§10 The first orthogonality relations

Def. G a finite group, K a field of characteristic zero.

- χ is irred if χ is the character of a simple KG-module.

- On the vector space of class functions $Cl_K(G)$ of G to K we define a scallar product

$$\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle_G := \frac{1}{|G|} \sum_{x \in G} \alpha(x) \beta(x^{-1})$$

Thm 1 (Frobenius reciprocity on class functions) G a finite group, $H \leq G$, K a field of char 0. $\varphi : G \to K$ and $\psi : H \to K$ class functions. We have:

$$\langle \psi, \mathrm{Res}^G_H(\varphi) \rangle_H = \langle \mathrm{Ind}^G_H(\psi), \varphi \rangle_G$$
$$\langle \mathrm{Res}^G_H(\varphi), \psi \rangle_H = \langle \varphi, \mathrm{Ind}^G_H(\psi) \rangle_G$$

Proof: exercise. □

Thm 2 (1st orthogonality relations) G a finite group, K a splitting field of char zero. The map sending simple KG-modules to their characters is a bijection between the isomorphism classes of

simple KG- modules and the set of irreducible characters $Irr(G)$ of G.
Moreover;

(i) $Irr_K(G)$ is a k-basis of $Cl_K(G)$

(ii) $\langle \chi, \chi \rangle = 1 \ \forall \chi \in Irr(G)$

(iii) $\langle \chi, \psi \rangle = 0 \ \forall \chi, \psi \in Irr(G)$, $\chi \neq \psi$.

First a technical lemma

Lemma 3 G a finite group, K a field of char zero, V a KG- module and χ a character of V, α a class function from G to K.
Then $\frac{1}{|G|} \sum_{x \in G} \alpha(x) x^{-1} \in Z(KG)$.

- In particular $F : V \longrightarrow V$
$$v \longmapsto \left(\frac{1}{|G|} \sum_{x \in G} \alpha(x) x^{-1}\right) v$$
is a KG-homomorphism and $tr_V(F) = \langle \alpha, \chi \rangle$.

Proof:
$$y\left(\sum_{x \in G} \alpha(x) x^{-1}\right) y^{-1} = \sum_{x \in G} \alpha(x)\, y x^{-1} y^{-1} =$$
$$= \sum_{x \in G} \alpha(yxy^{-1}) \left(y x y^{-1}\right)^{-1} =$$
$$= \sum_{x \in G} \alpha(x) x^{-1} \in Z(KG).$$

- Mult by an elem of the center is a KG- endom.

$$tr(F) = \frac{1}{|G|} \sum \alpha(x)\, tr(v \mapsto x^{-1} v) = \frac{1}{|G|} \sum \alpha(x) \chi(x^{-1}) = \langle \alpha, \chi \rangle_G$$
$\square$

Proof of Thm 2 (first try)

V, V' simple KG-modules with char χ, χ', $\{v_i\} \to K$-basis of V, $\{v_i'\} \to K$-basis of V'.

$\{a_{ij}(x)\}$, $\{a'_{ij}(x)\}$ $\to$ coeff of the left mult by x:

$$x v_i = \sum_j a_{ij}(x) v_j \qquad x v_i' = \sum_j a'_{ij}(x) v_j'$$

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{x \in G} \chi(x) \chi(x^{-1}) = \frac{1}{|G|} \sum_x \sum_i \sum_j a_{ii}(x) a'_{jj}(x^{-1})$$

Take $h_{ij}: V \longrightarrow V'$

$$h_{ij}(v_i) = v_j'$$
$$h_{ij}(v_k) = 0 \quad \forall\, k \neq i$$

Construction by "averaging" h_{ij}:

$f_{ij}(v) := \frac{1}{|G|} \sum_x x\, h_{ij}(x^{-1} v) \Rightarrow$

$$\Rightarrow f_{ij}(v_\ell) = \frac{1}{|G|} \sum_x \sum_k x\, \alpha_{\ell k}(x^{-1})\, h_{ij}(v_k)$$
$$= \sum_x x \cdot \alpha_{\ell i}(x^{-1})\, v_j'$$
$$= \sum_x \sum_k \alpha_{\ell i}(x^{-1})\, \alpha'_{jk}(x)\, v_k' \quad (*)$$

Now $\mathrm{Hom}_{KG}(V, V') = \{0\}$ if $V \not\cong V'$

$$\Rightarrow \forall\, k, \ \sum_x \alpha_{\ell i}(x^{-1})\, \alpha'_{jk}(x) = 0$$

and one can do that $\forall\, i, j, \ell$.

