X

CONTINUED FRACTIONS

10.1. Finite continued fractions. We shall describe the function

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\dots \atop {\qquad\qquad +\cfrac{1}{a_N}}}}} \tag{10.1.1}$$

of the $N+1$ variables

$$a_0,\ a_1,\dots,\ a_n,\dots,\ a_N,$$

as a finite continued fraction, or, when there is no risk of ambiguity, simply *as a continued fraction*. Continued fractions are important in many branches of mathematics, and particularly in the theory of approximation to real numbers by rationals. There are more general types of continued fractions in which the 'numerators' are not all 1's, but we shall not require them here.

The formula (10.1.1) is cumbrous, and we shall usually write the continued fraction in one of the two forms

$$a_0+\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_N}$$

or

$$[a_0, a_1, a_2,\dots, a_N].$$

We call a,, a,,..., a_N the *partial quotients*, or simply the *quotients*, of the continued fraction.

We find by calculation that†

$$[\text{a,}] = \frac{a_0}{1},\qquad [a_0, a_1] = \frac{a_1 a_0+1}{a_1},\qquad [a_0,\ a_1,\ a_2] = \frac{a_2 a_1 a_0+a_2+a_0}{a_2 a_1+1};$$

and it is plain that

$$[a_0, a_1] = a_0+\frac{1}{a_1}, \tag{10.1.2}$$

$$[a_0, a_1,\dots, a_{n-1}, a_n] = \left[a_0, a_1,\dots,a_{n-2}, a_{n-1}+\frac{1}{a_n}\right], \tag{10.1.3}$$

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† There is a clash between our notation here and that of § 6.11, which we shall use again later in the chapter (for example in § 10.5). In § 6.11, $[x]$ was defined as the integral part of x; while here [a,,] means simply a,. The ambiguity should not confuse the reader, since we use $[a_0]$ here merely as a special case of [a_0, a,,..., a_n]. The square bracket in this sense will seldom occur with a single letter inside it, and will not then be important.

$$(10.1.4)\quad [a_0, a_1, \dots, a_n] = a_0 + \frac{1}{[a_1, a_2, \dots, a_n]} = [a_0, [a_1, a_2, \dots, a_n]],$$

for $1 \leqslant n \leqslant N$. We could define our continued fraction **by (10.12)** and either (10.1.3) or (10.1.4). More generally

$$(10.1.5)\qquad [a_0, a_1, \dots, a,] = [a_0, a_1, \dots, a_{m-1}, [a_m, a_{m+1}, \dots, a_n]]$$

for $1 \leqslant m < n \leqslant N$.

10.2. Convergents to a continued fraction. We call

$$[a_0, a_1, \dots, a_n] \quad (0 \leqslant n \leqslant N)$$

the nth *convergent* to $[a_0, a_1, \dots, a_N]$. It is easy to calculate the convergents by means of the following theorem.

Theorem 149. *If p_n and q_n are defined by*

$$(10.2.1)\quad p_0 = a,, \quad p_1 = a_1 a_0 + 1, \quad p_n = a_n p_{n-1} + p_{n-2}\ (2 \leqslant n \leqslant N),$$

$$(10.2.2)\quad q_0 = 1, \quad q_1 = a,, \quad q_n = a_n q_{n-1} + q_{n-2}\ (2 \leqslant n \leqslant N),$$

then

$$(10.2.3)\qquad [a_0, a,, \dots, a,] = \frac{p_n}{q_n}.$$

We have already verified the theorem for $n = 0$ and $n = 1$. Let us suppose it to be true for $n \leqslant m$, where $m < N$. Then

$$[a,, a, , \dots, a_{m-1}, a,] = \frac{p_m}{q_m} = \frac{a_m p_{m-1} + p_{m-2}}{a_m q_{m-1} + q_{m-2}},$$

and $p_{m-1}, p_{m-2}, q_{m-1}, q_{m-2}$ depend only on

$$a_0, a_1, \dots, a_{m-1}.$$

Hence, using (10.1.3), we obtain

$$[a_0, a_1, \dots, a_{m-1}, a_m, a_{m+1}] = \left[a_0, a_1, \dots, a_{m-1}, a_m + \frac{1}{a_{m+1}}\right]$$

$$= \frac{\left(a_m + \frac{1}{a_{m+1}}\right)p_{m-1} + p_{m-2}}{\left(a_m + \frac{1}{a_{m+1}}\right)q_{m-1} + q_{m-2}}$$

$$= \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}}$$

$$= \frac{a_{m+1} p_m + p p_{m+1}.}{a_{m+1} q_m + q_{m-1}} - \frac{}{q_{m+1}},$$

and the theorem is proved by induction.

