Modular Representation Theory — **Homework 1**

Blok 4 — **Due date: Mai 9th 2008**

1. Let G be a cyclic group of order 3 and let k be a field of characteristic 2 having a subfield of 4 elements. Show that there is an algebra isomorphism $kG \simeq k \times k \times k$.

2. Let G be a cyclic group of order 3 and let $\mathbf{F}_2$ be the field of characteristic 2 having two elements. Show that the group algebra $\mathbf{F}_2 G$ is not isomorphic to $\mathbf{F}_2 \times \mathbf{F}_2 \times \mathbf{F}_2$.

3. Let $V_4 \simeq C_2 \times C_2$ be the Klein four group and k a field of characteristic 2. Determine the set of kV_4 submodules of kV_4 and their dimensions over k. Find the idempotents and the nilpotent elements in kV_4. Determine the endomorphisms of kV_4 as a kV_4-module and as a k-algebra.

4. Let k be a field and let $X = \{1, 2, 3, 4\}$. Let $M = k[X]$ be the 4-dimensional kS_4-module obtained from the natural action of the symmetric group S_4 on X. Show that M is not simple but that M has a 3-dimensional quotient which is a simple kS_4-module.

5. Let k be a field. Determine the dimension of the Jacobson radical $J(kS_3)$ of the group algebra kS_3 the symmetric group S_3. (Hint: the answer will depend on the characteristic of k).

6. Let n be a positive integer and let k be a field. Determine all 1-dimensional kS_n-modules (up to isomorphism).

7. Let G be a cyclic group of order 2. Determine the group $(\mathbf{Z}G)^\times$ of invertible elements in the group algebra $\mathbf{Z}G$ of G over the integers.

8. Give the definition of projective and injective modules over an algebra.

9. Let A be a finite-dimensional algebra over some field. Show that if A is simple then A has only one isomorphism class of simple modules.

10. Review the definition and basic properties of the tensor product.