11.5. Algebraic and transcendental numbers. An *algebraic number* is a number x which satisfies an *algebraic equation*, i.e. an equation

$$a_0x^n+a_1x^{n-1}+\dots+a_n = 0, \tag{11.51}$$

where a,,, a,,... are integers, not all zero.

A number which is not algebraic is called *transcendental.*

If x = a/b, then $bx-a$ = 0, so that any rational x is algebraic. Any quadratic surd is algebraic; thus $i = \sqrt{(-1)}$ is algebraic. But in this chapter we are concerned with real algebraic numbers.

An algebraic number satisfies any number of algebraic equations of different degrees; thus x = $\sqrt{2}$ satisfies $x^2-2 = 0$, x^4-4 = O,... . If x satisfies an algebraic equation of degree n, but none of lower degree, then we say that x is of degree n. Thus a rational is of degree 1.

A number is *Euclidean* if it measures a length which can be constructed, starting from a given unit length, by a Euclidean construction, i.e. a finite construction with ruler and compasses only. Thus $\sqrt{2}$ is Euclidean. It is plain that we can construct any finite combination of real quadratic surds, such as

$$\sqrt{(11+2\sqrt{7})}-\sqrt{(11-2\sqrt{7})} \tag{11.5.2}$$

by Euclidean methods. We may describe such a number as of real quadratic type.

Conversely, any Euclidean construction depends upon a series of points defined as intersections of lines and circles. The coordinates of each point in turn are defined by two equations of the types

$$lx+my+n = 0$$

or

$$x^2+y^2+2gx+2fy+c = 0,$$

where l, m, n, g, f, c are measures of lengths already constructed; and two such equations define x and y as real quadratic combinations of l, m,... . Hence every Euclidean number is of real quadratic type.

The number (11.5.2) is defined by

$$x = y-z, \qquad y^2 = 11+2t, \qquad z^2 = 11-2t, \qquad t^2 = 7$$

and we obtain

$$x^4-44x^2+112 = 0$$

on eliminating y, z, and t. Thus x is algebraic. It is not difficult to prove that any Euclidean number is algebraic, but the proof demands a little knowledge of the general theory of algebraic numbers.†

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† In fact any number defined by an equation $\alpha_0 x^n+\alpha_1 x^{n-1}+\dots+\alpha_n = 0$, where $\alpha_0, \alpha_1, \dots, \alpha_n$ are algebraic, is algebraic. For the proof see Hecke 66, or Hardy, *Pure mathematics* (ed. 9, 1944), 39.

11.6. The existence of transcendental numbers. It is not immediately obvious that there are any transcendental numbers, though actually, as we shall see in a moment, almost all real numbers are transcendental.

We may distinguish three different problems. The first is that of proving the existence of transcendental numbers (without necessarily producing a specimen). The second is that of giving an example of a transcendental number by a construction specially designed for the purpose. The third, which is much more difficult, is that of proving that some number given independently, some one of the 'natural' numbers of analysis, such as e or π, is transcendental.

We may define the *rank* of the equation (11.5.1) as

$$N = n+|a_0|+|a_1|+\ldots+|a_n|.$$

The minimum value of N is 2. It is plain that there are only a finite number of equations
$$E_{N,1},\ E_{N,2},\ \ldots,\ E_{N,k_N}$$
of rank N. We can arrange the equations in the sequence
$$E_{2,1},\ E_{2,2},\ldots,\ E_{2,k_2},\ E_{3,1},\ E_{3,2},\ldots,\ E_{3,k_3},\ E_{4,1},\ldots$$
and so correlate them with the numbers 1, 2, 3,... . Hence the aggregate of equations is enumerable. But every algebraic number corresponds to at least one of these equations, and the number of algebraic numbers corresponding to any equation is finite. Hence

THEOREM 189. *The aggregate of algebraic numbers is enumerable.*

In particular, the aggregate of real algebraic numbers has measure zero.

