IX.

THE REPRESENTATION OF NUMBERS BY DECIMALS

9.1. The decimal associated with a given number. There is a process for expressing any positive number ξ as a 'decimal' which is familiar in elementary arithmetic.

We write

(9.1.1) $$\xi = [\xi]+x = X+x,$$

where X is an integer and $0 \leqslant x < 1$,† and consider X and x separately.

If $X > 0$ and $$10^s \leqslant X < 10^{s+1},$$

and A_1 and X_1 are the quotient and remainder when X is divided by 10^s, then

$$X = A_1 . 10^s+X_1,$$

where $$0 < A_1 = [10^{-s}X] < 10, \qquad 0 \leqslant X_1 < 10^s.$$

Similarly

$$X_1 = A_2 . 10^{s-1}+X_2 \quad (0 \leqslant A_2 < 10,\ 0 \leqslant X_2 < 10^{s-1}),$$

$$X_2 = A_3 . 10^{s-2}+X_3 \quad (0 \leqslant A_3 < 10,\ 0 \leqslant X_3 < 10^{s-2}),$$

. . . .

$$X_{s-1} = A_s . 10+X_s \quad (0 \leqslant A_s < 10,\ 0 \leqslant X_s < 10),$$

$$X_s = A_{s+1} \quad (0 \leqslant A_{s+1} < 10).$$

Thus X may be expressed uniquely in the form

(9.1.2) $$X = A_1 . 10^s+A_2 . 10^{s-1}+...+A_s . 10+A_{s+1},$$

where every A is one of 0, 1, 2 ,..., 9, and A_1 is not 0. We abbreviate this expression to

(9.1.3) $$X = A_1 A_2 ... A_s A_{s+1},$$

the ordinary representation of X in decimal notation.

Passing to x, we write

$$x = f_1 \quad (0 \leqslant f_1 < 1).$$

We suppose that $a_1 = [10 f_1]$, so that

$$\frac{a_1}{10} \leqslant f_1 < \frac{a_1+1}{10};$$

a_1 is one of 0, 1,..., 9, and

$$a_1 = [10 f_1], \qquad 10 f_1 = a_1+f_2 \quad (0 \leqslant f_2 < 1).$$

† Thus $[\xi]$ has the same meaning as in § 6.11.

Similarly, we define $a_2, a_3, \ldots$ by

$$a_2 = [10f_2], \qquad 10f_2 = a_2+f_3 \quad (0 \leqslant f_3 < 1),$$
$$a_3 = [10f_3], \qquad 10f_3 = a_3+f_4 \quad (0 \leqslant f_4 < 1),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Every a, is one of 0, **1,** 2 ,..., 9. Thus

(9.1.4) $$x = x_n + g_{n+1},$$

where

(9.1.5) $$x_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n},$$

(9.1.6) $$0 \leqslant g_{n+1} = \frac{f_{n+1}}{10^n} < \frac{1}{10^n}.$$

We thus define a decimal $\cdot a_1 a_2 a_3 \ldots a_n \ldots$

associated with x. We call a,, $a_2, \ldots$ the first, second,... *digits* of the decimal.

Since a, < **10,** the series

(9.1.7) $$\sum_{1}^{\infty} \frac{a^n}{10^n}$$

is convergent; and since $g_{n+1} \to 0$, its sum is x. We may therefore write

(9.1.8) $$x = \cdot a_1 a_2 a_3 \ldots,$$

the right-hand side being an abbreviation for the series (9.1 .7).

If $f_{n+1} = 0$ for some ***n,*** i.e. if $10^n x$ is an integer, then

$$a_{n+1} = a_{n+2} = \ldots = 0.$$

In this case we say that the decimal ***terminates.*** Thus

$$\frac{17}{400} = \cdot 0425000\ldots,$$

and we write simply $$\frac{17}{400} = \cdot 0425.$$

It is plain that the decimal for x will terminate if and only if x is a rational fraction whose denominator is of the form $2^\alpha 5^\beta$.

