§13. Integrability and Burnside's Theorem.

Aims : For $\chi \in \mathrm{Irr}(G)$

- $\chi(1) \mid |G|$
- if $|G| = p^a q^b$ then G is solveable

Remark : any field of char 0 contains $\mathbb{Q}$.

Def : Let K be a field of char zero.
An element $\alpha \in K$ is called a algebraic integer if α is the root of a monic polynomial with coeff. in K ($x^n + a_{n-1}x^{n-1} + \dots + a_0$

Ex : Any n-th root of 1 is an algebraic integer ($\xi^n - 1 = 0$).

Prop 1 (Properties of algebraic integers)
Let K be a field of char zero and let S be the set of all alg. integers over K.

(i) Every subring R of K which is fin gene. as mod. over $\mathbb{Z}$ is contained in S

(ii) S is a subring of K

(iii) $S \cap \mathbb{Q} = \mathbb{Z}$.

Proof :

(i) Let $\gamma \in R \Rightarrow \gamma^n \in R \quad \forall\ n \geq 0$

Then $N = \langle \gamma^n \mid n \geq 0 \rangle$ is fin gene as a $\mathbb{Z}$-mod.

(R is fin. gene and $\mathbb{Z}$ is Noetherian)

$$\Rightarrow \gamma^m = \sum_{i=0}^{m-1} \alpha_i \gamma^i \Rightarrow \gamma \text{ is alg. integer.}$$

(ii) Take α, β two alg integers,

$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$

$\beta^m + b_{m-1}\beta^{m-1} + \cdots + b_0 = 0$

Then $N := \langle \alpha^i \beta^j \mid 0 \leq i < n,\ 0 \leq j < m \rangle$ contains all powers of α, β and their products. Thus N is a fin. gene subring of $K \Rightarrow N \subseteq S$.

In particular, $\alpha\beta$, $\alpha \pm \beta \in S$.

(iii) Suppose $\alpha = \frac{p}{q}$ with $\gcd(p,q) = 1$

Then $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0 \Leftrightarrow$

$$p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n = 0 \Rightarrow q \mid p \Rightarrow\Leftarrow .$$

$\square$

Rem : S need not be finitely generated $\mathbb{Z}$-module.

Prop 2. Let G be a finite group, K a splitting field of characteristic zero for G. Then:

(i) $\chi(x)$ is an algebraic integer.

If moreover $K \subseteq \mathbb{C}$

(ii) $|\chi(x)| \leq \chi(1)$

(iii) $|\chi(x)| = \chi(1) \Leftrightarrow x$ acts as $\lambda \mathrm{Id}$ on V

λ root of 1 in $\mathbb{C}$.

(iv) $Z(\chi) = \{ y \in G \mid |\chi(y)| = \chi(1) \}$
is normal sg. of G, $Z(\chi) = G \iff \chi(1) = 1$.

<u>Proof</u>:

(i) mult by x is repres by M
moreover once can consider that
M is triangular with $\lambda_1, \dots, \lambda_n$
on the diagonal.
$x^n = 1 \Rightarrow \lambda_i^n = 1 \ \forall i \Rightarrow \lambda_i$ is
an alg. integer $\Rightarrow tr(M) = \sum \lambda_i$ is
an alg integer. Hence $\chi(x)$ is also one.

(ii) $$\underbrace{|\sum \lambda_i|}_{= |\chi(x)|} \leq \sum |\lambda_i| = n = \chi(1)$$

(iii) "=" in the above inequality is
equivalent to $\lambda_1 = \lambda_2 = \dots = \lambda_n$.
Moreover if M is in standard Jordan
form then M is diagonal.
(the only Jordan block whose power is the
identity is the 1×1 matrix; we are in char 0)

So $M = \lambda \, Id_m$, $\lambda^m = 1$
The converse is trivial

(iv) From (iii) the mult by $y \in Z(\chi)$ is given
by $\lambda \, Id$, $\lambda^n = 1$. Thus the mult by yy'^{-1} is
given by $\lambda \lambda'^{-1} Id$. But $\lambda\lambda'^{-1}$ is also a
root of 1 so $yy'^{-1} \in Z(\chi)$. Moreover it
is clear that $Z(\chi)$ is normal in G as the

trace is preserved by conjugation.

~~Now~~ $\chi(1) = 1 \Rightarrow \chi(y)$ is mult by a root of 1 for all $y \in G \Rightarrow Z(\chi) = G$.

Conversely $Z(\chi) = G \Rightarrow$ the multiplication by any $y \in G$ on V is repres by $\lambda \cdot Id$.

But V is a simple module $\Rightarrow \dim_K V = 1 \Rightarrow \chi(1) = 1$.

$\square$

Prop 3 Let G be a finite group and K a splitting field of char zero for G. Then $\forall \chi \in Irr_K(G)$, the map
$$\omega : Z(KG) \longrightarrow K \qquad \text{is a } K\text{-alg homo.}$$
$$z \longmapsto \frac{\chi(z)}{\chi(1)}$$

Proof:
$$z \in Z(KG) \Rightarrow \begin{pmatrix} V \longrightarrow V \\ v \longmapsto zv \end{pmatrix} \in End_{KG}(V)$$
$$\Rightarrow = \lambda_z \cdot Id_V$$

$$\Rightarrow \omega ; Z(KG) \longrightarrow K \quad \text{is an algebra homomorphism.}$$
$$z \longmapsto \lambda_z$$

But $\dfrac{\chi(z)}{\chi(1)} = \dfrac{\lambda_z \dim_K V}{\dim_K V} = \lambda_z$ $\square$

Prop 4 : Let G be a finite group, K a splitting field for G of char zero. Then $\omega(c) = \dfrac{\chi(c)}{\chi(1)}$ is an algebraic integer where $c \in Z(KG)$ is the sum of all conjugates of $x \in G$.

