§6. Induction and Restriction

Notation $\mathrm{Mod}(A)$ – category of left A-modules;
$\mathrm{Mod}(A^{0})$ – category of right A-modules.

Definition: G a finite gp, $H < G$

- the restriction functor
$$\mathrm{Res}^{G}_{H} : \mathrm{Mod}(kG) \longrightarrow \mathrm{Mod}(kH)$$
$$M \longmapsto M \quad (\text{as } kH\text{-mod})$$
- the induction functor
$$\mathrm{Ind}^{G}_{H} : \mathrm{Mod}(kH) \longrightarrow \mathrm{Mod}(kG)$$
$$N \longmapsto kG \otimes_{kH} N.$$

More notations:

$$G = \coprod_{x \in [G/H]} xH = \coprod_{x \in [H\backslash G]} Hx$$

left and right H-coset repres.

$[H\backslash G/L]$ – double cosets representatives.

$$G = \coprod_{x \in [H\backslash G/L]} HxL$$

- $k[Hx]$ ← left kH-module
- $k[xH]$ ← right kH-module
- $k[HxL]$ ← ~~H-L~~-bimodule

Moreover $kG = \bigoplus_{x \in [H\backslash G]} k[Hx]$ ← as left kH-mod

- $kG = \bigoplus_{x\in[G/H]} k[xH]$ as right kH-modules
- $kG = \bigoplus_{x\in[H\backslash G/L]} k[HxL]$ as kH-kL-bimodules.

- kG is free as right kH and left kH-module of rank $[G:H]$.

- V a kH-module then

$$\mathrm{Ind}_H^G(V) = kG \otimes_{kH} V = \bigoplus_{x\in[G/H]} k[xH] \otimes_{kH} V$$
$$= \bigoplus_{x\in[G/H]} x \otimes_{kH} V$$

So, as kH-module, $\mathrm{Ind}_H^G(V)$ is a direct sum of isom copies of V.

<u>Ex</u>: 1) the regular kG-module kG is isom to $kG\otimes_k k$ so

$$kG = \mathrm{Ind}_1^G k.$$

In part. any free kG module W we have $W \simeq (kG)^n \simeq (\mathrm{Ind}_1^G k)^n$
$$\simeq \mathrm{Ind}_1^G\left((k)^n\right).$$

2) M a transitive G-set, G-finite

$M \xrightarrow{\sim} k[G/H]$ where $H = \mathrm{Stab}_G(m_0)$.

$g m_0 \longmapsto gH$

$\to k(G/H) \simeq \mathrm{Ind}_H^G(k)$ being a kG-mod.

$xH \longmapsto x \otimes 1_k$

$\lambda xH \longleftarrow x \otimes \lambda$

Hence in general:

A permutation kG-module M is isom to the direct sum over the orbits of M of $\mathrm{Ind}_{H_O}^G(k)$ where H_O is the stabilizer of an elem of the G-orbit in M.

<u>Theorem 1</u>: Ind_H^G and Res_H^G are exact functors (send short exact sequences to short exact sequences).

<u>Proof</u>: $\mathrm{Res}_H^G \longrightarrow$ trivial.
$\mathrm{Ind}_H^G \to$ in $\otimes_{kH}$ with a free kH module
$\Rightarrow$ this is also exact
<u>details</u>: exercise $\square$

<u>Theorem 2</u> (Frobenius reciprocity)
Let G be a finite group $H<G$, U a kG-mod
V a kH-module
(i) k-linear isomorphism:

$$\mathrm{Hom}_{kG}(\mathrm{Ind}_H^G(V), U) \simeq \mathrm{Hom}_{kH}(V, \mathrm{Res}_H^G(U))$$
$$\varphi \longmapsto (v \longmapsto \varphi(1_G \otimes v))$$

that is Ind_H^G is left adjoint to Res_H^G
(ii) k-linear isomorphism:

$$\mathrm{Hom}_{kH}(\mathrm{Res}_H^G(U), V) \simeq \mathrm{Hom}_{kG}(U, \mathrm{Ind}_H^G(V))$$
$$\psi \longmapsto (u \longmapsto \sum_{x \in [G/H]} x \otimes \psi(x^{-1}u))$$

that is Ind_H^G is right adjoint to Res_H^G.