Since $$\frac{a_{n+1}}{10^{n+1}} + \frac{a_{n+2}}{10^{n+2}} + \ldots = g_{n+1} < \frac{1}{10^n}$$

and $$\frac{9}{10^{n+1}} + \frac{9}{10^{n+2}} + \ldots = \frac{9}{10^{n+1}(1-\frac{1}{10})} = \frac{1}{10^n},$$

it is impossible that every a_n from a certain point on should be 9. With this reservation., every possible sequence (a,) will arise from some x. We define x as the sum of the series (9.1.7), and x,, and g_{n+1} as in (9.1.4) and (9.1.5). Then $g_{n+1} < 10^{-n}$ for every n, and x yields the sequence required.

Finally, if

$$\sum_{1}^{\infty} \frac{a_n}{10^n} = \sum_{1}^{\infty} \frac{b_n}{10^n}, \tag{9.1.9}$$

and the b_n satisfy the conditions already imposed on the a,, then a, = b_n for every n. For if not, let a_N and b_N be the first pair which differ, SO that $|a_N - b_N| \geqslant 1$. Then

$$\left|\sum_{1}^{\infty} \frac{a_n}{10^n} - \sum_{1}^{\infty} \frac{b_n}{10^n}\right| \geqslant \frac{1}{10^N} - \sum_{N+1}^{\infty} \frac{|a_n - b_n|}{10^n} \geqslant \frac{1}{10^N} - \sum_{N+1}^{\infty} \frac{9}{10^n} = 0.$$

This contradicts (9.1.9) unless there is equality. If there is equality, then all of $a_{N+1} - b_{N+1}, a_{N+2} - b_{N+2}, \ldots$ must have the same sign and the absolute value 9. :But then either a, = 9 and $b_n = 0$ for n > N, or else a, = 0 and $b_n = 9$, and we have seen that each of these alternatives is impossible. Hence a, = b_n for all n. In other words, different decimals correspond to different numbers.

We now combine (9.1.1), (9.1.3), and (9.1.8) in the form

$$\xi = X + x = A_1 A_2 \ldots A_{s+1} \cdot a_1 a_2 a_3 \ldots; \tag{9.1.10}$$

and we can sum up·our conclusions as follows.

THEOREM 134. *Any positive number ξ may be expressed as a decimal*

$$A_1 A_2 \ldots A_{s+1} \cdot a_1 a_2 a_3 \ldots,$$

where $0 \leqslant A_1 < 10,\ 0 \leqslant A_2 < 10, \ldots, 0 \leqslant a_n < 10,$

not all A and a are 0, and an infinity of the a, are iess than 9. If $\xi \geqslant 1$, then A, > 0. There is a (1, 1) correspondence between the numbers and the decimals, and

$$\xi = A_1 . 10^s + \ldots + A_{s+1} + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots.$$

In what follows we shall usually suppose that $0 \leqslant \xi < 1$, SO that X = 0, $\xi = x$. In this case all the A are 0. We shall sometimes save words by ignoring the distinction between the number x and the decimal which represents it, saying, for example, that the second digit of $\frac{17}{400}$ is 4.

9.2. Terminating and recurring decimals.

A decimal which does not terminate may *recur*. Thus

$$\tfrac{1}{3} = \cdot 3333\ldots, \qquad \tfrac{1}{7} = \cdot 14285714285714\ldots;$$

equations which we express more shortly as

$$\tfrac{1}{3} = \cdot\dot{3} \qquad \tfrac{1}{7} = \cdot\dot{1}4285\dot{7}.$$

These are pure recurring decimals in which the period reaches back to the beginning. On the other hand,

$$\tfrac{1}{6} = \cdot 1666\ldots = \cdot 1\dot{6},$$

a *mixed* recurring decimal in which the period is preceded by one non-recurrent digit.

We now determine the conditions for termination or recurrence.