So $\langle \chi, \chi' \rangle = 0$ (take $k = i$, $\ell = j$ and sum up)

$V \cong V' \Rightarrow f_{ij} = \lambda_{ij} Id_V \Rightarrow tr\, f = \lambda \dim_K(V)$

The k-morphism $x h_{ij}(x^{-1} v) : V \to V$ is the conjugated of h by the left multiplication by x so $tr\, h_{ij} = tr\, x h_{ij}(x^{-1} \cdot)$

and thus $tr\, f_{ij} = \frac{1}{|G|} \cdot |G|\, tr\, h_{ij} = tr\, h_{ij}$

Moreover $tr\, h_{ij} = 1$ if $i=j$ and 0 if $i \neq j$

Hence $\lambda_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{\dim_K(V)} & \text{if } i = j \end{cases}$

$$\sum_{x \in G} a_{li}(x^{-1}) a_{jk}(x) = 0, \quad \forall l, k, \quad \forall i \neq j$$

One the other hand, from $(*)$, trace of f_{ij} has an explicit form:

$$tr\, f_{ij} = \sum_x \sum_l \alpha_{li}(x^{-1}) \alpha_{jl}(x)$$

$$\Rightarrow \sum_x \sum_\alpha \alpha_{li}(x^{-1}) \alpha_{jl}(x) = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{\dim_K V} & \text{if } i = j \end{cases}$$

$$\underbrace{\sum_{i \neq j} \sum_{x \in G} a_{ij}(x^{-1}) a_{ji}(x)}_{=0} + \underbrace{\sum tr\, f_{ii}}_{=\frac{1}{\dim_K(V)}} = 1 = \sum_x \sum_i \sum_l \alpha_{li}(x^{-1}) \alpha_{il}(x) + \sum_{i \neq j} \sum_{x \in G} a_{ij}(x^{-1}) a_{ji}(x)$$

$$= \langle \chi, \chi \rangle + \sum_{l \neq i} \sum_{x \in G} a_{li}(x^{-1}) a_{il}(x) = \langle \chi, \chi \rangle$$

Now we showed that $Irr_K G$ is an orthonormal subset of $Cl_K(G)$, and thus linearly indep.

Suppose that $Irr_K(G)$ does not span $Cl_K(G)$. Then there exists $\alpha : G \to K$ $\alpha \in Cl_K(G)$ and $\langle \alpha, \chi \rangle = 0 \ \forall \chi \in Irr_K$

Apply lemma 3 to the regular KG-module KG.

$$F : KG \to KG$$
$$a \longmapsto \frac{1}{|G|} \sum \alpha(x) x^{-1} a$$

also F maps every KG-submodule of KG to itself. In particular
$$F : V \to V \quad \text{for any simple } FG\text{-mod.}$$

Thus $F|_V = \lambda \, Id_V \Rightarrow tr(F|_V) = \lambda \dim_K V$
also $tr(F|_V) = \langle \alpha, \chi \rangle = 0$ by the constr. of F,
implying $F|_V = 0$.

