XIX

PARTITIONS

19.1. The general problem of additive arithmetic. In this and the next two chapters we shall be occupied with the additive theory of numbers. The general problem of the theory may be stated as follows.

Suppose that A or
$$a_1,\ a_2,\ a_3,\ \ldots$$
is a given system of integers. Thus A might contain all the positive integers, or the squares, or the primes. We consider all possible representations of an arbitrary positive integer n in the form
$$n = a_{i_1}+a_{i_2}+\ldots+a_{i_s},$$
where s may be fixed or unrestricted, the a may or may not be necessarily different, and order may or may not be relevant, according to the particular problem considered. We denote by $r(n)$ the number of such representations. Then what can we say about $r(n)$? For example, is $r(n)$ always positive? Is there always at any rate one representation of every n?

19.2. Partitions of numbers. We take first the case in which A is the set 1, 2, 3,... of all positive integers, s is unrestricted, repetitions are allowed, and order is irrelevant. This is the problem of 'unrestricted partitions',

A *partition* of a number n is a representation of n as the sum of any number of positive integral parts. Thus
$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1$$
$$= 1+1+1+1+1$$
has 7 partitions.† The order of the parts is irrelevant, so that we may, when we please, suppose the parts to be arranged in descending order of magnitude. We denote by $p(n)$ the number of partitions of n; thus $p(5) = 7$.

We can represent a partition graphically by an array of dots or 'nodes' such as

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      . . . . . . .
      . . . .
A     . . .
      . . .
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† We have, of course, to count the representation by one part only.

the dots in a row corresponding to a part. Thus A represents the partition

$$7+4+3+3+1$$

of 18.

We might also read A by columns, in which case it would represent the partition

$$5+4+4+2+1+1+1$$

of 18. Partitions related in this manner are said to be *conjugate.*

A number of theorems about partitions follow immediately from this graphical representation. A graph with m rows, read horizontally, represents a partition into m parts; read vertically, it represents a partition into parts the largest of which is m. Hence

THEOREM 342. *The number of partitions of n into m parts is equal to the number of partitions of n into parts the largest of which is m.*

Similarly,

THEOREM 343. *The number of partitions of n into at most m parts is equal to the number of partitions of n into parts which do not exceed m.*

We shall make further use of 'graphical' arguments of this character, but usually we shall need the more powerful weapons provided by the theory of generating functions.

19.3. The generating function of *p(n)*. The generating functions which are useful here are power series†

$$F(x) = \sum f(n)x^n.$$

The sum of the series whose general coefficient is $f(n)$ is called the generating *function* of $f(n)$, and is said to ***enumerate*** $f(n)$.

The generating function of $p(n)$ was found by Euler, and is

$$F(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots} = 1+\sum_{1}^{\infty} p(n)x^n. \tag{19.3.1}$$

We can see this by writing the infinite product as

$$(1+x+x^2+\dots)$$
$$(1+x^2+x^4+\dots)$$
$$(1+x^3+x^6+\dots)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

and multiplying the series together. Every partition of n contributes just 1 to the coefficient of x^n. Thus the partition

$$10 = 3+2+2+2+1$$

† Compare § 17.10.

corresponds to the product of x^3 in the third row, $x^6 = x^{2+2+2}$ in the second, and x in the first; and this product contributes a unit to the coefficient of x^{10}.

This makes (19.3.1) intuitive, but (since we have to multiply an infinity of infinite series) some development of the argument is necessary.

Suppose that $0 < x < 1$, so that the product which defines $F(x)$ is convergent. The series

$$1+x+x^2+\ldots, \quad 1+x^2+x^4+\ldots, \quad \ldots, \quad 1+x^m+x^{2m}+\ldots$$

are absolutely convergent, and we can multiply them together and arrange the result as we please. The coefficient of x^n in the product is

$$p_m(n),$$

the number of partitions of n into parts not exceeding m. Hence

$$F_m(x) = \frac{1}{(1-x)(1-x^2)\ldots(1-x^m)} = 1+\sum_{n=1}^{\infty} p_m(n)x^n. \tag{19.3.2}$$

It is plain that

$$p_m(n) \leqslant p(n), \tag{19.3.3}$$

that

$$p_m(n) = p(n) \tag{19.3.4}$$

for $n \leqslant m$, and that

$$p_m(n) \to p(n), \tag{19.3.5}$$

when $m \to \infty$, for every n. And

$$F_m(x) = 1+\sum_{n=1}^{m} p(n)x^n+\sum_{m+1}^{\infty} p_m(n)x^n. \tag{19.3.6}$$

The left-hand side is less than $F(x)$ and tends to $F(x)$ when $m \to \infty$.

