We call this composition the *circle composition* in $\mathfrak{A}$. One verifies directly that it is associative; hence, $\mathfrak{A},\circ$ is a semi-group. Also clearly $a \circ 0 = a = 0 \circ a$; hence, 0 acts as identity in $\mathfrak{A},\circ$. It is now clear that the set of elements $\mathfrak{Q}$ that are *quasi-regular* (= left and right quasi-regular) is just the set of units of $\mathfrak{A},\circ$. Hence $\mathfrak{Q},\circ$ is a group.

The group $\mathfrak{Q},\circ$ is the analogue for an arbitrary ring of the group of units $\mathfrak{U}$ of a ring with an identity. In fact, if $\mathfrak{A}$ has an identity, then $\mathfrak{U}$ and $\mathfrak{Q}$ are isomorphic; for it is easy to see that the mapping $z \rightarrow 1 - z$ is an isomorphism of $\mathfrak{Q}$ onto $\mathfrak{U}$.

EXERCISES

1. Show that, if e is idempotent, then $e \circ e = e$. Hence prove that, if e is right quasi-regular, then $e = 0$.
2. Show that any nilpotent element belongs to $\mathfrak{Q}$.
3. (Kaplansky.) Establish the following characterization of a division ring: A ring in which every element with one exception has a right quasi-inverse.

4. Matrix rings. Let $\mathfrak{R}$ be an arbitrary ring. We shall now define the ring $\mathfrak{R}_n$ of $n \times n$ matrices with elements in $\mathfrak{R}$. The elements of $\mathfrak{R}_n$ are arrays or *matrices*

$$(a) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{4}$$

of n rows and columns with *elements* (*coefficients, coordinates*) a_{ij} in the base ring $\mathfrak{R}$. The element a_{ij} in the intersection of the ith row and jth column of (a) will be referred to as the (i,j) *element* of (a). Two matrices (a) and (b) are regarded as equal if and only if $a_{ij} = b_{ij}$ for every i,j, and the set $\mathfrak{R}_n$ is the complete set of matrices with elements in $\mathfrak{R}$.

We define addition of matrices by the formula

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nn} + b_{nn} \end{bmatrix}.$$

Thus to obtain the sum we add the elements a_{ij} and b_{ij} in the same position. It is easy to verify that $\mathfrak{R}_n$ and this addition composition form a commutative group. The 0 matrix is the matrix all of whose elements are 0 and the negative of (a) has $-a_{ij}$ in the (i,j)-position, that is, in the intersection of the ith row and the jth column. Multiplication of matrices is defined by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma a_{1k}b_{k1} & \Sigma a_{1k}b_{k2} & \cdots & \Sigma a_{1k}b_{kn} \\ \Sigma a_{2k}b_{k1} & \Sigma a_{2k}b_{k2} & \cdots & \Sigma a_{2k}b_{kn} \\ \cdot & \cdot & \cdots & \cdot \\ \Sigma a_{nk}b_{k1} & \Sigma a_{nk}b_{k2} & \cdots & \Sigma a_{nk}b_{kn} \end{bmatrix}.$$

The product $(p) = (a)(b)$ therefore has the element

$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

in the (i,j)-position. For example, in the ring I_3, I the ring of integers we have

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 5 & -2 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 2 & 5 & 1 \\ -1 & -6 & 2 \end{bmatrix} = \begin{bmatrix} -7 & -25 & 8 \\ 3 & 11 & -1 \\ 12 & 43 & 9 \end{bmatrix}.$$

Multiplication of matrices is associative. Thus consider the product $(a)[(b)(c)]$. The multiplication rule shows that the element in the (i,j)-position of this matrix is $\sum_{k,l} a_{ik}(b_{kl}c_{lj})$. Similarly, the element in the (i,j)-position of $[(a)(b)](c)$ is $\sum_{k,l} (a_{ik}b_{kl})c_{lj}$. Because of the associative law of multiplication in $\mathfrak{R}$, these elements are equal. Hence $(a)[(b)(c)] = [(a)(b)](c)$. The distributive laws hold; for the (i,j)-element of $(a)[(b) + (c)]$ is $\sum_k a_{ik}(b_{kj} + c_{kj})$ and the (i,j) element of $(a)(b) + (a)(c)$ is $\sum_k a_{ik}b_{kj} + \sum_k a_{ik}c_{kj}$. These elements are equal by the distributive law in $\mathfrak{R}$. Similarly we can verify the other distributive law.

