# BLOK 4

## Group Algebras
## and
## Modular Representation Theory

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# Part A. Introduction to group algebras.

## §1. Group Algebras

- $k$ a commutative ring
- $G$ a group

<u>Def</u> The <u>group algebra</u> $kG$ of the group $G$ is an algebra which is a free $k$-module with basis $\{x \mid x \in G\}$ and with multiplication induced $k$-bilinearly from the group multiplication in $G$.

Formally $kG = \{ \sum_{x \in G} \lambda_x x \mid \lambda_x \in k, \ \lambda_x = 0 \ \tilde{\forall} x \in G \}$

→ addition:

$$\sum_{x \in G} \lambda_x x + \sum_{x \in G} \mu_x x = \sum_{x \in G} (\lambda_x + \mu_x) x$$

→ scalar multiplication

$$\lambda \Big( \sum_{x \in G} \lambda_x x \Big) = \sum_{x \in G} (\lambda \lambda_x) x \quad , \ \lambda \in k$$

→ multiplication in $kG$

$$\Big( \sum_{x \in G} \lambda_x x \Big) \Big( \sum_{x \in G} \mu_x x \Big) = \sum_{y \in G} \Big( \sum_{\substack{x, x' \in G \\ x x' = y}} \lambda_x \mu_{x'} \Big) y$$

$$= \sum_{y \in G} \Big( \sum_{x \in G} \lambda_x \mu_{x^{-1} y} \Big) y$$

where $\tilde{\forall} x$, $\lambda_x = 0$, $\mu_x = 0$

Example 1 (group algebra of the cyclic group of order 2)

$G \simeq C_2 = \{e, z\}$ , $z^2 = e$

$k$ a field

$kG$ and $k \times k$ are both dimension 2 algebras over $k$.

Question $kG \simeq k \times k$ as $k$-algebras?
(= as rings + $k$-modules)

Answer: yes if char $k \neq 2$

If char $k = 2$ one can study the differences:

a) by counting idempotens.
($a \in A$ is an idempotent if $a^2 = a \overset{\neq}{} 0$)

- in $k \times k$ 3 idempotents: $(0,1), (1,0), (1,1)$
- in $kG$ $\lambda_e e + \lambda_z z$ is idempotent

$$\Leftrightarrow (\lambda_e e + \lambda_z z)^2 = \lambda_e e + \lambda_z z \Leftrightarrow$$

$$\Leftrightarrow (*) \begin{cases} 2\lambda_e \lambda_z = \lambda_z \\ \lambda_e^2 + \lambda_z^2 = \lambda_e \end{cases} \overset{\text{char } k = 2}{\Longleftrightarrow} \begin{cases} 0 = \lambda_z \\ \lambda_e^2 = \lambda_e \end{cases}$$

$\Rightarrow 1 \cdot e$ is the only idempotent

b) by counting nilpotent elements ($a^n = 0$)

- in $k \times k$ there are no nilpotent elements.
- in $kG$: $(e+z)^2 = e^2 + 2ez + z^2 =$
$= e + e = 0$ (char $k = 2$)

Hence, by either a) or b) $k \times k$ is not isomorphic to $kG$ if char $k = 2$.

Now if char $k \neq 2$ then there are two more idempotents in $kG$ (coming from (*)) : $\lambda_e = \frac{1}{2}$, $\lambda_z = \pm\frac{1}{2}$.

$$e_1 = \frac{1}{2}e + \frac{1}{2}z \qquad e_2 = \frac{1}{2}e - \frac{1}{2}z$$

One can define an algebra isomorphism

$$\varphi : k \times k \longrightarrow kG$$
$$(\alpha, \beta) \longmapsto \alpha\left(\frac{1}{2}e + \frac{1}{2}z\right) + \beta\left(\frac{1}{2}e - \frac{1}{2}z\right)$$
$$= \frac{\alpha+\beta}{2} \cdot e + \frac{\alpha-\beta}{2} z$$

(easy exercise : check that this is an isom.)

