§17. The Krull-Schmidt Theorem and lifting idempotents

Throughout this section: $\mathcal{O}$ is complete, commutative local Noetherian ring with residue field $k \simeq \mathcal{O}/J(\mathcal{O})$.

<u>Def</u>: Let A be an $\mathcal{O}$-algebra and let e be an idempotent of A. A <u>primitive decomposition</u> of e is a finite set I of pairwise orthogonal primitive idempotents of A such that $\sum_{i \in I} i = e$.

<u>Ex</u> (1) $$E_i = \begin{array}{c} \begin{array}{ccc} & i & \end{array} \\ \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} i \end{array} \qquad I_n = \sum_{i=1}^{n} E_i \text{ in } M_n(\mathcal{O}).$$

(2) U an A-module, A an $\mathcal{O}$-algebra $Id_U = \sum_{i \in I} \varphi_i$. Then $U = \bigoplus_{i \in I} \varphi_i(U)$ is a decomp of U in direct sum of indecomposable A-modules.

!!! Not every A-module admits such decomp. A sufficient condition for the existance of such decomposition is that U is fin. gene. as $\mathcal{O}$-module.

(3) G a finite group, K a splitting field of charact zero for G. For any $\chi \in Irr_K(G)$ set $e(\chi) = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1}) x$. Then $1 = \sum_{\chi \in Irr_K(G)} e(\chi)$ is a primitive decomposition of 1 in $Z(KG)$. $\left(KG \simeq \prod M_{n_i}(K) \right)$

Proposition 1 (properties of idempotents)

~~Let~~ A be an $\mathcal{O}$-algebra and ~~that~~ i, j be idemp. Then:

(i) If i, j are orthogonal then $A(i+j) = Ai \oplus Aj$

(ii) $\forall\, i \in A$, Ai is projective and.

(Ai is indecomposable if and only if i is prim)

(iii) if $A = U \oplus V$ then $\exists\, i, j$ s.th. $U = Ai$, $V = Aj$, $1 = i + j$.

(Moreover i, j are orthog).

(iv) Any direct summand of A as left module is eq. to Ai for some (not nec. unique) idemp i of A

(v) For any idemp $i \in A$ and any A-mod U we have an $\mathcal{O}$ linear isomorphism $\mathrm{Hom}_A(Ai, U) \simeq iU$ mapping $\varphi \in \mathrm{Hom}_A(Ai, U)$ to $\varphi(i)$

(vi) For any i, j, we have an $\mathcal{O}$ linear isomorphism $\mathrm{Hom}_A(Ai, Aj) \simeq iAj$ mapping $\varphi \in \mathrm{Hom}_A(Ai, Aj)$ to $\varphi(i)$.

(vii) For any idemp i, we have an $\mathcal{O}$-algebra isom. $\mathrm{End}_A(Ai) \simeq (iAi)^{op}$ mappind $\varphi \in \mathrm{End}_A(Ai)$ to $\varphi(i)$.

Proof:

(i) Let $a(i+j) \in A(i+j)$ and $bi + cj \in Ai + Aj$.

Clearly $A(i+j) \subseteq Ai + Aj$. Also $(bi+cj)(i+j) = bi + cj$ by orthogonality so $Ai + Aj \subseteq A(i+j)$.

Consider $bi = cj \in Ai \cap Aj$. Multply by i: $bi = cij = 0$

So $Ai \cap Aj = \{0\}$ and $Ai + Aj = Ai \oplus Aj$

(ii) $1 = i + (1-i)$ (orthog decomp) $\Rightarrow A = Ai \oplus A(1-i)$

$\Rightarrow Ai$ ~~$\oplus$~~ A is projective.

- Ai indecomposable implies i primitive by (i)
- i primitive $\Leftrightarrow iAi$ is local $\Leftrightarrow \mathrm{End}_A(Ai)$ is local $\Leftrightarrow Ai$ indecomposable.

