§18. Symmetry

Duality works particularly well for the class of finite dimensional k-algebra whose finitely generated projective and injective modules coincide.

<u>Def</u>: We say that a finite-dimensional k-algebra is <u>self-injective</u> if A is injective as left A-module

<u>Theorem 1</u>. Let k be a field and G a finite group. Then the group alg kG is self-injective

Proof:

- Solution 1: use that kG is symmetric. (see the rest of the section)

- Solution 2: <u>direct</u>

Prove that kG is injective as kG-module is equivalent to show that for any kG-module U and any monomorphism $f : kG \to U$, we have that f splits.

We start with a k-linear splitting for the restriction of f to k.
Let $g : U \to k$ be such splitting.
We have then $g \circ f(1) = 1$, $g \circ f(x) = 0$ for $x \in G \setminus 1$
By the averaging procedure we obtain
$\tilde{g} : U \to kG$ $\quad \tilde{g}(u) := \sum_{y \in G} y\, g(y^{-1} u)$

By standard arguments, $\tilde{g}$ is a kG-homomorphism.

Moreover $\tilde{g}(f(x)) = \sum_{y \in G} y g(y^{-1} f(x)) =$

$$= \sum_{y \in G} y g(f(y^{-1} x)) = x g(f(1)) = x$$

Thus $\tilde{g} \circ f(x) = x$ and f splits □

Theorem 2 (Properties of self-injective alg.)

1. TFAE
(i) A is self-injective (inj as left A-module)
(ii) A is injective as right A-module.
(iii) Every fin. gene. projective A-module is injective
(iv) Every fin. gene injective A-module is projective

2. For every fin. gene projective indecomposable A-module P, both soc P and $P/\text{rad} P$ are simple and the map $\text{soc}(P) \longmapsto P/\text{rad} P$ induces a bij between the classes of simple A-modules.

Proof:

1. We have $\{\text{proj indec}\}/_\sim \xleftrightarrow{\text{bij}} \{\text{inj indec}\}/_\sim$.
so (iii) ⇔ (iv)

Also (i) ⇔ (iii) as if P is proj A-module then $P = \bigoplus_{i=1}^{n} P_i$ where $P_i \in A$ and the direct sum of injective A-modules is an inj. A-module

To prove that (ii) is equivalent to ((iii) ⇔ (iv)) we first dualize ((iii) ⇔ (iv)).

2. By 1. the classes of fin. gen.
projective and injective modules coincide.
Moreover recall that if Q, P are projective indecomposable modules then $Q \simeq P \Leftrightarrow Q/\mathrm{rad}\,Q \simeq P/\mathrm{rad}\,P$.

Also if I, J are injective indecomposable modules then $I \simeq J \Leftrightarrow \mathrm{soc}(I) \simeq \mathrm{soc}(J)$. Hence:

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{isom classes of}\\ \text{simple } A\text{-modules}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{isom classes of indecomposable}\\ \text{projective } A\text{-modules}\end{array}\right\} \\
P/\mathrm{rad}(P) & \xleftrightarrow{\text{bij}} & P \\
\text{bijection} \updownarrow & & \text{bij} \updownarrow \quad \text{proj} \Longleftrightarrow \text{inj} \\
\mathrm{soc}(P) & \xleftrightarrow{\text{bij}} & P \\
\left\{\begin{array}{c}\text{isom classes of}\\ \text{simple } A\text{-modules}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{isom classes of indecomposable}\\ \text{injective } A\text{-modules}\end{array}\right\}
\end{array}$$

□

Symmetrizing form:

In general the bijection $P/\mathrm{rad}(P) \longleftrightarrow \mathrm{soc}(P)$ need not be the identity. But it is the identity if A is a $\boxed{\text{symmetric algebra}}$ (see def. below)

Recall
Given $\mathcal{O}$-algebras A, B and ${}_A M_B$ an A-B bimodule then the $\mathcal{O}$-dual ${}_B M^*_A := \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ is an B-A-bimodule. $[\, b \cdot \alpha \cdot a\,(m) = \alpha(a m b)\,]$

In particular A^* can be thought of as an A-A-bimodule.

