G-ODOMETERS AND THEIR ALMOST 1-1 EXTENSIONS.

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ABSTRACT. In this paper we recall the concepts of G-odometer and G-subodometer for G-actions, where G is a discrete finitely generated group, which generalize the notion of odometer in the case $G = \mathbb{Z}$. We characterize the G-regularly recurrent systems as the minimal almost 1-1 extensions of subodometers, from which we deduce that the family of the G-Toeplitz subshifts coincides with the family of the minimal symbolic almost 1-1 extensions of subodometers. We determine the continuous eigenvalues of these systems. When G is amenable and residually finite, a characterization of the G-invariant measures of these systems is given.

1. INTRODUCTION

It is known that an almost 1-1 extension of a minimal equicontinuous system always has this system as the maximal equicontinuous factor. This justifies the study of these almost 1-1 extensions. The aim of this paper is to study the extensions of a particular type of equicontinuous systems: the G-odometers, where G is a discrete finitely generated group, like for example a non abelian free group. The notion of G-odometer generalizes the notion of odometer, or adding-machine, in the case $G = \mathbb{Z}$.

An example of extensions of $\mathbb{Z}$-odometers are the Toeplitz flows, which were introduced by Jacobs and Keane in [JK]. Toeplitz flows have been extensively studied in different contexts and they have been used to provide series of examples with interesting dynamical properties (see for example [Do], [GJ], [Wi]). Markley and Paul characterize them in [MP] as the minimal almost 1-1 extensions of odometers and a proof of this theorem is given in [DL] by Downarowicz and Lacroix (see also [Au]). Let us mention also an example of F. Krieger in [Kr] where he constructs, for a residually finite and amenable group G, a G-Toeplitz sequence with an arbitrary entropy.

Following the work developed in [Co] for $G = \mathbb{Z}^d$, we prove that for a discrete finitely generated group G, the G-Toeplitz systems are the symbolic minimal almost 1-1 extensions of G-odometers. The main difficulties lie in the fact that we consider non-abelian groups and therefore the used techniques are not straight generalizations of the $\mathbb{Z}$-case. Unlike in the abelian case, there appear some degenerated systems that we call subodometers.

This paper is organized as follows: in Section 2, we give some basic definitions relevant for the study of topological dynamical systems. We recall also the generalized notions of odometer and subodometer and we identify the set of eigenvalues of these systems. In Section 3, we introduce the notions of regularly recurrent systems and strongly regularly recurrent systems. We characterize them as the minimal almost 1-1 extensions of subodometers and odometers respectively. In the particular case where G is amenable and residually finite, we show in Section 4 that the set of invariant probability measures of a G-regularly recurrent Cantor system can be represented as an inverse limit. In Section 5, in the case when G is a residually finite group, we introduce a notion of semicocycles and we show that an almost 1-1 extension of a G-subodometer is conjugated to the action of G on some semicocycle. Finally in Section

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6, we consider a particular family for a discrete group G: the G-Toeplitz arrays, which is a particular family of semicocycles when G is residually finite. We prove, by giving an explicit construction, that this family coincides with the family of symbolic almost 1-1 extensions of the G-subodometers.

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2. Basic definitions and background

In this article, by a *topological dynamical system* we mean a pair (X, G), where G is a topological group which acts, by homeomorphism, on a compact metric space (X, d). Given $g \in G$ and $x \in X$ we will identify g with the associated homeomorphism and we denote by $g.x$ the action of g on x. The dynamical system (X, G) is *free* if $g.x = x$ for some $x \in X$ implies $g = e$, where e is the neutral element in G. For a syndetic subgroup Γ of G, the Γ-*orbit* of $x \in X$ is $O_\Gamma(x) = \{\gamma.x : \gamma \in \Gamma\}$ and the Γ-*system associated* to x is $(\Omega_\Gamma(x), \Gamma)$, where $\Omega_\Gamma(x)$ is the closure of $O_\Gamma(x)$ and the action of Γ on $\Omega_\Gamma(x)$ is the restriction to Γ and $\Omega_\Gamma(x)$ of the action of G on X. The set of *return times* of $x \in X$ to $A \subseteq X$ is $T_A(x) = \{g \in G : g.x \in A\}$. The topological dynamical system (X, G) is *minimal* if the orbit of any $x \in X$ is dense in X, and it is said to be *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ satisfy $d(x, y) < \delta$ then $d(g.x, g.y) < \varepsilon$ for all $g \in G$. We say that (X, G) is an *extension* of (Y, G), or that (Y, G) is a *factor* of (X, G), if there exists a continuous surjection $\pi : X \to Y$ such that π preserves the action. We call π a *factor map*. When the factor map is bijective, we say that (X, G) and (Y, G) are *conjugate*. The factor map π is an *almost 1-1 factor map* and (X, G) is an *almost 1-1 extension* of (Y, G) by π if the set of points having one pre-image is residual (contains a dense G_δ set) in Y. In the minimal case it is equivalent to the existence of a point with one pre-image.
The set $\mathcal{M}_G(X)$ of *invariant probability measures* of X is the set of probability measures μ defined on $\mathcal{B}(X)$, the Borel σ-algebra of X, such that $\mu(g.B) = \mu(B)$ for all $g \in G$ and $B \in \mathcal{B}(X)$.

