§16. Projective covers and Injective Envelopes

Throughout the section:
$\mathcal{O}$ is complete, local, commutative, Noeth ring with $\mathcal{O}/J(\mathcal{O}) \simeq k$.

Thm 1 Let A be an $\mathcal{O}$-algebra, fin. generated as $\mathcal{O}$-module and let U, V be finitely generated A-module.

(i) If U is projective indecomposable then $U/\mathrm{rad}\,U$ is simple; in particular $\mathrm{rad}(U)$ is the unique maximal ideal of U

(ii) If U, V are projectives indecomposable then $U/\mathrm{rad}(U) \simeq V/\mathrm{rad}(V)$ if and only if $U \simeq V$

(iii) For every simple A-module S there is a proj. indec. A-module U s.th $S \simeq U/\mathrm{rad}(U)$.

Otherwise stated the map:

$$\begin{aligned} \mathrm{IndProj}(A) &\longrightarrow \mathrm{Irr}(A) \\ U &\longmapsto U/\mathrm{rad}(U) \end{aligned}$$

is a bijection between $\{\mathrm{IndProj}\}$ and $\{\mathrm{Simple}\}$.

Proof: (i) Consider the following diagram of A-modules:

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \arrow[d, "\pi"] \\ U/\mathrm{rad}\,U \arrow[r, "\overline{\varphi}"'] & U/\mathrm{rad}\,U \end{tikzcd}$$

where π is the canonical projection and φ is a

lift of $\bar{\varphi}$ or $\bar{\varphi}$ is a projection of φ.
Indeed any $\varphi \in \mathrm{End}_A(U)$ has the property $\varphi(\mathrm{rad}(U)) \subseteq \mathrm{rad}(U)$ so it passes to the projection $\bar{\varphi}$. Also the map $\theta : \mathrm{End}_A(U) \to \mathrm{End}_A(U/\mathrm{rad}(U))$ is a surjection as, for any $\bar{\varphi} \in \mathrm{End}_A(U/\mathrm{rad}(U))$ (and π is surj) given that U is projective, $\exists\ \varphi \in \mathrm{End}_A(U)$.

Now $\theta : \mathrm{End}_A(U) \longrightarrow \mathrm{End}_A(U/\mathrm{rad}(U))$ being surjective implies that $\mathrm{End}_A(U/\mathrm{rad}(U)) \cong \mathrm{End}_A(U)/\mathrm{Ker}(\theta)$.
So $\mathrm{End}_A(U)$ is local (as U is irreducible) implies $\mathrm{End}_A(U/\mathrm{rad}(U))$ is local and, hence, that $U/\mathrm{rad}(U)$ is irreducible.
But $U/\mathrm{rad}(U)$ is semisimple so it is in fact simple.

(ii) Same argument applied to the following diagram:

$$\begin{tikzcd}
U \arrow[r, "\varphi"] \arrow[d, "\pi_U"'] & V \arrow[d, "\pi_V"] \\
U/\mathrm{rad}(U) \arrow[r, "\bar{\varphi}"'] & V/\mathrm{rad}(V)
\end{tikzcd}$$

It is clear that if φ is an iso. then $\bar{\varphi}$ is also an isomorphism. Conversely, if $\bar{\varphi}$ is an iso. then it is in particular surjective so $\varphi(U) \notin \mathrm{rad}(V)$ implying that $\varphi(U) = V$ as $\mathrm{rad}(V)$ is a maximal submodule of V. Moreover V is projective so φ is split and hence an isomorphism as U is irreducible.

(iii) Take $s \in S - \{0\}$. Then the map $\psi: A \to S$ sending $a \longmapsto as$ is a surj. A-homomorphism. So $\exists\ U \ominus A$ such that $\psi(U) = S$ with U indecomposable (choose one whose image through φ is non-zero. So it has to be S). Since $U/\mathrm{rad}(U)$ is simple, we have that $U/\mathrm{rad}(U) \simeq S$. $\square$

• There is a dual version for socles (provided that socles exist) and injective module (provided that injective modules exist). Given the contraints, we formulate the theorem only for algebras over a field:

<u>Theorem 2</u> Let A be a finite-dimensional k-algebra and U, V finite dimensional k-modules.

(i) If U is injective indec. then $\mathrm{soc}(U)$ is simple. In particular $\mathrm{soc}(U)$ is the unique simple submodule of U.

(ii) If U, V are injective indecomposable ~~then~~ we have $\mathrm{soc}(U) \simeq \mathrm{soc}(V) \iff U \simeq V$.

(iii) For every simple A-module S there is an inj. indec. A-module U such that $S \simeq \mathrm{soc}(U)$.