Proof: $c_i c_j = \sum a_{ijk} c_k$ (Thm 10, §1.) , $a_{ijk} \geq 0$, c_i - ~~sum~~ over conj el.

$\Rightarrow \omega(c_i)\,\omega(c_j) = \sum a_{ijk}\,\omega(c_k)$ as ω is a K-alg homo.

But then the finitely generated $\mathbb{Z}$-module $N := \langle \omega(c_i) \mid 1 \le i \le h \rangle$ is a subring of K and thus contained in the alg. integer.

Remark: $c = \sum_{y \in [G : C_G(x)]} y x y^{-1}$

$\Longrightarrow \chi(c) = [G : C_G(x)] \cdot \chi(x)$

In particular $\dfrac{|G| \, \chi(x)}{|C_G(x)| \, \chi(1)}$ is an alg. integer.

Thm 5 (our 1st aim) Let G be a finite group K a splitting field of char 0 for G. Then $\chi(1) \mid |G|$.

Proof: $\{x_i, 1 \le i \le h\}$ set of repres of conj classes. $c_i := \sum_{y \in [G : C_G(x_i)]} y x y^{-1}$. Then:

$$|G| = \sum_{x \in G} \chi(x)\chi(x^{-1}) = \sum_{1 \le i \le h} [G : C_G(x_i)] \chi(x_i) \chi(x_i^{-1})$$

$$= \sum_{1 \le i \le h} \chi(c_i) \chi(x_i^{-1}) \iff$$

$\dfrac{|G|}{\chi(1)} = \sum_{1 \le i \le h} \omega(c_i) \chi(x_i^{-1})$ which is an alg. integer and a rational nb.

Thus $|G| / \chi(1) \in \mathbb{Z}$. $\square$

Rem:

- If $A \trianglelefteq G$ is abelian then $\chi(1) \mid [G : A]$
- Also $\chi(1) \mid [G : Z(\chi)]$

exercise!

<u>Theorem 6</u> (Burnside) Let G be a finite group, C the conj class of x in G and $\chi \in Irr_{\mathbb{C}}(G)$. Suppose that $(\chi(1), |C|) = 1$. Then $\chi(x) = 0$ or $|\chi(x)| = \chi(1)$.

$\longrightarrow a\chi(1) + b|C| = 1$

<u>Proof</u>:

$(\chi(1), |C|) = 1$, $\dfrac{\chi(x)|C|}{\chi(1)}$ is an alg. integer.

$$\frac{b\,\chi(x)\,|C|}{\chi(1)} + \frac{a\chi(x)\chi(1)}{\chi(1)} = \frac{\chi(x)}{\chi(1)}$$

Thus $\dfrac{\chi(x)}{\chi(1)}$ is an algebraic integer.

Consider the action of the of the Galois group of $K/\mathbb{Q}$ on $\chi(x)$, Where $K = \mathbb{Q}(\xi)$, $\xi^m = 1$, $m = \mathrm{Ord}(x)$.

$n := \prod_{\sigma \in Gal(\mathbb{Q}(\xi)/\mathbb{Q})} \left(\dfrac{\chi(x)}{\chi(1)}\right)$ is a rational number and an alg. integer $\Rightarrow$ an int

if $\left|\dfrac{\chi(x)}{\chi(1)}\right| < 1$ then $|n| < 1 \Rightarrow n = 0$.

<u>Theorem 7</u> Let G be a finite non-ab. simple group. Then $\{1\}$ is the only conj. class of order a power of a prime nb.

<u>Proof</u> Suppose $x \neq 1$ and $|C_x| = p^\alpha$

Let $\chi \in Irr_{\mathbb{C}}(G)$, $\chi \neq 1$.

- $|\chi(x)| \neq \chi(1)$ $\left(\Leftrightarrow x \notin Z(\chi)\right)$

<u>straight forward</u> as $Z(\chi) \trianglelefteq G$.

and $Z(\chi) \neq G$ as $\mathbb{1}$ is the only char of deg 1

$\left([G,G] = G \Rightarrow G/[G,G] = 1\right)$.

Now, by thm 6, if $p \nmid \chi(1)$ then $(\chi(1), |C|) = 1$ and given that $|\chi(x)| \neq \chi(1)$ we have $\chi(x) = 0$.

$$\text{Now } 0 = \rho(x) = \sum_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)} \chi(1)\chi(x) = 1 + \sum_{p \mid \chi(1)} \chi(1)\chi(x)$$

But then $-\frac{1}{p} = \sum_{p \mid \chi(1)} \frac{\chi(1)}{p} \chi(x)$ is an alg. int. $\Rightarrow \Leftarrow$

$\square$

<u>Thm 8 (Burnside).</u> Let G be a finite group, $|G| = p^a q^b$, $a, b \geq 0$. Then G is solveable.

<u>Proof</u>: Let G be a minimal counter example. (minimal = of minimal order)

<u>Then G is simple non-abelian.</u>

If $N \trianglelefteq G$ is proper $\Rightarrow G/N$, N are solveable

But then G is solveable $\Rightarrow \Leftarrow$.

Assume $a > 0$ and take $P \in \mathrm{Syl}_p(G)$, $y \in Z(P) \setminus \{1\}$. Then $|C_G(y)|$ contains P so the conj class of y has order $\frac{|G|}{|C_G(y)|}$, hence a power of q. But this contradicts theorem 7.