-3\}.$$

5.7. Thus, by induction on d and using Lemma 5.1, (C4) and (C5) are true for all $d \in \{1, \ldots, p-4\}$. In particular, (C5) for $d = p-4$, in which case $F_2 = F_{p-2}(n) = L(n)$, together with the last part of (C4), for any integer $d \in \{1, \ldots, p-4\}$, give the first two claims in Proposition 5.3.

5.8. The last ingredient we need to continue the induction on n is that the sequence $E(n) \to E(n+1) \to E(n+2)$ is exact. These maps are given by multiplication from the right by τ_φ in Yoneda's notation. To prove the exactness of the sequence at $E(n+1)$, we fix a decomposition $G = H \oplus Y$, where Y has order p, and we explicitly decompose $E(n+1)$ with respect to $E(m)$ for $m \leq n$ and to $E_H^G(m) := \sigma_{G/Y}^G \mathrm{Iso}_H^{G/Y} E_H(m)$ for $m \leq n+1$.

5.9. Recall that we have the following properties:

- $E(n+1) \cong L(n)\tau_{L,\varphi} \oplus \bigoplus_{X \in \mathcal{X}} E(n-1)\gamma_X \oplus E_H^G(n+1)$, where we have set $\mathcal{X} = K_G(H) \setminus \{Y\}$.
- the multiplication by γ_X induces an injective map from $E(m)$ to $E(m+2)$.
- $E(n) \to L(n) \to E(n)$ is exact and $\dim L(n) = \dim E(n)$.
- the multiplication by $\tau_{L,\varphi}$ induces an injective map from $L(m-1)$ to $E(m)$.

5.10. Let's concentrate on $L(n)\tau_{L,\varphi}$. It is easy to check that

$$\mathrm{Im}\big(E(n) \to L(n)\big)\tau_{L,\varphi} = E(n)\tau_\varphi \quad .$$

Moreover from the last diagram of the induction on d we have

$$L(n) = \mathrm{Im}\big(E(n-1) \to L(n)\big) + \mathrm{Im}\big(E(n) \to L(n)\big) \quad .$$

In terms of Yoneda's composition of extensions, this becomes

$$L(n)\tau_{L,\varphi} = E(n)\tau_\varphi + E(n-1)\tau_{\varphi,L}\tau_{L,\varphi} = E(n)\tau_\varphi + E(n-1)\big(\sum_{X \in K_G(H)} \gamma_X\big) \,.$$

The above sum is not direct but we have a description of the intersection of the two terms. This is given in the following proposition.

5.11. Proposition :

$$\begin{aligned} (\#) \quad &:= E(n)\tau_\varphi \cap E(n-1)s_\varphi\gamma_X \\ &= E(n-2)\tau_\varphi s_\varphi \\ &= E(n-2)s_\varphi\tau_\varphi \end{aligned}$$

Proof : By the induction started in 5.4, the kernel of the multiplication from the left by τ_φ from $E(n)$ to $E(n+1)$ is $E(n-1)\tau_\varphi$. Hence $\dim(E(n-2)s_\varphi\tau_\varphi) = \dim\big(E(n-2)s_\varphi\big) - \dim\big(E(n-1)\tau_\varphi \cap E(n-2)s_\varphi\big)$

Also by induction on n one can assume that $E(n-1)\tau_\varphi \cap E(n-2)s_\varphi = E(n-3)\tau_\varphi s_\varphi$ and that the multiplication by s_φ gives an injective map. Thus

$$\begin{aligned}\dim\big(E(n-2)s_\varphi\tau_\varphi\big) &= \dim\big(E(n-2)s_\varphi\big) - \dim\big(E(n-3)\tau_\varphi s_\varphi\big)\\ &= \dim E(n-2) - \dim\big(E(n-3)\tau_\varphi\big)\\ &= \dim E(n-2)\tau_\varphi\end{aligned}$$

where the last equality is given by the exactness of the sequence

$$E(n-3) \to E(n-2) \to E(n-1)$$

with maps given by multiplication from the right by τ_φ. Thus we get that the multiplication from the right by s_φ gives a bijection between $E(n-2)\tau_\varphi$ and $E(n-2)\tau_\varphi s_\varphi$.

Moreover $E(n-2)\tau_\varphi s_\varphi$ is contained in the intersection (#) which enables the following computation:

$$\begin{aligned}\dim\big(E(n-2)\tau_\varphi s_\varphi\big) &\le \dim(\#)\\ &= \dim E(n)\tau_\varphi + \dim E(n-1)s_\varphi - \dim L(n)\tau_{L,\varphi}\\ &= \dim E(n)\tau_\varphi + \dim E(n-1)s_\varphi - \dim L(n)\\ &= \dim E(n-1)s_\varphi - \dim E(n-1)\tau_\varphi\\ &\le \dim E(n-1) - \dim E(n-1)\tau_\varphi\\ &= \dim E(n-2)\tau_\varphi\end{aligned}$$

We proved that the first and the last expression in the sequence are equal so all the inequalities are equalities and we get (#) and, moreover, that $\dim E(n-1) = \dim E(n-1)s_\varphi$. □

5.12. Making use of the previous two propositions we get the following decomposition of $E(n+1)$ as a direct sum :

$$E(n+1) \cong [E(n)\tau_\varphi + E(n-1)(\sum_{X\in K_G(H)} \gamma_X)] \oplus \bigoplus_{X\in\mathcal{X}} E(n-1)\gamma_X \oplus E_H^G(n+1).$$

