III

FAREY SERIES AND A THEOREM OF MINKOWSKI

3.1. The definition and simplest properties of a Farey series. In this chapter we shall be concerned primarily with certain properties of the 'positive rationals' or 'vulgar fractions', such as $\frac{1}{2}$ or $\frac{7}{11}$. Such a fraction may be regarded as a relation between two positive integers, and the theorems which we prove embody properties of the positive integers.

The Farey series $\mathfrak{F}_n$ of order n is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n. Thus h/k belongs to $\mathfrak{F}_n$ if

$$0 \leqslant h \leqslant k \leqslant n, \quad (h,k) = 1; \tag{3.1.1}$$

the numbers 0 and 1 are included in the forms $\frac{0}{1}$ and $\frac{1}{1}$. For example, $\mathfrak{F}_5$ is

$$\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}.$$

The characteristic properties of Farey series are expressed by the following theorems.

THEOREM 28. *If h/k and h'/k'* are *two successive terms of $\mathfrak{F}_n$, then*

$$kh'-hk' = 1. \tag{3.1.2}$$

THEOREM 29. *If h/k, h''/k''*, and *h'/k' are three successive terms of $\mathfrak{F}_n$, then*

$$\frac{h''}{k''} = \frac{h+h'}{k+k'}. \tag{3.1.3}$$

We shall prove that the two theorems are equivalent in the next section, and then give three different proofs of both of them, in §§ 3.3, 3.4, and 3.7 respectively. We conclude this section by proving two still simpler properties of $\mathfrak{F}_n$.

THEOREM 30. *If h/k and h'/k' are two successive terms of $\mathfrak{F}_n$, then*

$$kfk' > n. \tag{3.1.4}$$

The 'mediant'

$$\frac{h+h'}{k+k'}\,[†]$$

of h/k and h'/k' falls in the interval

$$\left(\frac{h}{k}, \frac{h'}{k'}\right).$$

Hence, unless (3.1.4) is true, there is another term of $\mathfrak{F}_n$ between h/k and h'/k'.

[†] Or the reduced form of this fraction.

Theorem 31. ***If n > 1, then no two successive terms of $\mathfrak{F}_n$ have the same denominator.***

If $k > 1$ and h'/k succeeds h/k in $\mathfrak{F}_n$, then $h+1 \leqslant h' < k$. But then

$$\frac{h}{k} < \frac{h}{k-1} < \frac{h+1}{k} \leqslant \frac{h'}{k};$$

and $h/(k-1)$† cornes between h/k and h'/k in $\mathfrak{F}_n$, a contradiction,

3.2. The **equivalence** of the two characteristic properties. We now prove that each of Theorems 28 and 29 implies the other.

(1) ***Theorem 28 implies Theorem*** 29. If we assume Theorem 28, and solve the equations

$$kh''-hk'' = 1, \qquad k''h'-h''k' = 1 \tag{3.2.1}$$

for h'' and k'', we obtain

$$h''(kh'-hk') = h+h', \qquad k''(kh'-hk') = k\text{-}f\text{-}k'$$

and so (3.1.3).

(2) Theorem 29 implies Theorem 28. We assume that Theorem 29 is true generally and that Theorem 28 is true for $\mathfrak{F}_{n-1}$, and deduce that Theorem 28 is true for $\mathfrak{F}_n$. It is plainly sufficient to prove that the equations (3.2.1) are satisfied when h''/k'' belongs to $\mathfrak{F}_n$ but not to $\mathfrak{F}_{n-1}$, so that $k'' = n$. In this case, after Theorem 31, both k and k' are less than k'', and h/k and h'/k' are consecutive terms in $\mathfrak{F}_{n-1}$.

Since (3.1.3) is true *ex hypothesi*, and h''/k'' is irreducible, we have

$$h+h' = Ah'', \qquad k+k' = \lambda k'',$$

where λ is an integer. Since k and k' are both less than k'', λ must be 1. Hence

$$h'' = h+h', \qquad k'' = k+k',$$

$$kh''-hk'' = kh'-hk' = 1;$$

and similarly

$$k''h'-h''k' = 1.$$

3.3. First **proof** of Theorems 28 and 29. Our first proof is a natural development of the ideas used in § 3.2.

The theorems are true for n = 1; we assume them true for $\mathfrak{F}_{n-1}$ and prove them true for $\mathfrak{F}_n$.