2.1. G-odometers and G-subodometers.

In all the following, we will denote by G a discrete group generated by a finite family and by e its neutral element.

Definition 1. *A discrete finitely generated group G is called residually finite if and only if there exists a sequence $\Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_n \supset \ldots$ of subgroups Γ_n with finite index in G such that:*

$$\bigcap_n \Gamma_n = \{e\}.$$

A trivial example of a residually finite subgroup is the group of integers $\mathbb{Z}$, for example by taking the groups $\Gamma_n = n!\mathbb{Z}$. Less trivial examples are given by the fundamental groups of connected oriented compact graph. When $\pi : S_2 \to S_1$ is a finite covering of an oriented compact connected graph S_2 onto a compact graph S_1, the application π induces an homomorphism π_* from the fundamental group of S_2 to the fundamental group of S_1. The image of the morphism π_* is a subgroup of the fundamental group of S_1. The index of this subgroup is then the number of preimages of one point for the map π. Let us denote by $\widetilde{S_1}$ the universal cover of S_1. Consider a sequence $(S_n, \pi_n)_n$ of finite coverings $\pi_n : S_{n+1} \to S_n$ of compact connected and oriented graphs S_n such that for each n the injectivity radius of $\widetilde{S_1}$ onto S_n

goes to infinity when n goes to infinity. The sequence of fundamental groups of graphs S_n satisfies then the condition of Definition 1. More generally, we have the following result of Mal'cev [Ma]:

Theorem 1. [Ma] *For any integer n and any field $\mathbb{K}$ with characteristic null, every finitely generated subgroup of the group of invertible matrices $GL(n, \mathbb{K})$ is a residually finite group.*

In particular, the free groups $\mathbb{F}_n$ with n generators, the groups of surfaces and the braids group B_n generated by n elements are residually finite groups.
Let us denote, for a subgroup H of G, by G/H the set of right classes of H in G. It is important to note that G acts on G/H by left multiplication. Now we will prove the useful following lemma:

Lemma 1. *Let G be a group. If H is a subgroup of G with index in G equal to n (i.e. the cardinal of the quotient space G/H is n) then there exists a normal subgroup K of G contained in H such that the cardinality of G/K divides $n!$.*

Proof. The group G acts on G/H by left multiplication. This action defines an homomorphism ρ from G to the permutation group of n elements. The kernel of this application is a normal subgroup of G contained in H and its index in G divides the cardinal of permutations of n elements. □

As a corollary, G is a residually finite group if and only if there exists a sequence $H_1 \supset \ldots \supset H_n \supset \ldots$ of normal subgroups of G with finite index in G such that $\bigcap_n H_n = \{e\}$.

Let us consider a discrete group G generated by a finite family, and a decreasing sequence (for the inclusion) $(\Gamma_i)_{i\geq 0} \subseteq G$ of subgroups with finite index in G (we do not ask $\bigcap_{i\geq 0} \Gamma_i = \{e\}$) and let $\pi_i : G/\Gamma_{i+1} \to G/\Gamma_i$ be the function induced by the inclusion $\Gamma_{i+1} \subset \Gamma_i$, $i \geq 0$. Consider the inverse limit

$$\overleftarrow{G} = \lim_{\leftarrow i} (G/\Gamma_i, \pi_i).$$

More precisely, $\overleftarrow{G}$ is defined as the subset of the product $\Pi_{i\geq 0} G/\Gamma_i$ consisting of the elements $\mathbf{g} = (g_i)_{i\geq 0}$ such that $\pi_i(g_{i+1}) = g_i$ for all $i \geq 0$.
Every G/Γ_i is endowed with the discrete topology and $\Pi_{i\geq 0} G/\Gamma_i$ with the product topology. Thus $\overleftarrow{G}$ is a compact metrizable space whose topology is spanned by the cylinder sets

$$[i; a] = \{\mathbf{g} \in \overleftarrow{G} : g_i = a\}, \text{ with } a \in G/\Gamma_i \text{ and } i \geq 0.$$

The space $\overleftarrow{G}$ is a totally disconnected, it is a Cantor set when $G/\cap_{i\geq 0} \Gamma_i$ is infinite and a finite set when $G/\cap_{i\geq 0} \Gamma_i$ is finite.
The group G acts continuously on $\overleftarrow{G}$ by left multiplication, namely for $\mathbf{g} = (g_i)_i \in \overleftarrow{G}$ and $h \in G$,

$$h.\mathbf{g} = (h._i g_i)_i,$$

where $h._i$ denotes the action on G/Γ_i given by $h._i g\Gamma_i = hg\Gamma_i$, for every $h \in G$ and $g \in G$. Since for all $h \in G$ and for all cylinders $[i; a]$ we have

$$h.([i; a]) \subseteq [i; h._i a_i],$$

the topological dynamical system $(\overleftarrow{G}, G)$ is equicontinuous. Moreover, every orbit for this action is dense, then $(\overleftarrow{G}, G)$ is a minimal equicontinuous system.