THEOREM 190. *Almost all real numbers are transcendental.*

Cantor, who had not the more modern concept of measure, arranged his proof of the existence of transcendental numbers differently. After Theorem 189, it is enough to prove that *the continuum* $0 \leqslant x < 1$ *is not* enumerable. We represent x by its decimal
$$x = \cdot a_1 a_2 a_3 \ldots$$
(9 being excluded, as in § 9.1). Suppose that the continuum is enumerable, as $x_1, x_2, x_3, \ldots$, and let
$$x_1 = \cdot a_{11} a_{12} a_{13} \ldots$$
$$x_2 = \cdot a_{21} a_{22} a_{23} \ldots$$
$$x_3 = \cdot a_{31} a_{32} a_{33} \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
If now we define a_n by
$$a_n = a_{nn}+1 \quad \text{(if } a_{nn} \text{ is neither 8 nor 9)},$$
$$a_n = 0 \quad \text{(if } a_{nn} \text{ is 8 or 9)},$$
then $a_n \neq a_{nn}$ for any n; and x cannot be any of $x_1, x_2, \ldots$, since its decimal differs from that of any x_n in the nth digit. This is a contradiction.

11.7. Liouville's theorem and the construction of transcendental numbers. Liouville proved a theorem which enables us to produce as many examples of transcendental numbers as we please. It is the generalization to algebraic numbers of any degree of the negative half of Theorem **188**.

THEOREM **191**. *A real algebraic number of degree n is not approximable to any order greater than n.*

An algebraic number ξ satisfies an equation

$$f(f) = a_0\xi^n+a_1\xi^{n-1}+...+a_n = 0$$

with integral coefficients. There is a number $M(\xi)$ such that

$$|f'(x)| < M \quad (\xi-1 < x < \xi+1). \tag{11.7.1}$$

Suppose now that $p/q \neq \xi$ is an approximation to ξ. We may assume the approximation close enough to ensure that p/q lies in $(\xi-1, \xi+1)$, and is nearer to ξ than any other root of $f(x) = 0$, so that $f(p/q) \neq 0$. Then

$$\left|f\left(\frac{p}{q}\right)\right| = \frac{|a_0p^n+a_1p^{n-1}q+...|}{q^n} \geqslant \frac{1}{q^n}, \tag{11.7.2}$$

since the numerator is a positive integer; and

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right)-f(\xi) = \left(\frac{p}{q}-\xi\right)f'(x), \tag{11.7.3}$$

where x lies between p/q and ξ. It follows from (11.7.2) and (11.7.3) that

$$\left|\frac{p}{q}-\xi\right| = \frac{|f(p/q)|}{|f'(x)|} > \frac{1}{Mq^n} = \frac{K}{q^n},$$

so that ξ is not approximable to any order higher than n.

The cases n = 1 and n = 2 are covered by Theorems **186** and 188. These theorems, of course, included a positive as well as a negative statement.

(a) Suppose, for example, that

$$\xi = \cdot 110001000... = 10^{-1!}+10^{-2!}+10^{-3!}+...,$$

that $n > N$, and that ξ_n is the sum of the first n terms of the series. Then

$$\xi_n = \frac{p}{10^{n!}} = \frac{p}{q},$$

say. Also

$$0 < \xi-\frac{p}{q} = \xi-\xi_n = 10^{-(n+1)!}+10^{-(n+2)!}+... < 2 . 10^{-(n+1)!} < 2q^{-N}.$$

Hence ξ is not an algebraic number of degree less than N. Since N is arbitrary, ξ is transcendental.

(6) Suppose that

$$\xi = \frac{1}{10+}\,\frac{1}{10^{2!}+}\,\frac{1}{10^{3!}+\dots},$$

that n > N, and that $\frac{P}{4} = \frac{p_n}{qn'}$

the nth convergent to ξ. Then

$$\left|\frac{p}{q}-\xi\right| = \frac{1}{q_n q'_{n+1}} < \frac{1}{a_{n+1}q_n^2} < \frac{1}{a_{n+1}}.$$