5.13. Using this, by induction on n one can get the following direct sum decomposition, which has the advantage that for every term, the multiplications from the right by monomials of type $s_\varphi^\alpha \gamma_X^\delta \tau_\varphi^\varepsilon$, for $\alpha \geq 0$, for $\delta, \varepsilon \in \{0,1\}$, and $(\alpha, \delta) \neq (0,0)$, induce bijective maps from $E(n+1-2\alpha-2\delta-\varepsilon)$ to $E(n+1)$:

$$
\begin{array}{rclcl}
E(n+1) & \cong & E_H^G(n+1) & \oplus & E_H^G(n)\tau_\varphi \\
& \oplus & E_H^G(n-1)s_\varphi & \oplus & E_H^G(n-2)\tau_\varphi s_\varphi \\
& \oplus & E_H^G(n-3)s_\varphi^2 & \oplus & E_H^G(n-4)\tau_\varphi s_\varphi^2 \\
& \oplus & \dots & & \\
& \oplus & \displaystyle\bigoplus_{X \in \mathcal{X}} E(n-1)\gamma_X & \oplus & \displaystyle\bigoplus_{X \in \mathcal{X}} E(n-2)\gamma_X \tau_\varphi \\
& \oplus & \displaystyle\bigoplus_{X \in \mathcal{X}} E(n-3)s_\varphi \gamma_X & \oplus & \displaystyle\bigoplus_{X \in \mathcal{X}} E(n-4)s_\varphi \gamma_X \tau_\varphi \\
& \oplus & \displaystyle\bigoplus_{X \in \mathcal{X}} E(n-5)s_\varphi^2 \gamma_X & \oplus & \displaystyle\bigoplus_{X \in \mathcal{X}} E(n-6)s_\varphi^2 \gamma_X \tau_\varphi \\
& \oplus & \dots & &
\end{array}
$$

5.14. Now, for every group homomorphism $\psi : G \to k^+$ with $\mathrm{Ker}\,\psi \neq H = \mathrm{Ker}\,\varphi$, the restriction from $E(m)$ to $E_H(m)$ gives an isomorphism between the multiplication by τ_ψ on $E_H^G(m)$ and the multiplication by $\tau_{\mathrm{Res}_H^G(\psi)}$. Hence the short exact sequence $E_H(m) \to E_H(m+1) \to E_H(m+2)$ given by multiplication from the left by $\tau_{\mathrm{Res}_H^G(\psi)}$ induces a short exact sequence

$$E_H^G(m) \to E_H^G(m+1) \to E_H^G(m+2)$$

given by the multiplication from the left by τ_ψ. Summing all up we get an exact sequence $E(n) \to E(n+1) \to E(n+2)$ given by the multiplication from the left by τ_ψ.

5.15. The duality sends the multiplication from the left by τ_ψ to multiplication from the right by τ_ψ and we can exchange the roles of φ and ψ to get that the exact sequence $E(n) \to E(n+1) \to E(n+2)$ given by multiplication from the right by τ_φ is exact for any homomorphism $\varphi : G \to k^+$. This finishes the induction step on n and the proof of Proposition 5.3.

5.16. Remark that for $p = 3$ we have $\tau_\varphi^2 = s_\varphi$ thus, also in this case, τ_φ and s_φ commute. Now, using in Corollary 3.10 the fact that $L(n) = E(n)$ we get the following decomposition as a direct sum

$$E(n+1) \cong E(n)\tau_\varphi \oplus \bigoplus_{X \in \mathcal{X}} E(n-1)\gamma_X \oplus E_H^G(n+1) \quad .$$

Decomposing $E(n)$ in the same way we get

$$E(n+1) \cong E(n-1)s_\varphi \oplus \bigoplus_{X\in\mathcal{X}} E(n-2)\gamma_X\tau_\varphi \oplus E_H^G(n)\tau_\varphi \oplus \bigoplus_{X\in\mathcal{X}} E(n-1)\gamma_X \oplus E_H^G(n+1)$$

Continuing the replacement, we get the same decomposition as in 5.13.

5.17. Following the proof of Theorem 14.2 in [2] we get now the Poincaré series for $\mathrm{Ext}^*_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$ for all odd p. Indeed, the only ingredients needed in that proof are the exactness of

$$0 \longrightarrow L(n-1) \oplus \underset{X\in\mathcal{X}}{\oplus} E_G(n-2) \to E_G(n) \to E_H(n) \to 0 \ \ .$$

and that $\dim E_G(n) = \dim L(n)$. These facts are proved in Corollary 3.10, respectively in Proposition 5.3.

6. A presentation of the algebra $\mathcal{E}$

We are now able to give a presentation of the algebra $\mathcal{E} = \mathrm{Ext}^*_{\mathsf{M}_k^c(G)}(S_{\mathbf{1}}^G, S_{\mathbf{1}}^G)$. Our aim is to present an algbra $\mathcal{A}$ that, a priori, has $\mathcal{E}$ as quotient and then to show that the two algebra are isomorphic. Let r be the rank of G and $0 = H_0 < H_1 < \cdots < H_{r-1} = H < H_r = G$ be a maximal flag in G. We choose a direct sum decomposition of G

$$G = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_{r-1} \oplus Y_r \ \ ,$$

where Y_i is a complement of H_{i-1} in H_i. Let $\varphi_i : G \to \mathbb{F}_p$ be a non-trivial morphism with kernel $\bigoplus_{\substack{1\le j\le r\\ j\ne i}} Y_j$. For every $i \in \{1,\dots,r\}$ we define an atom $\hat{\tau}_i$ of degree 1, and for every $i \in \{1,\dots,r\}$ and for every subgroup $X \le G$ of order p, such that $XH_{i-1} = H_i$, we say that X *has position* i, and we define an atom $\hat{\gamma}_X$ of degree 2.

Consider the set $A = \{\hat{\tau}_i | 1 \le i \le r\} \cup \{\hat{\gamma}_X | X \le G,\, |X| = p\}$. A *word* with atoms in A is obtained in the natural way, by concatenation. The *degree* of a word is the sum of the degrees of its atoms. Denote by $\mathrm{pos}\, X$ the position of X. We say that a word is *of type* i if the atoms appearing in its decomposition are among $\hat{\tau}_i$ and $\hat{\gamma}_X$ with $\mathrm{pos}\, X = i$. The empty word is of type i for all $i \in \{1,\dots,r\}$.