Example 2 (cyclic group of order 3)

$G \cong C_3 = \{e, \rho, \rho^2\}$, $\rho^3 = e$.
$k = \mathbb{F}_2 = \{0, 1\}$.

$kG = \{0, e, \rho, \rho^2, e+\rho, e+\rho^2, \rho+\rho^2, e+\rho+\rho^2\}$
$kG$ is a vector space of dimension 3 over $k$

Idempotents : $e$, $\rho+\rho^2$, $e+\rho+\rho^2$

\!!!/ Remark: In general $f$ idempotent $\Leftrightarrow$ $e - f$ idempotent

$$(e-f)^2 = e^2 - 2ef + f^2$$
$$= e - 2f + f = e - f$$

$kG$ decomposes as a $kG$ module :

$$kG = k \cdot (e+\rho+\rho^2) \oplus k \cdot \{e+\rho, e+\rho^2\}.$$

In fact:

$$kG = \underbrace{kG\cdot(e+\rho+\rho^2)}_{\text{dim } 1} \oplus \underbrace{kG\cdot(\rho+\rho^2)}_{\text{dim } 2}$$

as algebras:

$kG\cdot(e+\rho+\rho^2) \simeq k \quad kG\cdot(\rho+\rho^2) \simeq \mathbb{F}_4$ (field with 4 elem.)

More general:
- $k$ a commutative ring
- $A$ a $k$-algebra, ~~(of finite dim. over $k$)~~
- $M$ an $A$-module.

Def: • $M$ is finitely generated if $\exists \{m_1, \dots, m_n\}$ s.th $\forall m \in M$.
$$m = \sum_{i=1}^{n} \alpha_i m_i \text{ for some } \alpha_i \in A$$

• $M$ is Noetherian if every ascending chain of submodules $U_1 \subseteq U_2 \subseteq \dots$ of $M$ becomes constant ($U_{i+1} = U_i \ \forall i > n_0$)

• $M$ is Artinian if every descending chain of submodules $U_1 \supseteq U_2 \supseteq \dots$ of $M$ becomes constant

Rem: $A$ finitely dimensional over $k$ + $M$ finitely generated $\Rightarrow$
$\Rightarrow$ $M$ finitly dimenisonal as $k$ vector space $\Rightarrow$ $M$ is Noetherian and Artinia

commutative!

<u>Def</u>: • $k$ is <u>Noetherian (Artinian)</u> if $k$ viewed as $k$-module is Noetherian (Artinian)

• a $k$-algebra $A$ (in general non-comm) is <u>left (right) Noetherian</u> if it is Noetherian as a left (right) $A$-module. $A$ is <u>Noetherian</u> if it is left <u>and</u> right Noetherian.

• same for Artinian.

<u>Proposition 3</u> Let $A$ be a $k$-algebra, $k$ a commutative ring, $M$ a finitely generated $A$-module. Then for all proper submodule $N$ of $M$, there exists a maximal submodule $U$ of $M$, $N \subseteq U$.

<u>Proof</u>: using Zorn's Lemma.

Consider the set $\mathcal{N}$ of all proper submodules of $M$ containing $N$. Aim: $\mathcal{N}$ has a maximal element.

Enough to prove that $\mathcal{N}$ is inductive: $\forall$ totaly ordered subset $T \subseteq \mathcal{N}$ has a sup in $\mathcal{N}$. Consider $V := \bigcup_{X \in \mathcal{N}} X$

$V$ is a submodule of $M$ containing $N$. It remains to see that it is proper: suppose that $M = V$ then $M = \sum_{i=1}^{n} \alpha_i m_i$ (finite gene, and thus $m_i \in V$ $\forall$ $1 \le i \le n$. But then there exists $T \in \mathcal{T}$ such that $m_i \in T$ $\forall 1 \le i \le n$. This leads to a contradiction as, by assumption $T \not\subseteq M$. $\square$

!!! • If $k$ is Noetherian (Artinian) and $A$ is a $k$-algebra, finitely generated as $k$-module (if $k$ is a field this is equiv to ask that $A$ is finite dimensional) then $A$ is left and right Noetherian (Artinian).