Thus
$$1+\sum_{n=1}^{m} p(n)x^n < F_m(x) < F(x),$$

which is independent of m. Hence $\sum p(n)x^n$ is convergent, and so, after (19.3.3), $\sum p_m(n)x^n$ converges, for any fixed x of the range $0 < x < 1$, uniformly for all values of m. Finally, it follows from (19.3.5) that

$$1+\sum_{n=1}^{\infty} p(n)x^n = \lim_{m\to\infty}\left(1+\sum_{n=1}^{\infty} p_m(n)x^n\right) = \lim_{m\to\infty} F_m(x) = F(x).$$

Incidentally, we have proved that

$$\frac{1}{(1-x)(1-x^2)\ldots(1-x^m)} \tag{19.3.7}$$

enumerates the partitions of n into parts which do not exceed m or (what is the same thing, after Theorem 343) into at most m parts.

We have written out the proof of the fundamental formula **(19.3.1)** in detail. We have proved it for $0 < x < 1$, and its truth for $|x| < 1$ follows at once from familiar theorems of analysis. In what follows we shall pay no attention to such 'convergence theorems',† since the interest of the subject-matter is essentially formal. The series and products with which we deal are all absolutely convergent for small x (and usually, as here, for $|x| < 1$). The questions of convergence, identity, and so on, which arise are trivial, and can be settled at once by any reader who knows the elements of the theory of functions.

19.4. Other generating functions. It is equally easy to find the generating functions which enumerate the partitions of n into parts restricted in various ways. Thus

(19.4.1) $$\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots}$$

enumerates partitions into odd parts;

(19.4.2) $$\frac{1}{(1-x^2)(1-x^4)(1-x^6)\ldots}$$

partitions into *even* parts;

(19.4.3) $$(1+x)(1+x^2)(1+x^3)\ldots$$

partitions into *unequal* parts;

(19.4.4) $$(1+x)(1+x^3)(1+x^5)\ldots$$

partitions into parts which are *both odd and unequal*; and

(19.4.5) $$\frac{1}{(1-x)(1-x^4)(1-x^6)(1-x^9)\ldots},$$

where the indices are the numbers $5m+1$ and $5m+4$, partitions into parts each of which is of one of these forms.

Another function which will occur later is

(19.4.6) $$\frac{x^N}{(1-x^2)(1-x^4)\ldots(1-x^{2m})}.$$

This enumerates the partitions of $n-N$ into even parts not exceeding $2m$, or of $\frac{1}{2}(n-N)$ into parts not exceeding m; or again, after Theorem 343, the partitions of $\frac{1}{2}(n-N)$ into at most m parts.

Some properties of partitions may be deduced at once from the forms

† Except once in § 19.8, where again we are concerned with a fundamental identity, and once in § 19.9, where the limit process involved is less obvious.

of these generating functions. Thus

(19.4.7) $$(1+x)(1+x^2)(1+x^3)\ldots = \frac{1-x^2}{1-x}\,\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\ldots$$
$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots}.$$

Hence

THEOREM 344. *The number of partitions of n into unequal parts is equal to the number of its partitions into odd parts.*

It is interesting to prove this without the use of generating functions. Any number l can be expressed uniquely in the binary scale, i.e. as

$$l = 2^a+2^b+2^c+\ldots \quad (0 \leqslant a < b < c\ldots).\dagger$$

Hence a partition of n into odd parts can be written as

$$n = l_1.1+l_2.3+l_3.5+\ldots$$
$$= (2^{a_1}+2^{b_1}+\ldots)1+(2^{a_2}+2^{b_2}+\ldots)3+(2^{a_3}+\ldots)5+\ldots;$$

and there is a $(1, 1)$ correspondence between this partition and the partition into the unequal parts

$$2^{a_1},\ 2^{b_1},\ldots,\ 2^{a_2}.3,\ 2^{b_2}.3\ ,\ldots,\ 2^{a_3}.5,\ 2^{b_3}.5,\ldots,\ \ldots.$$

19.5. Two theorems of Euler. There are two identities due to Euler which give instructive illustrations of different methods of proof used frequently in this theory.

THEOREM 345:

$$(1+x)(1+x^3)(1+x^5)\ldots$$
$$= 1+\frac{x}{1-x^2}+\frac{x^4}{(1-x^2)(1-x^4)}+\frac{x^9}{(1-x^2)(1-x^4)(1-x^6)}+\ldots.$$

THEOREM 346:

$$(1+x^2)(1+x^4)(1+x^6)\ldots$$
$$= 1+\frac{x^2}{1-x^2}+\frac{x^6}{(1-x^2)(1-x^4)}+\frac{x^{12}}{(1-x^2)(1-x^4)(1-x^6)}+\ldots.$$

In Theorem 346 the indices in the numerators are 1.2, 2.3, 3.4 ,... .