6.1. For all $n \ge 0$, let A_n be the set of words of degree n with the property that they don't contain any of the following sequences of two consecutive atoms:

(S1) $\hat{\tau}_i\hat{\tau}_j$, for $1 \le j \le i \le r$;

(S2) $\hat{\tau}_i\hat{\gamma}_X$, for $1 \leq \operatorname{pos} X < i \leq r$;

(S3) $\hat{\gamma}_{Y_i}\hat{\tau}_i$, for $1 \leq i \leq r$;

(S4) $\hat{\gamma}_X\hat{\gamma}_Y$, for $1 \leq \operatorname{pos} Y < \operatorname{pos} X \leq r$;

(S5) $\hat{\gamma}_X\hat{\tau}_i$, for $1 \leq i < \operatorname{pos} X \leq r$.

We call *pre-admissible* the words w that can be written as a product $w = w_1w_2\ldots w_r$ where, for $1 \leq i \leq r$, the word w_i is of type i. We call *admissible* the pre-admissible words that do not contain any of the sequences $\hat{\tau}_i\hat{\tau}_i$ and $\hat{\gamma}_{Y_i}\hat{\tau}_i$.

6.2. Proposition : *Let w be a word of degree n. Then w belongs to the union of the A_n's for all n if and only if it is admissible.*

Proof : The forbidden sequences (S1) for $i \neq j$, (S2), (S4) and (S5) imply that for $j < i$, the atoms $\hat{\tau}_i$ and $\hat{\gamma}_X$ with $\operatorname{pos} X = i$ only appear after $\hat{\tau}_j$ and $\hat{\gamma}_Y$ with $\operatorname{pos} Y = j$. This gives a pre-admissible form $w_1w_2\ldots w_r$ for w. Moreover, in each w_i, the sequences $\hat{\tau}_i\hat{\tau}_i$ and $\hat{\gamma}_{Y_i}\hat{\tau}_i$ are explicitly forbidden by (S1) for $i = j$ and (S3). □

6.3. Proposition : *Let p be an odd prime and $\mathcal{A}$ be the graded k-algebra with generators*

$$\{\hat{\tau}_i | 1 \leq i \leq r\} \text{ in degree 1, and } \{\hat{\gamma}_X | X \leq G,\ |X| = p\} \text{ in degree 2,}$$

subject to the relations

(R1) $\hat{\tau}_i\hat{\tau}_i = 0$, *if* $p \geq 5$, *or*
$\hat{\tau}_i\hat{\tau}_i = -\sum_{X \nleq \operatorname{Ker}\varphi_i} \hat{\gamma}_X$ *if* $p = 3$.

(R2) $\hat{\tau}_i\hat{\tau}_j + \hat{\tau}_j\hat{\tau}_i = 0$ *for* $1 \leq j \leq i \leq r$, *if* $p \geq 5$, *or*
$\hat{\tau}_i\hat{\tau}_j + \hat{\tau}_j\hat{\tau}_i = \sum_{X \nleq \operatorname{Ker}(\varphi_i+\varphi_j)} \hat{\gamma}_X - \sum_{X \nleq \operatorname{Ker}\varphi_i} \hat{\gamma}_X - \sum_{X \nleq \operatorname{Ker}\varphi_j} \hat{\gamma}_X$ *if* $p = 3$.

(R3) $[\hat{\tau}_i, \sum_{X \nleq \operatorname{Ker}\varphi_i} \hat{\gamma}_X] = 0$;

(R4) $[\varphi_j(x)\hat{\tau}_i - \varphi_i(x)\hat{\tau}_j, \hat{\gamma}_{\langle x\rangle}] = 0$, *for* $1 \leq i < j \leq 1$, $x \in G$;

(R5) $[\hat{\gamma}_X, \sum_{Y<Q} \hat{\gamma}_Y] = 0$, *for all* $X < Q \leq G$, $|X| = p$, $|Q| = p^2$.

Then for all $n \geq 1$, A_n is a basis of the k-vector space of elements of degree n in $\mathcal{A}$.

6.4. Remark : In the case $p = 3$, Relation (R3) is implied by Relation (R1).

The relations (R1) to (R5) are rather technical. This comes from the need to have a finite, easy to order set of generators, appropriate for induction arguments. We can give a more intuitive presentation for the algebra $\mathcal{A}$ if we allow ourselves to increase the set of generators, by taking a generator $\hat{\tau}_\varphi$ for every group homomorphism $\varphi : G \to \mathbb{F}_p^+$. Here we identify $\hat{\tau}_{\varphi_i}$ with $\hat{\tau}_i$. The relations are then given by

(L1) $\hat{\tau}_\varphi + \hat{\tau}_\psi = \hat{\tau}_{\varphi+\psi}$.

(L2) $\hat{\tau}_\varphi\hat{\tau}_\varphi = 0$ and $[\hat{\tau}_\varphi, \sum_{X \nleq \operatorname{Ker}\varphi} \hat{\gamma}_X] = 0$, if $p \geq 5$, or

$\hat{\tau}_\varphi\hat{\tau}_\varphi = -\sum_{X \nleq \operatorname{Ker}\varphi} \hat{\gamma}_X$ if $p = 3$.

(L3) $[\hat{\tau}_\varphi, \hat{\gamma}_X] = 0$, for $X \leq \operatorname{Ker}\varphi, |X| = p$.

(L4) $[\hat{\gamma}_X, \sum_{Y<Q} \hat{\gamma}_Y] = 0$, for all $X < Q \leq G$, $|X| = p$, $|Q| = p^2$.

We leave it as an easy exercise to the reader to verify that the two presentation lead to isomorphic algebras. These two presentations yield Theorem 1.3 and 1.2, respectively.

To prove Proposition 6.3, we need the following sequence of lemmas.

6.5. Lemma : *Let $l < m$ and let X and Y be subgroups of order p of G, such that* $\operatorname{pos} X = m$ *and* $\operatorname{pos} Y = l$. *Then we have $\hat{\gamma}_X\hat{\gamma}_Y = \hat{\gamma}_Y\hat{\gamma}_X + W$ where W is a sum of admissible words $\hat{\gamma}_{X'}\hat{\gamma}_{X''}$ of type m and degree* 4.

