§12. Character tables of finite groups

Let G be a finite group and let K be a splitting field of char zero for G.

Def The character table of G over K is the square matrix $(\chi(x))_{\substack{\chi \in \mathrm{Irr}_K(G) \\ x \in \mathcal{K}}}$

(recall that $\mathcal{K}$ is a complete set of representatives of conjugation classes in G). Remark that the matrix is defined up to perm. of rows and columns.

The character table of cyclic groups

Let $G \simeq C_n = \langle y \rangle$, $y^n = 1$.

- G is abelian so any simple KG-module is of dimension 1.
- so giving a character is equivalent to giving a group homomorphism $\chi : G \longrightarrow K^\times$
- χ is defined by its image on y. Moreover $y^n = 1$ so y is an n^{th} root of 1 in
- The complete list of characters is given by: $\chi_i(y) := \xi^i$ for $0 \le i \le n-1$
- Hence the character table is given by.

$$\left(\xi^{ij}\right)_{0 \le i,j \le n-1}$$

where ξ is a primitive root of 1.

Remark: the first row of the char table is the triv. char

The character table of a Klein ~~four~~ group

Let $G \simeq C_2 \times C_2 \simeq \langle s \rangle \times \langle t \rangle$, $s^2 = t^2 = 1$

- Any irred char is a group homomorphism $\chi : G \longrightarrow K^{\times}$ hence determined by its values on x and y. Indeed $\chi(1) = 1$ and $\chi(xy) = \chi(x)\chi(y)$ ~~is uniquely determined by the orthog. re~~
- $\chi(x)$ and $\chi(y)$ are 2^{nd} roots of 1 so 1 or -1.

char \ conj. cl.	1	x	y	xy
1	1	1	1	1
χ_1	1	-1	+1	-1
χ_2	1	+1	1	-1
χ_3	1	-1	-1	1

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The character table of an abelian group

If G is abelian then

$$G \simeq C_{m_1} \otimes C_{m_2} \times \cdots \times C_{m_t}$$

Let $(\xi_i^{je})_{0 \le j, e \le m_i - 1}$, $1 \le i \le t$ be the character tables for C_{m_i}.

We have $\mathrm{Hom}(G, K^{\times}) = \prod_{i=1}^{t} \mathrm{Hom}(C_{m_i}, K^{\times})$

Hence the character table for G is given by

$$\bigotimes_{i=1}^{t} \left(\xi_i^{je} \right)_{0 \le j, e \le m_i - 1}$$

where ξ_i is a primitive m_i^{th} root of 1.

"Recepe for cooking" parts of a character table when G is not abelian.

1. find the characters of degree 1.
 Such characters $\chi : G \longrightarrow K^{\times}$ are group homomorphism with image in an abelian group ($K^{\times}$) thus they factor uniquely through the abelianization $G/[G,G]$ of G (use the universal property of the abelianization:
 for any abelian group A and any group homomorphism $\varphi : G \to A$ there exists a unique group homo. $\overline{\varphi} : G/[G,G] \longrightarrow A$ such that $\varphi = \overline{\varphi} \circ \pi$

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & A \\ G/[G,G] \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

 where $\pi : G \longrightarrow G/[G,G]$ is the canonical proj.

Ex: $G = S_n$ then any morphism $\chi : G \longrightarrow S_n$ factors through.
$S_n/[S_n, S_n] = S_n/A_n \simeq C_2$

hence there are only 2 characters of dimension 1 for S_n:
→ trivial: $\chi(\sigma) = 1 \quad \forall \sigma \in S_n$
→ sign: $\chi(\tau) = 1 \quad \forall \tau \in A_n$
 $\chi((1,2)\tau) = -1$

Dihedral group

Let $D_{2n} := \langle x, t \mid x^n = t^2 = 1 \ , \ txt = x^{-1} \rangle$

- $[D_{2n}, D_{2n}] \subseteq \langle x \rangle$ as $D_{2n}/\langle x \rangle \simeq C_2$

Moreover $xtx^{-1}t^{-1} = x^2 \in [D_{2n}, D_{2n}]$
Hence $[D_{2n}, D_{2n}] = \langle x^2 \rangle$

If n is odd then $\langle x^2 \rangle = \langle x \rangle$ and $D_{2n}/_{[D_{2n}, D_{2n}]} \simeq C_2$
n is even then $\langle x^2 \rangle \subsetneq \langle x \rangle$ and $D_{2n}/_{[D_{2n}, D_{2n}]} \simeq C_2 \times C_2$

Thus ;

n odd :
- $\chi_1(x^j) = 1 \quad \chi_1(x^j t) = 1$ and
- $\chi_2(x^j) = 1 \quad \chi_2(x^j t) = -1$ are the 2 1-dim char.

n even
- $\chi_1(x^j) = 1, \ \chi_1(x^j t) = 1 \qquad (\chi(t) = 1 \ , \ \chi(x) = 1)$
- $\chi_2(x^j) = 1, \ \chi_2(x^j t) = -1 \qquad (\chi(t) = -1 \ , \ \chi(x) = 1)$
- $\chi_3(x^j) = (-1)^j, \ \chi_3(x^j t) = (-1)^j \qquad (\chi(t) = 1 \ , \ \chi(x) = -1)$
- $\chi_3(x^j) = (-1)^j, \ \chi_3(x^j t) = (-1)^{j+1} \qquad (\chi(t) = -1 \ , \ \chi(x) = -1)$

(2) construct simple KG-modules and compute the characters directly.