- A field is always Noetherian and Artinian
- $\mathbb{Z}$ is Noetherian but not Artinian

Theorem 4 Let $k$ be a commutative ring and $G$ a finite group. If $k$ is Noetherian (Artinian) then so is $kG$.

Proof: see the above remark.

⇝ group of invertible elements of a group algebra

If $A = kG$ then $G \subseteq \overset{\text{invertible elements.}}{A^{\times}}$, but in gene $G \neq A$

- In example 1: $\overset{\text{if char } k = 2.}{\checkmark}$ $(\lambda e + (1+\lambda) z)^2 = e$ so $\lambda e + (1+\lambda) z$ is invertible $\forall \lambda \in k$.

So we have the correspondances from the category of groups to the category of k-algebras and back: $G \longrightarrow kG \longrightarrow (kG)^\times$

They are in fact functorial corresp, and the second one is right adjoint to the first.

Theorem 5 Consider the corresp:

$$F_1 : \underline{\text{Groups}} \longrightarrow \underline{k\text{-Alg}}$$
$$G \longmapsto kG$$
$$F_2 : \underline{k\text{-Alg}} \longrightarrow \underline{\text{Groups}}$$
$$A \longmapsto A^\times$$

Then $F_1$, $F_2$ are functors and $F_2$ is right adjoint to $F_1$

Proof:

The group homomorphism $\varphi : G \longrightarrow H$ generates an algebra morph:

$$F_1(\varphi) : kG \longrightarrow kH$$
$$\sum_{x\in G} \lambda_x x \longmapsto \sum_{x\in G} \lambda_x \varphi(x)$$

and conversely the algebra morphism $\psi : A \longrightarrow B$ generates a group homomorphism $F_2(\psi) : A^\times \longrightarrow B^\times$ by restriction

To prove the adjunction, one has to show that there is a natural

bijection
$$\mathrm{Hom}_{\underline{\text{Groups}}}(G, A^*) \xrightarrow{\sim} \mathrm{Hom}_{\underline{kAlg}}(kG, A)$$
$$\varphi \longmapsto \left(\sum_{x \in G} \lambda_x x \to \sum \lambda_x \varphi(x)\right)$$
$$(x \mapsto \psi(x)) \longleftarrow\!\shortmid \psi$$

check that this is a bijection!

⇝ Augmentation

Def: $G$ a group, $k$ commutative ring.

- $\eta : kG \to k$, $\sum \lambda_x x \longmapsto \sum \lambda_x$ is the augmentation morphism over $k$

Remark: $\eta$ is the only $k$-algebra morphism such that $\eta(x) = 1$ $\forall x \in G$.

- $I(kG) := \ker(\eta)$ is the augmentation ideal of $kG$

Remark: $I(kG) = \left\{\sum_{x \in G} \lambda_x x \mid \sum_{x \in G} \lambda_x = 0\right\}$

- In Example 2 ($k = \mathbb{F}_2$, $G = C_3$)
  $I(kG) = kG \cdot (\rho + \rho^2)$

- If $|G| = n$ then $I(kG)$ is free and $\mathrm{rank}_k(I(kG)) = n-1$ (if $k$ is a field rank = dimension)

- $\{(x-1) \mid x \in G \setminus \{1\}\}$ is a $k$-basis of $I(kG)$ as $\sum_{x \in G} \lambda_x = 0 \Rightarrow \sum_{x \in G} \lambda_x x = \sum_{x \in G} \lambda_x (x-$

Abuse of notation $x - 1 = 1_k \cdot x - 1_k \cdot e$

!!! Through the augmentation morphism $k$ becomes a $kG$-module the trivial $kG$-module:

$$\Big(\sum_{x\in G}\lambda_x x\Big)\cdot\beta = \Big(\sum_{x\in G}\lambda_x\Big)\cdot\beta \quad , \quad \beta\in k$$

In particular, $\forall\, x\in G$, $x\cdot\beta=\beta$

Two statements on augmentation ideals

Proposition 6 Let $G=\langle y\rangle$ be a cyclic group and $k$ a commutative ring. Then $I(kG) = kG\cdot(y-1) = (y-1)\cdot kG$. In particular, $I(kG)$ is a principal ideal of $kG$.