Proof : Firstly we prove that all the subgroups of order p of $\langle X, Y\rangle$, different from Y, have position m. Indeed, if x is a generator of X and y is a generator of Y, then denote by X_c the subgroup of $\langle X, Y\rangle$ generated by $x + cy$ for $1 \leq c \leq p-1$. As $cy \in Y \leq H_{m-1}$ and $XH_{m-1} = H_m$ we get that $X_cH_{m-1} = H_m$ so X_c has position m.

Secondly we use Relation (R5) with $Q = \langle X, Y\rangle$ to get

$$\hat{\gamma}_X\hat{\gamma}_Y + \sum_{c=1}^{p-1} \hat{\gamma}_X\hat{\gamma}_{X_c} = \sum_{c=1}^{p-1} \hat{\gamma}_{X_c}\hat{\gamma}_X + \hat{\gamma}_Y\hat{\gamma}_X$$

Besides $\hat{\gamma}_X\hat{\gamma}_Y$, all the words appearing in the above equation are admissible and, excepting $\hat{\gamma}_Y\hat{\gamma}_X$, they are all of type m. The result follows. □

6.6. Lemma : *Let $l < m$ and X of order p with* $\operatorname{pos} X = m$. *Then*

$\hat{\gamma}_X\hat{\tau}_l = \hat{\tau}_l\hat{\gamma}_X + W$, where W is a linear combination of pre-admissible words of type m and degree 3.

Proof : Let x be a generator of X. If we set $\alpha := \varphi_l(x)$ and $\beta := \varphi_m(x)$, relation (R4) gives that

$$\hat{\gamma}_X(\beta\hat{\tau}_l - \alpha\hat{\tau}_m) = (\beta\hat{\tau}_l - \alpha\hat{\tau}_m)\hat{\gamma}_X$$

As $\hat{\tau}_l\hat{\gamma}_X$, $\hat{\gamma}_X\hat{\tau}_m$ and $\hat{\tau}_m\hat{\gamma}_X$ are all pre-admissible words of degree 3, the latter two being of type m, and $\beta \neq 0$ the result follows. The fact that we only get pre-admissibility (and not admissibility) comes from the fact that we might have $X = Y_m$. $\square$

6.7. Lemma : *Every word in $w \in \mathcal{A}$ can be written as a finite sum of pre-admissible words, i.e. $w = \sum_{j=1}^{s} w_{j1}w_{j2}\ldots w_{jr}$, with w_{ji} of type i for $1 \leq i \leq r$ (the w_{ji}'s are allowed to be the empty word). Moreover, if w is composed of atoms of position at least l, then the w_{ji}'s are empty for all $i < l$ and $j = 1, 2, \ldots, s$.*

Proof : We define a quadruple lexicographic order $\mathcal{O}$ on the set of words in $\mathcal{A}$ first by their degree then, for equal degrees, by the number of atoms of degree 1 (i.e. the $\hat{\tau}_i$'s) appearing in the word, then by the degree of the left term in the decomposition uv, where v is pre-admissible, of maximal possible degree, then, finally, by the number of atoms in v of higher position than the rightmost atom in u. We prove the lemma by induction on the order $\mathcal{O}$. We'll see in the proof that the order by the number of atoms of degree 1 appearing in the word is only needed for the case $p = 3$.

The lemma is trivial for words of degree 1 and straight forward, using Relations (R1) and (R2), for words of degree 2. Now take a word $w \in \mathcal{A}$ of degree $d \geq 3$. Write $w = uv$, a concatenation, with v pre-admissible, of maximal possible degree. If $\deg u = 0$, i.e. u is empty, then w is pre-admissible and we are done. If the number of atoms in v of higher position than the rightmost atom in u is 0 or the position of the leftmost atom in v is n then $w = v$ and we are again done. It is very important to remark that all the manipulations we make in the proof, using the relations in Proposition 6.3, are not increasing the number of atoms of order 1. Suppose now the lemma is true for all words smaller than w for the order $\mathcal{O}$. Let a be the rightmost atom of u i.e. $u = w'a$. If $u \neq a$ then av is of degree smaller than w so by the induction hypothesis on the degree $av = \sum_{j=1}^{s} w_j$ with w_j pre-admissible. Moreover $w'w_j$ is s maller than w with respect to the order $\mathcal{O}$, for all $j = 1, 2, \ldots, s$, so, by the induction hypothesis, $w'w_j$ is a sum of pre-admissible elements. Also by the induction hypothesis, the atoms appearing in the new words are of higher or equal position than the ones appearing in w. Thus we can suppose that $u = a$ is an atom of position m so $u = \hat{\tau}_m$ or $u = \hat{\gamma}_X$ with $\operatorname{pos} X = m$.

Case 1. Let $u = \hat{\gamma}_X$ with X of position m. We distinguish two sub-cases:

i) $v = \hat{\gamma}_Y v'$ with Y of position l and $l < m$. Then, using Lemma 6.5, we have $\hat{\gamma}_X v = \hat{\gamma}_X \hat{\gamma}_Y v' = \hat{\gamma}_Y \hat{\gamma}_X v' + W v'$. We have that $\hat{\gamma}_X v'$ is smaller than w, hence, by the induction hypothesis, it is equal to a sum of pre-admissible words, composed of atoms of position at most l. Thus the words obtained from these by multiplying to the left by $\hat{\gamma}_Y$ are still pre-admissible. The expression Wv' is a sum of words of type $\hat{\gamma}_{X'} \hat{\gamma}_{X''} v'$ with the first two atoms of position m. Again, by the induction hypothesis $\hat{\gamma}_{X''} v'$ is equal to a sum of pre-admissible words w_i, which, moreover, have less atoms of lower position than m than $\hat{\gamma}_Y v'$. Thus $\hat{\gamma}_{X'} w_i$ is smaller than w for the order $\mathcal{O}$ and, by the induction hypothesis, the former is equal to a sum of pre-admissible words.

ii) $v = \hat{\tau}_l v'$ with $l < m$. Then, using Lemma 6.6, there exists a linear combination W of words of type m such that $\hat{\gamma}_X v = \hat{\gamma}_X \hat{\tau}_l v' = \alpha \hat{\tau}_l \hat{\gamma}_X v' + W v'$, with $\alpha \in k$. Proceeding analogously to the part i) above we obtain that both $\hat{\tau}_l \hat{\gamma}_X v'$ and $W v'$ are equal to linear combinations of pre-admissible words.