(iii) $A = U \oplus V \Rightarrow 1 = i + j$ unique decomposition with respect to $\oplus$. Then $i = i^2 + ij \Rightarrow ij = 0$ and $i = i^2$. Same $ji = 0$ and $j = j^2 \in V$ (all by the unique decomposition in elements of U and V). Thus $A = Ai \oplus Aj$ by (i)

(iv) Let $U \trianglelefteq A$. Choose a complement $A = U \oplus V$. Then $U = Ai$, $V = Aj$ by (iii). The choice of complement is not necessary unique.

(v) The inverse map for $\operatorname{Hom}_A(Ai, U) \to iU$, $\varphi \longmapsto \varphi(i)$ is

$$iU \longrightarrow \operatorname{Hom}_A(Ai, U)$$
$$c \longmapsto (ai \longmapsto aic)$$

(vi) Special case where $U = Aj$

(vii) $(\psi \circ \varphi)(i) = \psi(\varphi(i)) = \psi(\varphi(i)\, i) = \varphi(i) \cdot \psi(i)$.
$\Rightarrow$ map of A-algebras □

<u>Proposition 2</u> (some kind of converse of (ii) above).
A an $\mathcal{O}$-algebra, fin. gene as an $\mathcal{O}$-module and U a proj. indecomposable A-module; then there exists an indempotent i, primitive, with $Ai \simeq U$.

<u>Proof</u>: Aim: U is a direct summand of A.
Let $S = U/\mathrm{rad}(U)$ and $\pi : U \to S$ the canonical projection. On the other hand consider
$$\rho : A \longrightarrow S \quad \text{for some } s \in S \setminus \{0\}.$$
$$a \longmapsto as$$
then ρ is surjective, and given the fact that A is projective the exist $\sigma : A \to U$ an A-homomorphism $\rho = \pi\sigma$.
Hence $\sigma(A) + \underset{\underset{\mathrm{rad}(U)}{\parallel}}{\ker(\pi)} = U$ and this

gives $\sigma(A) = U$ by Nakayama's lemma.
Given that U is proj we get that σ is split surjective so $U \mid A$. So $U \simeq Ai$ by (iv) and i is primitive by (ii) in the previous theorem. □

<u>Theorem 3</u>: Let A be an $\mathcal{O}$-algebra, fin. gene as an $\mathcal{O}$-module and let e be an idempotent in A. Then e has a primitive decomposition and any two primitive decompositions I, J are conjugated in A ($\exists\, u \in A^\times$ s.th. $J = uIu^{-1}$)

<u>Proof</u>: Enough to show the existence in the theorem for $\bar{A} = A/_{J(\mathcal{O})A}$ then lift the idempotents.

Existence comes from the fact that $\bar{A}$ is finite dimensional over k.

Now consider two decompositions $\sum_{i \in I} i = e = \sum_{j \in J} e$.
These give direct sum decompositions of Ae:
$\bigoplus_{i \in I} Ai = Ae = \bigoplus_{j \in J} Aj$
Quotient out the Jacobson radical:
$$\bigoplus_{i \in I} Ai/_{J(A)i} = Ae/_{J(A)e} = \bigoplus_{i \in J} Aj/_{J(A)j}$$
But $Ai/_{J(A)i}$ and $Aj/_{J(A)j}$ are simple $\forall i \in I$, $\forall j \in J$ (as Ai, resp Aj, are projective indecomp.)
By a corollary to Schurs lemma the decomp into simple submodules is unique up to permutation so $\exists\, \pi : I \to J$ bijective s.th. $Ai/_{J(A)i} \simeq A\pi(i)/_{J(A)\pi(i)}$ but that gives $Ai \simeq A\pi(i)$ as $Ai, A\pi(i)$ are proj. indec.

According to Proposition 1 (vi) the isomorphism $Ai \simeq A\pi(i)$ implies the existence of $c_i \in iA\pi(i)$ and $d_i \in \pi(i)Ai$ such that $c_i \cdot d_i$ represents the identity on Ai so $c_i \cdot d_i = i$, and $d_i \cdot c_i$ repres. the identity $A\pi(i)$ so $d_i \cdot c_i = \pi(i)$.

