§5. Wedderburn's Thm. and Splitting fields for Groups.

Def Let k be a commutative ring and A a k-algebra. A is called simple if it has no two-sided ideals other than $\{0\}$ and itself. A is called semi-simple if A is semi-simple as left A-module ($\Leftrightarrow A = \bigoplus$ simples)

Thm 1 (Artinian Algebras).
An Artinian k-algebra A is simple if and only if $A \simeq M_n(\Delta)$, the algebra of $n \times n$ matrices over some division k-algebra Δ. Moreover:

(i) A has a unique simple module S up to isom (the columns of $M_n(\Delta)$). Also $A \simeq \bigoplus_{i=1}^{n} S$ so A is semi-simple

(ii) We have $\Delta \simeq \mathrm{End}_A(S)^{op}$. Hence n, Δ are uniquely determined by the structure of A.

Proof

1. $M_n(\Delta)$ is simple: supp $0 \neq C \in M_n(\Delta)$ then $\exists\, c_{ij} \neq 0 \rightarrow E_{ki} C E_{j\ell}$ = matrix with c_{ij} in position $k\ell$ and 0 elsewhere. These matrices generate $M_n(\Delta)$ as Δ-module.

As Δ-module $M_n(\Delta) \simeq \Delta^{n^2}$ is free of rank n^2. Now Δ is Artinian as it is a division algebra (any submod is $\{0\}$ or Δ) hence so is $M_n(\Delta)$.

Conversely supp. A is Artinian, simple. Then A has a simple module $\underline{S}$.
[S exists as A is Artinian: Suppose not. Pick a submodule U_0 of A that is non-trivial. If U_0 is not simple then it has a proper, non-trivial submodule U_1. Continuing this procedure one gets a chain $U_0 \supsetneq U_1 \supsetneq U_2 \supsetneq \ldots$ which is a contradiction to A being Artinian]

Now Sa is also an A-module $\forall a \in A$
$\varphi_a : S \to Sa$ is an A-homomorphism.
$s \mapsto sa$
φ_a is surjective so either $Sa = \{0\}$ or $Sa \simeq S$.

S is a left A-module but not necessarily right A-module but $\sum_{a \in A} Sa$ is an A-A-bimodule so an ideal of A. As A is simple one gets: $\sum_{a \in A} Sa = A$

A is Artinian $\Rightarrow$ $A = \bigoplus_{i=1}^{n} S$ (inters. between 2 simple mod is $\{0\}$)
Now every simple module of A is a quotient of A, thus isomorphic to S

By general facts about algebras:

by sending $a \mapsto [b \mapsto ba]$

$$A \cong \mathrm{End}_A(A)^{op} \cong \mathrm{End}_A(S^n)^{op} \cong M_n(\mathrm{End}_A S)^{op} \cong M_n(\Delta) \quad \text{where} \quad \Delta \cong (\mathrm{End}_A S)^{op}$$

$M_n(\mathrm{End}_A S)^{op} \cong M_n((\mathrm{End}_A S)^{op})$ by sending a matrix to its transpose. $\square$

<u>Thm 2 (Wedderburn)</u> Let A be a finite di[m] <u>semi-simple algebra</u> over a field k. Let $\{S_i \mid 1 \le i \le n\}$ be the set of repres of simple A-modules. Set $\Delta_i = \mathrm{End}_A(S_i)$ and n_i be such $A = \bigoplus (S_i)^{n_i}$.
Then $A \cong \prod M_{n_i}(\Delta_i^{op})$

<u>Proof</u>: $\mathrm{End}_A(A) \cong A^{op}$ is an algebra homomorphism:

$$\varphi \longmapsto \varphi(1)$$

$$\psi(\varphi(1)) = \psi(1)\cdot\varphi(1)$$

$$[b \mapsto ba] \longleftarrow a$$

$$\mathrm{End}_A(A) = \mathrm{End}_A(\bigoplus (S_i)^{n_i}) \overset{\text{Schur}}{\cong} \prod \mathrm{End}_A((S_i)^{n_i}) \cong \prod M_{n_i}(\Delta_i)$$

where $\Delta_i = \mathrm{End}_A(S_i)$

$\square$

Rem Given any finite-dimensional k-algebra A, we can apply Wedderburn's Theorem to $A/J(A)$

- Wedderburn's Thm gives a bijection between simple left and right modules.
- Also there is a bijection between simple A-modules and simple $A/J(A)$-modules

Hence there is a bijection between the isom. classes of left and right simple A-modules, for any algebra A.

Remember, if k is algebraically closed then $End_A(S) \simeq k$. So $\Delta_i = k \ \forall i$

Def k - a field
A - fin. dim k-algebra

- A is split if $A/J(A) \simeq \prod_{i=1}^{s} M_{n_i}(k)$.
- An extension k' of k is a splitting field for A if $k' \otimes_k A$ is split
- k is a splitting field for kG, G a group, if kG is split

Thm 3 Let G be a finite gp and k a field. There is a finite field extension k'/k that is ~~the~~ splitting field for G.

