§4. The Jacobson Radical of a group algebra

Def: The Jacobson radical $J(A)$ of an algebra A over some commutative ring k is the intersection of anihilators of simple A-modules.

$$J(A) = \{a \in A \mid aS = \{0\},\ \forall S \text{ simple module of } A\}$$

!!! $J(A)$ is a two-sided ideal;

Remark: $a \in J(A)$, $b \in A$ $abS = a \cdot S = \{0\}$; $baS = b \cdot \{0\} = \{0\}$

- $J(A)$ is an ideal of A
- if M is an A-module then $J(A)M$ is a submodule of A
- If $U \subseteq M$ is a maximal submodule then M/U is simple $\Rightarrow J(A)(M/U) = 0$ $\Leftrightarrow J(A)M \subseteq U$.

Thm 1 (Nakayama's Lemma).
Let A be an algebra over a commutative ring k and M be a finitely generatig A-mod
If $N \subseteq M$ and $N + J(A)M = M$ then $M = N$.
In particular if $J(A)M = M$ then $M = 0$.

Proof:
One has to proove that if $N \subsetneq M$ than there exists $U \subseteq M$, maximal such that $N \subseteq U$.

- A is fin. dim over some field k. then take U proper, $N \subseteq U$ of max k-dimension in M.

- Otherwise use Zorn's Lemma:
  Let $\mathcal{U} = \{V \subsetneq M \mid N \subseteq V\}$ and $\mathcal{T} \subseteq \mathcal{U}$ a maximal sub-chain.
  Let $W := \bigcup_{V \in \mathcal{T}} V$. Then $W \in \mathcal{U}$
  as if $W = M$ then $\forall m \in M$ belongs to W
  $\Rightarrow m \in V$ for some $V \in \mathcal{T}$ $\Rightarrow \Leftarrow$.
  $\Rightarrow$ finite generating set of $M \in V$

Now if $N \subseteq U$ then $N + J(A)M \subseteq U$ $\Rightarrow\Leftarrow$.
as $J(A)M \subseteq U$, $\forall$ $U \subseteq M$ maximal.
For the last statement take $N = \{0\}$. $\square$

<u>Def</u>: An ideal I of an algebra A is called <u>nilpotent</u> iff $I^n = \{0\}$ for some n.
(Recall that $I^n = \{\sum_{a_i \in I} a_1 \cdots a_n\}$).

$\rightarrow$ Any element of a nilpotent ideal is nilpotent

$\rightarrow$ <u>But</u> there are non-nilpotent ideals consisting only of nilpotent elements:

<u>Ex</u>: $V_4 = \langle a, b \rangle$ $\quad x = a - 1$
k of char 2.
$x(kV_4)$ is a nilpotent ideal.

Thm 2 (characterization of $J(A)$)

Let A be an algebra over some commutative ring k. We have:

- $1+J(A) \subset A^{\times}$ (invertible elem)
- $J(A)$ is also:
  a) the intersection of all annihilators of right simple A-modules
  b) the intersection of all maximal left ideals of A.
  c) the intersection of all maximal right ideals of A.
- $J(A)$ contains any nilpotent ideal of A.

Proof:

- If $U \subseteq A$ maximal then

$$J(A)(A/U)=\{0\} \Leftrightarrow J(A) \subseteq U.$$

so $J(A)$ is contained in all maximal left ideals of A. $\Rightarrow J(A) \subseteq \bigcap_{U\text{-max}} U$

- Any simple A-module is a quotient of A: $S = A/\operatorname{Ker}\varphi$ $\left(\varphi_s : A \to S,\ a \mapsto as\right)$

$S = As$

$I_S := \bigcap_{s\in S\setminus\{0\}} \operatorname{Ker}\varphi_s = \{a \mid as = 0\ \forall s \in S\}$ ← the anihilator of S in A

But then $J(A) = \bigcap_{S\text{-simple}} I_S = \bigcap_{S\text{-simple}} \bigcap_{s\in S\setminus\{0\}} \operatorname{Ker}\varphi_s$

But $\operatorname{Ker}\varphi_s$ is maximal $\forall s \in S\setminus\{0\}$, $\forall S$-simple

$\Rightarrow \bigcap_{U\text{-max}} U \subseteq J(A)$

Hence $J(A)$ is equal to the intersection of all maximal left ideals of A.

- In an analogous way one gets that $J'(A)$, the intersection of the anihilators of all right simple A modules is equal to the intersection of all maximal right ideals of A.

