Modular Representation Theory **Homework 3**
Blok 4 **Due date: June 6th 2008**

1. Let G be a finite group, V a finite dimensional $\mathbf{C}G$-module and χ the character of V. Denote by $\rho : G \to \mathrm{GL}_{\mathbf{C}}(V)$ the structural group homomorphism sendings $x \in G$ to the automorphism $v \mapsto xv$ of V, where $v \in V$. Set

$$Z(\chi) = \{x \in G \,|\, \rho(x) = \lambda \mathrm{Id}_V \text{ for some } \lambda \in \mathbf{C}\}\,.$$

Set $\ker(x) := \ker(\rho) = \{x \in G \,|\, \rho(x) = \mathrm{Id}_V\}$. Prove the following statements:
(i) $\ker(\chi)$ is a normal subgroup of $Z(\chi)$ and $Z(\chi)/\ker(\chi)$ is cyclic.
(ii) $Z(\chi)/\ker(\chi) \subseteq Z(G/\ker(\chi))$.
(iii) If χ is irreducible then $Z(\chi)/\ker(\chi) = Z(G/\ker(\chi))$.

2. Let G be a finite group. Show that $Z(G) = \displaystyle\bigcap_{\chi \in \mathrm{Irr}_{\mathbf{C}}(G)} Z(\chi)$.
(This shows that the character table of G determines $Z(G)$.)

3. Let G be a finite group and let H be a subgroup of G. Let χ be a character of G over $\mathbf{C}$ and denote by $\mathrm{Res}^G_H(\chi)$ its restriction to the subgroup H. Show that $< \mathrm{Res}^G_H(\chi), \mathrm{Res}^G_H(\chi) >_H \leq [G : H] < \chi, \chi >_G$ and that the equality holds if and only if χ is zero on $G \setminus H$.

4. Let G be a finite group and let $\chi \in \mathrm{Irr}_{\mathbf{C}}(G)$. Show that $\chi(1)^2 \leq [G : Z(\chi)]$, with equality if and only if χ vanishes on $G \setminus Z(\chi)$.

5. Let G be a finite group, let $\chi \in \mathrm{Irr}_{\mathbf{C}}(G)$ and suppose that $G/Z(\chi)$ is abelian. Show that $[G : Z(\chi)] = \chi(1)^2$. (Show that a vanishes on $G \setminus Z(\chi)$, then use problem 1.(iii) to observe that for $x \in G \setminus Z(\chi)$ there is $g \in G$ such that $xyx^{-1}y^{-1} \notin \ker(\chi)$.

6. Let G be a finite group and let K be a splitting field of characteristic zero for G. Use the orthogonality relations to prove that for any $\chi \in \mathrm{Irr}_{\mathbf{C}}(G)$ there is a unique element $e(\chi) \in KG$ such that $e(\chi)$ acts as identity on every simple KG-module having χ as character and as zero on every simple KG-module having a character different of χ. Prove the following:
(i) $e(\chi) = \dfrac{\chi(1)}{|G|} \displaystyle\sum_{x \in G} \chi(x^{-1})x$.
(ii) $e(\chi) \in Z(KG)$.
(iii) $e(\chi) = e(\chi)^2$ for any $\chi \in \mathrm{Irr}_K(G)$; that is $e(\chi)$ is an idempotent in $Z(KG)$.
(iv) $e(\chi)e(\chi') = 0$ for any two different $\chi, \chi \in \mathrm{Irr}_K(G)$.
(v) $\sum e(\chi) = 1$.
(vi) $e(\chi)$ is a primitive idempotent in $Z(KG)$ for every $\chi \in \mathrm{Irr}_K(G)$.
(This shows that the set $\{e(\chi) \,|\, \chi \in \mathrm{Irr}_K(G)\}$ is a *primitive decomposition* of the unit element in $Z(KG)$.

7. Let G be a finite group and let K be a splitting field of characteristic zero for G. Let V, V' be finite-dimensional KG-modules with characters χ, χ', respectively. Show that $< \chi, \chi' >= \dim_K(\mathrm{Hom}_{KG}(V, V'))$.

The point of the following problem is to see that the notions of restriction and induction make sense for arbitrary class functions, not just modules and their characters, and that some of the formal properties of induction and restriction of modules, such as Frobenius reciprocity, have their counterpart for class functions.

8. Let G be a finite group, let H be a subgroup of G and let K be a field of characteristic zero. For a class function $\phi \in \mathrm{Cl}_K(G)$ we denote by $\mathrm{Res}^G_H(\phi)$ the class function in $\mathrm{Cl}_K(H)$ obtained from restricting ϕ to G. For a class function $\psi \in \mathrm{Cl}_K(H)$ we define a class function $\mathrm{Ind}^G_H(\psi) \in \mathrm{Cl}_K(G)$ as follows:

$$\mathrm{Ind}^G_H(\psi)(x) = \frac{1}{|H|} \sum_{y \in G} \psi^0(yxy^{-1}) \, ,$$

where, $\psi^0(y) = \psi(y)$ if $y \in H$ and $\psi^0(y) = 0$ if $y \in G \setminus H$. Prove the following:
(a) The map $\mathrm{Ind}^G_H(\psi)$ does indeed belong to $\mathrm{Cl}_K(G)$.
(b) $\mathrm{Ind}^G_H(\psi)(1) = [G : H]\phi(1)$.
(c) $\mathrm{Ind}^G_H(\psi)(x) = \sum_{y \in [H \setminus G]} \psi^0(yxy^{-1})$ for every $x \in G$.
(d) For any $\psi \in \mathrm{Cl}_K(H)$ and $\phi \in \mathrm{Cl}_K(G)$ we have

$$< \psi, \mathrm{Res}^G_H(\phi) >_H = < \mathrm{Ind}^G_H(\psi), \phi >_G \ .$$

(e) Show that if ψ is the character of a finite-dimensional KH-module V then $\mathrm{Ind}^G_H(\psi)$ is the character of $\mathrm{Ind}^G_H(V)$.