Modular Representation Theory **Homework 4**
Blok 4 **Due date: June 13th 2008**

1. Let $\mathcal{O}$ be a discrete valuation ring and let $\pi \in \mathcal{O}$ such that $J(\mathcal{O}) = \pi\mathcal{O}$. Prove the following:
(i) For every $\lambda \in \mathcal{O} \setminus \{0\}$ there is a unique maximal integer $\nu(\lambda)$ such that $\lambda \in \pi^{\nu(\lambda)}\mathcal{O}$.
(ii) For any λ, $\mu \in \mathcal{O} \setminus \{0\}$ we have $\nu(\lambda\mu) = \nu(\lambda) + \nu(\mu)$ and $\nu(\lambda + \mu) \geq \min\{\nu(\lambda), \nu(\mu)\}$.
(iii) Every non zero ideal in $\mathcal{O}$ is of the form $\pi^n\mathcal{O}$ for some unique integer $n \geq 0$.

2. Let k be a field, let A be a finite-dimensional k-algebra and let M be a finite-dimensional non-zero A-module. Recall that a *composition series* of M is a finite decreasing sequence of submodules $M = M_0 \supset M_1 \supset \cdots \supset M_n \supset M_{n+1} = \{0\}$ such that M_i/M_{i+1} is a simple A-module for $0 \leq i \leq n$. We say that M is *uniserial* if M has a unique composition series. Prove that TFAE:
(i) M is uniserial.
(ii) The sequence $M \supset J(A)M \supset \cdots \supset J(A)^nM \supset \{0\}$ is a composition series of M, where n is the smallest positive integer such that $J(A)^{n+1}M = \{0\}$.
(iii) Every non-zero submodule of M has a unique maximal submodule.
(iv) For any two submodules U, V of M we have $U \subseteq V$ or $V \subseteq U$.

3. Let k be a field of positive characteristic p and let P be a cyclic finite p-group. Show that the regular kP-module kP is uniserial.

4. Let V be a Klein four group and let k be a field of characteristic 2. Show that the regular kV-module is not uniserial.

5. Let p be an odd prime and let G be the dihedral group of order $2p$ given by generators and relations $< y, t \,|\, y^p = 1 = t^2, tyt = y^{-1} >$. Let P be the cyclic subgroup of G generated by y. Let K be a field of characteristic 0 containing a primitive p-th root of unity and k be a field of characteristic p.
(i) Show that G has two characters of degree 1 in $\mathrm{Irr}_K(G)$.
(ii) Show that for any non-identity $\eta \in \mathrm{Irr}_K(P)$ the character $\mathrm{Ind}_P^G(\eta)$ is in $\mathrm{Irr}_K(G)$.
(iii) Determine $\mathrm{Irr}_K(G)$.
(iv) Show that we have $kG = U \oplus V$ as left $kG-$modules where U, V are non-isomorphic projective indecomposable kG-modules.
(v) Find idempotents i, j in kG such that $U \simeq kGi$ and $V \simeq kGj$.
(vi) Show that every projective indecomposable kG-module is uniserial.

6. Let $\mathcal{O}$ be a complete commutative local Noetherian ring and let A be an $\mathcal{O}$-algebra which is finitely generated as $\mathcal{O}$-module. Let U, V, W be finitely generated A-modules. Show that if $U \oplus W \simeq V \oplus W$ then $U \simeq V$.

7. Let A be an algebra over some commutative ring k. An algebra automorphism α of A is called inner automorphism if there is an element $x \in A$ such that $\alpha(a) = xax^{-1}$ for all $a \in A$. Show that the set $\mathrm{Inn}(A)$ of all inner automorphisms of A is a normal subgroup of the group $\mathrm{Aut}(A)$ of all algebra automorphism of A.

8. Let k be a field and n a positive integer. Show that every algebra automorphism of the matrix algebra $M_n(k)$ is inner.
(Hint: recall what is $\mathrm{End}_{M_n(k)}(V)$ where V is the simple $M_n(k)$-module.)

9. Let G be a finite group and let K be a splitting field of characteristic zero for G.
(i) Let $\chi \in \mathrm{Irr}_K(G)$ and $\alpha \in \mathrm{Aut}(KG)$. Show that $\chi \circ \alpha \in \mathrm{Irr}_K(G)$ (χ is viewed as extended to KG in the natural way).
(ii) Show that an algebra automorphism α of KG is inner if and only if $\chi \circ \alpha = \chi$ for all $\chi \in \mathrm{Irr}_K(G)$.