Now K has characteristic zero so KG is the sum of its simple modules. Hence $F = 0$. In particular $F(1) = 0$ implying that $\frac{1}{|G|} \sum \alpha(x) x^{-1} = 0 \Rightarrow\Leftarrow$

$\square$

<u>Prop 4</u> Let G, H be finite groups and K a field, V a fin dim KG-mod, W a fin. dim KH-mod. Then $Hom_K(V, W)$ is a $K(H \times G)$-module via $((y, x) \cdot f)(v) = y f(x^{-1} v)$.
Then the character of $Hom_K(V, W)$ as $K(H \times G)$-module is $\chi_{Hom_K(V,W)}(y, x) = \chi_W(y) \chi_V(x^{-1})$

Proof: $\mathrm{Hom}_K(V,W) \cong W \otimes_K V^*$
then use prop 2, previous section. □

Proof of Thm 2 (second - and better - try)

Consider the element $d = \sum_x (x,x) \in K(G\times G)$

Recall $\mathrm{Hom}_K(V',V) \simeq V \otimes (V')^*$ as $K(G\times G)$-module.
Multiplication by d on $\mathrm{Hom}_K(V',V)$:
$(d\cdot\varphi)(v') \stackrel{\mathrm{def}}{=} \sum_{x\in G} x\,\varphi(x^{-1}v') \quad \forall v' \in V'$
Thus $d\cdot\varphi \in \mathrm{Hom}_{KG}(V',V)$ ("averaging")

$\mathrm{Trace}(d\cdot \text{ in } \mathrm{Hom}_K(V',V)) = \chi_{\mathrm{Hom}(V',V)}(d) = \sum_{x\in G} \chi(x)\chi'(x^{-1}) = |G|\langle\chi,\chi'\rangle_G$

- if $V \not\cong V'$ then $\mathrm{Hom}_{KG}(V',V) = \{0\} \Rightarrow d\varphi = 0,\ \forall \varphi \in \mathrm{Hom}_K(V',V)$
  $\Longrightarrow 0 = \chi_{\mathrm{Hom}_K(V',V)}(d) = |G|\langle\chi,\chi'\rangle_G$

- if $V = V'$ then
  $d\cdot \mathrm{End}_K(V) \subseteq \mathrm{End}_{KG}(V) = K\cdot \mathrm{Id}_V$

$tr(d\cdot\varphi) = |G|\cdot tr(\varphi)$ (in $\sum x\varphi(x^{-1}\cdot)$ every term has same trace)

Also $d\cdot\varphi = \lambda\,\mathrm{Id}_V \Rightarrow tr(d\cdot\varphi) = \lambda \dim_K(V)$

This implies that if $tr(\varphi) = 0$ then $d\cdot\varphi = 0$
To study the mult. by d on $\mathrm{Hom}_K(V,V)$
choose a basis $\{\mathrm{Id}_V, \varphi_1, \dots, \varphi_{s-1}\}$
where φ_i are matrices of trace zero.

Remark that we can always choose such a basis because if $tr(\varphi) \neq 0$ then $tr(\varphi - tr(\varphi)\cdot Id_V) = 0$.

The multiplication by d sends Id_V on $|G|\cdot Id_V$ and φ_i on zero $\forall\, i = 1, \dots, s-1$.

Thus $\chi_{Hom_K(V,V)}(d) = |G|$

But $\chi_{Hom_K(V,V)}(d) = |G| \langle \chi, \chi \rangle$ $\Rightarrow \langle \chi, \chi \rangle = 1$

Hence $Irr_K(G)$ is orthonormal and the characters of non-isomorphic simple modules are different.

Now $KG \cong \prod_{1 \le i \le h} End_K(V_i)$ (where V_i are repres of isom. classes of simple modules)

$\dim_K Cl_K(G) = |\text{conj classes}| = \dim_K Z(KG)$

$Z(KG) \cong \prod_{i=1}^{h} Z(End_K(V_i)) = \prod_{i=1}^{h} K \cdot Id_{V_i}$

$\Rightarrow \dim_K Z(KG) = h = |Irr_K(G)|$ (in bij with $\{V_1, \dots, V_h\}$)

Thus $Irr_K(G)$ spans $Cl_K(G)$ and is a K-basis of $Cl_K(G)$.

$\square$

<u>Thm 5</u> Let G a finite group, K a splitting field for G such that $\frac{1}{|G|} \in K$ and $\chi \in Irr_K(G)$. Define $e(\chi) = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1})\, x$.