It follows from (10.2.1) and (10.2.2) that

(10.2.4) $$\frac{p_n}{q_n} = \frac{a_n p_{n-1}+p_{n-2}}{a_n q_{n-1}+q_{n-2}}.$$

Also

$$p_n q_{n-1} - p_{n-1} q_n = (a_n p_{n-1}+p_{n-2})q_{n-1} - p_{n-1}(a_n q_{n-1}+q_{n-2})$$
$$= (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}).$$

Repeating the argument with n-l, n-2,..., 2 in place of n, we obtain

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}(p_1 q_0 - p_0 q_1) = (-1)^{n-1}.$$

Also

$$p_n q_{n-2} - p_{n-2} q_n = (a_n p_{n-1}+p_{n-2})q_{n-2} - p_{n-2}(a_n q_{n-1}+q_{n-2})$$
$$= a_n(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = (-1)^n a_n.$$

THEOREM 150. *The functions p_n and q_n satisfy*

(10.2.5) $$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$$

or

(10.24 $$\frac{p_n}{q_n} - \frac{p_{n-1}}{q\,n-1} = \frac{(-1)^{n-1}}{qn\text{-}1\; q_n}.$$

THEOREM 151. *They also satisfy*

(10.2.7) $$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$$

or

(10.2.8) $$\frac{p_n}{q_n} \quad \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^n a_n}{q_{n-2} q_n}.$$

10.3. Continued fractions with positive quotients. We now assign numerical values to the quotients a,, and so to the fraction **(10.1.1)** and to its convergents. We shall always suppose that

(10.3.1) $$a_1 > 0, \quad \dots \quad a_N > 0,\dagger$$

and usually also that a, is *integral,* in which case the continued fraction is said to be ***simple.*** But it is convenient first to prove three theorems (Theorems 152-4 below) which hold for all continued fractions in which the quotients satisfy **(10.3.1).** We write

$$x_n = \frac{p_n}{qn}, \qquad x = x_N,$$

so that the value of the continued fraction is x_N or x.

It follows from **(10.1.5)** that

(10.3.2) $$x = [a_0, a_1, \dots, a_N] = [a_0, a_1, \dots, a_{n-1}, [a_n, a_{n+1}, \dots, a_N]]$$
$$= \frac{[\text{a,, a,,,,...,}\ a_N]p_{n-1}+p_{n-2}}{[a_n, a_{n+1}, \dots, a_N]q_{n-1}+q_{n-2}}$$

for 2 ⩽ n ⩽ N.

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† a_0 may be negative.

THEOREM 152. *The even* convergents x_{2n} *increase strictly with n, while the odd convergents* x_{2n+1} *decrease strictly.*

THEOREM 153. *Every odd convergent* is *greater than any even convergent.*

THEOREM 154. *The value of the continued* fraction *is greater than that of any of its even convergents and less than that of any of its odd convergents (except that it* is *equal to the last convergent, whether this be even or odd).*

In the first place every q_n is positive, so that, after (10.2.8) and (10.3.1), x_n-x_{n-2} has the sign of $(-1)^n$. This proves Theorem 152.

Next, after (10.2.6), x_n-x_{n-1} has the sign of $(-1)^{n-1}$, so that

$$x_{2m+1} > x_{2m}. \tag{10.3.3}$$

If Theorem 153 were false, we should have $x_{2m+1} \leqslant x_{2\mu}$ for some pair m, μ. If $\mu <$ m, then, after Theorem 152, $x_{2m+1} < x_{2m}$, and if $\mu >$ m, then $x_{2\mu+1} < x_{2\mu}$; and either inequality contradicts **(10.3.3).**

Finally, x = x_N is the greatest of the-even, or the least of the odd convergents, and Theorem 154 is true in either case.

