

§8. Duality

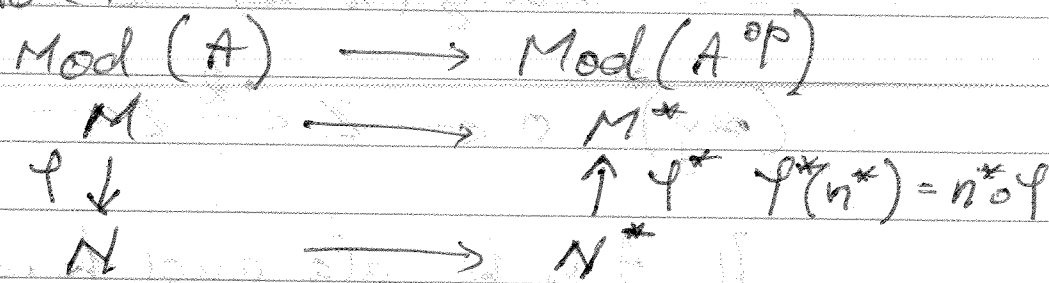
Let A be an algebra over a commutative ring k . For any A -module M , denote by $M^* := \text{Hom}_k(M, k)$

M^* becomes a right A -module by setting $(m^* \cdot a)(m) := m^*(am)$ for $m^* \in M^*, m \in M, a \in A$

Similarly if W is a right A -module, W^* becomes a left A -module through $(a \cdot w^*)(w) := w^*(wa)$ for $w^* \in W^*, w \in W, a \in A$.

So if B is another k -algebra and U is an A - B -bimodule, then U^* is a B - A -bimodule.

III The k -duality is a contravariant functor.



More generally the k -duality is a contravariant functor:

$$\text{Mod}(A \otimes_k B^{op}) \longrightarrow \text{Mod}(B \otimes_k A^{op})$$

(with the convention that an

A - B -bimodule V is an $A \otimes B^{\text{op}}$ -module

$$\begin{aligned} a_2 a_1 m b_1 b_2 &= (a_2 \otimes b_2) \cdot (a_1 \otimes b_1) m \\ &= (a_2 a_1 \otimes b_2 b_1) m \\ (a_2 a_1 \otimes b_1 b_2) \cdot m &= (a_2 a_1 \otimes b_1 b_2) m \end{aligned}$$

$$\begin{aligned} \text{Hom}_A(U, V) &\longrightarrow \text{Hom}_{\text{top}}(V^*, V^*) \\ \downarrow &\longmapsto (V^* \mapsto V^* \circ \varphi) \\ &\text{is } k\text{-linear.} \end{aligned}$$

Properties

- $(U \oplus V)^* = U^* \oplus V^*$
- φ is surj $\Rightarrow \varphi^*$ is injective

BUT • φ is inj $\not\Rightarrow \varphi^*$ is surjective:

$$\text{(ex)} \quad 0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}, \quad \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \{0\}$$

$$\text{(ex)}^* \quad 0 \leftarrow \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \leftarrow 0 \leftarrow 0$$

If A is k -alg and k is a field then the k -duality is exact:

(k -field) • φ is inj $\Rightarrow \varphi^*$ is surjective.

Indeed: let $f: U \rightarrow V$ be an injective A -homomorphism and $f^*: V^* \rightarrow U^*$

$$v^* \mapsto v^* \circ f$$

Want to show: $\forall u^* \in \text{Hom}_k(U, k) = U^*$
 $\exists v^* \in \text{Hom}_k(V, k) = V^*$, $v^* \circ f = u^*$

k is a field so U, V are k -vector spaces
Let $\{u_i | i \in I\}$ be a basis for U .

Then $\{f(u_i) | i \in I\} \cup \{v_j | j \in J\}$ is
a k -basis of V as f is injective.

Define $v^*(f(u_i)) := u^*(u_i)$ and
 $v^*(v_j) := 0 \quad i \in I, j \in J$
Then $v^* \in V^*$ and $v^* \circ f = u^*$.

Taking duality twice:

$$\begin{array}{ccccc}
 \text{Mod}(A) & \longrightarrow & \text{Mod}(A^*) & \longrightarrow & \text{Mod}(A) \\
 U & \longrightarrow & U^* & \longrightarrow & U^{**} \\
 f \downarrow & & \uparrow f^* & & \downarrow f^{**} \\
 V & \longrightarrow & V^* & \longrightarrow & V^{**}
 \end{array}$$

$$U^* = \{ f: U \rightarrow k \}$$

$$U^{**} = \{ \alpha: U^* \rightarrow k \}$$

$$(U \rightarrow k) \rightarrow k$$

(78)

We have two functors:

$$\text{Mod}(A) \xrightarrow{\text{Id}} \text{Mod}(A) \quad \text{"dual dual"}$$

$$(U \rightarrow U^{**}) \leftarrow \text{Notation: } \Delta \Delta_{\text{Mod}(A)}$$

and a natural transformation between them:

$$\eta: \text{Id}_{\text{Mod}(A)} \rightarrow \Delta \Delta_{\text{Mod}(A)}$$

$$\eta_U: U \rightarrow U^{**}$$

$$u \mapsto (u^* \mapsto u^*(u))$$

Verify:

$$\begin{array}{ccc} U & \xrightarrow{\eta_U} & U^{**} \\ \varphi \downarrow & \nearrow & \downarrow \varphi^{**} \\ V & \xrightarrow{\eta_V} & V^{**} \end{array} \quad \text{commutes}$$

$$\begin{aligned} \varphi^{**}(\eta_U(u)) &= \varphi^{**}(u^* \mapsto u^*(u))(v^*) \\ &= (u^* \mapsto u^*(u)) \circ \varphi^*(v^*) \\ &= (u^* \mapsto u^*(u))(v^* \circ \varphi) \\ &= v^* \circ \varphi(u) \end{aligned}$$

$$\begin{aligned} \eta_V(\varphi(u))(v^*) &= (u^* \mapsto u^* \circ \varphi(u))(v^*) \\ &= v^* \circ \varphi(u) \end{aligned}$$

In general η is not an equivalence (there is no θ s.t. $\eta_U \circ \theta_U = \text{Id}_U$)

BUT if k is a field then η is an equivalence on the subcategory $\text{mod}(A)$ of finite-generated A -modules.

when A is a finite-dimensional k -alg.