Set $u := 1 - e + \sum_{i \in I} d_i \quad v := 1 - e + \sum_{i \in I} c_i$

We see that $u \cdot v = v \cdot u = 1$ and $u i v = \pi(i)$ (we use that $c_i \cdot d_j \in iA\pi(i)\pi(j)Aj = \{0\}$ if $i \neq j$) ☐

<u>Theorem 4</u> (Krull-Schmidt) Let A be an $\mathcal{O}$-algebra which is finitely generated as $\mathcal{O}$-module and let U be a finitely generated A-module. Then U is a direct sum of finitely many indecomposable modules and this sum is unique up to permutation:

$$\bigoplus_{1 \le i \le n} U_i = U = \bigoplus_{1 \le j \le m} V_j \Rightarrow m = n \text{ and } U_i \simeq V_{\pi(i)}.$$

<u>Proof</u>: Giving a primitive decomposition of Id_U in $End_A(U)$ is equivalent to give a direct sum decomposition of U as an A-module. So the existence of the decomposition is proved.

Let $U = \bigoplus_{i \in I} U_i$ corresponding to the primitive decomposition $Id_U = \sum_{i \in I} \varphi_i$ and $U = \bigoplus_{j \in J} U_j$ corresponding to $Id_U = \sum_{j \in J} \psi_j$

But then by thm 3 $\exists\ \alpha \in End_A(U)^{\times}$

such that $\varphi_i = \alpha \psi_{\pi(i)} \alpha^{-1}$ for some permutation $\pi : I \to J$. But then $U_i \simeq V_{\pi(i)}$ through α.

$$\varphi_i : U \to U_i \qquad \varphi_i(U_i) = U_i$$
$$\psi_j : U \to V_j \qquad \psi_j(V_j) = V_j$$

$$U_i = \alpha \psi_{\pi(i)} \alpha^{-1}(U) = \alpha \psi_{\pi(i)}(V) = \alpha(V_{\pi(i)}).$$

$\square$

<u>Corollary 5.</u> Let A be an $\mathcal{O}$ algebra, finitely generated as an $\mathcal{O}$-module an let i, j be idempotents in A. We have $Ai \simeq Aj$ as A-modules if and only if i, j are conj in A.

<u>Proof</u>: Supp $Ai \cong Aj$. Given that $A = Ai \oplus A(1-i)$ it follows that $A(1-i) \cong A(1-j)$. Thus,

$$Ai \cong Aj \iff \exists\ c \in iAj,\ d \in jAi \quad cd = i,\ dc = j$$
$$A(1-i) \cong A(1-j) \iff \exists\ c' \in (1-i)A(1-j),\ d' \in (1-j)A(1-i),$$
$$c'd' = (1-i),\ d'c' = (1-j).$$

Let $a = c + c'$ then $aa' = 1 = a'a$
$a' = d + d'$ $\qquad (cd' \in iAj(1-j)A(1-i) = \{0\})$

Moreover $aja' = i$ and $a' = a^{-1}$

Conversly if $aja^{-1} = i$ then

$$Ai \longrightarrow Aj \qquad \text{is an isom.}$$
$$bi \longmapsto (bi)\cdot a = baj$$

of inverse
$$Aj \longrightarrow Ai$$
$$cj \longmapsto (cj)a^{-1} = cai.$$

$\square$

Corollary 6 : Let A be an $\mathcal{O}$-algebra, finitely generated as an $\mathcal{O}$-module. The

(i) The map $\{i \mid i \text{ primitive idemp}\} \to \{U \mid U \text{ proj indec}\}$

$i \longmapsto Ai$

is a bij between the conj. classes of primitive idempotents and the isomorphism classes of projective indecomp. A-modules.