- <u>Aim</u> $J(A) = J'(A)$

- First $J(A) + 1 \subseteq A^{\times}$

Let $a \in J(A)$ $A = Aa + A(1-a)$
$= J(a) + A(1-a)$

$(1-b)$ has a right inverse

$\Rightarrow A = A(1-a) \Rightarrow \exists b \in A \quad 1 = (1-b)(1-a)$

But then $b = (b-1)a \in J(A)$

Doing the same with b instead of a one gets that $(1-b)$ has also a left inverse which must be $(1-a)$.

Hence $J(A) + 1 \subseteq A^{\times}$

- Same way $J'(A) + 1 \subseteq A^{\times}$

- Suppose $J(A) \not\supseteq J'(A)$ then $\exists$ a left maximal ideal U s.th $J'(A) \not\subseteq U$

$\Rightarrow J'(A) + U = A$ (as left A-modules)

$\Rightarrow \exists\, b \in J'(A), x \in U$ s. th $b + x = 1$

$\Leftrightarrow x = 1 - b \in 1 + J'(A) \subseteq A^{\times} \Rightarrow \in$

Thus $J'(A) \subseteq J(A)$ and similarly $J(A) \subseteq J$

So we get $J(A) = J'(A)$.

• Let I be a nilpotent ideal and S a simple A-module then $IS \subseteq S$. So either $IS = S$ or $IS = \{0\}$. The first case is impossible as
$IS = S \Rightarrow I^n S = S \;\; \forall n \Rightarrow S = \{0\}$
Hence $IS = \{0\}$ for any simple A-module $S \Rightarrow I \subseteq J(A)$

$\square$

Some more specific properties of $J(A)$ when A is a finite-dimensional k-algebra and k is a field.

Thm 3 Let k be a field and A a finite dimensional k-algebra. Then $J(A)$ is the unique maximal nilpotent ideal in A

Proof: Consider the chain $J(A) \supseteq J(A)^2 \supseteq \dots$
Given that A is finite dimensional over k we have that $\exists n$ s.th $J(A)^n = J(A)^{n+1}$ but then $J(A)^n = 0$ by Nakayma's lemma

Moreover every nilpotent ideal of A is contained in $J(A)$ so $J(A)$ is the maximal nilpotent ideal of A $\square$

<u>Thm 4</u>: Let k be a commutative ring and A a k-algebra, finitely generated as k-module. Then $J(k)A \subseteq J(A)$.

<u>Proof</u>: Let S be a simple A-module. Then S is also a k-module and

no 1 inside $\rightarrow$ $J(k)S$ is a <u>proper</u> k-submodule of S.

But $J(k)S$ is also an A-module (any element of A commutes with any element of k). Thus $J(k)S = \{0\}$ as S is simple

So $J(k)$ annihilates any simple A-module and thus $J(k)A \subseteq J(A)$. $\square$

<u>Thm 5</u> Let A be an algebra over k, k commutative ring. Then for every ideal I of A with $I \subseteq J(A)$ we have $J(A/I) = J(A)/I$

<u>Proof</u>:

$I \subseteq J(A)$ so every simple A-module is a simple A/I-module. Thus

$$a \in J(A) \iff a + I \in J(A/I)$$

Hence $J(A/I) = J(A)/I$

$\square$

Rem : • $J(A)M$ is contained in all maximal submodules of M. Thus $J(A)M$ is contained in the intersection of maximal submodules of M.
• When M is finitely generated we have even more:

Thm 6. Let k be a field, A a finitely dimensionale k-algebra and M a finitely generated A module. Then
(i) M is semi-simple iff $J(A)M = \{0\}$; in particular, $J(A) = \{0\}$ iff A is semi-simple as left or right A-module.
(ii) $J(A)M$ is the intersection of all maximal submodules of M.

Proof :

(i) if M is semi-simple then $J(A)M = \{0\}$ by definition of $J(A)$. The converse is more difficult to prove

first: $A/J(A)$ is semi-simple

A is finite-dimensional so $\exists$ n, and M_i maximal ideals of A, $1 \leq i \leq n$ s.th. $J(A) = \bigcap_{i=1}^{n} M_i$

aim: $A/J(A)$ is a submodule of $\bigoplus_{i=1}^{n} A/M_i$

Indeed: the canonical map

$$A \longrightarrow \bigoplus_{i=1}^{n} A/M_i$$
$$a \longmapsto (a+M_1, a+M_2, \dots, a+M_n)$$

has kernel $\bigcap_{i=1}^{n} M_i = J(A)$.

Now if $J(A)M = 0$, M can be seen as a $A/J(A)$-module. Moreover it is finitely generated and thus a quotient of $(A/J(A))^m$ for some m. But this is a semi-simple A-module and so is M.