Case 2. Let $u = \hat{\tau}_m$. We distinguish two sub-cases.

i) $v = \hat{\gamma}_Y v'$ with $\operatorname{pos} Y = l$ and $l < m$. Then $Y \leq \operatorname{Ker} \varphi_m$ and, using Relation (R4), we have $\hat{\tau}_m \hat{\gamma}_Y v' = \hat{\gamma}_Y \hat{\tau}_m v'$. Moreover $\hat{\tau}_m v'$ is of degree smaller than w so, by the induction hypothesis, $\hat{\tau}_m v'$ is a linear combination of pre-admissible words having only atoms of position greater or equal than l. So multiplying by $\hat{\gamma}_Y$ to the left still keeps these words pre-admissible.

ii) $v = \hat{\tau}_l v'$ with $l < m$. Then, for $p \geq 5$, using Relation (R2) we have $\hat{\tau}_m v = \hat{\tau}_m \hat{\tau}_l v' = -\hat{\tau}_l \hat{\tau}_m v'$ and the result as in part i) of Case 2. When $p = 3$ the proof is more difficult. Using also Relation (R2), we have $\hat{\tau}_m v = \hat{\tau}_m \hat{\tau}_l v' = \sum\limits_{X \not\leq \operatorname{Ker}(\varphi_m + \varphi_l)} \hat{\gamma}_X v' - \sum\limits_{X \not\leq \operatorname{Ker} \varphi_m} \hat{\gamma}_X v' - \sum\limits_{X \not\leq \operatorname{Ker} \varphi_l} \hat{\gamma}_X v' - \hat{\tau}_l \hat{\tau}_m v'$. Moreover $\operatorname{pos} X > l$ for all $\hat{\gamma}_X$ appearing in the right hand side term of this equality. Now $\tau_m v'$ has smaller degree than w, so, by the induction hypothesis, it is equal to a linear combination of pre-admissible words. The other words appearing in the right hand side term have all a smaller number number of atoms of degree 1 than w, so, by the induction hypothesis, they are all equal to linear combinations of pre-admissible words. Remark here that Case 1 and Case 2 i) are complet ely solving the case of words having at most one atom of degree 1, without needing the Case 2 ii). So there is no problem in starting the induction process.

□

Proof : [of Proposition 6.3] Firstly we prove that A_n is a set of generators in degree n. Using Lemma 6.7 we have that any word of degree n can be written as a linear combination of pre-admissible words $w_1w_2\dots w_r$, i.e. with w_i of type i for $1 \leq i \leq r$. Let $w = w_1w_2\dots w_r$ be one of these words. By Lemma 6.2, it remains to show that, possibly using new linear combinations, we also eliminate the expressions $\tau_i\tau_i$ and $\hat{\gamma}_{Y_i}\hat{\tau}_i$ that might appear in the words w_i. Let i be lowest index such that w_i has forbidden sequences, with the convention that $i = r+1$ is w contains no forbidden sequence. We define a triple lexicographic order $\mathcal{O}$ on the words, given by decreasing order on the index i defined previously, then by the number of atoms of position i and, finally, by the decreasing order of the degree of the its admissible prefix of maximal degree. We proceed by induction on $\mathcal{O}$. If the degree of the word is at most one or there are no atoms of degree 1 or the word is already admissible, then there is nothing to prove. Else, denote $w_{<i} := w_1w_2\dots w_{i-1}$ and by $w_{>i} := w_{i+1}w_{i+2}\dots w_r$. Let f be the leftmost forbidden sequence in w_i. There are two kinds of forbidden sequences to consider.

i) $f = \hat{\tau}_i\hat{\tau}_i$. Write $w = u_1fu_2$. Using Relation (R1), we have $f = 0$ when $p \geq 5$ and $f = \sum_{X \nleq \operatorname{Ker}\varphi_i} \hat{\gamma}_X$ when $p = 3$. So, the case $p \geq 5$ is trivial ,and in the following we study the case $p = 3$. Now for $X \nleq \operatorname{Ker}\varphi_i$ we have $\operatorname{pos} X \geq i$. If $\operatorname{pos} X = i$ then the word $w_{<i}u_1\hat{\gamma}_X$ doesn't have any forbidden sequence, and, by induction hypothesis, $\hat{\gamma}_X u_2 w_{>i}$ is equal to a linear combination of admissible words. Suppose now that $\operatorname{pos} X > i$. Using Lemma 6.7, $\hat{\gamma}_X u_2 w_{>i}$ is equal to a linear combination of pre-admissible words with atoms of position at least i. Let $v_i \dots v_r$ be one of these words. Then $w_{<i}u_1v_i\dots v_r$ is a pre-admissible word with less atoms position i than w. By the induction hypothesis, $w_{<i}u_1v_i\dots v_r$ is equal to a linear combination of admissible words.

ii) $f = \hat{\gamma}_{Y_i}\hat{\tau}_i$. Write $w = u_1fu_2$. Relation (R3) gives

$$\hat{\gamma}_{Y_i}\hat{\tau}_i = \hat{\tau}_i \sum_{X \nleq \operatorname{Ker}\varphi_i} \hat{\gamma}_X + \sum_{X \nleq \operatorname{Ker}\varphi_i - \{Y_i\}} \hat{\gamma}_X\hat{\tau}_i\,.$$

As $H_{i-1} \leq \operatorname{Ker}\varphi_i$ we have that any $X \nleq \operatorname{Ker}\varphi_i$ is of position at least i. The words $w_{<i}u_1\hat{\tau}_i\hat{\gamma}_{Y_i}$, $w_{<i}u_1\hat{\tau}_i\hat{\gamma}_X$ and $w_{<i}u_1\hat{\gamma}_X\hat{\tau}_i$ with X of position i contain no forbidden sequence and are of degree bigger than $w_{<i}u_1$, so,

by the induction hypothesis these words multiplied by $w_{>i}$ are equal to linear combinations of admissible words. In the case of the $u_1\hat{\tau}_i\hat{\gamma}_X$ and $u_1\hat{\gamma}_X\hat{\tau}_i$ with X of position greater than i, using Lemma 6.7, $\hat{\gamma}_X\hat{\tau}_i u_2 w_{>i}$ and $\hat{\tau}_i\hat{\gamma}_X u_2 w_{>i}$ are equal to a linear combination of pre-admissible words with atoms of position at least i. Let $v_i \dots v_r$ be one of these words. Then $w_{<i} u_1 v_i \dots v_r$ has less atoms of position i than w, hence, by the induction hypothesis, it is equal to a linear combination of admissible words.