10.4. Simple continued fractions. We now suppose that the a, are integral and the fraction simple. The rest of the chapter will be concerned with the special properties of simple continued fractions, and other fractions will occur only incidentally. It is plain that p_n and q_n are integers, and q_n positive. If

$$[a_0, a_1, a_2, \dots, a_N] = \frac{p_N}{q_N} = x,$$

we say that the number x (which is necessarily rational) is represented by the continued fraction. We shall see in a moment that, with one reservation, the representation is unique.

THEOREM 155. $q_n \geqslant q_{n-1}$ *for n* $\geqslant$ **1,** *with* inequality *when n >* **1.**

THEOREM 156. $q_n \geqslant$ n, with *inequality when n > 3.*

In the first place, $q_0 = 1$, $q_1 = a_1 \geqslant 1$. If n $\geqslant$ 2, then

$$q_n = a_n q_{n-1}+q_{n-2}\ 3\ q_{n-1}+1,$$

so that $q_n > q_{n-1}$ and $q_n \geqslant$ n. If n > 3, then

$$q_n \geqslant q_{n-1}+q_{n-2} > q_{n-1}+1 \geqslant n,$$

and so $q_n > n$.

A more important property of the convergents is

THEOREM 157. *The convergents to a simple continued fraction are in their lowest terms.*

For, by Theorem 150,

$$d\,|\,p_n \cdot d\,|\,q_n \rightarrow d\,|\,(-1)^{n-1} \rightarrow d\,|\,1.$$

10.5. The representation of an irreducible rational fraction by a simple continued fraction.

Any simple continued fraction [a,, a,, . . ., a_N] represents a rational number

$$\text{x} = x_N.$$

In this and the next section we prove that, conversely, every positive rational x is representable by a simple continued fraction, and that, apart from one ambiguity, the representation is unique.

Theorem 158. *If x is representable by a simple continued fraction with an odd (even) number of convergents, it is also representable by one with an even (odd) number.*

For, if a, $\geqslant 2$,

$$[a_0, a_1, \ldots, \text{a,}] = [a_0, a_1, \ldots, a_n - 1, 1],$$

while, if a, = 1,

$$[a_0, \text{a,,} \ldots, a_{n-1}, 1] = [a_0, \text{a,,} \ldots, a_{n-2}, a_{n-1} + 1].$$

For example $$[2, 2, 3] = [2, 2, 2, 1].$$

This choice of alternative representations is often useful.

We call $$a'_n = [\text{a,, a,,,,} \ldots, a_N] \quad (0 \leqslant n \leqslant N)$$

the *n-th complete quotient* of the continued fraction

$$[a_0, a_1, \ldots, \text{a,,} \ldots, a_N].$$

Thus $$x = a'_0, \qquad x = \frac{a'_1 a_0 + 1}{a'_1}$$

and

$$x = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}} \quad (2 \leqslant \text{n} \leqslant \text{N}). \tag{10.5.1}$$

Theorem 159. a, = $[a'_n]$, *the integral part of* a'_n,† *except that*

$$a_{N-1} = [a'_{N-1}] - 1$$

when $a_N = 1$.

If $N = 0$, then a, = $a'_0 = [a'_0]$. If $N > 0$, then

$$a'_n = a_n + \frac{1}{a'_{n+1}} \quad (0 \leqslant \text{n} \leqslant \text{N-l}).$$

Now $$a' , , ,) \ 1 \quad (0 \leqslant n \leqslant N - 1)$$

except that $a'_{n+1} = 1$ when n = N- 1 and $a_N = 1$. Hence

$$\text{a,} < a'_n < a_n + 1 \quad (0 \leqslant n \leqslant \text{N-1}) \tag{10.52}$$

and $$a_n = [a'_n] \quad (0 \leqslant \text{n} \leqslant \text{N-1})$$

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† We revert here to our habitual use of the square bracket in accordance with the definition of § 6.11.

except in the case specified. And in any case

$$a_N = a'_N = [a'_N].$$

Theorem 160. *If two simple continued fractions*

$$[a_0, a_1, \dots, a_N], \qquad [b_0, b_1, \dots, b_M]$$

have the same value x, and $a_N > 1$, $b_M > 1$, then $M = N$ and the fractions are identicul.