Proof: $kG(y-1) = (y-1)kG$ as $kG$ is commut.

Clearly $(y-1)kG \subseteq I(kG)$ as $y-1\in I(kG)$
Now let $\sum_{x\in G}\lambda_x x\in I(kG)$ then

$$\sum_{x\in G}\lambda_x x = \sum_{x\in G}\lambda_x(x-1) = \sum_{i=1}^{n}\lambda_{y^i}(y^i-1)$$

$$=\sum_{i=1}^{n}(y-1)(y^{i-1}+\dots+1) \in (y-1)kG. \quad \square$$

Proposition 7. (kernel of a group algebra morphism induced by a group homomorphism) Let $\varphi : G \longrightarrow H$ be a group homomorphism and $k$ a commutative ring. Then $\ker(F_1\varphi) = I(kN)kG$, where $N = \ker\varphi$.

Proof:

$N$ is a normal subgroup of $G$ so $\forall x \in G, \forall y \in N$ $xyx^{-1} \in N$. Now $\sum_{y\in N} \lambda_y y \in I(kN) \iff \sum_{y \in N} \lambda_y = 0$

$\Rightarrow \sum_{y\in N} \lambda_y xyx^{-1} \in I(kN)$. Hence $xI(kN) = I(kN)x$ $\forall x \in G$.

Denote $\psi := F_1\varphi$. Then $\psi\left(\sum_{y\in N} \lambda_y y\right) = \sum_{y\in N} \lambda_y$ and thus $I(kN) \subseteq \ker\psi$.

Take $\sum_{x\in G} \lambda_x x \in \ker\psi$. But $\psi\left(\sum_{x\in G} \lambda_x x\right) = \sum_{x\in G} \lambda_x \varphi(x)$

$$= \sum_{t\in H}\Big(\sum_{\substack{x\in G\\ \varphi(x)=t}} \lambda_x\Big) t \overset{\substack{\text{fix an } x\\ \text{for every } t}}{=} \sum_{t\in H}\Big(\sum_{\substack{s\in N\\ \varphi(x)=t}} \lambda_{sx}\Big) t$$

But then $\sum_{s\in N} \lambda_{sx} = 0 \quad \forall x, \forall s$
(the sum is taken over the elements of $Nx$)

$$\Rightarrow \sum_{x\in G} \lambda_x x = \sum_{x \in [N\backslash G]} \sum_{s\in N} \lambda_{sx} sx = \sum_{x\in[N\backslash G]} \Big(\underbrace{\sum_{s\in N} \lambda_{sx} s}_{\in I(kN)}\Big) x$$

$$\Rightarrow \sum_{x\in G} \lambda_x x \in I(kN)kG.$$

$\longrightarrow$ opposite algebra $A^{op}$ or $A^{o}$

Def: Let $k$ be a commutative ring and $A$ a $k$-algebra. The opposite algebra $A^{op}$ or $A^{o}$ is a $k$-algebra that is the same a $A$, as $k$-module and whose multiplication is given by $a \circ b = a \cdot b$

Remarks
- $(A^o)^o = A$
- Every left $A$-module $M$ can be seen as a right $A^o$-module through:
  $m \circ a = a m \quad , a \in A, \ m \in M$
- If $U, V$ are right $A$-modules then we write $\mathrm{Hom}_{A^o}(U,V)$ for the homomorphism space from $U$ to $V$ as right $A$-module (same as the hom space from $U$ to $V$ as left $A^o$-modules)