Hence $\mathfrak{R}_n$ is a ring. Even if $\mathfrak{R}$ is commutative, $\mathfrak{R}_n$ will not be commutative if $n > 1$ (cf. ex. 3 below). Also $\mathfrak{R}_n$ contains zero-divisors $\neq 0$ if $n > 1$.

EXERCISES

1. Calculate

$$\begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -5 & 6 \\ 7 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

2. Give examples to show that I_2 is not commutative and that it has zero-divisors $\neq 0$.

3. Prove that, if $\mathfrak{R} \neq 0$ and $n > 1$, then $\mathfrak{R}_n$ has zero-divisors $\neq 0$ and that, if $\mathfrak{R}$ contains elements a, b such that $ab \neq 0$, then $\mathfrak{R}_n$, $n > 1$, is not commutative.

If $\mathfrak{R}$ has an identity 1, then it is clear that the element

$$(5) \qquad 1 = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

is the identity in the ring $\mathfrak{R}_n$. We assume now that $\mathfrak{R}$ is commutative and we propose to determine the multiplicative group of units of $\mathfrak{R}_n$. For this purpose we make use of the determinant of a matrix. We assume that the reader is acquainted with the elementary theory of determinants of any order. The usual treatments in textbooks on elementary algebra or geometry are valid for determinants of matrices with elements in any commutative ring.

We recall here the definition of the *determinant* of a matrix. If (a) is as in (4) its determinant det (a) is

$$(6) \qquad \sum_P \pm a_{1i_1}a_{2i_2}\cdots a_{ni_n}$$

where the summation is taken over all permutations $(i_1, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$ and the sign + or − is taken according as the permutation is even or odd. The *cofactor* of the element a_{ij} in (4) is $(-1)^{i+j}$ times the determinant of order $n - 1$ that is obtained by striking out the ith row and jth column of (a). It is well known that the sum of the products of the elements of any row (column) by their cofactors has the value det (a). Thus if A_{ij} is the cofactor of a_{ij}, then

$$(7) \qquad \begin{aligned} a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} &= \det(a) \\ a_{1i}A_{1i} + a_{2i}A_{2i} + \cdots + a_{ni}A_{ni} &= \det(a). \end{aligned}$$

Also it is known that the sum of the products of the elements of any row (column) by the cofactors of the elements of another row (column) is 0:

$$(8) \qquad \begin{aligned} a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} &= 0, \quad i \neq j \\ a_{1i}A_{1j} + a_{2i}A_{2j} + \cdots + a_{ni}A_{nj} &= 0, \quad i \neq j. \end{aligned}$$

These relations lead us to define the *adjoint* of the matrix (a) to be the matrix whose (i,j) element $\alpha_{ij} = A_{ji}$. Using this definition it is immediate that the rules (7) and (8) are equivalent to the matrix equations

$$(9) \qquad (a)\mathrm{adj}(a) = \begin{bmatrix} \det(a) & & & 0 \\ & \det(a) & & \\ & & \ddots & \\ 0 & & & \det(a) \end{bmatrix} = [\mathrm{adj}(a)](a).$$

It follows that if $\Delta = \det(a)$ is a unit in $\mathfrak{R}$, then the matrix (b), $b_{ij} = \alpha_{ij}\Delta^{-1}$ satisfies

$$(10) \qquad (a)(b) = 1 = (b)(a).$$

We have therefore proved the sufficiency part of the following

Theorem 1. *If $\mathfrak{R}$ is a commutative ring with an identity, a matrix $(a) \varepsilon \mathfrak{R}_n$ is a unit if and only if its determinant is a unit in $\mathfrak{R}$.*

To prove the necessity we require the fundamental multiplication rule