Proposition 1: Let k be a field, A a finitely dimensional k -algebra and U a finitely generated A -module. Then:

(i) U is (semi-)simple A -module if and only if U^* is (semi-)simple right A -module.

(ii) U indecomposable A -module if and only if U^* is indecomposable right A -module.

(iii) U is projective (injective) iff U^* is injective (projective).

(iv) $(U/\text{rad } U)^* \cong \text{soc}(U^*)$ and $(\text{soc}(U))^* \cong U^*/\text{rad}(U^*)$.

Proof: Remark that $U \cong U^{**}$ as A -modules.

Consider the short exact sequence

$$0 \rightarrow V \rightarrow U \rightarrow U/V \rightarrow 0$$

and its dual

$$0 \leftarrow V^* \leftarrow U^* \leftarrow (U/V)^* \leftarrow 0$$

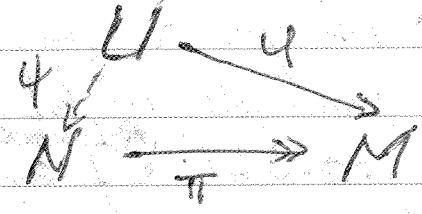
which is also exact as k is a field.

(i) if U not simple then $\exists 0 \neq V \subsetneq U \Rightarrow \exists 0 \neq (U/V)^* \subsetneq U^* \Rightarrow U^*$ is not simple.

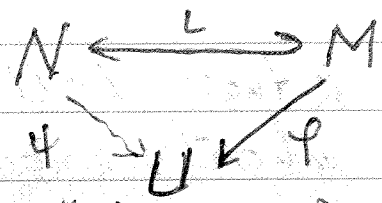
For the converse apply again the duality. Also the duality commutes with \oplus .

(ii) as the ℓ_2 -duality commutes with \otimes
 (iii)

U is projective if for any epimorphism $\pi: N \twoheadrightarrow M$ and homomorphism $\varphi: U \rightarrow M$, $\exists \psi: U \rightarrow N$ s.t. $\pi \circ \psi = \varphi$



V is injective if for any monomorphism $\iota: M \hookrightarrow N$ and homomorphism $\varphi: M \rightarrow U$, $\exists \psi: N \rightarrow U$ s.t. $\psi \circ \iota = \varphi$



To prove (iii) one just uses that ℓ_2 -duality switches between monomorphisms and epimorphisms.

(iv) $\text{rad}(U) \subseteq V \iff U/V$ is semi-simple.

$(U/V)^* \subseteq \text{soc}(U^*) \iff (U/V)^*$ is semi-simple

$$\Rightarrow (U/\text{rad}(U))^* \cong \text{soc}(U^*)$$

Change $U \leftrightarrow U^*$ and apply duality $\Rightarrow (U^*/\text{rad}(U^*)) \cong (\text{soc}(U))^*$

□

Proposition 2 Let A, B be two algebras over ring k .
let V be an A -module and W a B -module.

If k is a field no need for this condition

Suppose that one of V, W is finitely generated projective as k -modules.
There is a natural isomorphism of B - A -bimodules

$$\begin{cases} W \otimes_k V^* \cong \text{Hom}_k(V, W) \\ w \otimes v^* \mapsto (v \mapsto v^*(v)w) \end{cases}$$

$$v \in V, v^* \in V^*, w \in W.$$

Proof

B - A -bimodule structure:

- on $W \otimes_k V^* \quad b(w \otimes v^*)a = b w \otimes v^* a$
- on $\text{Hom}_k(V, W) \quad (b \cdot \varphi \cdot a)(v) = b \cdot \varphi(a \cdot v)$

Check • well defined

$$b(w \otimes v^*)a \mapsto (v \mapsto v^* a(v) b w)$$

$$b \cdot \underbrace{(v^*(a \cdot v))}_{\in k} \cdot w$$

- isomorphism $\Leftrightarrow k$ -linear isom
- true for $V=k$ or $W=k$

$$k \otimes_k V^* \cong V^* \cong \text{Hom}_k(V, k)$$

$$W \otimes_k \underbrace{k^*}_{=k} \cong W \cong \text{Hom}_k(k, W)$$

• then do it for direct sums and summands □

Rem in Prop 2 is V and W are free of finite rank then $\{w_j \otimes v_i^*\}$ is a k -basis for $W \otimes V^*$ and $\{v \mapsto v_i^*(v) w_j\}$ is a k -basis for $\text{Hom}_k(W, W)$

Similarly one can prove:

Proposition 3 A, B -algebras over a commutative ring k , V an A -module W a B -module. Suppose that V or W are projective of finite rank over k . Then there is a natural isomorphism of B - A -bimodules:

$$\begin{aligned} W^* \otimes_k V^* &\cong (V \otimes W)^* \\ (w^* \otimes v^*) &\longmapsto (v \otimes w \longmapsto v^*(v) w^*(w)) \end{aligned}$$

Proof

For $V=k$ or $W=k$ this is trivially isomorphic to V^* or W^* .

Then use direct sums and summands

□

Rem U is projective as k -module \Leftrightarrow
 $\Leftrightarrow U$ is the direct summand of a free k -module.