PART B CHARACTER THEORY

§ 9. Characters of finite groups

Def: The trace of a square matrix with coefficients in a commutative ring is the sum of its diagonal elem.
Notation: $tr(M)$, $M \in M_n(k)$

Properties
- $tr(NM) = tr(MN)$
- $tr(NMN^{-1}) = tr(M)$ if N is invert.

Given a k-module V of finite rank n this defines a trace map:

$$tr_V : End_k(V) \to k$$
$$\varphi \longmapsto tr(\varphi)$$

where $tr(\varphi)$ is the trace of a matrix of the homomorphism in a given basis. Given that if we change the basis we have to conjugate this matrix by the change of base matrix, the trace will be constant under this operation.

Def: Let G be a group, k a commutative ring. Let V be a kG-module which is free of finite rank over k. The character of V is the k-linear map

$$\chi_V : kG \longrightarrow k$$
$$a \longmapsto tr\begin{pmatrix} V \longrightarrow V \\ v \longmapsto av \end{pmatrix}.$$

if $V = kG$ then χ_V is the character of G over k

Remarks :

- $\chi = tr_V \circ \rho \qquad kG \xrightarrow{\rho} End_k(V) \xrightarrow{tr_V} k$
  where $\rho(a) : V \longrightarrow V$
  $\qquad v \longmapsto av$
- χ_V is determined as k-linear map by the values on G
- $\chi_V(1) = rank_k(V)$
  (multiplication by 1 is given by Id_V)
- $\chi_V(ab) = \chi_V(ba) \quad a, b \in kG$
  ($tr_V(MN) = tr_V(NM)$)
- $\chi_V(aba^{-1}) = \chi_V(b)$ for $a \in kG^\times$, $b \in kG$

$\rightsquigarrow$ thus $\chi_V(x)$ only depend on the conjugacy class of x in G

- χ_V is a class function on G
  (i.e. k-linear map from $kG \longrightarrow k$ that is constant on the conjugacy classes of G)

Proposition 1 Suppose that $K \subseteq \mathbb{C}$ is a subfield of $\mathbb{C}$. Let χ be a character of a fin.gr. G over K. Then $\chi(x^{-1}) = \overline{\chi(x)}$, $\forall x \in G$.

Proof :

$x \in G$, $|G| = m < \infty \Rightarrow x^m = 1$

Thus the multiplication by x is repr. by a matrix M such that $M^m = Id$. Changing the basis, M can be taken to be upper triangular.

Moreover if λ_i is an element on the diagonal then $\lambda_i^m = 1$, i.e. λ_i is an m^{th} root of unity. But then $(\lambda_i^{-1}) = \overline{\lambda_i}$. Hence

$$tr(M^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{tr(M)}$$

$\square$

- If A is an algebra over a commutative ring k. Something special for group algebras $kG \cong (kG)^\circ$ by sending x to x^{-1}.

Thus if V is a left ~~kG~~-module then $V^* = Hom_k(V,k)$ can also be seen as a left kG-module via

$(x \cdot \mu)(v) = \mu(x^{-1} v)$

- If V, W are left ~~kG~~-modules then $V \oplus W$ is a left kG-module via

$x \cdot (v, w) = (xv, xw)$

(here we ~~can~~ replace kG by any k-algebra A and the result doesn't change)

- V and W kG-modules then $V \otimes_k W$ is a $kG \otimes_k kG$-module via
$(x \otimes y)(v \otimes w) = xv \otimes yw$
Recall that $kG \otimes_k kG \cong k(G \times G)$
$(x \otimes y) \longmapsto (x, y)$
Moreover we have
$\Delta : kG \longrightarrow k(G \times G)$ the diagonal map
$x \longmapsto (x, x)$
Via Δ a $kG \otimes_k kG$-module becomes a kG-module:
$x \cdot (v \otimes w) := (x \otimes x)(v \otimes w) = xv \otimes xw$

Prop 2 (computation with characters)
Let k be a field, G - finite group, V, W - fin. dim k-mod.
(i) V^* has a kG-mod structure given by $(x\mu)(v) = \mu(x^{-1}v)$ and $\forall x \in G$ $\chi_{V^*}(x) = \chi_V(x^{-1})$
(ii) $V \oplus W$ has a kG-mod structure via $x(v, w) = (xv, xw)$ and $\forall x \in G$ $\chi_{V \oplus W}(x) = \chi_V(x) + \chi_W(x)$
(iii) $V \otimes_k W$ has a kG-mod structure given by $x(v \otimes w) = (xv \otimes xw)$ and $\chi_{V \otimes W}(x) = \chi_V(x)\chi_W(x)$.