This proves that the algebra $\mathcal{A}$ is generated as k-module in degree n by A_n.

Secondly, to prove the k-linear independence of the words in A_n, we construct k-algebra homomorphism Θ from $\mathcal{A}$ onto $\mathcal{E}$, by setting: $\Theta(\hat{\tau}_i) := \tau_{\varphi_i}$, $\Theta(\hat{\gamma}_X) := \gamma_X$. This is indeed a surjective algebra homomorphism given that the relations in $\mathcal{A}$ between the k-generators are all satisfied by their images through Θ inside $\mathrm{Ext}(S_{\mathbf{1}}, S_{\mathbf{1}})$ as showed in Lemma 4.9, Lemma 4.10, Lemma 4.13 and Lemma 4.14 in this paper, and in [2, Proposition 12.9]. □

6.8. This gives a presentation by generators and relations of $\mathcal{E}$ as stated in Conjecture 3.7.

6.9. Theorem : *The morphism Θ is an algebra isomorphism from $\mathcal{A}$ to* $\mathrm{Ext}^*_{\mathsf{M}^c_k(G)}(S^G_{\mathbf{1}}, S^G_{\mathbf{1}})$.

Proof : $\Theta : \mathcal{A} \to \mathrm{Ext}^*_{\mathsf{M}^c_k(G)}(S^G_{\mathbf{1}}, S^G_{\mathbf{1}})$ is surjective and, in every degree n, the algebra $\mathrm{Ext}^n_{\mathsf{M}^c_k(G)}(S^G_{\mathbf{1}}, S^G_{\mathbf{1}})$ has at least the dimension over k of A^n. Hence $\mathrm{Ext}^n_{\mathsf{M}^c_k(G)}(S^G_{\mathbf{1}}, S^G_{\mathbf{1}})$ and A^n have the same dimension over k and Θ is an isomorphism. □

A. Arithmetics of extensions in an abelian category

This Appendix contain a series of classical results on computations in the graded algebra of extensions in an abelian category. We present the general framework and include the proofs of those results.

Let $\mathcal{A}$ be an abelian category. Consider following exact sequences in $\mathcal{A}$

$$0 \longrightarrow X \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n \longrightarrow Y \longrightarrow 0$$

and

$$0 \longrightarrow X \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \cdots \longrightarrow B_n \longrightarrow Y \longrightarrow 0$$

representing the elements $\eta, \xi \in \mathrm{Ext}^n_{\mathcal{A}}(Y, X)$. Then $\eta + \xi$ is represented by the exact sequence

$$0 \longrightarrow X \longrightarrow C_1 \longrightarrow A_2 \oplus B_2 \longrightarrow \cdots \longrightarrow A_{n-1} \oplus B_{n-1} \longrightarrow C_n \longrightarrow Y \longrightarrow 0$$

where C_1, respectively C_n, is pushout, respectively pullback, of the following diagrams.

$$\begin{tikzcd} X \oplus X \arrow[r] \arrow[d, "\pi_1+\pi_2"'] & A_1 \oplus B_1 \arrow[d] \\ X \arrow[r] & C_1 \end{tikzcd} , \qquad \begin{tikzcd} A_n \oplus B_n \arrow[r] & Y \oplus Y \\ C_n \arrow[u] \arrow[r] & Y \arrow[u, "\Delta"'] \end{tikzcd} .$$

When $n = 1$ there is an ambiguity of this construction, given by the order in which we take the pullback and the pushout. In fact, both choices lead to equivalent exact sequences. To show this we need the following technical lemma in abelian categories. Recall that if X is the pullback of the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"] & T \end{tikzcd}$$

one also says that the square (X, Y, Z, T) is cartesian and we have that X is isomorphic to the kernel of $Y \oplus Z \xrightarrow{f-g} T$. Analogously, if T is the pushout of the diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"] & Y \arrow[d] \\ Z \arrow[r] & T \end{tikzcd}$$

one also says that the square (X, Y, Z, T) is co-cartesian and and we have that T is isomorphic to the cokernel of $X \xrightarrow{s \oplus t} Y \oplus Z$.

A.1. Lemma : *Let $\mathcal{A}$ be an abelian category, and let*

$$\begin{tikzcd}
& 0 \arrow[rr] & & A \arrow[rr, "\alpha"] \arrow[dl, "a"'] & & B \arrow[rr, "\beta"] \arrow[dl, "b"'] & & C \arrow[rr] \arrow[dl, "c"'] & & 0 \\
0 \arrow[rr] & & D \arrow[rr, "\delta"] & & E \arrow[rr, "\varepsilon"] & & F \arrow[rr] & & 0 & \\
& 0 \arrow[rr] & & L \arrow[uu, "\varphi"] \arrow[rr, "\lambda"] \arrow[dl, "l"'] & & M \arrow[uu, "\psi"] \arrow[rr, "\mu"] \arrow[dl, dashed, "m"'] & & N \arrow[uu, "\chi"] \arrow[rr] \arrow[dl, "n"'] & & 0 \\
0 \arrow[rr] & & P \arrow[uu, "\tau"] \arrow[rr, "\pi"] & & Q \arrow[uu, "\sigma"] \arrow[rr, "\theta"] & & R \arrow[uu, "\rho"] \arrow[rr] & & 0 &
\end{tikzcd}$$

be a commutative diagram with exact rows, such that the square (A, B, D, E) is co-cartesian, (Q, R, E, F) and (M, N, B, C) are cartesian, and the morphism $n : N \to R$ is an isomorphism.