(i) We first prove these theorems by Euler's device of the introduction of a second parameter a.

Let

$$K(a) = K(a,x) = (1+ax)(1+ax^3)(1+ax^5)\ldots = 1+c_1 a+c_2 a^2+\ldots,$$

† This is the arithmetic equivalent of the identity

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots = \frac{1}{1-x}.$$

where $c_n = c_n(x)$ is independent of a. Plainly

$$K(a) = (1+ax)K(ax^2)$$

or

$$1+c_1a+c_2a^2+\ldots = (1+ax)(1+c_1ax^2+c_2a^2x^4+\ldots).$$

Hence, equating coefficients, we obtain

$$c_1 = x+c_1x^2,\ c_2 = c_1x^3+c_2x^4,\ldots,\ c_m = c_{m-1}x^{2m-1}+c_mx^{2m},\ldots,$$

and so

$$c_m = \frac{x^{2m-1}}{1-x^{2m}}c_{m-1} = \frac{x^{1+3+\ldots+(2m-1)}}{(1-x^2)(1-x^4)\ldots(1-x^{2m})} = \frac{x^{m^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2m})}.$$

It follows that

$$(19.51)\quad (1+ax)(1+ax^3)(1+ax^5)\ldots = 1+\frac{ax}{1-x^2}+\frac{a^2x^4}{(1-x^2)(1-x^4)}+\ldots,$$

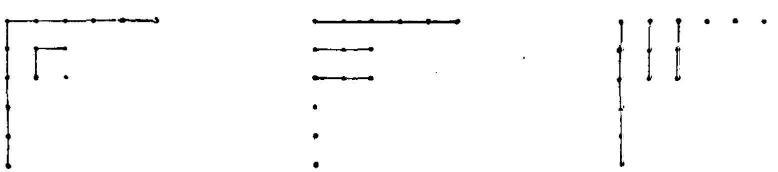
and Theorems 345 and 346 are the special cases a = 1 and a = x.

(ii) The theorems can also be proved by arguments independent of the theory of infinite series. Such proofs are sometimes described as 'combinatorial'. We select Theorem 345.

We have seen that the left-hand side of the identity enumerates partitions into odd and unequal parts: thus

$$15 = 11+3+1 = 9+5+1 = 7+5+3$$

has 4 such partitions. Let us take, for example, the partition $11+3+1$, and represent it graphically as in B, the points on one bent line corresponding to a part of the partition.

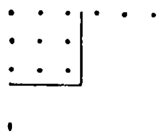


B C D

We can also read the graph (considered as an array of points) as in C or D, along a series of horizontal or vertical lines. The graphs C and D differ only in orientation, and each of them corresponds to another partition of 15, viz. $6+3+3+1+1+1$. A partition like this, symmetrical about the south-easterly direction, is called by Macmahon a *self-conjugate* partition, and the graphs establish a (1, 1) correspondence between self-conjugate partitions and partitions into odd and unequal parts. The left-hand side of the identity enumerates odd and unequal partitions, and therefore the identity will be proved if we can show that its right-hand side enumerates self-conjugate partitions.

Now our array of points may be read in a fourth way, viz. as in E.

E



Here we have a square of 3^2 points, and two 'tails', each representing a partition of $\frac{1}{2}(15-3^2) = 3$ into 3 parts at most (and in this particular case all 1's). Generally, a self-conjugate partition of n can be read as a square of m^2 points, and two tails representing partitions of

$$\tfrac{1}{2}(n-m^2)$$

into m parts at most. Given the (self-conjugate) partition, then m and the reading of the partition are fixed; conversely, given n, and given any square m^2 not exceeding n, there is a group of self-conjugate partitions of n based upon a square of m^2 points.

Now
$$\frac{x^{m^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2m})}$$
is a special case of (19.4.6), and enumerates the number of partitions of $\frac{1}{2}(n-m^2)$ into at most m parts, and each of these corresponds as we have seen to a self-conjugate partition of n based upon a square of m^2 points. Hence, summing with respect to m,

$$1+\sum_{1}^{\infty}\frac{x^{m^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2m})}$$

enumerates all self-conjugate partitions of n, and this proves the theorem.

Incidentally, we have proved

THEOREM 347. *The number of partitions of n into odd and unequal parts is equal to the number of its self-conjugate partitions.*

Our argument suffices to prove the more general identity (19.5.1), and show its combinatorial meaning. The number of partitions of n into just m odd and unequal parts is equal to the number of self-conjugate partitions of n based upon a square of m^2 points. The effect of putting a = 1 is to obliterate the distinction between different values of m.