!!! In general A^* is not isomorphic to A as an A-A-bimodule (even if it is isomorphic to A as a left and right $\mathcal{O}$-module)

<u>Definition</u> Let A be an $\mathcal{O}$-algebra. A is proj, fin gen as $\mathcal{O}$-mod.
We say that A is <u>symmetric</u> if $A \simeq A^*$ as A-A-bimodules. ~~They~~ Any isomorphism of A-A-bimod $\alpha : A \xrightarrow{\sim} A^*$ induce a <u>symmetrising form</u>
$\alpha(1_A) : A \longrightarrow \mathcal{O}$

* Since 1_A generates A as left and right module, any symmetrising form $s : A \to \mathcal{O}$ generates A^* both as left and right modules.

<u>Theorem 3</u> Let G be a finite group and $\mathcal{O}$ a commutative ring. There is an isomorphism of $\mathcal{O}G$-$\mathcal{O}G$ bimodules
$$\beta : (\mathcal{O}G)^* \longrightarrow \mathcal{O}G$$
$$f \longmapsto \sum_{x \in G} f(x^{-1}) x$$
This map induces an $\mathcal{O}$-linear isomorphism: $\underbrace{\{\text{class functions} : G \to \mathcal{O}\}}_{Cl_{\mathcal{O}}(G)} \longleftrightarrow Z(\mathcal{O}G)$
The symmetrising form corresponding to β is $s : \mathcal{O}G \to \mathcal{O}$ defined by: $s\left(\sum_{x \in G} \lambda_x x\right) = \lambda_1$.

<u>Proof</u>:
Inverse of β: $\alpha : \mathcal{O}G \longrightarrow \mathcal{O}G^*$
$$x \longmapsto \left(\sum \lambda_y y \longmapsto \lambda_{x^{-1}}\right)$$
Now $\sum f(x^{-1}) x \in Z(\mathcal{O}G) \iff f(x) = f(y)$ if $x \sim_G y$.
$\iff f \in Cl_{\mathcal{O}}(G)$.
The symmetrising form corresponding to β is $s = \alpha(1) = \left(\sum_{y \in G} \lambda_y y \longmapsto \lambda_1\right)$ $\square$

Remark:

- the previous theorem gives a standard symmetrising form $s : \mathcal{O}G \longrightarrow \mathcal{O}$
$$\sum \lambda_x x \longmapsto \lambda_1$$

- two different isomorphisms
$\beta, \beta' : A^* \xrightarrow{\simeq} A$ as A-A-bimodule
differ by an automorphism of A as A-A-bimodule:
BUT $\mathrm{End}_{A \otimes A^{op}}(A) \simeq Z(A)$
(in fact $\mathrm{End}_A(A) \simeq Z(A)$)
$(a \longmapsto za) \longleftarrow z$ ← and this is an A-A isom
Moreover $\boxed{\mathrm{Aut}_{A \otimes A^{op}}(A) \simeq Z(A)^*}$

Theorem 4 Let A be an $\mathcal{O}$ algebra, finitely generated as an $\mathcal{O}$-module. Then
(i) A is symmetric if and only if A has a symmetrising form s and s has the property $s(ab) = s(ba)$, $\forall\, a, b \in A$
(ii) A linear map $s \in A^*$ is symmetrising form of A if and only if the map
$$A \longrightarrow A^* \quad \text{is an isom. of } A\text{-}A\text{-bimod.}$$
$$a \longmapsto a \cdot s$$
(iii) If s, s' are two symmetrising forms of A then there is a unique element $z \in Z(A)^\times$ such that $s' = z . s$

Proof : i) Clearly A is symmetric if and only if it has a symmetrizing form.

Also $s(\sum \lambda_x x \cdot \sum \mu_x x) = \sum_{x \in G} \lambda_x \mu_{x^{-1}} = \sum \mu_x \lambda_{x}^{-1} = s(\sum \mu_x x \sum \lambda_x x)$

In general, if s is a symmetrizing form, there exists $\Phi : A \xrightarrow{\sim} A^*$ s.th $\Phi(a) = s$

Thus $a \cdot s = a \cdot \Phi(1_A) = \Phi(a) = \Phi(1_A) \cdot a = s \cdot a$

So $s(ab) = \Phi(1_A)(ab) = \Phi(1_A \cdot a)(b) =$
$= \Phi(a \cdot 1_A)(b) = \Phi(1_A)(ba) = s(ba)$.

(ii) If s is a symmetrizing form for A then $a \longmapsto a \cdot s$ is given by $a \mapsto a \cdot \Phi(1_A) = \Phi(a)$ so it's exactly the A-A isomorphism Φ.

If $a \longmapsto a \cdot s$ is an A-A-isomorphism then take $\Phi : A \longrightarrow A^*$ defined by $\Phi(a) = a \cdot s$. Here again we get $s = \Phi(1_A)$ so s is a symmetrizing form for A.