When we say that two continued fractions are identical we mean that they are formed by the same sequence of partial quotients.

By Theorem 159, $a_0 = [x] = b_0$. Let us suppose that the first *n* partial quotients in the continued fractions are identical, and that a'_n, b'_n are the nth complete quotients. Then

$$x = [a_0, a,, \dots, a_{n-1}, a'_n] = [a,, a,, \dots, a_{n-1}, b'_n].$$

If n = 1, then

$$a_0 + \frac{1}{a'_1} = a_0 + \frac{1}{b'_1},$$

$a'_1 = b'_1$, and therefore, by Theorem 159, a, = b,. If n > 1, then, by (10.5.1),

$$\frac{a'_n p_{n-1} + p_{n-2}}{a'_n\, q_{n-1} + q_{n-2}} = \frac{b'_n p_{n-1} + p_{n-2}}{b'_n\, q_{n-1} + q_{n-2}},$$

$$(a'_n - b'_n)(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = 0.$$

But $p_{n-1} q_{n-2} - p_{n-2} q_{n-1} = (-1)^n$, by Theorem 150, and so $a'_n = b'_n$. It follows from Theorem 159 that a, = b_n.

Suppose now, for example, that $N \leqslant M$. Then our argument shows that

$$a, = b_n$$

for $n \leqslant N$. If $M > N$, then

$$\frac{p_N}{q_N} = [a,, a_1, \dots, a_N] = [a_0, a, , \dots, a_N, b_{N+1}, \dots, b_M] = \frac{b'_{N+1} p_N + p_{N-1}}{b'_{N+1}\, q_N + q_{N-1}},$$

by (10.5.1); or

$$p_N\, q_{N-1} - p_{N-1}\, q_N = 0,$$

which is false. Hence $M = N$ and the fractions are identical.

10.6. The continued fraction algorithm and Euclid's algorithm. Let x be any real number, and let a, = $[x]$. Then

$$x = a_0 + \xi_0, \qquad 0 \leqslant \xi_0 < 1.$$

If $\xi_0 \neq 0$, we can write

$$\frac{1}{\xi_0} = a'_1, \qquad [a'_1] = a_1, \qquad a'_1 = a_1 + \xi_1, \qquad 0 \leqslant \xi_1 < 1.$$

If $\xi_1 \neq 0$, we can write

$$\frac{1}{\xi_1} = a'_2 = a_2 + \xi_2, \qquad 0 \leqslant \xi_2 < 1,$$

and so on. Also $a_n' = 1/\xi_{n-1} > 1$, and so a, $\geqslant$ 1, for n $\geqslant$ 1. Thus

$$x=[a_0,a_1'] = a\ a\Big[+\frac{1}{0'} =_1\Big[a\frac{1}{a_2'}\Big]\iota_1, a_2'] = [a_0,a_1,a_2, a_3'] = \ . \ . \ . \ .$$

where a,, a,,... are integers and

$$a_1 > 0, \qquad a_2 > 0, \ldots .$$

The system of equations

$$\begin{aligned} x &= a_0+\xi_0 && (0 \leqslant \xi_0 < 1),\\ \frac{1}{\xi_0} &= a_1' = a_1+\xi_1 && (0 \leqslant \xi_1 < 1),\\ \frac{1}{\xi_1} &= a_2' = a_2+\xi_2 && (0 \leqslant \xi_2 < 1),\\ &. \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

is known *as* the ***continued fraction algorithm.*** The algorithm continues so long as $\xi_n \neq 0$. If we eventually reach a value of n, say N, for which $\xi_N = 0$, the algorithm terminates and

$$x = [a_0, a_1, a_2,\ldots, a_N].$$

In this case x is represented by a simple continued fraction, and is rational. The numbers a_n' are the complete quotients of the continued fraction.