In particular, the map sending an injective indecomposable module U on its socle $\mathrm{soc}(U)$ is a bijection between the isom classes of injective indecomposable A-modules and the set of isomorphism classes of simple A-modules.

<u>Proof</u>: We dualize the arguments in the proof of the previous theorem:

(i) $\tilde{\varphi} \in \mathrm{End}_A(U)$ sends $\mathrm{soc}(U)$ on $\mathrm{soc}(U)$ so it restricts to $\varphi \in \mathrm{End}_A(\mathrm{soc}\, U)$.

$$\begin{array}{ccc} U & \xrightarrow{\tilde{\varphi}} & U \\ \uparrow \iota & & \uparrow \iota \\ \mathrm{soc}(U) & \xrightarrow{\varphi} & \mathrm{soc}(U) \end{array}$$

Moreover the map $\mathrm{End}_A(U) \longrightarrow \mathrm{End}_A(\mathrm{soc}(U))$

$\tilde{\varphi} \longmapsto \varphi$

is a surjective map as U is injective, and
$\mathrm{End}_A(U)$ is a local algebra $\Rightarrow \mathrm{End}_A(\mathrm{soc}(U))$ is also local $\Longleftrightarrow \mathrm{soc}(U)$ is indecomposable.
But $\mathrm{soc}(U)$ is semi-simple $\Rightarrow \mathrm{soc}(U)$ is simple.

(ii) It is clear that if $\tilde{\varphi}$ is an isomorphism then φ is also an isomorphism. Conversely if φ is an isomorphism then

$$\begin{tikzcd} U \arrow[r, "\tilde{\varphi}"] & V \\ \mathrm{soc}(U) \arrow[u, hook, "\iota_U"] \arrow[r, "\varphi"] & \mathrm{soc}(V) \arrow[u, hook, "\iota_V"'] \end{tikzcd}$$

$\ker \tilde{\varphi} \cap \mathrm{soc}(U) = \{0\}$. But $\mathrm{soc}(U)$ contains all simple submodules of $\ker\tilde{\varphi}$. Thus $\ker \tilde{\varphi} = \{0\}$.
Hence $\tilde{\varphi}$ is injective, and it is also split as U is injective. Now V is indecomposable so $\tilde{\varphi}$ is an isomorphism.

(iii) Let S be a simple A-module. Then $S^* = \mathrm{Hom}_k(S, k)$ is a simple right A-module. By the previous theorem, there exists a proj. indec. module V with $V/\mathrm{rad}(V) \simeq S^*$. This is equivalent to the short exact sequence:

$$0 \longrightarrow \mathrm{rad}(V) \longrightarrow V \longrightarrow \overset{S^*}{\cancel{\mathrm{rad}(V)}} \longrightarrow 0$$

Given that $\mathrm{Hom}_k(-, k)$ is an exact contravariant functor we get the short exact sequence

$$0 \longrightarrow S \longrightarrow V^* \longrightarrow \mathrm{rad}(V^*) \longrightarrow 0$$

Hence S is the submodule of an injective indecomposable module V^* (it's the dual of a projective indecomposable module!).

$\square$

Prop 3 : Let A be an $\mathcal{O}$-algebra, fin. gene as an $\mathcal{O}$-module and let U, V be finitely generated A-modules. Let $f: U \to V$ be an A-homomorphism. Then

- $f(\mathrm{rad}(U)) \subseteq f(\mathrm{rad}(V))$
- $\bar{f}: U/\mathrm{rad}(U) \longrightarrow V/\mathrm{rad}(V)$ is surjective if and only if f is surjective

Proof:

- $\mathrm{rad}(U) = J(\mathcal{O})U \Rightarrow f(J(\mathcal{O})U) = J(\mathcal{O})f(U) \subseteq \mathrm{rad} V$.
- if f is surjective then clearly $\bar{f}$ is surjective. if $\bar{f}$ is surjective then $V = f(U) + \mathrm{rad}(V)$ $\Rightarrow V = f(U)$ by Nakayama's lemma. $\square$

Prop 4 : (dual of Prop 3, for $\mathcal{O} = k$)
Let A be a k-algebra, fin dimensional, and let U, V be finite dimensional A-modules. Let $f: U \to V$ be an A-homomorphism. Then

- $f(\mathrm{soc}(U)) \subseteq \mathrm{soc}(V)$
- $f|_{\mathrm{soc}(U)}$ is injective if and only if f is injective.