Now $a_{n+1} = 10^{(n+1)!}$ and

$$q_1 < a_1+1, \qquad \frac{q_{n+1}}{q_n} = a_{n+1}+\frac{q_{n-1}}{q_n} < a_{n+1}+1 \quad (n \geqslant 1);$$

so that

$$\begin{aligned} q_n &< (a_1+1)(a_2+1)\dots(a_n+1) \\ &< \left(1+\frac{1}{10}\right)\left(1+\frac{1}{10^2}\right)\dots\left(1+\frac{1}{10^n}\right)a_1a_2\dots a_n \\ &< 2a_1a_2\dots a, = 2.10^{1!+\dots+n!} < 10^{2(n!)} = a_n^2, \end{aligned}$$

$$\left|\frac{p}{q}-\xi\right| < \frac{1}{a_{n+1}} = \frac{1}{a_n^{n+1}} < \frac{1}{a_n^n} < \frac{1}{q_n^{\frac{1}{2}n}} < \frac{1}{q_n^{\frac{1}{2}N}}.$$

We conclude, as before, that ξ is transcendental.

THEOREM 192. *The numbers*

$$\xi = 10^{-1!}+10^{-2!}+10^{-3!}+\dots$$

and

$$\xi = \frac{1}{10^{1!}+}\,\frac{1}{10^{2!}+}\,\frac{1}{10^{3!}+\dots}$$

are transcendental.

It is plain that we could replace 10 by other integers, and vary the construction in many other ways. The general principle of the construction is simply that a *number defined by a sufficiently rapid sequence of rational approximations is necessarily transcendental.* It is the simplest irrationals, such as $\sqrt{2}$ or $\frac{1}{2}(\sqrt{5}-1)$, which are the least rapidly approximable.

It is much more difficult to prove that a number given 'naturally' is transcendental. We shall prove e and π transcendental in §§ 11.13-14. Few classes of transcendental numbers are known even now. These classes include, for example, the numbers

$$e,\ \pi,\ \sin 1,\ J_0(1),\ \log 2,\ \frac{\log 3}{\log 2},\ e^{\pi},\ 2^{\sqrt{2}}$$

but not 2^e, 2^π, π^e, or Euler's constant y. It has never been proved even that any of these last numbers are irrational.

11.8. The measure of the closest approximations to an arbitrary irrational. We know that every irrational has an infinity of approximations satisfying (11.1. 1), and indeed, after Theorem 183 of Ch. X, of rather better approximations. We know also that an algebraic. number, which is an irrational of a comparatively simple type, cannot be 'too rapidly' approximable, while the transcendental numbers of Theorem 192 have approximations of abnormal rapidity.

The best approximations to ξ are given, after Theorem **181,** by the convergents p_n/q_n of the continued fraction for ξ; and

$$\frac{p_n}{q_n}-\xi\Big| = \frac{1}{q_n q'_{n+1}} < \frac{1}{a_{n+1}q_n^2},$$

so that we get a particularly good approximation when a_{n+1} is large. It is plain that, to put the matter roughly, ξ will or will not be rapidly approximable according as its continued fraction does or does not contain a sequence of rapidly increasing quotients. The second ξ of Theorem 192, whose quotients increase with great rapidity, is a particularly instructive example.

One may say, again very roughly, that the structure of the continued fraction for ξ affords a measure of the 'simplicity' or 'complexity' of ξ. Thus the second ξ of Theorem 192 is a 'complicated' number. On the other hand, if a, behaves regularly, and does not become too large, then ξ may reasonably be regarded as a 'simple' number; and in this case the rational approximations to ξ cannot be too good. From the point of view of rational approximation. *the simplest numbers are the worst.*

The 'simplest' of all irrationals, from this point of view, is the number

$$\xi = \tfrac{1}{2}(\surd 5-1) = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+\dots}, \tag{11.8.1}$$

in which every a, has the smallest possible value. The convergents to this fraction are

$$\frac{0}{1},\ \frac{1}{1},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{5},\ \frac{6}{8},\dots$$

so that $q_{n-1} = p_n$ and $\frac{qn-1}{qn} = \frac{Pn}{q_m} \to \xi.$

Hence

$$\left|\frac{p_n}{q_n}-\xi\right| = \frac{1}{q_n q'_{n+1}} = \frac{1}{q_n\{(1+\xi)q_n+q_{n-1}\}}$$

$$= \frac{1}{q_n^2}\left(1+\xi+\frac{q_{n-1}}{q_n}\right)^{-1} \sim \frac{1}{q_n^2}\,\frac{1}{1+2\xi} = \frac{1}{q_n^2\surd 5}$$

when $n \to \infty$.