THEOREM 161. ***Any rational number can be represented by a finite simple*** continued ***fraction.***

If x is an integer, then $\xi_0 = 0$ and x = a,. If x is not integral, then

$$x = \frac{h}{k},$$

where *h* and *k* are integers and $k > 1$. Since

$$\frac{h}{k} = a_0+\xi_0, \qquad h = a_0k+\xi_0k,$$

a, is the quotient, and $k_1 = \xi_0 k$ the remainder, when *h* is divided by *k*.†

If $\xi_0 \neq 0$, then
$$a_1' = \frac{1}{\xi_0} = \frac{k}{k_1}$$

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† The 'remainder', here and in what follows, is to be non-negative (here positive). If a, $\geqslant$ 0, then x and h are positive and k_1 is the remeinder in the ordinary sense of arithmetic. If a, < 0, then x and h are negative and the 'remainder' is

$$(x-[x])k.$$

Thus if $h = -7$, $k = 5$, the 'remeinder' is

$$(-\tfrac{7}{5}-[-\tfrac{7}{5}])5 = (-\tfrac{7}{5}+2)5 = 3.$$

and $$\frac{k}{k_1} = a_1+\xi_1, \qquad k = a_1k_1+\xi_1k_1;$$

thus a, is the quotient, and $k_2 = \xi_1 k_1$ the remainder, when k is divided by k,. We thus obtain a series of equations

$$h = a_0k+k_1, \qquad k = a_1k_1+k_2, \qquad k_1 = a_2k_2+k_3, \qquad \ldots$$

continuing so long as $\xi_n \neq 0$, or, what is the same thing, so long as $k_{n+1} \neq 0$.

The non-negative integers k, k_1, k,, . . .form a strictly decreasing sequence, and so $k_{N+1} = 0$ for some N. It follows that $\xi_N = 0$ for some N, and that the continued fraction algorithm terminates. This proves Theorem161.

The system of equations

$$\begin{array}{ll} h = a_0k+k_1 & (0 < k_1 < k), \\ k = a_1k_1+k_2 & (0 < k_2 < k_1), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ k_{N-2} = a_{N-1}k_{N-1}+k_N & (0 < k_N < k_{N-1}), \\ k_{N-1} = a_Nk_N & \end{array}$$

is known as *Euclid's algorithm*. The reader will recognize the process as that adopted in elementary arithmetic to determine the greatest common divisor k_N of h and k.

Since $\xi_N = 0$, a'_N = a,; also

$$0<\frac{1}{a_N}=\frac{1}{a'_N}=\xi_{N-1}<1,$$

and so $a_N \geqslant 2$. Hence the algorithm determines a representation of the type which was shown to be unique in Theorem 160. We may always make the variation of Theorem 158.

Summing up our results we obtain

THEOREM 162. *A rational number can be expressed as a finite simple continued fraction in just two ways, one with an even and the other with an odd number of convergents. In one form the last partial quotient is* 1, *in the other it is* **greater** *than* 1.

10.7. The difference between the fraction and its convergents.

Throughout this section we suppose that N > 1 and n > 0. By (10.5.1)

$$x = \frac{a'_{n+1}p_n+p_{n-1}}{a'_{n+1}q_n+q_{n-1}},$$

for $1 \leqslant n \leqslant N-1$, and so

$$x-\frac{p_n}{q_n} = -\frac{p_nq_{n-1}-p_{n-1}q_n}{q_n(a'_{n+1}q_n+q_{n-1})} = \frac{(-1)^n}{q_n(a'_{n+1}q_n+q_{n-1})},$$

Also
$$x-\frac{p_0}{q_0} = x-a_0 = \frac{1}{a_1'}.$$

If we write

(10.7.1) $$q_1' = a_1', \qquad q_n' = a_n' q_{n-1}+q_{n-2} \quad (1 < n \leqslant N)$$

(so that, in particular, $q_N' = q_N$), we obtain

THEOREM 163. *If* $1 \leqslant n \leqslant N-1$, *then*
$$x-\frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n+1}'}.$$

This formula gives another proof of Theorem **154.**

Next,
$$a_{n+1} < a_{n+1}' < a_{n+1}+1$$
for $n \leqslant N-2$, by (10.5.2), except that
$$a_{N-1}' = a_{N-1}+1$$
when $a_N = 1$. Hence, if we ignore this exceptional case for the moment, we have