(1) If
$$x = \frac{p}{q} = \frac{p}{2^{\alpha}5^{\beta}},$$
where $(p, q) = 1$, and

(9.2.1)
$$\mu = \max(\alpha, \beta),$$

then $10^n x$ is an integer for $n = \mu$ and for no smaller value of n, so that x terminates at a_μ. Conversely,

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_\mu}{10^\mu} = \frac{P}{10^\mu} = \frac{p}{q},$$

where q has the prime factors 2 and 5 only.

(2) Suppose next that $x = p/q$, $(p,q) = 1$, and $(q, 10) = 1$, so that q is not divisible by 2 or 5. Our discussion of this case depends upon the theorems of Ch. VI.

By Theorem 88, $\qquad 10^\nu \equiv 1 \pmod{q}$

for some ν, the least such ν being a divisor of $\phi(q)$. We suppose that ν has this smallest possible value, i.e. that, in the language of § 6.8, 10 belongs to $\nu \pmod{q}$ or ν is the order of $10 \pmod{q}$. Then

(9.2.2)
$$10^\nu x = \frac{10^\nu p}{q} = \frac{(mq+1)p}{q} = mp + \frac{p}{q} = mp + x,$$

where m is an integer. But

$$10^\nu x = 10^\nu x_\nu + 10^\nu g_{\nu+1} = 10^\nu x_\nu + f_{\nu+1},$$

by (9.1.4). Since $0 < x < 1$, $f_{\nu+1} = x$, and the process by which the decimal was constructed repeats itself from $f_{\nu+1}$ onwards. Thus x is a pure recurring decimal with a period of at most ν figures.

On the other hand, a pure recurring decimal $\cdot\dot{a}_1 a_2 \ldots \dot{a}_\lambda$ is equal to

$$\left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_\lambda}{10^\lambda}\right)\left(1 + \frac{1}{10^\lambda} + \frac{1}{10^{2\lambda}} + \ldots\right)$$
$$= \frac{10^{\lambda-1}a_1 + 10^{\lambda-2}a_2 + \ldots + a_\lambda}{10^\lambda - 1} = \frac{p}{q},$$

when reduced to its lowest terms. Here $q \; 10^\lambda - 1$, and so $\lambda \geqslant \nu$. It follows that if $(q, 10) = 1$, and the order of 10 (mod q) is ν, then x is a pure recurring decimal with a period of just ν digits; and conversely.

(3) Finally, suppose that

$$x = \frac{p}{q} = \frac{p}{2^\alpha 5^\beta Q}, \tag{9.2.3}$$

where $(p, q) = 1$ and $(Q, 10) = 1$; that μ is defined as in (9.2.1); and that ν is the order of 10 (mod Q). Then

$$10^\mu x = \frac{p'}{Q} = X + \frac{P}{Q},$$

where p', X, P are integers and

$$0 \leqslant X < 10^\mu, \qquad 0 < P < Q, \qquad (P, Q) = 1.$$

If $X > 0$ then $10^s \leqslant X < 10^{s+1}$, for some $s < \mu$, and $X = A_1 A_2 \ldots A_{s+1}$; and the decimal for P/Q is pure recurring and has a period of ν digits. Hence

$$10^\mu x = A_1 A_2 \ldots A_{s+1} \cdot \dot{a}_1 a_2 \ldots \dot{a}_\nu$$

and

$$x = \cdot b_1 b_2 \ldots b_\mu \dot{a}_1 a_2 \ldots \dot{a}_\nu, \tag{9.2.4}$$

the last $s+1$ of the b being $A_1, A_2, \ldots, A_{s+1}$, and the rest, if any, 0.

