§3. Semi-simplicity and Maschke's Theorem

k-comm. ring, A a k-algebra

Def: • An A-module M is called *simple* if its only submodules are M and $\{0\}$.

• An A-module M is called *semi-simple* if it is the (direct) sum of simple submodules
← not needed (see below).

Def: • A as an A-module is called *the regular A-modul*.

Proposition 1 Let A be an algebra over a commutative ring k. Then every simple A-module is a quotient of the regular A-mo

Proof:

Let S be a non-trivial simple A-module and take $s \in S \setminus \{0\}$. Consider the map.

$$\pi : A \longrightarrow S$$
$$a \longmapsto a \cdot s$$

The map π:

• is an A-homomorphism: $\forall b \in A$

$$b \cdot a \xmapsto{\pi} (b \cdot a) \cdot s = b \cdot (a \cdot s) = b \cdot \pi(a)$$

• is surjective $\pi(A) \subseteq S$ is a non-trivial submodule of S so $\pi(A) = S$

Thus $S \cong A/\ker \pi$

$\square$

Theorem 2 (Schur's Lemma). Let A be an algebra over a commutative ring. For any simple A-module S, $End_A(S)$ is a division algebra. For any two non-isomorphic A-modules S, T, $Hom_A(S,T) = \{0\}$.

Proof:

Let $\varphi : S \longrightarrow T$ be an A-homomorphism with S, T simple modules.
Suppose $\varphi \neq \{0\}$ then this is equivalent to the following:

- $Im\, \varphi \neq \{0\} \Longleftrightarrow Im\, \varphi = T$
- $ker\, \varphi \neq S \Longleftrightarrow ker\, \varphi = S$

Hence φ is an isomorphism. □

III/ If A is a finite dimensional algebra over some field k, then every simple A-module S is finite dimensional over k and, thus $End_A(S)$ is finite dimensional over k. If, moreover k is alg. closed, $End_A(S) = \{\lambda \cdot Id_S \mid \lambda \in k\}$. In fact the "algebraically closed" hypothesis is not needed in its full strength:

Corollary 3 Let A be a finite-dimensional algebra over a field k and S an simple A-mod. Suppose that the characteristic polynomial for any $\varphi \in End_A(S)$ has one root in k. Then $End_A(S) \simeq \{\lambda Id_S \mid \lambda \in k\}$.

Proof:

Let $\varphi: S \to S$ be a non-zero morphism. If $\lambda \in k$ is a ~~root of the~~ characteristic polynomial of φ then $\varphi(v) = \lambda v$ for some eigenvector $v \in S$. Thus $\ker(\varphi - \lambda \mathrm{Id}_S : S \to S)$ is not zero and, by Schur's lemma $\varphi = \lambda \mathrm{Id}_S$. $\square$

!!! Stay tunned for description of such k in the case where $A = k.G$.

Thm 4: (characterization of semi-simple modules). Let A be an algebra over a commutative ring k. If M is semisimple then so is every quotient and submodule of M. Moreover the following are equivalent:

(a) M is semi-simple

(b) M is a direct sum of simple modules

(c) Every submodule $U \subseteq M$ has a complement ($\exists\, V \subseteq M$ s.th $M = U \oplus V$).

Proof:

M is the sum of its simple modules

$\Rightarrow$ any quotient has the same property:

$$M = \sum S \quad \Rightarrow \quad M/N = \sum \underbrace{S/_{S \cap N}}_{\cong \{0\} \text{ or } S}$$

• Also property (c) passes to submodules:

→ Suppose $\forall U \subseteq M$ $\exists V$ s.th $U \oplus V = M$.
Take $N \subseteq M$ and $W \subseteq N$. But then $M = W \oplus V$ (for some V) and

• $N = W + (V \cap N)$ as $\forall m \in N$, $\exists! w \in W, v \in V$ s.th. $m = w + v$. But then $v = m - w \in N$.

• $W \cap (V \cap N) \subseteq W \cap V = \{0\}$.

Hence the semi-simplicity passes to submodules.

Schema for the rest of the proof:

a) ⇒ b) ⇒ c) ⇒ a).

a) ⇒ b)

Take $\mathcal{S}$ a maximal set of simple submodules of M such that $U = \bigoplus_{S \in \mathcal{S}} S \subseteq M$
$\mathcal{S}$ exists by Zorn's Lemma.
(in the finite dimensional case, take U of max dim with this property).

