§11. The second orthogonality relations

Theorem 1 Let G be a finite group and K a splitting field, $\frac{1}{|G|} \in K$. For any two elem. $x, y \in G$ we have

- $$\sum_{\chi \in Irr_K(G)} \chi(x)\chi(y^{-1}) = |C_G(x)| \text{ if } x = zyz^{-1}, z$$

- $$\sum \chi(x)\chi(y^{-1}) = 0 \text{ if } x \neq zyz^{-1} \; \forall z \in G$$

Proof:

"Characteristic class function for y":

Let
$$\gamma : G \longrightarrow K$$
$$t \longmapsto 1 \text{ if } t = zyz^{-1} \text{ for some } z \in G$$
$$t \longmapsto 0 \text{ otherwise.}$$

Clearly γ is a class function, so $\gamma = \sum_{\chi \in Irr_K(G)} \alpha_\chi \chi$.

Now
$$\sum_{x \in G} \underbrace{\chi(x^{-1})\gamma(x)}_{=0 \text{ unless } x \sim_G y \; (x = zyz^{-1})} = |G| \langle \gamma, \chi \rangle = \alpha_\chi \cdot |G|$$

Thus
$$\alpha_\chi = \frac{1}{|G|} \cdot \underbrace{\# \{x \in G \mid x \sim_G y\}} \cdot \chi(y^{-1})$$

$$= \frac{1}{|G|} \cdot [G : C_G(y)] \cdot \chi(y^{-1}) \qquad \chi(x^{-1}) = \chi(y^{-1})$$

$$= \frac{1}{|C_G(y)|} \cdot \chi(y^{-1})$$

So we get:

$$\rho(x) = \sum_{\chi \in \mathrm{Irr}_K(G)} \alpha_\chi \chi(x) = \sum_{\chi \in \mathrm{Irr}_K(G)} \frac{1}{|C_G(y)|} \chi(y^{-1}) \cdot \chi(x)$$

So $\displaystyle\sum_{\chi \in \mathrm{Irr}_K(G)} \frac{1}{|C_G(y)|} \chi(y^{-1}) \cdot \chi(x) = \begin{cases} 1 & \text{if } x \sim_G y \\ 0 & \text{if } x \not\sim_G y \end{cases}$

The result follows. $\square$

<u>Thm 2</u> Let G be a finite group, K a splitting field for G, $\frac{1}{|G|} \in K$. Let ρ be the character of the regular KG-module KG. Then:

$$\rho = \sum_{\chi \in \mathrm{Irr}_K(G)} \chi(1)\chi$$

In particular: $\displaystyle |G| = \sum_{\chi \in \mathrm{Irr}_K(G)} \chi^2(1)$.

<u>Proof</u>: We have $\rho = \sum_{\chi \in \mathrm{Irr}_K(G)} a_\chi \chi$ as $\mathrm{Irr}_K(G)$ is a K-basis of $Cl_K(G)$.
Moreover $a_\chi = \langle \rho, \chi \rangle$ by the first orthogonality relations. Hence $a_\chi = \frac{1}{|G|} \sum_{x \in G} \rho(x)\chi(x^{-1}) = \chi(1)$.

Shorter, more conceptual proof of Thm 1 and Thm 2 can be given using the following observation;

Prop 3. Let G be a finite group and K a splitting field for G, $\frac{1}{|G|} \in K$. Let $\{V_i\}_{1\leq i\leq h}$ be a complete set of representatives of isomorphism classes of simple KG-modules. We have an isomorphism of $K(G\times G)$-modules

$$KG \simeq \bigoplus_{i=1}^{h} V_i \otimes_K V_i^*$$

In particular, the character β of KG as a $K(G\times G)$-module is given by:

$$\beta(x,y) = \sum_{\chi \in \operatorname{Irr}_K(G)} \chi(x)\chi(y^{-1})$$

Moreover $\beta(x,y) = |C_G(x)|$ if $x \sim_G y$ and $\beta(x,y) = 0$ if $x \not\sim_G y$.

Proof:

By Wedderburn's theorem the structural homomorphisms
$$KG \longrightarrow \operatorname{End}_K(V_i)$$
$$a \longmapsto (v_i \mapsto a v_i)$$
induces an algebra isomorphism
$$KG \xrightarrow{\simeq} \prod_{1\leq i\leq h} \operatorname{End}_K(V_i) \simeq \prod_{1\leq i\leq h} V_i \otimes V_i^*$$

Now KG is also a permutation $K(G\times G)$-module through the action $(x,y)\cdot z = xzy^{-1}$.

So $\beta(x,y) = \#\{z\in G \mid \underbrace{xzy^{-1} = z}_{\Leftrightarrow\ x = zyz^{-1} \Rightarrow\ x \sim_G y}\}$

Hence $\beta(x,y) \neq 0$ if and only if $x \sim y$.
Moreover $zxz^{-1} = z'xz'^{-1} \iff z^{-1}z' \in C_G(x)$
So if $x \sim_G y$ then $\beta(x,y) = |C_G(x)|$. □

Rem: • Thm 1 is an immediate consequence of Thm 3. For Thm 2, restrict $\beta(x,y)$ to $G \times 1 \leq G \times G$.