Thm 8 (symmetry of group algebras)
Let $k$ be a commutative ring and $G$ a group. Then there exists a $k$-algebra isomorphism:

$$\tau : kG \xrightarrow{\sim} (kG)^o$$
$$x \longmapsto x^{-1}$$

Moreover $\tau$ is canonical and $\tau \circ \tau = \mathrm{Id}_{kG}$

Proof: We have $(xy)^{-1} = y^{-1}x^{-1}$ and thus $\tau(xy) = \tau(y)\tau(x)$
$$= \tau(x) \circ \tau(y).$$
$\tau \circ \tau = \mathrm{Id}_{kG}$

$\square$

Other important feature:

Thm 9 There is a unique $k$-algebra morphism $\Delta : kG \longrightarrow k(G\times G)$
$$x \longmapsto (x,x).$$

Remarks • $k(G\times H) \cong kG \otimes_k kH$
$$(x,y) \longmapsto x\otimes y.$$

• $k(G\times H) \cong kG\otimes_k kH \cong \underbrace{kG\otimes_k (kH)^{\circ}}_{}$

kG-kH-bimodule.

Explicitely if $M$ is a $kG$-$kH$-bimodule then we can see $M$ as a $k(G\times H)$-module via $(x,y)\cdot m = x m y^{-1}$

$\rightsquigarrow$ the center of a group algebra.

• $k$ commutative ring, $G$ finite group
$$\sum_{x\in G} \lambda_x x \in Z(kG) \Leftrightarrow \lambda_x = \lambda_{yxy^{-1}} \;\forall x,y\in G$$
the coeff. are const. on the conj cla

Theorem 10 (description of $Z(kG)$)
Let $k$ be comm. ring, $G$ a finite group.
Let $\mathcal{C}$ be the set of conj. classes in $G$. For every $C\in\mathcal{C}$ denote $\Sigma C := \sum_{x\in C} x$. Then $\{\Sigma C \mid C\in\mathcal{C}\}$ is a $k$-basis of $Z(kG)$.

Proof: see above.

Remark: denote:

$$(\sum C)\cdot(\sum C') =: \sum_{D\in \mathcal{C}} a_D (\sum D) \quad \text{then } a_D \geq 0,\ a_D \in \mathbb{Z}$$

Let $z \in D$ then

$$a_D = \#\{(x,y) \mid xy = z\},$$

$\rightsquigarrow$ dual of a group algebra.

Def Let $A, B$ be $k$-algebras and $M$ a $A$-$B$-bimodule. The $k$-dual of $M$ is $M^* := \mathrm{Hom}_k(M,k)$.

Remark: $M^*$ is a $B$-$A$-bimodule through:

$$(b\cdot\alpha\cdot a)(m) = \alpha(amb) \quad \text{for } a\in A, b\in B, m\in M, \alpha\in M^*$$

In particular, $A^*$ is again an $A$-$A$-bimodule

BUT $A$ need not be isom to $A^*$ as $A$-$A$-bimodule. This is "fixed" for $A = kG$

Theorem 11 (isomorphism $kG \cong (kG)^*$)

Let $k$ be a commutative ring and $G$ a finite group. There is an isomorphism of $kG$-$kG$-bimodules

$$\theta : (kG)^* \longrightarrow kG$$

$$\alpha : kG \to k \longmapsto \sum_{x\in G} \alpha(x^{-1})\, x$$

Proof: $\theta$ is a $k$-linear isomorphism:

$$\theta\left(\alpha_x : \begin{array}{l} x \longmapsto 1 \\ y \longmapsto 0 \ \forall y \neq x \end{array}\right) = x^{-1}$$

- morphism of left $kG$-modules:

$$\theta(y\cdot\alpha) = \sum \alpha(x^{-1}y)\, x = \sum \alpha(x^{-1})\, yx$$
$$= y \sum \alpha(x^{-1})\, x = y\cdot\theta(\alpha)$$

- morphism of right $kG$-modules

$$\theta(\alpha\cdot y) = \sum \alpha(yx^{-1})\, x = \sum \alpha(x^{-1})\, xy$$
$$= \left(\sum \alpha(x^{-1})\, x\right) y = \theta(\alpha)\cdot y$$

$\square$