Proof

(i) the inverse is given by

$(x \otimes v \mapsto x f(v)) \longleftarrow\!\!\shortmid \; \varphi$

(ii) the inverse is given by

$(u \mapsto y v_y) \longleftarrow\!\!\shortmid \; \varphi$

where $y \in H$ $\qquad f(u) = \sum_{x \in [G/H]} x \otimes v_x$

(can as well be considered $y = 1$) $\quad$ there is a unique y such that $y \in H$

Exercise: check details.

Rem:

More structural proof for (ii)

$kG = kH \oplus k[G \setminus H]$ as kH-kH-bimod. ($G \setminus H$: set '$-$')

$\Rightarrow kG \otimes_{kH} V \simeq V \oplus k[G \setminus H] \otimes_{kH} V$

This direct sum decomposition gives a proj:

$\pi_V : \mathrm{Res}^G_H \, \mathrm{Ind}^G_H V \longrightarrow V$ of $\mathrm{Ker} = k[G \setminus H] \otimes_{kH} V$

So now the inverse in the proof of Thm 2 is given by $\pi_V \circ \mathrm{Res}^G_H(\varphi) \longleftarrow\!\!\shortmid \; \varphi$

General principle: A, B k-algebras.

$\rightarrow M \otimes_B - : \mathrm{Mod}(B) \to \mathrm{Mod}(A)$ for an A-B-bimodule M

$\Gamma \; V \longmapsto M \otimes_B V$

$\quad \Psi \longmapsto \mathrm{Id}_M \otimes_B \Psi. \;\lrcorner$

$$\longrightarrow \mathrm{Hom}_A(M,-) : \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(B)$$

$$\begin{aligned} U &\longmapsto \mathrm{Hom}_A(M,U) \qquad b\cdot\theta(m) = \theta(m \\ \varphi &\longmapsto [\mu \mapsto \varphi\circ\mu] \end{aligned}$$

<u>Thm 3.</u> A, B - k-algebras and M an A-B-bimodule then we have a natural k-isomorphism, where U is an A-mod, V a B-mod.

$$\mathrm{Hom}_A(M\otimes_B V, U) \simeq \mathrm{Hom}_B(V, \mathrm{Hom}_A(M,U))$$

$$\varphi \longmapsto (v \mapsto (m \mapsto \varphi(m\otimes v)))$$

$$(m\times v \mapsto \psi(v)(m)) \longleftarrow\!\shortmid\ \psi$$

In other words $M\otimes_B -$ is left adj of $\mathrm{Hom}_A(M,-)$

<u>Proof</u> exercise.

<u>Rem</u> Thm 2. (i) : $A = kG$, $B = kH$ $M = {}_{kG}kG_{kH}$.

(ii) more complicated.

involves the fact that kG, kH are symmetric algebras

(!!!) in general it is not true that $M\otimes_B -$ is also a right adjoint of $\mathrm{Hom}_A(M,-)$.

What happens is we do $\mathrm{Res}^G_H \mathrm{Ind}^G_L$?

(iii) the right kH module $k[xH]$ can be seen as a left $k(xHx^{-1})$ module:
$k[xH] = k[(xHx^{-1})x]$. $\quad {}^xH := xHx^{-1}$
so in fact $k[xH]$ is a ${}^xH - H$ - bimodule

$\rightsquigarrow$ V is a kH-module then
${}^xV := k[xH] \otimes_{kH} V$ is a $k{}^xH$-module.
(transport of structure).

$\Longleftrightarrow$ $V^x = V$ as k-module. — same $k{}^xH$-mod structure.
$\forall y \in {}^xH \quad y \cdot v := (x^{-1}yx) \cdot v$

Thm 4 (Mackey's formula ~ reinterpretation of $G = \coprod_{x \in [H\backslash G/L]} HxL$.)

G a finite group, $L, H \leq G$

(i) $kG = \bigoplus_{x \in [H\backslash G/L]} k[HxL]$ as kH-kL-bimod.