• !!! 1st Orthog. Relations $\iff$ 2nd Orthog. Relations
Indeed let $\mathcal{K}$ be ~~the~~ set of repres. of conj classes of elements in G.
Take $\chi, \chi' \in \mathrm{Irr}_K(G)$. Use that
- χ, χ' are class functions
- nb of elem equal by conj to x in G is $|C_G(x)|$.
- nb of elem in the conj class of x is $|G : C_G(x)|$

$$\frac{1}{|G|} \sum \chi(x)\chi'(x^{-1}) = \sum_{x \in \mathcal{K}} \frac{1}{|C_G(x)|} \cdot \chi(x) \cdot \chi'(x^{-1})$$

We have $|\mathcal{K}| = |\mathrm{Irr}_K(G)|$. Let

$$A := \left(\frac{\chi(x)}{|C_G(x)|} \right)_{\substack{x \in \mathcal{K} \\ \chi \in \mathrm{Irr}_K(G)}} \quad \text{and } B := \left(\chi(x^{-1}) \right)^t_{\substack{x \in \mathcal{K} \\ \chi \in \mathrm{Irr}_K(G)}}$$

$AB = \mathrm{Id} \iff$ 1st orthogonality relations

$$\Updownarrow \qquad {0 \atop 1\,(\chi = \chi')} = \frac{1}{|G|}\sum_{x \in G} \chi(x)\chi'(x^{-1}) = \frac{1}{|C_G(x)|} \sum_{x \in \mathcal{K}} \chi(x)\chi'(x^{-1})$$

$BA = \mathrm{Id} \iff$ 2nd orthogonality relations

$$\frac{1}{C_G(x)} \sum_{\chi \in \mathrm{Irr}_K(G)} \chi(x)\chi'(y^{-1}) = {0 \atop 1\,(x \sim_G y)}$$

□

Thm 4. G a finite group, K a splitting field for G, $\frac{1}{|G|} \in K$ (of char zero). The group G is abelian if and only if $\chi(1) = 1$ for every irred. char. χ, or, equivalently, if and only if every simple KG-module has K-dimension 1.

Proof: $|\mathrm{Irr}_K(G)| = |\mathcal{K}| = |G|$

$$\Leftrightarrow \sum_{\chi \in \mathrm{Irr}_K(G)} \chi(1)^2 = |\mathcal{K}| \Leftrightarrow \chi(1) = 1 \ \forall \chi \in \mathrm{Irr}_K(G).$$

□

Thm 5 (back on permutation modules)
Let G be a finite group, M a G-set and $|M| < \infty$. Let K be a field of characteristic zero. Denote $\pi : G \longrightarrow K$ the character of the permutation KG-module KM. Then:

(i) $\sum \pi(x) = t|G|$ where t is the nb of G-orbits in M

(ii) $\langle \pi, 1 \rangle = 1$ if and only if G is transitive on M

(iii) Supp (ii), $m \in M$, $H := \mathrm{Stab}_G(m)$. Let s be the nb of H-orbits in M. Then
$$\sum_{x \in G} \pi(x)^2 = s|G|$$

(iv) Supp (ii). then G is 2-transitive if and only if $\pi = 1 + \chi$ for some $\chi \in \mathrm{Irr}_K(G) \setminus \{1\}$. (here 1 is the constant character on G equal to $1 \ \forall x \in G$.)

<u>Proof</u> (i) if G is transitive on M,
then $\forall\ m_1, m_2\ \exists\ x \in G,\ m_2 = x m_1$.
But $\sum_{x\in G}\pi(x) = \sum_{x \in G} |Fix(x)| = |G|$ as they are both equal to $|\{(m, x) \mid xm = m\}|$
For the general case consider
$$M = M_1 \amalg M_2 \amalg \dots \amalg M_t$$
Where M_i is an orbit under the action of G so G acts transitively:
$$\sum_{x\in G} \pi(x) = \sum_{x\in G} |Fix(x)| =$$
$$= \sum_{x\in G} |Fix_1(x)| + \dots + \sum_{x \in G} |Fix_t(x)|$$
$$= t \cdot |G|.$$
(ii) we have that $\sum_{x\in G} \pi(x) = \langle \pi, 1\rangle |G|$ so
Hence $\langle \pi, 1\rangle = 1 \iff t = 1 \iff G$ is trans. on M.
(iii) $\sum \pi(x)^2 = \sum_{x\in G} |\{(m, n) \mid xm = m,\ xn = n\}|$
$$= \sum_{m\in M} \sum_{x \in G_m} |\{n \mid xn = n\}|$$
$$= \sum_{m\in M} s\,|G_m| = s\,|G| \qquad \left(|M| = \frac{|G|}{|G_m|} \right)$$
where s is the nb of G_m orbits in M.

(iv) $\pi = \sum a_\chi \chi$ with $a_\chi = \langle \pi, \chi\rangle$
By (ii) $a_1 = \langle \pi, 1\rangle = 1$
But G is 2-transitive $\iff 2 = \langle \pi, \pi\rangle$
(as G_m has 2 orbits in M: $\{m\}$ and $M\setminus\{m\}$)

Thus $2 = \langle \pi, \pi\rangle = 1 + \sum_{\chi \in Irr_K(G)\setminus\{1\}} a_\chi^2 \implies$ there exists a unique $\chi \in Irr_K(G)\setminus\{1\}$ such that $a_\chi \neq 0$ and, moreover, $a_\chi = 1$. $\square$