Suppose that h/k and h'/k' are consecutive in $\mathfrak{F}_{n-1}$ but separated by h''/k'' in $\mathfrak{F}_n$.‡ Let

$$kh''-hk'' = r > 0, \qquad k''h'-h''k' = s > 0. \tag{3.3.1}$$

---

† Or the reduced form of this fraction.

‡ After Theorem 31, h''/k'' is the only term of $\mathfrak{F}_n$ between h/k and h'/k'; but we do not assume this in the proof.

Solving these equations for h'' and k'', and remembering that

$$kh'-hk' = 1,$$

we obtain

$$h'' = sh+rh', \qquad k'' = sk+rk'. \tag{3.3.2}$$

Here $(r,s) = 1$, since $(h'', k'') = 1$.

Consider now the set S of all fractions

$$\frac{H}{K} = \frac{\mu h+\lambda h'}{\mu k+\lambda k'} \tag{3.3.3}$$

in which λ and μ are positive integers and $(\lambda,\mu) = 1$. Thus h''/k'' belongs to S. Every fraction of S lies between h/k and h'/k', and is in its lowest terms, since any common divisor of H and K would divide

$$k(\mu h+\lambda h')-h(\mu k+\lambda k') = \lambda$$

and

$$h'(\mu k+\lambda k')-k'(\mu h+\lambda h') = \mu.$$

Hence every fraction of S appears sooner or later in some $\mathfrak{F}_q$; and plainly the first to make its appearance is that for which K is least, i.e. that for which $\lambda = 1$ and $\mu = 1$. This fraction must be h''/k'', and so

$$h'' = h+h', \qquad k'' = k+k'. \tag{3.3.4}$$

This proves Theorem 29. It is to be observed that the equations (3.3.4) are not generally true for three successive fractions of $\mathfrak{F}_n$, but are (as we have shown) true when the central fraction has made its first appearance in $\mathfrak{F}_n$.

3.4. Second proof of the theorems. This proof is not inductive, and gives a rule for the construction of the term which succeeds h/k in $\mathfrak{F}_n$.

Since $(h, k) = 1$, the equation

$$kx-hy = 1 \tag{3.4.1}$$

is soluble in integers (Theorem 25). If x_0, y_0, is a solution then

$$x_0+rh, \qquad y_0+rk$$

is also a solution for any positive or negative integral r. We can choose r so that

$$n-k < y_0+rk \leqslant n.$$

There is therefore a solution (x, y) of (3.4.1) such that

$$(x,y) = 1, \qquad 0 \leqslant n-k < y \leqslant n. \tag{3.4.2}$$

Since x/y is in its lowest terms, and $y \leqslant n$, x/y is a fraction of $\mathfrak{F}_n$. Also

$$\frac{x}{y} = \frac{h}{k}+\frac{1}{ky} > \frac{h}{k},$$

so that x/y comes later in $\mathfrak{F}_n$ than h/k. If it is not h'/k', it comes later than h'/k', and

$$\frac{x}{y}-\frac{h'}{k'}=\frac{k'x-h'y}{k'y}\geqslant\frac{1}{k'y};$$

while

$$\frac{h'}{k'}-\frac{h}{k}=\frac{kh'-hk'}{kk'}\geqslant\frac{1}{kk'}.$$

Hence

$$\frac{1}{ky}=\frac{kx-hy}{ky}=\frac{x}{y}-\frac{h}{k}\geqslant\frac{1}{k'y}+\frac{1}{kk'}=\frac{k+y}{kk'y}$$

$$>\frac{n}{kk'y}\geqslant\frac{1}{ky},$$

by (3.4.2). This is a contradiction, and therefore x/y must be h'/k', and $kh'-hk'=1$.

Thus, to find the successor of $\frac{4}{9}$ in $\mathfrak{F}_{13}$, we begin by finding some solution (x_0, y_0) of $9x-4y=1$, e.g. $x_0=1$, $y_0=2$. We then choose r so that $2+9r$ lies between $13-9=4$ and 13. This gives $r=1$, $x=1+4r=5$, $y=2+9r=11$, and the fraction required is $\frac{5}{11}$.