Proof: (Schur's Lemma revisited)

Let $\overline{k}$ be the alg. closure of k.
We have now $\overline{k}G$ is split.
$\Rightarrow \overline{k}G = \prod M_{n_i}(\overline{k})$
Let S be a simple $\overline{k}G$-module and $\{s_1, \ldots, s_n\}$ a $\overline{k}$-basis for S.
The action of $x \in G$ is given by a matrix $(\sigma_{ij}^S)_{1 \le i \le n} \in M_n(\overline{k})$.

Let $k' := k(\sigma_{ij}^S, \forall i, \forall j, \forall S)$. Then k' is a finite extension $k \subseteq k' \subseteq \overline{k}$

Let S' such that $S = \overline{k} \otimes_{k'} S'$
(in fact $S' = \sum k' s_i$).
Given that all the coeff. of the action of $x \in G$ on S' are in k' we have that S' is a $k'G$-module.

Moreover S' is simple as if $S' = V' \oplus U'$ then $S = (\overline{k} \otimes_{k'} V') \oplus (\overline{k} \otimes_{k'} U')$.

We want to prove $\mathrm{End}_{k'G}(S') \cong k'$
Let $\varphi \in \mathrm{End}_{k'G}(S')$. Then $\mathrm{Id}_{\overline{k}} \otimes_{k'} \varphi$ belongs to $\mathrm{End}_{\overline{k}G}(S) \cong \overline{k}$. Thus there exists $\lambda \in \overline{k}$, $\mathrm{Id}_{\overline{k}} \otimes_{k'} \varphi = \lambda \, \mathrm{Id}_S$
But then $\lambda S' \subseteq S'$ (elementwise)
$\Rightarrow \lambda \in k' \Rightarrow \mathrm{End}_{k'G}(S') \cong k'$

$\square$

<u>Examples</u>:

1. Let P be a finite p-group and k a field of characteristic p. Then $J(kP) = I(kP)$ and

$$kP/J(kP) \simeq k \Rightarrow k \text{ is a splitting field for } P.$$

2. Consider the cyclic group C_3.

i) $k = \mathbb{F}_2$ $\quad kC_3 \simeq k \oplus M$ ← as kC_3-mod (M dim 2.)

$\quad kC_3 \simeq k \times \mathbb{F}_4$ ← as algebra

ii) $k = \mathbb{F}_4$ $\quad kC_3 \simeq k \oplus k \oplus k$ ← as kC_3-mod

$\quad \simeq k \times k \times k$ ← as alg.

<u>Def</u>: the structural morphism:

$$\theta_M : A \longrightarrow \mathrm{End}_k(M)$$
$$a \longmapsto (m \mapsto am)$$

(in general this is neither injective nor surjective)

Let A be a split, fin.-dim alg over k.

<u>Thm 4</u> The structural morphism $\theta_S : A \to \mathrm{End}_k(S)$ is surjective if and only if S is simple.

<u>Proof</u>: Suppose $S = U \oplus V$ not simple. Let $\varphi \in \mathrm{End}_k(S)$, $\varphi(U) \not\subseteq U$. Then

φ is not in the image of Θ_S, as all.

- Suppose S is simple.

Let I_S be the annihilator of S.

In particular $I_S = J(A) \cdot (\prod_{S_i \neq S} M_{n_i}(\underbrace{\mathrm{End}_{A/J(A)}(S_i)}_{\simeq k})$

$$A/_{I_S} \simeq M_{n_{i_0}}(\mathrm{End}_{A/J(A)}(S))$$

$$\simeq M_{n_{i_0}}(k) \simeq \mathrm{End}_k(S)$$

$\uparrow$ the k-dimension of S. $\square$

<u>Definition</u> (socle / radical).

A an algebra over a commutative ring k.

M an A-module.

- <u>The socle</u> of M $\quad soc(M) = \sum\limits_{\substack{S\text{-simple} \\ S \subseteq M}} S$

$soc(M) = \{0\}$ if there are no simple mod of M

- The <u>radical of</u> M $\quad rad(M) = \bigcap\limits_{\substack{\underline{m}\text{ - maximal} \\ \underline{m} \subseteq M}} \underline{m}$

$rad(M) = M$ if M has no max submod.

<u>Rem</u> - $soc(M)$ = largest semi-simple submodule of M

- $rad(M)$ = smallest submodule of M such that $M/rad(M)$ is semi-simple.

- if A is fin-dim over a field k and M is fin. generated then.
  $\Rightarrow$ $soc(M)$ is the largest s-mod of M anihilated by $J(A)$
  $\Rightarrow$ $rad(M) = J(A)M$.

- <u>radical series</u>
  $$M \supseteq J(A)M \supseteq J(A)^2 M \supseteq \cdots$$
- <u>socle series</u>
  $$\{0\} \subseteq soc^1(M) \subseteq soc^2(M) \subseteq \cdots$$

where $soc^{n+1}(M) = \pi_n^{-1}\left(soc\left(M/soc^n(M)\right)\right)$
$\pi_n : M \longrightarrow M/soc^n(M)$ the canonical projection.

<u>Rem</u> For both series the quotients between successive terms are semi-simple A-modules.