§8. Duality

Let A be an algebra over a commutative ring k. For any A-module M, denote by $M^* := \mathrm{Hom}_k(M, k)$

M^* becomes ~~an~~ right A-module by setting $(m^* \cdot a)(m) := m^*(am)$ for $m^* \in M^*$, $m \in M$, $a \in A$

Similarly if W is a right A-module, W^* becomes ~~a~~ left A-module through
$$(a \cdot w^*)(w) = w^*(wa)$$
for $w^* \in W^*$, $w \in W$, $a \in A$.

So if B is another k-algebra and U is an A-B-bimodule, then U^* is a B-A-bimodule.

$\sqrt{\frac{|||}{000}}$ The k-duality is a contravariant functor.

$$\begin{array}{ccc} \mathrm{Mod}(A) & \longrightarrow & \mathrm{Mod}(A^{op}) \\ M & \longrightarrow & M^* \\ \varphi \downarrow & & \uparrow \varphi^* \quad \varphi^*(n^*) = n^* \circ \varphi \\ N & \longrightarrow & N^* \end{array}$$

More generaly the k-duality is a contravariant functor:
$$\mathrm{Mod}(A \otimes_k B^{op}) \longrightarrow \mathrm{Mod}(B \otimes_k A^{op})$$

(with the convention that an A-B-bimodule M is an $A \otimes B^{op}$-module.

$$\begin{aligned} a_2 a_1 m b_1 b_2 &= (a_2 \otimes b_2) \cdot (a_1 \otimes b_1)\, m \\ &= (a_2 a_1 \otimes b_2 \circ b_1)\, m \end{aligned}$$

$$\| $$

$$(a_2 a_1 \otimes b_1 b_2) \cdot m = (a_2 a_1 \otimes b_1 b_2)\, m \quad)$$

$$\begin{aligned} \mathrm{Hom}_A(U, V) &\longrightarrow \mathrm{Hom}_{A^{op}}(V^*, U^*) \\ \varphi &\longmapsto (v^* \longmapsto v^* \circ \varphi) \end{aligned}$$

is k-linear.

Properties

- $(U \oplus V)^* = U^* \oplus V^*$
- φ is surj $\Rightarrow$ φ^* is injective

BUT • φ is inj $\not\Rightarrow$ φ^* is surjective:

(seq) $\quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}, \quad \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) = \{0\}$$

$(\text{seq})^*$ $\quad 0 \longleftarrow \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \longleftarrow 0 \longleftarrow 0$

If A is k-alg and k is a field then the k-duality is exact:

(k-field) • φ is inj $\Rightarrow$ φ^* is surjective.

Indeed: let $\varphi: U \to V$ be an injective A-homomorphism and $\varphi^*: V^* \to U^*$

$v^* \mapsto v^* \circ \varphi$.

Want to show: $\forall\, u^* \in \mathrm{Hom}_k(U, k) = U^*$
$\exists\, v^* \in \mathrm{Hom}_k(V, k) = V^*$, $v^* \varphi = u^*$

k is a field so U, V are k-vector spaces
Let $\{u_i \mid i \in I\}$ be a basis for U.
Then $\{\varphi(u_i) \mid i \in I\} \amalg \{v_j \mid j \in J\}$ is a k-basis of V as φ is injective.

Define $v^*(\varphi(u_i)) := u^*(u_i)$ and
$v^*(v_j) := 0 \qquad i \in I, j \in J.$
Then $v^* \in V^*$ and $v^* \circ \varphi = u^*$.

Taking duality twice:

$$\begin{array}{ccccc} \mathrm{Mod}(A) & \longrightarrow & \mathrm{Mod}(A^*) & \longrightarrow & \mathrm{Mod}(A) \\ U & \longrightarrow & U^* & \longrightarrow & U^{**} \\ \varphi \downarrow & & \uparrow \varphi^* & & \downarrow \varphi^{**} \\ V & \longrightarrow & V^* & \longrightarrow & V^{**} \end{array}$$

$U^* = \{f : U \to k\}$

$U^{**} = \{\alpha : U^* \to k\}$
$(U \to k) \to k$.

We have two functors:

"dubbledual"

$$\begin{tikzcd} \mathrm{Mod}(A) \arrow[r, shift left, "Id"] \arrow[r, shift right, "(U \to U^{**})"'] & \mathrm{Mod}(A) \end{tikzcd}$$

$(U \to U^{**}) \leftarrow$ Notation: $\delta\delta_{\mathrm{Mod}(A)}$

and a natural transformation between them:

$$\eta : Id_{\mathrm{Mod}(A)} \longmapsto \delta\delta_{\mathrm{Mod}(A)}$$

$$\eta_U : U \longrightarrow U^{**}$$
$$u \longmapsto (u^* \longmapsto u^*(u))$$

Verify:

$$\begin{tikzcd} U \arrow[r, "\eta_U"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & U^{**} \arrow[d, "\varphi^{**}"] \\ V \arrow[r, "\eta_V"'] & V^{**} \end{tikzcd}$$

commutes

$$\begin{aligned} \varphi^{**}(\eta_U(u))(v^*) &= \varphi^{**}(u^* \longmapsto u^*(u))(v^*) \\ &= (u^* \longmapsto u^*(u)) \circ \varphi^*(v^*) \\ &= (u^* \longmapsto u^*(u))(v^* \circ \varphi) \\ &= v^* \circ \varphi(u) \end{aligned}$$

$$\begin{aligned} \eta_V(\varphi(u))(v^*) &= (v^* \longmapsto v^* \circ \varphi(u))(v^*) \\ &= v^* \circ \varphi(u) \end{aligned}$$

In general η is not an equivalence (there is no θ s.th $\eta_U \circ \theta_U = Id_U$)

<u>BUT</u> if k is a field then η is an equivalence on the subcategory mod(A) of finite-generated A-modules,

when A is a finite-dimensional k-alg.