(ii) <u>Recall</u>: for any $U \subseteq M$ maximal one has $M/U = S$ is a simple module and thus $J(A)(M/U) = \{0\} \Rightarrow U \supseteq J(A)M$
So one can replace M by $M/J(A)M$ and U by $U/J(A)M$.

Now the aim is to prove that $\bigcap_{U \text{ max}} U = \{0\}$.
Given that $M/J(A)M$ is semi-simple (by part ii) using that $J(A)(M/J(A)M) = \{0\}$,

$$M/J(A)M = \bigoplus_{i=1}^{n} S_i, \quad S_i \text{ - simple}$$
(we use here that M is finitely generated)

But then one can take:

$M_j := \bigoplus_{\substack{i=1 \\ i\neq j}}^{n} S_i$ which are maximal submodules of $M/J(A)M$.

It is straightforward that $\bigcap_{j=1}^{n} M_j = \{0\}$

Thus $J(A)M = \{0\} = \bigcap_{\substack{U\text{-max} \\ U \subseteq M/J(A)M}} U$

$\square$

<u>Rem</u> If k is not a field the previous theorem is false in general.
Take $k = \mathbb{Z}$. The simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$, p a prime number

Thus $J(\mathbb{Z}) = \bigcap_{p\text{-prime}} p\mathbb{Z} = \{0\}$.

But $\mathbb{Z}$ is not semi-simple as $\mathbb{Z}$-module.
All the submodules of $\mathbb{Z}$ are of type $n\mathbb{Z}$, $n \in \mathbb{Z}$ ($\mathbb{Z}$ is principal I D).
But $n\mathbb{Z} \cap m\mathbb{Z} = (n, m)\mathbb{Z} \neq \{0\}$
$\Rightarrow$ no direct decomp, and $\mathbb{Z}$ is not simple

<u>Corollary 7</u>: Let A be a finite-dimensional algebra over a field k. TFAE:

(i) $J(A) = 0$

(ii) A is semi-simple as left A-module

(iii) A is semi-simple as right A-module

(iv) Every left or right fin. gene A-module is semi-simple.

Proof exercise (very easy).

Theorem 8. (Maschke's Thm revisited)
Let G be a finite group, k a field.
(i) If either char $k = 0$ or char $k = p \nmid |G|$ then $J(kG) = \{0\}$.
(ii) If char $k = p \mid |G|$ then $\sum_{x \in G} x \in J(kG)$.
In particular $J(kG) \neq \{0\}$

Proof
(i) $\Rightarrow$ kG is semi-simple $\Rightarrow J(kG) = 0$

(ii) Let $z = \sum_{x \in G} x$. Then $xz = z \ \forall x \in G$.
$z^2 = |G| z = 0$. Thus z is nilpotent and $z(kG) = (kG)z$ is a nilpotent ideal hence contained in $J(kG)$. $\square$

Thm 9 (on Jacobson radical of kP, where P is finite p-group, k of char p)
Let k be a field of char p and P a finite p-group. The Jacobson radical $J(kP)$ of kP is equal to $I(kP)$, the augmentation ideal and the trivial module k is, up to isomorphism, the unique simple kP-mo

Proof

$I(kP)$ is a maximal ideal as $kP/I(kP) \simeq k$. So enough to prove that $I(kP)$ is nilpotent. ($J(kP)$ contains all nilpotent ideals of kP and is contained in all max ideals)

By induction on $|P|$.
$|P| = 1$ ✓ / $|P|=p \Rightarrow I(kP) = \underbrace{(z-1)kP}_{\text{nilpotent}}$
$|P| > 1 \Rightarrow |Z(P)| \neq 1$ (center of P)
$\Rightarrow \exists\ \langle z\rangle = Z \leq Z(P)$ and $z^p = 1$
$|Z| = p$.
Canonical projection $\pi : P \longrightarrow P/Z$ induces an algebra homomorphism:
$\tilde{\pi} : k(P) \longrightarrow k(P/Z)$ whose kernel is:
$$\mathrm{Ker}(\tilde{\pi}) = \underbrace{I(kZ)}_{\text{nilpotent by above.}} \cdot kP$$
By ind $I(k(P/Z))$ is nilpotent

The homomorphism $\tilde{\pi}$ sends $I(kP)$ to $I(k(P/Z))$:

$$\tilde{\pi}\left(\sum \lambda_x x\right) = \sum \lambda_x \bar{x}$$

sum coeff: $\sum \lambda_x \qquad \sum \lambda_x$

$\Rightarrow I^n(kP) \subseteq \ker \tilde{\pi} = I(kZ)\,kP$

But $|Z| = p \Rightarrow (I(kZ))^p = 0$ □

Rk One gets moreover: $(I(kP))^{|P|} = 0$.