(iii) $s = \Phi(1_A) \quad s' = \Phi'(1_A) \Rightarrow \Phi^{-1} \circ \Phi' \in \mathrm{Aut}_{A \otimes A^{op}}(A)$

$\Rightarrow \exists\, z \in Z(A)^\times$ such that $\Phi^{-1} \circ \Phi'(a) = z \cdot a$

Hence $\Phi'(a) = z \cdot \Phi(a)$ implying $s' = zs$. $\square$

Theorem 5 : Let A be a symmetric k-algebra. Then

(i) A is self injective.

(ii) for any P proj indecomposable $\mathrm{soc}(P) \simeq P/\mathrm{rad}(P)$

(iii) the Cartan matrix of A is a symmetric matrix.

Proof:

(i) We have $A \simeq A^*$ as A-A-bimodule. Given that A is right projective module we get that the dual A^* is a left injective A-module. Using the isomorphism $A \cong A^*$ we get that A is a left injective A-module.

(ii) As U is projective indecomposable and A is self injective, we get that U is also injective indecomposable.

U proj indec $\Rightarrow U/\mathrm{rad}(U)$ simple

U inj indec $\Rightarrow \mathrm{soc}(U)$ simple

First we isolate the block of A having U as summand and no other proj. indec.

Recall $U \cong Ai$ for some primitive idempotent $i \in A$. Take $e := \sum_{U \cong Ai} i$. Then $Ae = \oplus Ai$ and $\forall\ V \in A(1-e)$, $V \not\cong U$.

On the other hand take $T := \sum_{S \simeq \mathrm{soc}(U)} S$

We know that $S \not\subseteq A(1-e)$ for all $S \simeq \mathrm{soc}(U)$, as otherwise $\exists\ V$ injective indec. with $V \subseteq A(1-e)$ and $\mathrm{soc}(V) \simeq S \Rightarrow V \cong U \Rightarrow\Leftarrow$

Hence $T \subseteq Ae$. <u>Remark:</u> T is a 2-sided ideal

- Now $s((1-e)ae) = s(ae(1-e)) = s(0) = 0$ as s is symmetric. Thus $(1-e)Ae \subseteq \mathrm{Ker}\, s$.

But $T \subseteq Ae$ so $(1-e)T \subseteq \operatorname{Ker} S$

- Aim: prove that $(1-e)T = \{0\}$

Suppose that there exists $x \in (1-e)T \setminus \{0\}$.

Then $xA \subseteq (1-e)T \subseteq \operatorname{Ker} S$.

$$S(x \cdot a) = 0 \iff x \cdot \Phi(1_A)(a) = 0 \ \forall a$$

$$\iff x \cdot \Phi(1_A) = 0$$

$$\iff \Phi(x) = 0 \implies \Leftarrow \ x \neq 0.$$

Hence $(1-e)T = \{0\}$. or equiv. $\operatorname{Hom}_A(A(1-e), T) = \{0\}$.

- Now $\operatorname{Hom}_A(A,T) = \operatorname{Hom}_A(Ae \oplus A(e-1), T) = \operatorname{Hom}_A(Ae, T) \neq \{0\}$.

But any non-zero morphism from Ae to a semi-simple module (T in our case) factors through $Ae/J(A)e$

Hence we get a non-zero morphism between $Ae/J(A)e$ and T so we have a non-zero morphism between a simple submodule of $Ae/J(A)e$ and a simple module of T, and thus an isomorphism.

But $Ae/J(A)e \simeq \oplus U/\operatorname{rad} U$ and $T \simeq \oplus \operatorname{soc} U \implies U/\operatorname{rad}(U) \simeq \operatorname{soc}$

(iii) Let I be a set of representatives of conj. classes of primitive idempotents in A. Then the coeff of the Cartan matrix are: $c_{ij} = \dim_k(iAj)$. Aim: $c_{ij} = \dim_k(jAi)$

- $A \simeq A^*$ as A-A-bimod $\implies iAj \simeq iA^*j$ as k vect sp

But $\tau \in A^* \implies i\tau j(a) = \tau(jai) \implies i\tau j$ determined by the restriction of τ to $jAi \implies iA^*j \simeq (jAi)^*$

Hence $c_{ij} = \dim_k(iAj) = \dim_k((jAi)^*) = \dim_k(jAi) = c_{ji}$

the vector spaces are finite dim

$\square$