(10.7.2) $$q_1 = a_1 < a_1' < a_1+1 \leqslant q_2$$

and

(10.7.3) $$q_{n+1}' = a_{n+1}' q_n + q_{n-1} > a_{n+1} q_n + q_{n-1} = q_{n+1},$$

(10.7.4) $$q_{n+1}' < a_{n+1} q_n + q_{n-1} + q_n = q_{n+1}+q_n \leqslant a_{n+2} q_{n+1} + q_n = q_{n+2},$$

for $1 \leqslant n \leqslant N-2$. It follows that

(10.7.5) $$\frac{1}{q_{n+2}} < |p_n - q_n x| < \frac{1}{q_{n+1}} \quad (n \leqslant N-2),$$

while

(10.7.6) $$|p_{N-1}-q_{N-1}x| = \frac{1}{q_N}, \qquad p_N - q_N x = 0.$$

In the exceptional case, **(10.7.4)** must be replaced by
$$q_{N-1}' = (a_{N-1}+1)q_{N-2}+q_{N-3} = q_{N-1}+q_{N-2} = q_N$$
and the first inequality in **(10.7.5)** by an equality. In any case **(10.7.5)** shows that $|p_n - q_n x|$ decreases steadily as n increases; *a fortiori*, since q_n increases steadily,
$$\left|x-\frac{p_n}{q_n}\right|$$
decreases steadily.

We may sum up the most important of our conclusions in

THEOREM 164. *If* $N > 1$, $n > 0$, *then the differences*
$$x-\frac{p_n}{q_n}, \qquad q_n x - p_n$$

decrease steadily in absolute value as n increases. Also

$$q_n x - p_n = \frac{(-1)^n \delta_n}{q_{n+1}},$$

where $0 < \delta_n < 1 \quad (1 \leqslant n \leqslant N\text{-}2), \qquad \delta_{N-1} = 1,$
and

$$\left| x - \frac{p_n}{q_n} \right| \leqslant \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2} \tag{10.7.7}$$

for n $\leqslant$ *N- 1, with inequality in both places except when n* = N-l.

10.8. Infinite simple continued fractions. We have considered so far only finite continued fractions; and these, when they are simple, represent rational numbers. The chief interest of continued fractions, however, lies in their application to the representation of irrationals, and for this *infinite* continued fractions are needed.

Suppose that a,,, a,, a_2,... is a sequence of integers satisfying (10.3. 1), so that

$$x_n = [a_0, a_1, \ldots, a_n]$$

is, for every n, a simple continued fraction representing a rational number x,. If, as we shall prove in a moment, x_n tends to a limit x when n → ∞, then it is natural to say that *the simple continued fraction*

$$[a_0, a_1, a_2, \ldots] \tag{10.8.1}$$

converges to the value x, and to write

$$x = [a_0, a_1, a_2, \ldots]. \tag{10.8.2}$$

Theorem 165. *If* a,, a,, a_2 ,... *is a sequence* of integers *satisfying* (10.3.1), *then* $x_n = [a_0, a_1, \ldots,$ *a,] tends to a limit* x *when n* → ∞.

We may express this more shortly as

Theorem 166. *All infinite simple continued fractions are convergent.*

We write
$$x_n = \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n],$$
as in § 10.3, and call these fractions the convergents to **(10.8.1).** We have to show that the convergents tend to a limit.

If N $\geqslant$ **n,** the convergent x_n is also a convergent to [a,, a,,..., a_N]. Hence, by Theorem 152, the even convergents form an increasing and the odd convergents a decreasing sequence.

Every even convergent is less than x_1, by Theorem 153, so that the increasing sequence of even convergents is bounded above; and every odd convergent is greater than x_0, so that the decreasing sequence of odd convergents is bounded below. Hence the even convergents tend to a limit ξ_1, and the odd convergents to a limit ξ_2, and $\xi_1 \leqslant \xi_2$.