Then there exists a unique morphism $m : M \to L$ such that $n\mu = \theta m$ and $b\psi = \sigma m$. Moreover, the square (L, M, P, Q) is commutative, and co-cartesian, and the morphisms τ, φ and c are isomorphisms.

Proof : Via the universal properties of pullback/pushout and using that the horizontal sequences are exact, one gets that φ, τ and c are isomorphisms, (A, B, D, E) is cartesian and, (Q, R, E, F) and (M, N, B, C) are co-cartesian. Now set $u = b\psi$ and $v = n\mu$. Then

$$\varepsilon u = \varepsilon b\psi = c\beta\psi = c\chi\mu = \rho n\mu = \rho v \quad .$$

Since the square (Q, R, E, F) is cartesian, there exists a unique morphism $m : M \to Q$ such that

$$u = \sigma m \ \text{ and } \ v = \theta m \quad .$$

Now setting $s = m\lambda$ and $t = \pi l$, we have

$$\sigma s = \sigma m\lambda = u\lambda = b\psi\lambda = b\alpha\varphi = \delta a\varphi = \delta\tau l = \sigma\pi l = \sigma t \quad .$$

Moreover

$$\theta s = \theta m\lambda = v\lambda = n\mu\lambda = 0 \quad ,$$

and

$$\theta t = \theta\pi l = 0 \quad .$$

Thus $\theta s = \theta t$, and $\sigma s = \sigma t$. As the square (Q, R, E, F) is cartesian, this implies $s = t$, hence the square (L, M, P, Q) is commutative.

Now let X be any object of $\mathcal{A}$, and let $f : X \to M$ and $g : X \to P$ be morphisms such that $\pi g = mf$. Setting $w = \psi f$ and $r = \tau g$, we have that

$$bw = b\psi f = \sigma m f = \sigma\pi g = \delta\tau g = \delta r \quad .$$

As (A, B, D, E) is cartesian, there is a unique morphism $y : X \to A$ such that

$$w = \alpha y \quad \text{and} \quad r = ay \quad .$$

Setting $z = \varphi^{-1} y : X \to L$, we have that

$$lz = l\varphi^{-1} y = \tau^{-1} a y = \tau^{-1} r = g \quad .$$

Similarly

$$\psi\lambda z = \psi\lambda\varphi^{-1} y = \alpha\varphi\varphi^{-1} y = \alpha y = w = \psi f \quad .$$

Moreover

$$\mu\lambda z = 0 \quad ,$$

and

$$n\mu f = \theta m f = \theta\pi g = 0 \quad .$$

On the other hand

$$\chi\mu f = \beta\psi f = \beta\alpha y = 0 \quad .$$

As the morphism $(\chi, n) : N \to C \oplus R$ is an monomorphism, it follows that $\mu f = 0 = \mu\lambda z$. Since moreover $\psi\lambda z = \psi f$, and since (M, N, B, C) is cartesian, it follows that $\lambda z = f$.

If there is another morphism $z' : X \to L$ such that $lz' = g$ and $\lambda z' = f$, then the morphism $x = z - z'$ is such that $lx = 0$ and $\lambda x = 0$. Then

$$\alpha\varphi x = \psi\lambda x = 0 \quad ,$$

and

$$a\varphi x = \tau l x = 0 \quad .$$

Since (A, B, D, E) is cartesian, it follows that $\varphi x = 0$, hence $x = 0$.

This shows that the square (L, M, P, Q) is cartesian. Thus, to show that this square is also co-cartesian it is enough to have that $P \oplus M \xrightarrow{m+\pi} Q$ is an epimorphism. To prove this, suppose that we have $Y \in \mathcal{A}$ and $f : Q \to Y$ such that $f\pi = 0$ and $fm = 0$. Then f factors through the cokernel of π so there exist $g : R \to Y$ such that $f = g\theta$. Then $0 = fm = g\theta m = gn\mu$. As μ is an epimorphism and n is an isomorphism, the morphism $n\mu$ is also an epimorphism, so $g = 0$ and, thus $f = 0$. $\square$

When considering $\mathcal{A}$ as a module category we have an explicit description of the sum in $\mathrm{Ext}^1_{\mathcal{A}}(Y, X)$.

A.2. Lemma : *Let $\eta, \xi \in \mathrm{Ext}^1_{\mathcal{A}}(Y, X)$. When constructing a representative for $\eta + \xi$ both choices for the order pullback, pushout lead to the same construction. Moreover, if* $0 \longrightarrow X \xrightarrow{i} A \xrightarrow{s} Y \longrightarrow 0$ *is a representative for η and* $0 \longrightarrow X \xrightarrow{j} B \xrightarrow{t} Y \longrightarrow 0$ *is a representative of ξ. Then a representative for $\eta + \xi$ is given by* $0 \longrightarrow X \longrightarrow C \longrightarrow Y \longrightarrow 0$ *with*

$$C \simeq \{(a, b, x) \in A \oplus B \oplus X | s(a) = t(b)\} / \{(i(x_1), j(x_2), -x_1 - x_2) | x_1, x_2 \in X\}\,.$$

Proof : To simplify notation, in the proof will be understood that a, b, x, y run through A, B, X, respectively Y. Also let

$$I = \{(i(x_1), j(x_2), -x_1 - x_2) | x_1, x_2 \in X\}\,.$$

Consider $0 \longrightarrow X \longrightarrow C_{ob} \longrightarrow Y \longrightarrow 0$ be the representative for $\eta + \xi$ given by taking first the pushout and then the pullback. We have

$$\begin{tikzcd} X \oplus X \arrow[r, "i \oplus j"] \arrow[d, "\pi_1 + \pi_2"'] & A \oplus B \arrow[d] \\ X \arrow[r] & C' \end{tikzcd}\ , \qquad \begin{tikzcd} C' \arrow[r, "\widetilde{s \oplus t}"] & Y \oplus Y \\ C_{ob} \arrow[u] \arrow[r] & Y \arrow[u, "\Delta"'] \end{tikzcd}\ .$$

with $C' \simeq A \oplus B \oplus X / I$ and, thus,

$$C_{ob} \simeq \{(c, y) | c \in C', y \in Y, \widetilde{s \oplus t} = \Delta(y)\} \simeq \{(a, b, x) | s(a) = t(b)\} / I\,.$$

