§2. Representations and modules

Def: A <u>representation of a group G over a commutative ring k</u> is a group homomorphism $\rho : G \to GL(V)$ where V is a k-module and $GL(V)$ is the group of k-linear automorphisms of V. If $\mathrm{rank}_k V = n$ then $GL(V) \simeq GL_n(k)$.

$$\begin{array}{c}\text{Representations of } G\\ \text{over } k\end{array} \longleftrightarrow kG\text{-modules}$$

$$\rho : G \to GL(V) \longmapsto \begin{pmatrix} V \text{ with mult} \\ x \cdot v := \rho(x)(v) \end{pmatrix}$$

$$\left(\rho : x \mapsto (m \mapsto x \cdot m)\right)_{G \to GL(M)} \longleftarrow M$$

$\rightsquigarrow$ permutation representations and permutation modules.

Def • We say that a set M is a <u>G-set</u> if G acts on M $\begin{pmatrix} G \times M \to M \\ (g, m) \mapsto gm \end{pmatrix}$

• If M is a G-set one can define $k[M]$ to be the free k-module on M, with mult from G given by $x \cdot \sum_{m \in M} \lambda_m m = \sum \lambda_m xm$, $\tilde{\forall} \lambda_m = 0$

Hence if we extend bilinearly the multiplication from G, $k[M]$ becomes a kG-mod.

$$\left(\sum \lambda_x x\right)\left(\sum \mu_m m\right) = \sum_{m\in M}\left(\sum_{\substack{(x,n)\in G\times M\\ xn=m}} \lambda_x \mu_n\right) m$$

kG-mod $\to$ V is called a <u>permutation module</u> if $V \simeq k[M]$ for some G-set M.

$\iff$ V is k-free and has a G-stable k-basis.

<u>Example 1</u> $G = S_n$, $M = \{1,2,\dots,n\}$

$k[M]$ = natural permutation module of S_n (also denoted kM) $\quad \sigma\cdot\sum\lambda_i i = \sum\lambda_i\sigma(i)$

Natural source of examples: <u>transitive G-action</u>

<u>Proposition 2</u> Let M be a transitive G-set; let $m\in M$ and $H = \mathrm{Stab}(m) = \{x\in G \mid xm = x\}$ the stabiliser of m in G. The map $f: G/H \to M$ sending x to xm induces an isomorphism of G-sets $G/H \cong M$, and hence, an isomorphism of kG-modules $k[G/H] \cong k[M]$.

<u>Proof</u>: f is a bijection:

injective & well-defined $\Big[$ $xm = ym \iff y^{-1}xm = m \iff y^{-1}x \in H \iff [x] = [y]$ in G/H

surjective: the action of G is transitive: $\forall m'\in M\ \exists x'\in G$ s.th $m' = x'm$

$\implies f(x') = m'$.

Thus the action of G on G/H and M is the same $\square$.

Proposition 3. Let G be a finite group, k a commutative ring and let M be a finite G-set. The permutation kG-module $k[M]$ has a trivial submodule: $k \cdot \sum_{m \in M} m$. In particular if $|M| \geq 2$ then $k[M]$ is not simple.

Proof: The action of G permutes the ellments of $M \Rightarrow \forall x \in G \quad x \cdot \sum_{m \in M} m = \sum_{m \in M} m$
$\Rightarrow k \sum_{m \in M} m$ is a kG-module $\square$

Question: Is $\left(kM \Big/ \sum_{m \in M} m \right)$ a simple kG-module?

Answer: Proposition 4. Let G be a finite group and k a field of characteristic p.
Assume that M is a finite G-set with 2-transitive G-action and that $p = 0$ or $p > \frac{Stab(m)}{2} + 1$
$\left(kM \Big/ \sum_{m \in M} m \right)$ is a simple kG-module.

Proof

Recall that 2-transitive G-action means that the stabiliser $Stab(m)$ for $m \in M$ acts transitively on $M \setminus \{m\}$

We know from previous proposition that $k\sum_{m\in M} m$ is a kG-module.