(ii) The map $\{i \mid i \text{ primitive idemp}\} \to \{S \mid S \text{ simple}\}$

$i \longmapsto Ai/J(A)i$

is a bij° between the conj classes of primitive idempotents and the isomorphism classes of simple A-modules.

Proof : (exercise)

Theorem 7 (Lifting idempotents)

Let A, B be $\mathcal{O}$-algebras finitely generated as $\mathcal{O}$-modules and let $f : A \to B$ be a surjective algebra homomorphism. Then

(i) f sends $J(A)$ on $J(B)$ and $A^\times$ on $B^\times$

(ii) $\forall$ prim. idemp i in A, either $i \in \operatorname{Ker}(f)$ or $f(i)$ is prim. idemp in B

(iii) for any primitive idem. j in B there exists a primitive idemp i in A such that $f(i) = j$.

(iv) $i, i' \notin \operatorname{Ker}(f)$ primitive idempotents in A are conj in A if and only if $f(i), f(i')$ are conj in B.

Proof :

(i) • The preimage through f of any maximal left ideal of B is a maximal left ideal of A. So $f(J(A)) \subseteq J(B)$.

By Wedderburn's Theorem $A/J(A) \simeq \prod M_{n_i}(\Delta_i)$

Now $B/f(J(A))$ is a quotient algebra of $A/J(A)$

Because $A/J(A)$ is a product of simple algebras $B/f(J(A))$ is also of the same type. So $J(B) \subseteq f(J(A))$ otherwise the image of $J(B)$ in $B/f(J(A))$ would be a non-zero nilp id.

• Consider the diagram :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A/J(A) \arrow[r, "\bar{f}"] & B/J(B) \end{tikzcd}$$

where $\bar{f}(a + J(A)) = f(a) + J(B)$.

f is surjective so $\bar{f}$ is also surjective

Recall $A/J(A) \cong \prod_{i \in I} M_{n_i}(\Delta_i)$ and

$B/J(B) \cong \prod_{i \in J} M_{n_i}(\Delta_i)$ for some $J \subseteq I$.

Clearly if $x \in A^\times$, $f(x) \in B^*$

Conversly take $y \in B^\times$. Then $\bar{y}$ is invertible in $B/J(B)$. Write $y = (y_i)_{i \in J}$ in $\prod_{i \in J} M_{n_i}(\Delta_i)$.

Define $\bar{x} \in A/J(A)$ by $(x_i)_{i \in I}$ $x_i = y_i$ for $i \in J$ and $x_i = 1_{n_i}$ for $i \in I \setminus J$. Clearly $\bar{x} \in (A/J(A))^\times$. But by a standard argument $x \in A^\times$:

$\bar{x} \in (A/J(A))^\times \iff (A/J(A))\bar{x} = A/J(A)$

$\iff Ax + J(A) = A \overset{\text{Nakayama}}{\iff} Ax = A \iff x \in A^\times$

Now $\bar{f}(\bar{x}) = \bar{y}$ so $f(x) - y \in J(B)$.
Thus $\exists$ $c \in J(A)$ such that $f(c) = f(x) - y$,
or, equivalently, $f(x-c) = y$.
Since $\overline{x-c} = \bar{x}$ the same argument
implies that $x - c \in A^\times$.

(ii) i is a primitive idempotent in A
such that $f(i) \neq 0$ then iAi is a local
algebra $\Rightarrow$ $f(iAi) = \underbrace{f(i)\,B\,f(i)}_{\text{quot. alg.}}$ is also
a local algebra and thus $f(i)$ is
primitive

(iii) I a primitive decomp of 1_A in A
Then (i) $\Rightarrow$ $f(I) - \{0\}$ is a primitive
decomp of 1_B in B

So if j is another primitive idempotent
then one has: $B = \bigoplus_{i \in I, f(i) \neq 0} B f(i) = Bj \oplus B(1_B - j)$

So, by Krull-Schmidt theorem $\exists$ $v \in B^\times$ such
that $v j v^{-1} = f(i)$ for some i.