Finally, by Theorems 150 and 156,

$$\left|\frac{p_{2n}}{q_{2n}}-\frac{p_{2n-1}}{q_{2n-1}}\right| = \frac{1}{q_{2n}q_{2n-1}} \leqslant \frac{1}{2n(2n-1)} \to 0,$$

so that $\xi_1 = \xi_2 =$ x, say, and the fraction (10.8.1) converges to x.

Incidentally we see that

THEOREM 167. *An infinite simple continued fraction is less than any of its odd convergents and greater than any of its even convergents.*

Here, and often in what follows, we'use 'the continued fraction' as an abbreviation for 'the value of the continued fraction'.

10.9. The representation of an irrational number by an infinite continued fraction. We call

$$a'_n = [a_n, a_{n+1},\ldots]$$

the *n-th complete quotient* of the continued fraction

$$x = [a_0, a_1,\ldots].$$

Clearly

$$a'_n = \lim_{N\to\infty} [\text{a,, a,+,,...,}\ a_N]$$

$$= \text{a,+} \lim_{N\to\infty} \frac{1}{[a_{n+1},\ldots, a_N]} = a_n + \frac{1}{a'_{n+1}},$$

and in particular

$$\text{x} = a'_0 = a_0 + \frac{1}{a'_1}.$$

Also

$$a'_n > a_n, \qquad a'_{n+1} > a_{n+1} > 0, \qquad 0 < \frac{1}{a'_{n+1}} < 1;$$

and so a, = $[a'_n]$.

THEOREM 168. *If [a,,, a_1, a_2 ,...] $= x$, then*

$$a_0 = [x], \qquad a_n = [a'_n]\ (n \geqslant 0).$$

From this we deduce, as in § 10.5,

THEOREM 169. *Two infinite* simple *continued* **fractions** *which have the same value* are *identical.*

We now return to the continued fraction algorithm of § **10.6.** If x is irrational the process cannot terminate. Hence it defines an infinite sequence of integers

$$a_0, a_1, a_2,\ldots,$$

and as before

$$x = [a_0, a'_1] = [a_0, a_1, a'_2] = \ldots = [a_0, a_1, a_2,\ldots, a_n, a'_{n+1}],$$

where

$$a'_{n+1} = a_{n+1} + \frac{1}{a'_{n+2}} > a_{n+1}.$$

Hence
$$x = \frac{a'_{n+1} p_n + p_{n-1}}{a'_{n+1} q_n + q_{n-1}},$$
by (10.5.1), and so
$$x - \frac{p_n}{q_n} = \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n(a'_{n+1} q_n + q_{n-1})} = \frac{(-1)^n}{q_n(a'_{n+1} q_n + q_{n-1})},$$
$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n(a_{n+1} q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}} \leqslant \frac{1}{n(n+1)} \to 0$$
when $n \to \infty$. Thus
$$x = \lim_{n\to\infty} \frac{p_n}{q_n} = [a_0, a_1, \dots, a_n, \dots],$$
and the algorithm leads to the continued fraction whose value is x, and which is unique by Theorem 169.

THEOREM 170. *Every irrational number can be expressed in just one way as an infinite simple continued fraction.*

Incidentally we see that the value of an infinite simple continued fraction is necessarily irrational, since the algorithm would terminate if x were rational.

We define
$$q'_n = a'_n q_{n-1} + q_{n-2}$$
as in § 10.7. Repeating the argument of that section, we obtain

THEOREM 171. *The results of Theorems 163 and 164 hold also (except for the references to N) for infinite continued fractions. In particular*

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{10.9.1}$$

10.10. A lemma. We shall need the theorem which follows in § 10.11.

THEOREM 172. *If*
$$x = \frac{P\zeta + R}{Q\zeta + S},$$
where $\zeta > 1$ and P, Q, R, and S are integers such that
$$Q > S > 0, \qquad PS - QR = \pm 1,$$
then R/S and P/Q are two consecutive convergents to the simple continued fraction whose value is x. If R/S is the $(n-1)$th convergent, and P/Q the n-th, then ζ is the $(n+1)$th complete quotient.

We can develop P/Q in a simple continued fraction

$$\frac{P}{Q} = [a_0, a_1, \dots, a_n] = \frac{p_n}{q_n}. \tag{10.10.1}$$