Proof

- $f(\text{simple}) = \text{simple}$ or $\{0\} \Rightarrow f(\mathrm{soc}(U)) \subseteq \mathrm{soc}(V)$
- if f is injective then $f|_{\mathrm{soc}(U)}$ is inj. Conversely if $f|_{\mathrm{soc}(U)}$ is inj. then $\mathrm{soc}(U) \cap \mathrm{Ker} f = \{0\}$. But then $\mathrm{Ker}(f)$ is zero as it doesn't have any non-trivial simple submodule. $\square$

<u>Definition</u> Let A be an $\mathcal{O}$-algebra.
A <u>projective cover of</u> an A-module U is a pair (P_U, π_U) consisting of a proj. A-module P_U and a surjective A-morphism $\pi_U : P_U \longrightarrow U$ which induces an isom $P_U/\mathrm{rad}(P_U) \simeq U/\mathrm{rad}(U)$

<u>Theorem 5</u> (Existence and uniqueness of proj. covers)
Let A be an $\mathcal{O}$-algebra which is fin. generated as an $\mathcal{O}$-module and let U be a finitely generated A-mod.
(i) U has a proj. cover (P_U, π_U)
(ii) We have $\mathrm{Ker}(\pi_U) \subset \mathrm{rad}(P_U)$
(iii) (universal property) for any other pair (P, π) consisting of a proj. mod P and a surjective A-homomorphism $\pi : P \to U$, there exists a split surjection $\sigma : P \to P_U$, $\pi_U \sigma = \pi$.
(iv) if (P'_U, π'_U) is another proj. cover then there is an isom $\sigma : P'_U \longrightarrow P_U$ such that $\pi_U \sigma = \pi'_U$.

<u>Proof</u>:
(i) Let S be a simple ~~A~~ module. Then $\mathrm{rad}(S) = 0$ and by Thm 1 (iii) there exists a projective indecomposable P_S s.th $P_S/\mathrm{rad}(P_S) \simeq S$.
Thus P_S is a projective cover of S.
The property extends to semi-simple fin. generated A-module by taking direct sums of projective covers.
Hence for any fin. generated A-module U,

the exists a projective cover of $U/\mathrm{rad}(U)$, say (P_U, π)

So we have the following diagram:

$$\begin{tikzcd} & & U \arrow[d, "\varphi"] \\ P_U \arrow[rru, dashed, "\pi_U"] \arrow[rr, "\pi"] & & U/\mathrm{rad}(U) \end{tikzcd}$$

where φ is the canonical proj. Given that P_U is projective, there exists a morphism

$$\pi_U : P_U \longrightarrow U \text{ s.th } \varphi \circ \pi_U = \pi.$$

Now π is surj, so $\pi_U(P_U) + \mathrm{rad}(U) = U$ which, by Nakayama's lemma implies that $\pi_U(P_U) = U$ so that π_U is surjective. Moreover we have $P_U/\mathrm{rad}(U) \cong U/\mathrm{rad}(U)$

(ii) We have $\operatorname{Ker}\pi = \mathrm{rad}(P_U)$ so $\operatorname{Ker}\pi_U \subset \ker \pi = \mathrm{rad}(P_U)$

(iii)

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "\tau"'] & U \\ P_U \arrow[ru, "\pi_U"'] & \end{tikzcd}$$

As P is projective, there exists $\tau : P \to P_U$ s.th. $\pi_U \tau = \pi$

Again $\tau(P) + \operatorname{Ker}\pi_U = P_U$.

$$\begin{array}{c} \operatorname{Ker}\pi_U \\ \cap \\ \tau(P) + \mathrm{rad}(P_U) \end{array}$$

Hence $P_U = \tau(P)$ by Nakayama's lemma. Moreover τ is split as P_U is proj.

(iv) apply (iii) twice

$\square$

<u>Def</u>: Let A be a fin-dim k-algebra. An <u>injective envelope</u> of an A-module U is is a pair (I_U, ι_U) consisting of an injective A-module I_U and an injective A-homomorphism

$\iota_U : U \hookrightarrow I_U$ which induces an isomorphism $soc(U) \simeq soc(I_U)$.

Theorem 6 (Dual of Thm 5, for $\theta = k$).
Let A be a finite-dimensional k-algebra and U a finite dimensional A-module. Then
(i) U has an injective enveloppe. (I_U, ι_U)
(ii) $soc(I_U) \subseteq Im(\iota_U)$
(iii) for any (I, ι) where I is an injective A-module and $\iota : U \to I$ there is a split injective A-homomorphism $\tau : I_U \longrightarrow I$ such that $\iota = \tau \iota_U$.
(iv) for any other ~~minimal~~ injective enveloppe (I'_U, ι'_U) there is an isomorphisme $\tau : I_U \to I'_U$ such that $\iota'_U = \tau \iota_U$.

Proof: By theorem 5 one finds a projective cover (P_{U^*}, π_{U^*}) for the dual U^*. Dualizing one gets an injective envelope $((P_{U^*})^*, (\pi_{U^*})^*)$ of $U^{**} \simeq U$. $\square$