3.5. The integral lattice. Our third and last proof depends on simple but important geometrical ideas.

Suppose that we are given an origin 0 in the plane and two points P, Q not collinear with 0. We complete the parallelogram $OPQR$, produce its aides indefinitely, and draw the two systems of equidistant parallels of which OP, QR and OQ, PR are consecutive pairs, thus dividing the plane into an infinity of equal parallelograms. Such a figure is called a *lattice* (*Gitter*).

A lattice is a figure of lines. It defines a figure of points, viz. the system of points of intersection of the lines, or lattice points. Such a system we call a point-lattice.

Two different lattices may determine the same point-lattice; thus in Fig. **1** the lattices based on OP, OQ and on OP, OR determine the same system of points. Two lattices which determine the same point-lattice are said to be *equivalent.*

It is plain that any lattice point of a lattice might be regarded as the origin 0, and that the properties of the lattice are independent of the choice of origin and symmetrical about any origin.

One type of lattice is particularly important here. This is the lattice which is formed (when the rectangular coordinate axes are given) by parallels to the axes at unit distances, dividing the plane into unit squares. We call this the *fundamental lattice* L, and the point-lattice which it determines, viz. the system of points (x, y) with integral coordinates, the *fundamental point-lattice* Λ.

Any point-lattice may be regarded as a system of numbers or vectors, the complex coordinates $x+iy$ of the lattice points or the vectors to these points from the origin. Such a system is plainly a modulus in the sense of $2.9. If ***P*** and Q are the points (x_1, y_1) and (x_2, y_2), then the coordinates of any point S of the lattice based upon ***OP*** and OQ are

$$x = mx_1+nx_2, \qquad Y = my_1+ny_2,$$

where m and n are integers; or if z_1 and z_2 are the complex coordinates of ***P*** and Q, then the complex coordinate of S is

$$z = mz_1+nz_2.$$

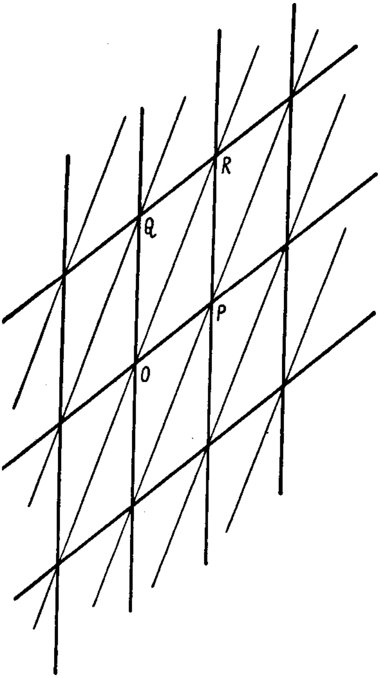

FIG. 1

3.6. Some simple properties of the fundamental lattice. (1) We now consider the transformation defined by

(3.6.1) $x' = ax+by, \quad y' = cx+dy,$

where a, ***b,*** c, ***d*** are given, positive or negative, integers. It is plain that any point (x,y) of A is tran.sformed into another point (x', y') of A.

Solving (3.6.1) for x and y, we obtain

$$x = \frac{dx' - by'}{ad-bc}, \qquad y = -\frac{cx'-ay'}{ad-bc}. \tag{3.6.2}$$

If

(3.6.3) A = ***ad-bc*** = fl,

then any integral values of x' and y' give integral values of x and y, and every lattice point (x',y') corresponds to a lattice point (x,y). In this case Λ is transformed into itself.

Conversely, if Λ is transformed into itself, every integral (x', y') must give an integral (x, y). Taking in particular (x', y') to be **(1,0)** and (0, **1**), we see that

$$\Delta \mid d, \qquad \Delta \mid b, \qquad \Delta \ c, \qquad \Delta \mid a,$$

and so

$$\Delta^2 \ ad\text{-}bc, \qquad \Delta^2 \mid \Delta.$$

Hence A = fl.