Suppose $U \neq M$. Then $\exists S$ simple submodule of M s.th $S \not\subseteq U$. Given that S is simple this is equivalent to $U \cap S = \{0\}$.
Hence $U \oplus S \subseteq M$ contradicting the maximality of $\mathcal{S}$.
Thus $U = M$.

b) ⇒ c) Let $U \subseteq M$.
Take V a max submodule of M s.th $U \cap V = \{0\}$. This exists by Zorn's Lemma (if dim is finite the take V of max dim).

Suppose $U+V \neq M$. Then $\exists S$ simple such that $(U+V)\cap S = \{0\}$. But then $U \cap (V \oplus S) = \{0\}$ contradicting the maximality of V. Thus $M = U \oplus V$.

c) $\Rightarrow$ a)

Take $U = \sum_{\substack{S \text{ simple} \\ S \subseteq M}} S$. Suppose $U \neq M$

Then $\exists V \neq \{0\}$ $M = U \oplus V$. Let $v \in V \setminus \{0\}$
~~Take~~ W maximal, $W \subseteq V \setminus \{v\}$. Property c) passes to submodules so $V = W \oplus S$
But then $M = U \oplus W \oplus S \Rightarrow S$ is not simple again $S = S_1 \oplus S$.
Now $(W \oplus S_1) \cap (W \oplus S_2) = W$:

$$\Big\lceil \; w_1 + s_1 = w_2 + s_2 \iff w_1 - w_2 = s_1 - s_2 \;\Big/\; W \oplus S$$
$$\iff w_1 - w_0 = 0 = s_1 - s_2$$
$$\Rightarrow s_1 = s_2 \;\Big/\; S_1 \oplus S_2 \Rightarrow s_1 = 0 = s_2 \Big\rfloor$$

So either $v \notin W \oplus S_1$ or $v \notin W_1 \oplus S_2$, in both cases contradicting the maximality of W. Thus $M = U$. $\square$

Thm 5 (Maschke's Theorem) Let G be a finite group and k a field. Suppose that the characteristic of k is zero or does not divide $|G|$. Then any kG-module is semi-simple

Lemma 6 (averaging process):
G a finite group, k a commutative ring, M, N kG-modules and $\varphi : M \to N$ a k-linear morphism. Then $\rho : M \to N$ defined by $\rho(m) = \sum_{x \in G} x\,\varphi(x^{-1}m)$ is a kG-homomorphism.

Proof:
Clearly ρ as defined is a k-linear map as π is k-linear. Moreover ρ commutes with the action of G: $y \in G$

$$y \cdot \rho(m) = y \cdot \sum_{x \in G} x\,\varphi(x^{-1}m) =$$
$$= \sum yx\,\varphi(x^{-1}y^{-1}ym) =$$
$$= \sum yx\,\varphi((yx)^{-1}ym) = \rho(ym)$$

(we used that the left multiplication by y induces a bijection $G \xrightarrow{\sim} G$) $\square$

Proof of Thm 5
Aim: prove that $\forall U \leq M$ has a kG-complement
Steps:
(1) Choose a k-linear projection $\pi : M \to U$
(2) Transform it into a kG-homomorphism ρ
(3) Show that $\operatorname{Im}\rho = U$, $\operatorname{Ker}\rho \cap U = \{0\}$
(1) + (2) + (3) $\Rightarrow M = U \oplus \operatorname{Ker}\rho$ as kG-modules

(1) U is a k-vector space contained in M so it admits a k-complement W

as k-modules : $M = U \oplus W \Rightarrow m = u + w$ in an unique way. Take $\pi : M \to U$.
$$u + v \mapsto u$$

This is a k-linear surjectif morphism.

(2) Construct $\rho : M \to U$
$$m \mapsto \frac{1}{|G|} \sum_{x \in G} x\, \pi(x^{-1} m)$$

(by hypothesis $\frac{1}{|G|}$ exists in k)

by Lemma 6 ρ is a kG-homomorphism

(3) • $\rho|_U = id_U$: U is a kG-module so if $m \in U$ then $xm \in U$, $\forall x \in G$

Take $m \in U$

$$\rho(m) = \frac{1}{|G|} \sum_{x \in G} x \underbrace{\pi(x^{-1} m)}_{\in U} = \frac{1}{|G|} \sum_{x \in G} x x^{-1} m$$
$$= \frac{1}{|G|} \cdot |G| m = m$$