Conversely, it is plain that any decimal (9.2.4) represents a fraction (9.2.3). We have thus proved

THEOREM 135. *The decimal for a rational number p/q between 0 and 1 is terminating or recurring, and any terminating or recurring decimal is equal to a rational number. If $(p, q) = 1$, $q = 2^\alpha 5^\beta$, and $\max(\alpha, \beta) = \mu$, then the decimal terminates after μ digits. If $(p, q) = 1$, $q = 2^\alpha 5^\beta Q$, where $Q > 1$, $(Q, 10) = 1$, and ν is the order of 10 (mod Q), then the decimal contains μ non-recurring and ν recurring digits.*

9.3. **Representation of numbers in other scales.** There is no reason except familiarity for our special choice of the number 10; we may replace 10 by 2 or by any greater number r. Thus

$$\frac{1}{8} = \frac{0}{2} + \frac{0}{2^2} + \frac{1}{2^3} = \cdot 001,$$

$$\frac{2}{3} = \frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{0}{2^4} + \ldots = \cdot \dot{1}\dot{0},$$

$$\frac{2}{3} = \frac{4}{7} + \frac{4}{7^2} + \frac{4}{7^3} + \ldots = \cdot \dot{4},$$

the first two decimals being 'binary' decimals or 'decimals in the scale of 2', the third a 'decimal in the scale of 7'.† Generally, we speak of 'decimals in the scale of r'.

The arguments of the preceding sections may be repeated with certain changes, which are obvious if r is a prime or a product of different primes (like 2 or 10), but require a little more consideration if r has square divisors (like 12 or 8). We confine ourselves for simplicity to the first case, when our arguments require only trivial alterations. In § 9.1, 10 must be replaced by r and 9 by r-l. In § 9.2, the part of 2 and 5 is played by the prime divisors of r.

THEOREM 136. *Suppose that r is a prime or a product of different primes. Then any positive number ξ may be represented uniquely as a decimal in the scale of r. An infinity of the digits of the decimal are less than $r-1$; with this reservation, the correspondence between the numbers and the decimals is (1,1).*

Suppose further that

$$0 < x < 1, \qquad x = \frac{p}{q}, \quad (p,q) = 1.$$

If

$$q = s^{\alpha}t^{\beta}...u^{\gamma},$$

where s, t,..., u are the prime factors of r, and

$$\mu = \max(\alpha, \beta, ..., \gamma),$$

then the decimal for x terminates at the μth digit. If q is prime to r, and ν is the order of r (mod q), then the decimal is pure recurring and has a period of ν digits. If

$$q = s^{\alpha}t^{\beta}...u^{\gamma}Q \quad (Q > 1),$$

Q is prime to r, and ν is the order of r (mod Q), then the decimal is mixed recurring, and has μ non-recurring and ν recurring digits.‡

9.4. Irrationals defined by decimals.

It follows from Theorem 136 that a decimal (in any scale||) which neither terminates nor recurs must represent an irrational number. Thus

$$x = \cdot 0100100010...$$

† We ignore the verbal contradiction involved in the use of 'decimal'; there is no other convenient word.

‡ Generally, when $r = s^{A}t^{B}...u^{C}$, we must define μ as

$$\max\left(\frac{\alpha}{A}, \frac{\beta}{B}, ..., \frac{\gamma}{C}\right)$$

if this number is an integer, and otherwise as the first greater integer.

|| Strictly, any 'quadratfrei' scale (scale whose base is a prime or a product of different primes). This is the only case actually covered by the theorems, but there is no difficulty in the extension.

(the number of 0's increasing by 1 at each stage) is irrational. We consider some less obvious examples.

THEOREM 137: $\cdot 011010100010\ldots,$

where the digit a_n is 1 if n is prime and 0 otherwise, is irrational.

Theorem 4 shows that the decimal does not terminate. If it recurs, there is a function $An+B$ which is prime for all n from some point onwards; and Theorem 21 shows that this also is impossible.

This theorem is true in any scale. We state our next theorem for the scale of 10, leaving the modifications required for other scales to the reader.

THEOREM 138 : $\cdot 2357111317192329\ldots,$

where the sequence of digits is formed by the primes in ascending order, is irrational.

The proof of Theorem 138 is a little more difficult. We give two alternative proofs.