The reader will find it instructive to give a combinatorial proof of Theorem 346. It is best to begin by replacing x^2 by x, and to use the decomposition $1+2+3+...+m$ of $\frac{1}{2}m(m+1)$. The square of (ii) is replaced by an isosceles right-angled triangle.

19.6. Further algebraical identities. We can use the method (i) of § **19.5** to prove a large number of algebraical identities. Suppose, for example, that

$$K_j(a) = K_j(a,x) = (1+ax)(1+ax^2)...(1+ax^j) = \sum_{m=0}^{j} c_m a^m.$$

Then
$$(1+ax^{j+1})K_j(a) = (1+ax)K_j(ax).$$

Inserting the power series, and equating the coefficients of a^m, we obtain

$$c_m+c_{m-1}\,x^{j+1} = (c_m+c_{m-1})x^m$$

or
$$(1-x^m)c_m = (x^m-x^{j+1})c_{m-1} = x^m(1-x^{j-m+1})c_{m-1},$$

for $1 \leqslant m \leqslant j$. Hence

THEOREM 348:

$$(1+ax)(1+ax^2)...(1+ax^j) = 1+ax\frac{1-x^j}{1-x}+a^2x^3\frac{(1-x^j)(1-x^{j-1})}{(1-x)(1-x^2)}+$$
$$+...+a^m x^{\frac{1}{2}m(m+1)}\frac{(1-x^j)...(1-x^{j-m+1})}{(1-x)...(1-x^m)}+...+a^j x^{\frac{1}{2}j(j+1)}.$$

If we write x^2 for x, $1/x$ for a, and make $j \to \infty$, we obtain Theorem 345. Similarly we can prove

THEOREM 349:

$$\frac{1}{(1-ax)(1-ax^2)...(1-ax^j)} = 1+ax\frac{1-x^j}{1-x}+a^2x^2\frac{(1-x^j)(1-x^{j+1})}{(1-x)(1-x^2)}+....$$

In particular, if we put $a = 1$, and make $j \to \infty$, we obtain

THEOREM 350:

$$\frac{1}{(1-x)(1-x^2)...} = 1+\frac{x}{1-x}+\frac{x^2}{(1-x)(1-x^2)}+....$$

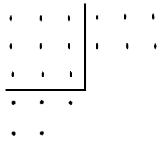
19.7. Another formula for $F(x)$. As a further example of 'combinatorial' reasoning we prove another theorem of Euler, viz.

THEOREM 351:

$$\frac{1}{(1-x)(1-x^2)(1-x^3)...}$$
$$= 1+\frac{x}{(1-x)^2}+\frac{x^4}{(1-x)^2(1-x^2)^2}+\frac{x^9}{(1-x)^2(1-x^2)^2(1-x^3)^2}+....$$

The graphical representation of any partition, say

F



contains a square of nodes in the north-west corner. If we take the largest such square, called the 'Durfee square' (here a square of 9 nodes), then the graph consists of a square containing i^2 nodes and two tails; one of these tails represents the partition of a number, say l, into not more than i parts, the other the partition of a number, say m, into parts not exceeding i; and

$$n = i^2+l+m.$$

In the figure $n = 20$, $i = 3$, $l = 6$, $m = 5$.

The number of partitions of l (into at most i parts) is, after § 19.3, the coefficient of x^l in

$$\frac{1}{(1-x)(1-x^2)...(1-x^i)},$$

and the number of partitions of m (into parts not exceeding i) is the coefficient of x^m in the same expansion. Hence the coefficient of x^{n-i^2} in

$$\left\{\frac{1}{(1-x)(1-x^2)...(1-x^i)}\right\}^2,$$

or of x^n in
$$\frac{x^{i^2}}{(1-x)^2(1-x^2)^2...(1-x^i)^2},$$

is the number of possible pairs of tails in a partition of n in which the Durfee square is i^2. And hence the total number of partitions of n is the coefficient of x^n in the expansion of

$$1+\frac{x}{(1-x)^2}+\frac{x^4}{(1-x)^2(1-x^2)^2}+...+\frac{x^{i^2}}{(1-x)^2(1-x^2)^2...(1-x^i)^2}+....$$

This proves the theorem.

There are also simple algebraical† proofs.

† We use the word 'algebraical' in its old-fashioned sense, in which it includes elementary manipulation of power series or infinite products. Such proofs involve (though sometimes only superficially) the use of limiting processes, and are, in the strict sense of the word, 'analytical' ; but the word 'analytical' is usually reserved, in the theory of numbers, for proofs which depend upon analysis of a deeper kind (usually upon the theory of functions of a complex variable).