Then $e(\chi)$ is a primitive idemp in $Z(KG)$ which acts as the identity on every simple module with character χ and annihilates any other simple KG-module with character $\neq \chi$. Moreover $e(\chi)$ is a primitive decomposition of 1 in $Z(KG)$.

Proof: χ is a class function so $e(\chi) \in Z(KG)$ by Lemma 3. Thus the left multiplication by $e(\chi)$ is a KG-homomorphism:

$$(v \mapsto e(\chi)v) \in \mathrm{End}_{KG}(V) = K \cdot \mathrm{Id}_V$$
$$\Rightarrow e(\chi)v = \lambda v \quad \forall v \in V.$$

Let ψ be the character of V.

$$\psi(e(\chi)) = \lambda \dim_K(V) = \lambda \psi(1)$$
$$\parallel \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1}) \psi(x) = \chi(1) \langle \psi, \chi \rangle$$

Thus $\psi \neq \chi \Rightarrow \lambda = 0$ || by the orthog
$\psi = \chi \Rightarrow \lambda = 1$ || relations.

Also $KG \cong \prod_{i=1}^{h} \mathrm{End}_K(V_i)$ $\Rightarrow$ $a \in KG$ is
$a \longmapsto ((v_i \mapsto a v_i))_{i=1,\dots,h}$ uniquely determ by its action on V_i's

$\Rightarrow e(\chi)$ is uniquely charact by its action.
but $e(\chi)^2$ has the same properties $\Rightarrow e(\chi) = e(\chi)^2$

Moreover $e(\chi)$ = id elem on V_i and zero in the other matrix algebras

$\Rightarrow e(\chi)$ is primitive and $1 = \sum_{\chi \in \mathrm{Irr}(G)} e(\chi)$ □

<u>Corollary 6.</u> Let G be a finite group and K a splitting field for G. Suppose that $\frac{1}{|G|} \in K$. We have $KG = \prod_{\chi \in \mathrm{Irr}_K(G)} KGe(\chi)$ and for each $\chi \in \mathrm{Irr}_K(G)$ the algebra $KG\,e(\chi)$ is isom to $\mathrm{End}_K(V)$, where V is the simple KG-module of character χ.

<u>Proof:</u> The family $\{e(\chi)\}_{\chi \in \mathrm{Irr}_K(G)}$ is mapped to the family $\{\mathrm{Id}_{V_i}\}_{1 \le i \le h}$ where $\{V_i\}_{1 \le i \le h}$ is a set of representatives of the isomorphism classes of simple KG-modules through the algebra isom $KG \simeq \prod_{1 \le i \le h} \mathrm{End}_K(V_i)$ (see Thm. 5). $\square$

Any element of the "truncated" algebra $KG\,e(\chi)$ can be expressed back into the K-basis of KG which is G itself

<u>Prop 7.</u> Let G be a finite group, K a splitting field for G, $\frac{1}{|G|} \in K$ and $\chi \in \mathrm{Irr}_K(G)$. For any $s \in KG\,e(\chi)$ we have:

$$s = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1}s)\, x$$

<u>Proof:</u> s is a linear combination of elements of type $y \cdot e(\chi)$ with $y \in G$. So it is enough to prove Prop 7 for $s = y \cdot e(\chi)$, $y \in G$.

Now $e(\chi)$ acts as identity on any simple module with character χ. so

$$\chi(x^{-1}y\, e(\chi)) = \chi(x^{-1}y)$$

(the multiplication by $a\, e(\chi)$ on V_χ is the same as the multiplication by a on V_χ)

Thus:

$$\sum_{x\in G} \chi(x^{-1}y\, e(\chi))\, x = \sum_{x\in G} \chi(x^{-1}y)\, x$$

$$= \sum_{x\in G} \chi(x^{-1}y)\, x\, y^{-1}\, y$$

$$= \underbrace{\sum_{x\in G} \chi(yx^{-1})(yx^{-1})^{-1}}_{\frac{|G|}{\chi(1)} e(\chi)}\, y$$

Multiplying by $\frac{\chi(1)}{|G|}$ yields the result

$\square$