By (i) we get $u \in A^\times$, $f(u) = v$ and thus
$u^{-1} i u$ is a primitive idempotent in A whose
image is $v^{-1} f(i) v = j$.

(iv) let i, i' be primitive idempotents in $A \setminus \ker(f)$
such that $f(i)$, $f(i')$ are conjugate in B.
Since f is surj, f induces surj map $Ai \to Bf(i)$
which, in turn induces surj map

$$Ai/J(A)i \longrightarrow B(f(i))/J(B)f(i)$$

This map is in fact an isomorphism since
$Ai/J(A)i$ and $Bf(i)/J(B)f(i)$ are simple modules

(we consider $Bf(i)/_{\mathcal{J}(B)f(i)}$ as an A-module through a multiply by $f(a)$)

Hence $Ai/_{\mathcal{J}(A)i} \simeq Bf(i)/_{\mathcal{J}(B)f(i)}$

si $f(i), f(i')$ are conj

$Ai'/_{\mathcal{J}(A)i'} \simeq Bf(i')/_{\mathcal{J}(B)f(i')}$

Hence $Ai \simeq Ai'$ and i is conj to i' in A. $\square$

<u>Corollary 8</u> Let A, B be $\mathcal{O}$ algebras, fin. gene as $\mathcal{O}$-modules. Let I be an ideal in A and J an ideal in B. Suppose that we have an $\mathcal{O}$-algebra homomorphism $f: A \to B$ such that $f(I) = J$. Then

(i) For any primitive idempotent i in A either $i \in \ker(f)$ or $f(i)$ is a prim. idemp of B

(ii) The map sending $i \in A$ idemp to the A-mod $Ai/\mathcal{J}(A)i$ induces a bij between the set of conj cl of prim idemp and the isom of simple A-modules.

(iii) $\forall j \in B$ idemp, $j \in J$ there exists $i \in I$ idemp. such that $f(i) = j$

(iv) $i, i' \in A$ idemp are conj if and only if $f(i), f(i') \in B$ are conj.

<u>Proof</u>: replace B by $f(A)$ in previous thm. $j \in f(A)$ prim. idemp. remains primitive in B: $j = j_1 + j_2$ then $j \cdot j_1 = j_1 \in J$ and $j \cdot j_2 = j_2 \in J$ then apply previous thm $\square$

Corollary 9 Let A be an $\mathcal{O}$-algebra, finitely generated as $\mathcal{O}$-module. Let $a \in A$ and let i be an idempotent in A. Then

(i) We have $a \in A^\times$ if and only if $a + J(A) \in \left(A/J(A)\right)^\times$

(ii) i is primitive in A if and only if $i + J(A)$ is primitive in $A/J(A)$

Proof:

(i) if $a \in A^\times$ then clearly $a + J(A) \in \left(A/J(A)\right)^\times$
Conversely let $a + J(A) \in \left(A/J(A)\right)^\times$. Then $Aa + J(A) = A$ and by Nakayma's Lemma $Aa = A$

(ii) $J(A)$ contains no idempotents + apply Theorem 7.

Recall • $J(A)$ contains all nilpotent left and right ideals of A.

• If A is a finite dimensional k-algebra and M is finitely generated A-module then $\operatorname{rad} M = J(A)M$

• If A is a finite dimensional k-algebra then $J(A)$ is nilpotent. Moreover this implies that every element of $J(A)$ is nilpotent as $Ia \subseteq J(A)$ and $(Ia)^n \subseteq J(A)^n = \{0\}$ for some n. Thus $J(A)$ contains no idempotent.

• Now any idempotent i of $J(A)$, in the case where A is a finitely generated $\mathcal{O}$-algebra, $\mathcal{O}$ a commutative, complete, Noetherian, local ring, has an image in $J(A)/J(\mathcal{O})A = J\left(A/J(\mathcal{O})A\right) = J(\bar{A})$

But, by the previous remark, $J(\bar{A})$ doesn't have any idempotent element so $\bar{i} = 0$ meaning that $i \in J(\theta)A$
(This can also be seen by saying that
$J(A)^n \subseteq J(\theta)A$).