(ii) $k[HxL] \simeq kH \otimes_{k(H \cap {}^xL)} k[xL]$ as kH-kL-bimod

(iii) $\mathrm{Res}^G_H \mathrm{Ind}^G_L (W) \simeq \bigoplus_{x \in [H\backslash G/L]} \mathrm{Ind}^H_{H\cap {}^xL} \mathrm{Res}^{{}^xL}_{H \cap {}^xL} ({}^xW)$

Proof: exercise
[for (iii) apply $- \otimes_{kH} W$ to (i) and (ii)]

<u>Corollary 5</u> G a finite group, $H \leq G$, V a kH-module. Then V is isom to a direct summand ~~of~~ of $\mathrm{Res}^G_H \mathrm{Ind}^G_H V$.

<u>Proof</u>:

$\rightsquigarrow$ By Thm 4 $\mathrm{Res}^G_H \mathrm{Ind}^G_H V = \bigoplus_{x \in [H \backslash G / H]} \mathrm{Ind}^H_{H \cap {}^xH} \mathrm{Res}^{{}^xH}_{H \cap {}^xH} {}^xV$

for $x \in H$ $\mathrm{Ind}^H_{H \cap {}^xH} \mathrm{Res}^{{}^xH}_{H \cap {}^xH} {}^xV \cong V$.

$\rightsquigarrow$ (2nd proof) $kG = kH \oplus k[G \setminus H]$

$\Rightarrow \mathrm{Ind}^G_H V = kG \otimes_{kH} V \cong V \oplus k[G \setminus H] \otimes_{kH} V$ as kH-modules

isom to V. □

<u>Corollary 6</u> G - finite, $N \trianglelefteq G$, V a kN-module then

$$\mathrm{Res}^G_N \mathrm{Ind}^G_N V = \bigoplus_{x \in [G/N]} {}^xV \text{ as } kN\text{-mod.}$$

<u>Proof</u> $xN = Nx$ $\forall x \in G$ as $N \trianglelefteq G$.

$$N \overset{\|}{\cap} {}^xN = N$$

$\Rightarrow \mathrm{Ind}^N_{N \cap {}^xN} \mathrm{Res}^N_{N \cap {}^xN} ({}^xV) = {}^xV$ and $[N \backslash G / N] = [G/N]$

~~2nd proof~~

$\rightsquigarrow$

$kG \cong \bigoplus_{x \in [G/N]} k[xN]$ as kN-kN-bimod.

$\Rightarrow \mathrm{Ind}^G_N (V) \cong \bigoplus_{x \in [G/N]} k[xN] \otimes_{kN} V$

$\cong \bigoplus_{x \in [G/N]} x \otimes V \cong \bigoplus_{x \in [G/N]} {}^xV$

□

Thm 7. Let G be a finite group, let N be a normal subgroup of G, k a splitting field for both G and N, S a simple kN-mod, $[G:N]^{-1}$ exists in k. Then $\mathrm{Ind}_N^G(S)$ is simple if and only if $S \ncong {}^xS$ for all $x \in G \setminus N$.

Proof: From corollary 7

$$\mathrm{Res}_N^G \mathrm{Ind}_N^G S = \bigoplus_{x \in [G/N]} {}^xS \text{ is semi-simple.}$$

$[G:N]$ is invertible in k so $\mathrm{Ind}_N^G S$ is also semi-simple.

So $\mathrm{Ind}_N^G S$ is simple if and only if $\mathrm{End}_{kG}(\mathrm{Ind}_N^G S)$ has k-dimension 1

But $\mathrm{End}_{kG}(\mathrm{Ind}_N^G S) = \mathrm{Hom}_{kG}(\mathrm{Ind}_N^G S, \mathrm{Ind}_N^G S)$

$$\overset{\text{Frob. reciprocity}}{\cong} \mathrm{Hom}_{kN}(S, \underbrace{\mathrm{Res}_N^G \mathrm{Ind}_N^G S}_{\cong \bigoplus {}^xS})$$

The last term has k-dim 1 if and only if $S \ncong {}^xS \ \forall x \notin N$

$\square$