We have thus proved

THEOREM 32. *A necessary and sufficient condition that the transformation (3.6.1) should transform A into itself is that* A = **fl.**

We call such a transformation *unimodular.*

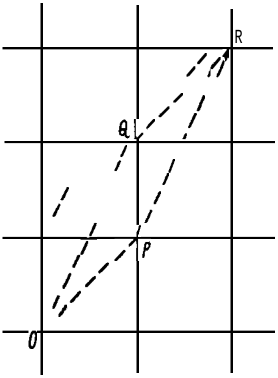

FIG. 2a

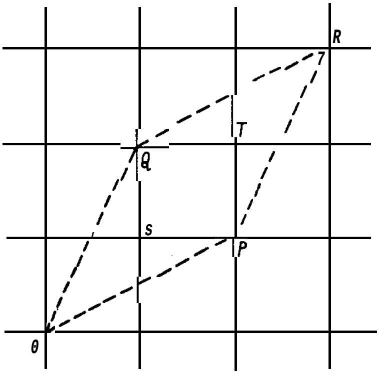

FIG. 2b

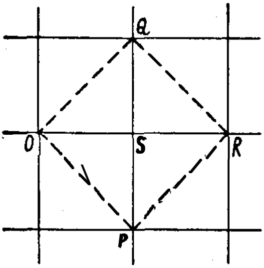

FIG. 20

(2) Suppose now that *P* and Q are the lattice points (a,c) and (b,d) of A. The area of the parallelogram defined by *OP* and OQ is

$$\delta = \pm(ad-bc) = |ad-bc|,$$

the sign being chosen to make δ positive. The points (x',y') of the lattice Λ' based on *OP* and OQ are given by

$$x' = xa+yb, \qquad y' = xc+yd,$$

where x and y are arbitrary integers. After Theorem 32, a necessary and sufficient condition that Λ' should be identical with A is that $\delta = 1$.

THEOREM 33. *A necessary and sufficient condition that the lattice L′ based upon OP and OQ should be equivalent to L is that the area of the parallelogram defined by OP and OQ should be unity.*

(3) We call a point P of Λ *visible* (i.e. visible from the origin) if there is no point of A on OP between 0 and P. In order that (x, y) should be visible, it is necessary and sufficient that x/y should be in its lowest terms, or $(x, y) = 1$.

THEOREM 34. *Suppose that P and Q are visible points of Λ, and that δ is the area of the parallelogram J defined by OP and OQ. Then*

(i) *if $\delta = 1$, there is no point of Λ inside J;*

(ii) *if $\delta > 1$, there is at least one point of* A *inside J, and, unless that point is the intersection of the diagonals of J, at least two, one in each of the triangles into which J is divided by PQ.*

There is no point of A inside J if and only if the lattice L' based on OP and OQ is equivalent to L, i.e. if and only if $\delta = 1$. If $\delta > 1$, there is at least one such point S. If R is the fourth vertex of the parallelogram J, and RT is parallel and equal to OS, but with the opposite sense, then (since the properties of a lattice are symmetrical, and independent of the particular lattice point chosen as origin) T is also a point of A, and there are at least two points of A inside J unless T coincides with S. This is the special case mentioned under (ii).

The different cases are illustrated. in Figs. 2 a, 2 b, 2 c.

3.7. Third proof of Theorems 28 and 29. The fractions h/k with

$$0 \leqslant h \leqslant k \leqslant n, \qquad (h, k) = 1$$

are the fractions of $\mathfrak{F}_n$, and correspond to the visible points (k, h) of A inside, or on the boundary of, the triangle defined by the lines $y = 0$, $y = x$, $x = n$.

If we draw a ray through 0 and rotate it round the origin in the counter-clockwise direction from an initial position along the axis of x, it will pass in turn through each point (k, h) representative of a Farey fraction. If P and P' are points (k, h) and (k', h') representing consecutive fractions, there is no representative point inside the triangle OPP' or on the join PP', and therefore, by Theorem 34,

$$kh'-hk' = 1.$$

3.8. The Farey dissection of the continuum. It is often convenient to represent the real numbers on a circle instead of, as usual, on a straight line, the object of the circular representation being to eliminate integral parts. We take a circle C of unit circumference, and an arbitrary point 0 of the circumference as the representative of 0, and represent x by the point P_x whose distance from 0, measured round the circumference in the counter-clockwise direction, is x. Plainly all