Corollary 10 p a prime, k a field of char p, P a finite p-group. Then (kP) is indec as kP-module and for all $Q \le P$ $k(P/Q)$ is indecomposable.

Proof: We show that kP has a unique maximal ideal (as if decomp then more than one max ideal).
Max id of $kP = I(kP) \Rightarrow kP$ indec.

$$\tilde{\pi} : kP \longrightarrow k(P/Q) \rightarrow \text{non-zero}$$
$$x \longmapsto \bar{x}$$

$\Rightarrow \ker \tilde{\pi} \neq kP \Rightarrow \ker \tilde{\pi} \subseteq I(kP)$.

$\Rightarrow \tilde{\pi}(I(kP))$ is ~~the~~ the unique max module of $k(P/Q)$. □

Thm 11: Let G be a finite group, p a prime and $P \trianglelefteq G$ (a normal p-subgroup of G). Let k be a field of characteristic p. Then all elements in P act trivially on every simple kG-module. Moreover $I(kP)kG \subseteq J(kG)$

Proof: By Clifford's Theorem, every simple kG-module S restricts to a semi-simple kP-module.

But the only simple kP-module is k itself. So any semi-simple kP module is a direct sum of copies of k, and, thus kP acts trivially on S. This is the case for any simple kG module S.

Now $I(kP) = \bigoplus_{y \in P \setminus \{1\}} k \cdot (y-1)$ and given that y acts trivially on S, $(y-1)$ anihilates S. So $I(kP)$ anihilates any simple kG-module $\Rightarrow$ $I(kP)\,kG \subseteq J(kG)$ □

$\rightarrow$ (other proof) $I(kP)$ is nilpotent $P \trianglelefteq G \Rightarrow I(kP)kG = kG\, I(kP)$ is a nilpotent ideal of $kG \Rightarrow kG\, I(kP) \subseteq J(kG)$. Since $J(kG)$ anihilates any simple kG-modules and $I(kP) = \bigoplus_{y \in P \setminus \{1\}} k\,(y-1)$ we have that $y-1$ anihilates any simple kG-module. But this is equivalent to the fact y acts trivially on every simple kG-module

□

<u>Corollary 12</u> Same hyp as in Thm 11 + P is a Sylow p-sg of G. Then $I(kP)kG = J(kG)$.

<u>Proof</u>: Since G/P is a p'-group we have that $J(k(G/P)) = 0$. But $\tilde{\pi}: kG \to k(G/P)$ sends $J(kG)$ to $J(k(G/P))$.
So $J(kG) \subseteq \ker \tilde{\pi} = I(kP)kG$

By Thm 11 we also have $J(kG) \supseteq I(kP)kG$
so $J(kG) = I(kP)kG$.

<u>Proposition 13</u> Let G be a finite group, $H \leq G$ and k a field of char not dividing $|G/H|$. Let M be a kG-mod whose restriction to kH is semi-simple. Then M is a semi-simple kG-module.

<u>Proof</u>:
Want to prove that M is a semi-simple kG-module. Then we have to prove that every kG-submodule U of M has a kG-complement

As in Maschke's theorem, take a kH complement V of U (for Maschke's Thm one has $H = 1$ and char $k \nmid |G|$).

Let $\pi: M \to M$ be the proj on U of kernel V. Construct $\varphi(m) = \frac{1}{|G/H|} \cdot \sum_{x \in [G/H]} x\pi(x^{-1}m)$

As in Maschke's Thm:

- $\psi|_U = id_U$
- ψ is a kG-morphism.
- $\mathrm{Im}\,\psi = U$ and $M = \mathrm{Im}\,\psi \oplus \mathrm{Ker}\,\psi$.

Thus M is semi-simple. $\square$

<u>Corollary 14</u> Let G be a finite group, $N \trianglelefteq G$, k a field of char $p \nmid [G/N]$. Then $J(kG) = J(kN)kG = kG\,J(kN)$.

<u>Proof</u>:

$N \trianglelefteq G \Rightarrow J(kN)kG = kG\,J(kN)$

kN - fin-dim k-algebra $\Rightarrow J(kN)$ is nilpotent $\Rightarrow J(kN)kG$ is nilpotent $\Rightarrow J(kN)kG \subseteq J(kG)$.

Now $J(kN)kG = J(kG) \Leftrightarrow kG/J(kN)kG$ is semi-simple as kG-module.

Given that $p \nmid [G/N]$ it is enough by Prop. 13 to prove that $kG/J(kN)kG$ is semi-simple as kN-mod.

But this is straight forward as $J(kN)$ anihilates $kG/J(kN)kG$.

$\square$