But now if $i \in (J(\theta)A)$ then $i^2 \in (J(\theta)A)^2 \subseteq J(\theta)^2 A$
(the multiplication is θ-bilinear)
So $i \in J(\theta)^n A \;\; \forall\, n \geq 1$ as i is idempotent
hence $i \in \bigcap_{n \geq 1} J(\theta)^n A = \{0\}$. $\perp$

So to resume: If $J(A)$ has an idempotent the its image in $J(\bar{A}) = J(A/J(\theta)A)$ has to be zero, and thus it is contained in $J(\theta)A$, which is a contradiction as $J(\theta)A$ has no idempotent elements $\square$

<u>Corollary 10</u>

Let A be an θ-algebra of finite generation as an θ-module and suppose that B is a subalgebra of A such that $A = B + J(A)$.
<u>Then</u>
- every primitive idempotent i of B is primitive in A
- i, j idempotents primitive in B are conjugate in B if and only if they are conjugate in A.

<u>Proof</u>:

(!!!) B is not a submodule of A so one cannot conclude that $B = A$!

Compose the surjection $A \longrightarrow A/J(A)$ with

the isomorphism $A/J(A) \cong B/J(A)\cap B$.
Also we have $B \cap J(A) \subseteq J(B)$. So we are in position to apply Thm 9 again $\square$

We now use the material developped in this section to give an alternate proof of Jordan-Hölder thm. <u>Recall</u>: S is simple $\iff \exists$ a primitive idempotent i in A such that $S \cong Ai/J(A)i$

<u>Thm 11</u> (Jordan Hölder revisited)
Let A be a finite dimensional split algebra. Let U be a finitely generated A-module, S a simple A-module and i a primitive idempotent such that $S \cong Ai/J(A)i$.
The number of composition factors isomorphic to S in any composition series of U is equal to $\dim_k(iU)$. In particular this number does not depend on the composition series.

<u>Proof</u>:
<u>Recall</u>: We have a k-linear isomorphism
$iU \cong \mathrm{Hom}(Ai, U)$
$iu \longmapsto (ai \longmapsto aiu)$
We do the proof by induction on the number of terms in a composition series of U (or on $\dim_k U$)
• if there is only one term then U is simple
Then $\mathrm{Hom}_A(Ai, U) = \{0\}$ if $U \not\cong S$
$\mathrm{Hom}_A(Ai, U) = \mathrm{Hom}_A(S, U) \cong k$.

Indeed, any morphism $Ai \to U$ is either zero or has kernel $J(A)i$ and the last choice happens only if $Ai/J(A)i \cong S$.

Now let $U = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = \{0\}$ be a composition series for U. The second term in the sequence gives the short exact sequence:

$$0 \to U_1 \to U \to U/U_1 \to 0$$

Multiplying by i gives rise to the short exact sequence:

$$0 \to iU_1 \to iU \to i(U/U_1) \to 0$$

Now
$$\dim_k(iU) = \underbrace{\dim_k(iU_1)}_{\substack{= \#\text{ of factors isomorphic to } S \\ \text{in a composition series of } U_1}} + \underbrace{\dim_k(i(U/U_1))}_{\begin{cases} 0 & \text{if } U/U_1 \not\cong S \\ 1 & \text{if } U/U_1 \cong S \end{cases}}$$

The result follows $\square$

<u>Def</u>: Let I be a set of representatives of conj. classes of primitive idempotents in A (A is a fin-dim k alg.). The <u>Cartan Matrix</u> is given by

$$\left(\dim_k(iAj)\right)_{i,j \in I}$$

$\| $ nb of comp factors in Aj isomorphic to $S_i := Ai/J(A)i$

Aj: repres of isom classes of proj. indecomp A-modules

S_i: repres of isom classes of simple A-modules