Proposition 1: Let k be a field, A a finitely dimensional k-algebra and U a finitely generated A-module. Then:

(i) U is (semi-) simple A-module if and only if U^* is (semi-) simple right A-modul.

(ii) U indecomposable A-module if and only if U^* is indecomposable right A-module

(iii) U is projective (injective) iff U^* is injective (projective)

(iv) $(U/\mathrm{rad}\,U)^* \cong \mathrm{soc}(U^*)$ and $(\mathrm{soc}(U))^* \cong U^*/\mathrm{rad}(U^*)$.

Proof: Remark that $U \cong U^{**}$ as A-modules. Consider the short exact sequence

$$0 \longrightarrow V \longrightarrow U \longrightarrow U/V \longrightarrow 0$$

and its dual

$$0 \longleftarrow V^* \longleftarrow U^* \longleftarrow (U/V)^* \longleftarrow 0$$

which is also exact as k is a field.

(i) if U not simple then $\exists\, 0 \neq V \subsetneq U \Rightarrow \{0\} \neq (U/V)^* \subsetneq U^* \Rightarrow U^*$ is not simple. For the converse apply again the duality. Also the duality commutes with $\oplus$

(ii) ← as the k-duality commutes with $\oplus$

(iii)

U is projective if for any epimorphism $\pi : N \twoheadrightarrow M$ and homomorphism $\varphi : U \to M$, $\exists\, \psi : U \to N$ s.th. $\pi \circ \psi = \varphi$

$$\begin{tikzcd} & U \arrow[dl, dashed, "\psi"'] \arrow[dr, "\varphi"] & \\ N \arrow[rr, two heads, "\pi"'] & & M \end{tikzcd}$$

V is injective is for any monomorphism $\iota : M \hookrightarrow N$ and homomorphism $\varphi : M \to U$, $\exists\, \psi : N \to U$ s.th. $\psi \circ \iota = \varphi$

$$\begin{tikzcd} N \arrow[dr, dashed, "\psi"'] & & M \arrow[ll, hook', "\iota"'] \arrow[dl, "\varphi"] \\ & U & \end{tikzcd}$$

to prove (iii) one just uses that k-duality switches between monomorphisms and epimorphisms.

(iv) $\mathrm{rad}(U) \subseteq V \iff U/V$ is semi-simple.

$\Updownarrow$

$(U/V)^* \subseteq \mathrm{soc}(U^*) \iff (U/V)^*$ is semi-simple

$\Rightarrow \left(U/\mathrm{rad}(U)\right)^* \cong \mathrm{soc}(U^*)$

Change $U \leftrightarrow U^*$ and apply duality $\Rightarrow \left(U^*/\mathrm{rad}(U^*)\right) \cong \left(\mathrm{soc}(U)\right)^*$

$\square$

<u>Proposition 2.</u> Let A, B be two algebras/k ring.
Let V be an A-module and W a B-module.

If k is a field no need for this condition: Suppose that one of V, W is finitely generated projective as k-modules.

There is a natural isomorphism of B-A-bimod

$$\begin{cases} W \otimes_k V^* \cong \mathrm{Hom}_k(V, W) \\ w \otimes v^* \longmapsto (v \longmapsto v^*(v)\, w) \end{cases}$$

$v \in V$, $v^* \in V^*$, $w \in W$.

<u>Proof</u>

B-A-bimodule structure:

- on $W \otimes_k V^*$ $\quad b(w \otimes v^*)a = bw \otimes v^* a$
- on $\mathrm{Hom}_k(V, W)$ $\quad (b \cdot \varphi \cdot a)(v) = b\varphi(a v)$

Check
- well defined
- $b(w \otimes v^*)a \longmapsto (v \longmapsto v^*a(v)\, bw)$

$$\overset{\shortparallel}{} \quad b \cdot (\underbrace{v^*(av)}_{\in k} \cdot w)$$

- isomorphism $\Leftrightarrow$ k-linear isom
- true for $V = k$ or $W = k$

$$k \otimes_k V^* \simeq V^* \simeq \mathrm{Hom}_k(V, k)$$

$$W \otimes_k \underbrace{k^*}_{=k} \simeq W \simeq \mathrm{Hom}_k(k, W).$$

- then do it for direct sums and summands

$\square$

<u>Rem</u> in Prop 2 if V and W are free of finite rank then $\{w_j \otimes v_i^*\}$ is a k-basis for $W \otimes V^*$ and $\{v \longmapsto v_i^*(v)\, w_j\}$ is a k-basis for $\mathrm{Hom}_k(V, W)$

<u>Similarly one can prove:</u>

<u>Proposition 3</u> A, B - algebras over a commutative ring k, V an A-module W a right B-module. Suppose that V or W are projective of finite ~~rank~~ over k. Then there is a natural isomorphism of B-A-bimodules:

$$
\begin{aligned}
W^* \otimes_k V^* &\cong (V \otimes_k W)^* \\
(w^* \otimes v^*) &\longmapsto (v \otimes w \longmapsto v^*(v)\, w^*(w))
\end{aligned}
$$

<u>Proof</u>

For $V = k$ or $W = k$ this is trivially isomorphic to V^* or W^*.

Then use direct sums and summands

$\square$

<u>Rem</u> U is projective as k-module $\Longleftrightarrow$ $\Longleftrightarrow$ U is the direct summand of a free k-module.