Take $U \subseteq kM$ strictly containing $k\sum_{m\in M} m$.
The aim is to prove that $U = kM$.
The fact that $U \neq k\sum_{m\in M} m$ implies that there exists an element $\sum_{i=1}^{n} \lambda_i m_i$ with $\lambda_i \in k$ not all equal.
(Here we use the notation $M = \{m_1, \ldots, m_n\}$ for some positive integer n)

Now $x = \sum_{i=1}^{n} \lambda_i m_i - \lambda_n \sum_{i=1}^{n} m_i \in U \setminus \{0\}$
W.L.O.G. one can suppose that $\lambda_1 - \lambda_n \neq 0$ and, dividing by this number, that U has an element with the coefficient of m_1 being 1 and the coefficient of m_n, 0

$$m_1 + \sum_{i=1}^{n-1} \lambda_i m_i$$

Let $H := \mathrm{Stab}(m_1)$. Because of the 2-transitive action of G on M we have that H acts transitively on $\{m_2, \ldots, m_n\}$. Moreover the stabilisers of m_i in H have all the same cardinality. Consider the sum

$$\sum_{y\in H} y \cdot \left(m_1 + \sum_{i=2}^{n-1} \lambda_i m_i\right) = |H| m_1 + |\mathrm{Stab}_H(m_2)| \left(\sum_{i=2}^{n-1} \lambda_i\right) \cdot \left(\sum_{i=2}^{n} m_i\right) = (*$$

$$(*) = |H|\, m_1 - |\mathrm{Stab}_H(m_2)| \left(\sum \lambda_i\right) m_1 + \underbrace{|\mathrm{Stab}_H(m_2)| \left(\sum \lambda_i\right) \left(\sum_{i=1}^{n} \ldots\right.}_{\in k \sum_{i=1}^{n} m_i \in U}$$

$$\Rightarrow \left(|H| - |\mathrm{Stab}_H(m_2)| \left(\sum \lambda_i\right)\right) m_1 \in U$$

Case 1: $|H| - |\mathrm{Stab}_H(m_2)| \left(\sum \lambda_i\right) \neq 0$

$\Rightarrow m_1 \in U \Rightarrow m_i \in U \; \forall i$ by transitivity of the G-action $\Rightarrow U = kM$

Case 2: $\sum \lambda_i = \dfrac{|H|}{|\mathrm{Stab}_H(m_2)|} < |H|$ ← an integer $(1+1+\cdots+1)$ $\neq 0$, even ≥ 1

Let $K := \mathrm{Stab}_G(m_n)$. Again we have $|\mathrm{Stab}_K(m_i)|$ is constant $\forall\; 1 \leq i \leq n-1$ given the transitive K-action on $M \setminus \{m_n\}$. Consider the sum:

$$\sum_{y \in K} \left(y \cdot \left(m_1 + \sum_{i=2}^{n-1} \lambda_i m_i\right)\right) = |\mathrm{Stab}_K(m_1)| \cdot \sum_{i=1}^{n-1} m_i +$$

$$+ |\mathrm{Stab}_K(m_1)| \left(\sum_{i=2}^{n-1} \lambda_i\right) \left(\sum_{i=1}^{n-1} m_i\right)$$

$$= |\mathrm{Stab}_K(m_1)| \left(\left(\sum_{i=2}^{n-1} \lambda_i\right) + 1\right) \left(\sum_{i=1}^{n-1} m_i\right)$$

Case 2.1 $\left(\sum_{i=2}^{n-1} \lambda_i\right) + 1 \neq 0 \Rightarrow \sum_{i=1}^{n-1} m_i \in U$

$\Rightarrow m_n \in U$ ✓

Case 2.2 $\left(\sum_{i=2}^{n-1} \lambda_i\right) + 1 = 0 \Rightarrow$

$$\Rightarrow \frac{|H|}{|\mathrm{Stab}_H(m_2)|} + 1 = 0 \Rightarrow ⚡$$

$\square$

<u>Definition</u>: • Let A be an algebra over some commutative ring k. A <u>composition series</u> of an A-module M is a finite chain $M=M_0 \supseteq M_1 \supseteq \dots \supseteq M_n=\{0\}$ such that M_i/M_{i+1} is simple $\forall\ 0\le i\le n-1$. The simple factors M_i/M_{i+1} are called <u>composition factors</u> of this series an n is called the length of the series.

• Two series are called <u>equivalent</u> if they have same length n and there is a bijection between the two sets of composition factors.