Similarly, consider $0 \longrightarrow X \longrightarrow C_{bo} \longrightarrow Y \longrightarrow 0$ be the representative for $\eta + \xi$ given by taking first the pullback and then pushout. We have

$$\begin{tikzcd} A \oplus B \arrow[r, "s \oplus t"] & Y \oplus Y \\ C'' \arrow[u] \arrow[r] & Y \arrow[u, "\Delta"'] \end{tikzcd}\ , \qquad \begin{tikzcd} X \oplus X \arrow[r, "\widetilde{i \oplus j}"] \arrow[d, "\pi_1 + \pi_2"'] & C'' \arrow[d] \\ X \arrow[r] & C_{bo} \end{tikzcd}\ .$$

with $C'' \simeq \{(a, b, y) | s(a) = y = t(b)\} \simeq \{(a, b) | s(a) = t(b)\}$ and, thus, $C_{bo} \simeq \{(a, b, x) | s(a) = t(b)\} / I$. Hence we obtain $C_{ob} = C_{bo}$ and there is no ambiguity in the construction of a representative of $\eta + \xi$. $\square$

We introduce here a notation we extensively use in the paper. Let $X \xrightarrow{f} A \xrightarrow{g} Y$ be an exact sequence at A. Then we write $A \simeq \begin{pmatrix} \mathrm{Coker}\, f \\ \mathrm{Ker}\, g \end{pmatrix}$.

In particular, we have a short exact sequence

$$0 \longrightarrow X \longrightarrow \begin{pmatrix} Y \\ X \end{pmatrix} \longrightarrow Y \longrightarrow 0 \ .$$

and a representative in $\mathrm{Ext}^2_{\mathcal{A}}(Y, X)$ can be written as

$$0 \longrightarrow X \longrightarrow \begin{pmatrix} Z \\ X \end{pmatrix} \longrightarrow \begin{pmatrix} Y \\ Z \end{pmatrix} \longrightarrow Y \longrightarrow 0 \ .$$

With this notation we have an easy way to compute the exact sequence representing the sum in $\mathrm{Ext}^2_{\mathcal{A}}(Y, X)$. The computation is an easy consequence of the following technical lemma whose proofs are left to the reader.

A.3. Lemma : *Let $\mathcal{A}$ be an abelian category, and let*

$$\begin{tikzcd}
0 \arrow[r] & X \oplus X \arrow[r, "\alpha"] \arrow[d, "a"] & \begin{pmatrix} A \\ X \end{pmatrix} \oplus \begin{pmatrix} B \\ X \end{pmatrix} \arrow[r, "\beta"] \arrow[d, "b"] & A \oplus B \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & X \arrow[r, "\delta"] & C \arrow[r, "\varepsilon"] & A \oplus B \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram with exact rows, such that the square on the left is co-cartesian. Then c is an isomorphism and, hence, $C \simeq \begin{pmatrix} A \oplus B \\ X \end{pmatrix}$.

and its dual

A.4. Lemma : *Let $\mathcal{A}$ be an abelian category, and let*

$$\begin{tikzcd}
0 \arrow[r] & A \oplus B \arrow[r, "\alpha"] & \begin{pmatrix} Y \\ A \end{pmatrix} \oplus \begin{pmatrix} Y \\ B \end{pmatrix} \arrow[r, "\beta"] & Y \oplus Y \arrow[r] & 0 \\
0 \arrow[r] & A \oplus B \arrow[r, "\delta"] \arrow[u, "f"] & C' \arrow[r, "\varepsilon"] \arrow[u, "g"] & Y \arrow[r] \arrow[u, "h"] & 0
\end{tikzcd}$$

be a commutative diagram with exact rows, such that the square on the right is cartesian. Then f is an isomorphism and, hence, $C' \simeq \begin{pmatrix} Y \\ A \oplus B \end{pmatrix}$.

Remark that, in the above notation, $\begin{pmatrix} A \oplus B \\ X \end{pmatrix}$ has $\begin{pmatrix} A \\ X \end{pmatrix}$ and $\begin{pmatrix} B \\ X \end{pmatrix}$ as subobjects and $\begin{pmatrix} Y \\ A \oplus C \end{pmatrix}$ has $\begin{pmatrix} Y \\ A \end{pmatrix}$ and $\begin{pmatrix} Y \\ C \end{pmatrix}$ as factors. We conclude with the computation of the exact sequence representing the sum in $\mathrm{Ext}^2_{\mathcal{A}}(Y, X)$.

A.5. Lemma : *Let $\eta, \xi \in \mathrm{Ext}^2_{\mathcal{A}}(Y, X)$ be represented by*

$$0 \longrightarrow X \longrightarrow \begin{pmatrix} Z_1 \\ X \end{pmatrix} \longrightarrow \begin{pmatrix} Y \\ Z_1 \end{pmatrix} \longrightarrow Y \longrightarrow 0 \ ,$$

respectively by

$$0 \longrightarrow X \longrightarrow \begin{pmatrix} Z_2 \\ X \end{pmatrix} \longrightarrow \begin{pmatrix} Y \\ Z_2 \end{pmatrix} \longrightarrow Y \longrightarrow 0 \ .$$

Then $\eta + \xi$ is represented by

$$0 \longrightarrow X \longrightarrow \begin{pmatrix} Z_1 \oplus Z_2 \\ X \end{pmatrix} \longrightarrow \begin{pmatrix} Y \\ Z_1 \oplus Z_2 \end{pmatrix} \longrightarrow Y \longrightarrow 0 \ .$$

Proof : Apply Lemma A.3 and Lemma A.4. □

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