For KD_{2n} there are simple modules M_i, $0 \le i \le n-1$ of dimension 2, given in matrix form by the group homomorphisms:

$$\rho_i : D_{2n} \longrightarrow M_2(K)^*$$

defined by the following. Choose ξ a primitive n^{th} root of 1

$$\rho_i(x) := \begin{pmatrix} \xi^i & 0 \\ 0 & \xi^{-i} \end{pmatrix} \quad \rho_i(t) := \begin{pmatrix} 0 & \xi^{-i} \\ \xi^i & 0 \end{pmatrix}$$

$$(\rho_i(x))^n = Id_2 \quad (\rho_i(t))^2 = Id_2$$

$$\rho_i(t)\,\rho_i(x)\,\rho_i(t) = \begin{pmatrix} 0 & \xi^{-2i} \\ \xi^{2i} & 0 \end{pmatrix}\begin{pmatrix} 0 & \xi^{-i} \\ \xi^i & 0 \end{pmatrix} = \begin{pmatrix} \xi^{-i} & 0 \\ 0 & \xi^i \end{pmatrix}$$
$$= (\rho_i(x))^{-1}$$

This verification shows that ρ_i is a group homomorphism, $\forall\ 0 \le i \le n-1$
The character of M_i is thus given by
$\chi_i(x^k) = \xi^{ik} + \xi^{-ik}$ and $\chi_i(x^k t) = 0$

- for $0 < i < \frac{n}{2}$ the modules M_i are simple and pairwise non-isomorphic:

$$\langle \chi_i, \chi_i \rangle = \frac{1}{2n}\sum_{k=0}^{n-1} \underbrace{\xi^{2ki}}_{\Sigma=0} + 2 + \underbrace{\xi^{-2ki}}_{\Sigma=0} = 1$$
unless $2i = n \Rightarrow \Leftarrow$

$$\langle \chi_i, \chi_j \rangle = \frac{1}{2n}\sum_{k=0}^{n-1} \underbrace{\xi^{k(i+j)}}_{\Sigma=0} + \underbrace{\xi^{k(i-j)}}_{\Sigma=0} + \underbrace{\xi^{k(j-i)}}_{\Sigma=0} + \underbrace{\xi^{-(i+j)}}_{\Sigma=0} = 0$$

- for the simplicity of the M_i's one can also see that there is no KD_{2n} invariant subspace of dimension 1 of M_i
$\{ z(\lambda,\lambda) \mid \lambda \in K \}$ is the only invariant subspace by $\chi_i(t)$ but this is not invariant under $\chi_i(x)$ unless $\xi^i = \xi^{-i} \Leftrightarrow 2i = n \Rightarrow \Leftarrow$

Now we can see that we have all the modules as the squares of the dimension add up to $2n$:

- n even: $4 \cdot 1 + (\frac{n}{2} - 1)4 = 2n$
- n odd: $2 \cdot 1 + (\frac{n-1}{2})4 = 2n$

3) obtain new simple modules using induction.

- an alternative way to describe the 2-dim simple KD_{2n}-modules is via induction as follows:

Let W_i be the 1-dimensional $K\langle x\rangle$ module with character $\chi_i : x \longmapsto \xi^i$

Set $M_i := \mathrm{Ind}_{\langle x\rangle}^{D_{2n}}(W_i)$. We show that M_i is the same module as the one previously defined:

- the action of t is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ w.r.t. $\{1\otimes 1, t\otimes 1\}$
- the action of x:
  - on $1\otimes w$: $x\cdot(1\otimes w) = x\otimes w = 1\otimes x w$
    $= \xi^i(1\otimes w)$
  - on $t\otimes w$: $x\cdot(t\otimes w) = xt\otimes w = tx^{-1}\otimes w$
    $= t\otimes x^{-1} w = \xi^{-i}(t\otimes w)$

So the action of x is given by $\begin{pmatrix} \xi^i & 0 \\ 0 & \xi^{-i} \end{pmatrix}$

Remarks. • t acts as ~~t~~ before.

- Simple as ${}^{t}W_i \not\cong W_i$, and $\langle x\rangle \trianglelefteq D_{2n}$
- One doesn't always need to construct representations:

$S_3 \cong D_6$ → non-ab so at least 1 char of deg > 1
→ two char χ_1, χ_2 of deg 1
→ $6 = 1^2 + 1^2 + 2^2$ ⇒ one more char ψ of deg 2
→ $\rho = \chi_1 + \chi_2 + 2\psi$ (ρ = regular charact.)
$\Rightarrow \psi = \frac{1}{2}(\rho - \chi_1 - \chi_2)$