<u>Theorem 5.</u> (Jordan - Hölder) Let A be an algebra over some commutative ring k and let M be an A-module. Then M has a composition series if and only if M is both Artinian and Noetherian. In that case the two composition series of M are equiv

<u>Lemma 6 (to help in Thm 5)</u>. Let M be an A-module, A a k-alg over a comm ring k, $U, N \subseteq M$, $V \overset{max}{\subseteq} U$ ($\Leftrightarrow U/V$ is simple).

$$U+N = V+N \iff V\cap N \subsetneq U\cap N$$

~~Also~~ $U/V \cong U+N/V+N$

Otherwise $U/V \cong U\cap N / V\cap N$

Proof of Lemma 6:

$$A+B/_{B} \simeq A/_{A\cap B}$$

We have

$$V \subseteq (U\cap N)+V \subseteq U$$

V is maximal so either

A) $V = (U\cap N)+V \Leftrightarrow U\cap N \subseteq V \Leftrightarrow U\cap N = V\cap N$

$$\overset{(V+N)+U}{\overbrace{U+N}}/_{V+N} = U/_{U\cap(V+N)} = U/_{V}$$

B) $(U\cap N)+V = U \Leftrightarrow V+N = U+N$

$$U\cap N/_{\underset{\substack{\| \\ (U\cap N)\cap V}}{V\cap N}} = (U\cap N)+V/_{V} = U/_{V}.$$

$\square$

Proof of Theorem 5

If M has a composition series of length n, then M has a maximal submodule (M') and M' has a composition series of length n-1. By induction, Both M' and M/M' are Artinian and Noetherian (a simple module is always Artinian and Noetherian) and, hence, M is Artinian and Noetherian.

Conversly suppose that M is Noetherian and Artinian

- M is Notherian $\Rightarrow$ M is finitely generated A-module $\Rightarrow$ any submodule of M has maximal submodules

Take $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ where M_{i+1} is maximal in M_i. Now

- M is Artinian $\Rightarrow$ $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ becomes constant. But given that any non-zero A-module has a ~~proper~~ maximal submodule, the last term of the sequence has to be $M_n = \{0\}$.

Let $M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_n = \{0\}$ and $M = N_0 \supsetneq N_1 \supsetneq \dots \supsetneq N_k = \{0\}$ be two different composition series of M.

<u>Aim</u>: prove that the two series are equivalent.

If $n \leq 1$ $\Rightarrow$ $M = \{0\}$ or M is simple so we can consider $n > 1$. Take $N := N_1$

Consider the chain:

(*) $M = M_0 + N \supseteq M_1 + N \dots \supseteq M_n + N = N = M_0 \cap N \supseteq \dots \supseteq M_n \cap N =$

Given that N is max there exists a unique index i such that

$$M = M_0 + N \supsetneq \dots \supsetneq M_i + N \supsetneq M_{i+1} + N = \dots = M_0 + N = \dots$$

By Lemma 6 this is the same i for which

$$M_i \cap N = M_{i+1} \cap N.$$

Hence the chain $(*)$ becomes

$$M = M_i + N \supsetneq M_{i+1} + N = \underbrace{N = M_0 \cap N \supsetneq \dots \supsetneq M_i \cap N = M_{i+1} \cap N \supsetneq \dots M_n \cap \dots}_{(**)}$$

Remark that the chain has length n so, by deleting the first term, one gets a composition series for N $\left({}^{M_i+N}/_{M_{i+1}+N} \simeq {}^{M_i}/_{M_{i+1}} \right.$ or ${}^{M_i \cap N}/_{M_{i+1} \cap N} \simeq {}^{M_i}/_{M_{i+1}}$ by Lemma 6$\left. \right)$.
By induction on the length of the composition series of N, $(n-1) = (k-1)$ and. $(**)$ is equiv to $N \supsetneq N_2 \supsetneq \dots \supsetneq N_n = \{0\}$.

The only missing composition factor in $(**)$ which is in $M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_n = 0$ is M_i/M_{i+1}. But, also by Lemma 6 applied to M, N, M_i and M_{i+1}, $M_i + N \neq M_{i+1} + N$ we have

$$M/N \simeq {}^{M_i+N}/_{M_{i+1}+N}$$

$\square$