• $\rho \circ \rho = \rho$ (ou ρ est ~~vu~~ comme un endomorphisme de M)

$$\rho \circ \rho(m) = \frac{1}{|G|} \sum_{x \in G} \rho(x \pi(x^{-1} m)) = \frac{1}{|G|} \sum x \rho(\underbrace{\pi(x^{-1} m)}_{\in U})$$
$$= \frac{1}{|G|} \sum_{x \in G} x \pi(x^{-1} m) = \rho(m)$$

• $\pi(x^{-1} m) \in U, \forall x \overset{U \text{ is } kG\text{-mod.}}{\Rightarrow} x \pi(x^{-1} m) \in U, \forall x$
$$\Rightarrow \frac{1}{|G|} \sum_{x \in G} x \pi(x^{-1} m) \in U \Rightarrow \mathrm{Im}\,\rho \subseteq U$$

Given that $\rho|_U = id_U$ we also have $U \subseteq \mathrm{Im}\,\rho$. Thus $U = \mathrm{Im}\,\rho$.

- $\mathrm{Ker}\,\rho \cap U = 0$ as $\rho|_U = id_U$

- $M = \mathrm{Ker}\,\rho + U$: $m = \underbrace{m - \rho(m)}_{\in ?} + \underbrace{\rho(m)}_{\in \mathrm{Im}\,\rho = U}$

$$\rho(m - \rho(m)) = \rho(m) - \rho(\rho(m)) = \underbrace{\rho(m) - \rho(m)}_{=0}$$

$\Rightarrow m - \rho(m) \in \mathrm{Ker}\,\rho$.

Collecting all the dots: $M = U \oplus \mathrm{Ker}\,\rho$ as kG-modules, $\forall\, U \subseteq M$.

$\Rightarrow M$ is semi-simple $\square$

<u>Remark</u>: The converse of Maschke's Theorem is also true:
if $\mathrm{char}(k) \mid |G|$ then kG is not semi-simple

Indeed $k \cdot \left(\sum_{x \in G} x\right)$ does not have a kG-complement. Suppose this is the case. Then $kG = k\left(\sum_{x \in G} x\right) \oplus M$

Consider $p : kG \to k\left(\sum_{x \in G} x\right)$ the kG-projector given by this decomposition. Then
$\sum_{x \in G} x = p\left(\sum_{x \in G} x\right) = \sum x \cdot p(1_G) = \left(\sum x\right) \cdot \mu\left(\sum x\right)$

$$= \left(\sum x\right)^2 \cdot \mu = \mu \cdot |G| \cdot \left(\sum x\right) = 0 \Rightarrow \Leftarrow$$

(we use here that $\mathrm{Im}(\rho) \subseteq k \sum_{x \in G} x$
so $\rho(1_G) = \mu \sum_{x \in G} x$ for some $\mu \in k$). $\square$

→ More structural approach makes sense of Jacobson Radical (see next section).

Two more results on semi-simple modules:

1) In General if one restricts a simple A-module to a subalgebra B, one does not get a semi-simple module BUT:

Theorem 7 (Clifford) Let G be a finite group and N a normal subgroup de G. Let k be a field. For any simple kG-module S, the restriction of S to kN is semi-simple

Proof: Let T be a simple kN-submodule of the restriction of S to kN. Let $x \in G$.
→ Then xT is again a simple kN-module (in general it is not the case that xT is also a kN-module if N is not normal in G)
Indeed $n x T = x n' T \subseteq x T$
It is clear that xT is also simple.
→ Now $\sum_{x \in G} xT$ is a kG-module.

and a submodule of S. Thus
$S = \sum_{x \in G/kN} xT$ and S is semi-simple
as kN-module (it is the sum
of its simple submodules). $\square$

Rem: G acts transitively on the simple
kN-submodules of S.

2) proof of the fact that
$M = \bigoplus_{i=1}^{n} S_i$ is invariant up to permutation
using Schur's lemma.

Theorem 8 Let A be a k-algebra,
k a commutative ring and M an A-module.
Suppose that $M = \bigoplus_{i=1}^{n} S_i$, S_i simple A-module. Then the number
of the S_i's isomorphic to a given simple
A-module S only depend on M.

Proof:
$$Hom_A(M,S) = Hom_A\left(\bigoplus_{i=1}^{n} S_i, S\right) = \bigoplus_{i=1}^{n} Hom_A(S_i, S)$$

$$Hom_A(S_i,S) = \begin{cases} 0 & \text{if } S_i \not\cong S \\ End_A(S) & \text{if } S_i \cong S \end{cases}$$

Thus $Hom_A(M,S)$ is a free $End_A(S)$-module
of dimension the number of S_i's isomorphic
to the simple module S. $\square$