Proof: Let $\{v_1, \dots, v_n\}$ be a k-basis of V
$\{w_1, \dots, w_m\}$ be a k-basis of W
Define v_i^* by $v_i^*(v_i) = 1$ and $v_i^*(v_j) = 0$ $\forall j \neq i$
$\Rightarrow \{v_1^*, \dots, v_n^*\}$ is a basis of V^* (dual basis to $\{v_1, \dots, v_n\}$)

Consider the multiplication by x^{-1} in V
$x^{-1}v_i = \sum \lambda_{ij} v_j \quad \Rightarrow \chi_V = \sum \lambda_{ij}$

Consider now the multiplication by x in V^*:

$$x \cdot v_i^*(v_j) = v_i^*(x^{-1} v_j) = $$
$$= v_i^*\left(\sum \lambda_{j\ell} v_\ell\right) =$$
$$= \sum \lambda_{j\ell} v_i^*(v_\ell) = \lambda_{ji}$$

We would like to write λ_{ij} as a linear combination of v_ℓ^* evaluated at v_j

$$\lambda_{ji} = \sum \lambda_{ki} v_k^*(v_j) \Rightarrow x v_i^* = \sum \lambda_{ki} v_k^*$$

and $\chi_{V^*}(x) = \sum \lambda_{ii} = \chi_V(x^{-1})$

(ii) $\{v_1, \dots, v_n, w_1, \dots, w_m\}$ is a basis of $V \oplus W$.
Consider the multiplication by x:

$$x \cdot v_i = \sum \lambda_{ij} v_j \qquad x \cdot w_k = \sum \mu_{kj} w_j$$

Thus $\chi_{V \oplus W}(x) = \sum v_{ii} + \sum w_{ii} = \chi_V(x) + \chi_W(x)$.

(iii) $\{v_i \otimes w_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis for $V \otimes W$
Consider the multiplication by x:

$$x \cdot (v_i \otimes w_j) = x v_i \otimes x w_j = \left(\sum \lambda_{i\ell} v_\ell\right) \otimes \left(\sum \mu_{jk} w_k\right)$$
$$= \sum \lambda_{i\ell} \cdot \mu_{jk} (v_\ell \otimes v_k)$$

diagonal $\Rightarrow i = \ell, \ j = k$

$$\Rightarrow \chi_{V \otimes W}(x) = \sum_{i,j} \lambda_{ii} \mu_{jj} = \left(\sum \lambda_{ii}\right)\left(\sum \mu_{jj}\right) =$$
$$= \chi_V(x) \cdot \chi_W(x)$$

$\square$.

Rem. Let G be a finite group, k a field then the character of the trivial kG module k is 1

Prop 3 Let G be a finite group, k a field and M a finite G-set. Then $\chi_{k[M]}(x) = \underbrace{|\{m \in M \mid xm = m\}|}_{\text{nb of fixed points for } x}$

Proof: $M = \{m_1, \dots, m_t\}$ is a basis of $k[M]$.
Multiplication by x: $x \cdot m_i = m_{\sigma(i)}$
$\Rightarrow$ Matrix with only 1 value 1 in every line and column $\Rightarrow$ on the diag $\Leftrightarrow i = \sigma(i)$
$\Leftrightarrow x \cdot m_i = m_i$ □

Prop 4: G a finite group, k a field; kG, the regular kG-module has the following character: $\chi(1) = |G|$, $\chi(x) = 0$, $x \in G \setminus 1$.