(1) Let us assume that *any arithmetical progression of the form*

$$k.10^{s+1}+1 \quad (k = 1, 2, 3, \ldots)$$

contains primes. Then there are primes whose expressions in the decimal system contain an arbitrary number s of O's, followed by a 1. Since the decimal contains such sequences, it does not terminate or recur.

(2) Let us assume that *there is a prime between N and $10N$ for every* N $\geqslant$ 1. Then, given s, there are primes with just s digits. If the decimal recurs, it is of the form

(9.4.1) $$\cdot\ldots a_1 a_2 \ldots a_k | a_1 a_2 \ldots a_k | \ldots,$$

the bars indicating the period, and the first being placed where the first period begins. We can choose $l > 1$ so that all primes with $s = kl$ digits stand later in the decimal than the first bar. If p is the first such -prime, then it must be of one of the forms

$$p = a_1 a_2 \ldots a_k | a_1 a_2 \ldots a_k | \ldots | a_1 a_2 \ldots a_k$$

or $$p = a_{m+1} \ldots a_k | a_1 a_2 \ldots a_k | \ldots | a_1 a_2 \ldots a_k | a_1 a_2 \ldots a_m$$

and is divisible by $a_1 a_2 \ldots a_k$ or by $a_{m+1} \ldots a_k a_1 a_2 \ldots a_m$; a contradiction.

In our first proof we assumed a special case of Dirichlet 's Theorem 15. This special case is easier to prove than the general theorem, but we shall not prove it in this book, so that (1) will remain incomplete. In (2) we assumed a result which follows at once from Theorem 418 (which we shall prove in Chapter XXII). The latter theorem asserts that, for every $N \geqslant 1$, there is at least one prime satisfying $N < p \leqslant 2N$. It follows, a fortiori, that $N < p < 10N$.

9.5. Tests for divisibility. In this and the next few sections we shall be concerned for the most part with trivial but amusing puzzles.

There are not very many useful tests for the divisibility of an integer by particular integers such as 2, 3, 5,... . A number is divisible by 2 if its last digit is even. More generally, it is divisible by 2^ν if and only if the number represented by its last ν digits is divisible by 2^ν. The reason, of course, is that 2^ν 10^ν; and there are similar rules for 5 and 5^ν.

Next
$$10^\nu \equiv 1 \pmod 9$$
for every v, and therefore
$$A_{,}.\ 10^s+A_2.10^{s-1}+\ldots+A_s.10+A_{s+1} \equiv A_1+A_2+\ldots+A_{s+1} \pmod 9.$$
A fortiori this is true mod 3. Hence we obtain the well-known rule 'a number is divisible by 9 (or by 3) if and only if the sum of its digits is divisible by 9 (or by 3)'.

There is a rather similar rule for 11. Since $10 \equiv -1 \pmod{11}$, we have
$$10^{2r} \equiv 1, \quad 10^{2r+1} \equiv -1 \pmod{11},$$
so that
$$A_{,}.\ 10^s+A_2.10^{s-1}+\ldots+A_s.10+A_{s+1} \equiv A_{s+1}-A_s+A_{s-1}-\ldots \pmod{11}.$$
A number is divisible by 11 if and only if the difference between the sums of its digits of odd and even ranks is divisible by 11.

We know of only one other rule of any practical use. This is a test for divisibility by any one of 7, 11, or 13, and depends on the fact that 7.11.13 = 1001. Its working is best illustrated by an example: if 29310478561 is divisible by 7, 11 or 13, so is
$$561-478+310-29 = 364 = 4.7.13.$$
Hence the original number is divisible by 7 and by 13 but not by 11.

9.6. Decimals with the maximum period. We observe when learning elementary arithmetic that
$$\tfrac{1}{7} = \cdot\dot{1}4285\dot{7}, \qquad \tfrac{2}{7} = \cdot\dot{2}8571\dot{4}, \quad \ldots, \quad \tfrac{6}{7} = \cdot\dot{8}5714\dot{2},$$
the digits in each of the periods differing only by a cyclic permutation.