Proof: as in prop 3, $G = \{e, g_1, \dots, g_{n-1}\}$ is a k-basis for kG.
mult. by 1: all the basis is elementwise fixe $\Rightarrow \chi(1) = |G|$
mult by $x \neq 1$: no fixed points $\Rightarrow \chi(x) = 0$. □

Def: G a finite group, $H \leq G$, k a commutative ring, $\varphi: G \to k$ a class function, $\psi: H \to k$ another class function.
$\mathrm{Res}^G_H(\varphi): H \to k$ $\quad \mathrm{Res}^G_H(\varphi)(y) = \varphi(y)$
$\mathrm{Ind}^G_H(\psi): G \to k$ $\quad \mathrm{Ind}^G_H(\psi)(x) = \frac{1}{|H|} \sum_{\substack{y \in G \\ yxy^{-1} \in H}} \psi(yxy^{-1})$
(here need $\frac{1}{|H|} \in k$)

Prop 5: G a finite group, $H \leq G$, k a commutative ring, $\frac{1}{|H|} \in k$ and $\varphi : H \to k$ a class function. Then

(i) $\mathrm{Ind}_H^G(\varphi)$ is a class function

(ii) $\mathrm{Ind}_H^G(\varphi)(x) = \sum_{y \in [H \backslash G]} \varphi^\circ(yxy^{-1}) \quad \forall x \in G.$

where $\varphi^\circ(x) = \varphi(x)$ if $x \in H$ and $\varphi(x) = 0$ if $x \in G \setminus H$.

(iii) $\mathrm{Ind}_H^G(\varphi)(1) = [G:H]\,\varphi(1)$.

Proof

(i) $\mathrm{Ind}_H^G(\varphi)(xyx^{-1}) = \frac{1}{|H|} \sum_{g \in G} \varphi^\circ(gxyx^{-1}g^{-1})$

$= \frac{1}{|H|} \sum_{g \in G} \varphi^\circ(y) = \mathrm{Ind}_H^G(\varphi)(y).$

(ii) φ is a class function $\Rightarrow \varphi^\circ$ is also a class function.

but then $\varphi^\circ(yxy^{-1}) = \varphi^\circ(hyxy^{-1}h^{-1})$

and $yxy^{-1} \in H \iff hyxy^{-1}h^{-1} \in H$, $h \in H$

Thus $\sum_{h \in H} \varphi_0(hyxy^{-1}h^{-1}) = |H| \cdot \varphi_0(yxy^{-1})$.

(iii) $\mathrm{Ind}_H^G \varphi(1) = \frac{|G|}{|H|}\varphi(1) = [G:H]\,\varphi(1).$

□

Rem: by (ii) we could change the definition of $\mathrm{Ind}_H^G(\varphi)$ in order to extend it to the case where $|H|$ is not invertible.

<u>Thm 6</u>: G a finite group, $H \leq G$,
k a commutative ring.
M a k-free kH-module of finite rank$/k$.
Let $\psi : H \to k$ be a char of M.
Then $\mathrm{Ind}_H^G(\psi)$ is the character of $\mathrm{Ind}_H^G(M)$.

<u>Proof</u>: $\mathrm{Ind}_H^G(M) = \bigoplus_{y \in [G/H]} y \otimes M$

$\left(kG = \bigoplus_{y \in [G/H]} y \cdot kH \text{ as right } kH\text{-mod}\right)$

Study the action of $x \in G$ on $\mathrm{Ind}_H^G(M)$
this sends $y \otimes M$ to $xy \otimes M = y' \otimes M$ where $y'h = xy$ for some $h \in H$.

This will give a contribution on the diagonal only if
$y = y' \iff yh = xy \iff y^{-1}xy \in H$

Moreover the contribution is the trace of the multiplication by $h = y^{-1}xy$ on M which is $\psi(y^{-1}xy)$.

Summing up the contributions we get that the trace of the mult by x on M is $\sum_{y \in [G/H]} \psi^0(y^{-1}xy) = \mathrm{Ind}_H^G(\psi)(x)$

□

<u>Thm 7</u> (Frobenius reciprocity for class functions)
$\langle \psi, \mathrm{Res}_H^G(\varphi) \rangle_H = \langle \mathrm{Ind}_H^G(\psi), \varphi \rangle_G$
$\langle \mathrm{Res}_H^G(\varphi), \psi \rangle_H = \langle \varphi, \mathrm{Ind}_H^G(\psi) \rangle_G$