Definition 2. *We call $(\overleftarrow{G}, G)$ a G-subodometer system* or simply a subodometer. If in addition, every Γ_i is normal, we say that $(\overleftarrow{G}, G)$ is a G-odometer system or simply an odometer.*

It is straightforward to show that for a point $\mathbf{g} = (g_i)_i$ of a subodometer $\overleftarrow{G}$, its stabilizer for the G-action is the group $\bigcap_i \tilde{g}_i \Gamma_i \tilde{g}_i^{-1}$, where $\tilde{g}_i$ is a representing element of the class $g_i \in \mathbf{G}/\Gamma_i$ in G, for $i \geq 0$. Hence, when G is a residually finite group and $\bigcap_{i\geq 0} \Gamma_i = \{e\}$, for $\mathbf{e} = (e_i)_i \in \overleftarrow{G}$, where e_i is the projection of the neutral element of G on G/Γ_i, its stabilizer is trivial. This does not mean necessarily that the action of G on $\overleftarrow{G}$ is free. If furthermore, all the groups Γ_i are normal subgroups of G, then the stabilizer of every point of a G-odometer is trivial and the action of G is free. For this reason, when G is residually finite and $\bigcap_{i\geq 0} \Gamma_i = \{e\}$, the G-odometer $\lim_{\leftarrow i}(G/\Gamma_i, \pi_i)$ will be called a *free G-odometer.*

If $(\overleftarrow{G}, G)$ is an odometer then the set $\overleftarrow{G}$ is a group equipped with the multiplication defined by

$$\mathbf{g}.\mathbf{h} = (g_i \cdot_i h_i)_{i\geq 0},$$

where $\cdot_i$ denotes the multiplication operation induced on G/Γ_i by the multiplication on G. notice that for a free odometer $(\overleftarrow{G}, G)$, the group G is then a dense subgroup of $\overleftarrow{G}$.
Notice that for all $\mathbf{g}$ in a cylinder set $[i; a]$ of an odometer $\overleftarrow{G} = \lim_{\leftarrow i}(G/H_i, \pi_i)$, the set of return times of $\mathbf{g}$ to $[i; a]$ is H_i. Throughout this paper we will use this property and we will identify $\overleftarrow{G}$ with $(\overleftarrow{G}, G)$.

Lemma 2. *Let $\overleftarrow{G}_j = \lim_{\leftarrow i}(G/H_i^j, \pi_i)$ be two subodometers ($j = 1, 2$). Let $\mathbf{e_j}$ $(j = 1, 2)$ be the element $(e_i^j)_i \in \overleftarrow{G}_j$ where e_i^j denotes the class of the neutral element $e \in G$ in G/H_i^j. There exists a factor map $\pi : (\overleftarrow{G}_1, G) \to (\overleftarrow{G}_2, G)$ such that $\pi(\mathbf{e_1}) = \mathbf{e_2}$ if and only if for every H_i^2 there exists some H_k^1 such that $H_k^1 \subseteq H_i^2$.*

Proof. If $\pi : \overleftarrow{G}_1 \to \overleftarrow{G}_2$ is a factor map then by continuity, given $i \geq 0$ and e_i^2 in G/H_i^2, there exists $k \geq 0$ such that $[k; e_k^1] \subseteq \pi^{-1}[i; e_i^2]$. Let $v \in H_k^1$. We have that $v.\mathbf{g} \in [k; e_k^1]$ for all $\mathbf{g} \in [k; e_k^1]$, which implies that

$$\pi(v.\mathbf{g}) = v.\pi(\mathbf{g}) \in [i; e_i^2].$$

Since $\pi(\mathbf{g}) \in [i; e_i^2]$ and $T_{[i;e_i^2]}(\pi(\mathbf{g})) = H_i^2$, we get $v \in H_i^2$.
Suppose that for every $i \geq 0$ there exists $H_{n_i}^1 \subseteq H_i^2$. Since the sequences $(H_i^j)_{i\geq 0}$, $j = 1, 2$, are decreasing, we can take $n_i \leq n_{i+1}$ for all $i \geq 0$. The function $\pi : \overleftarrow{G}_1 \to \overleftarrow{G}_2$ defined by $\pi((g_i)_{i\geq 0}) = (j_{n_i}(g_{n_i}))_{i\geq 0}$ where $j_{n_i} : G/H_{n_i}^1 \to G/H_i^2$ is the function induced by the inclusion $H_{n_i}^1 \subseteq H_i^2$, is a factor map. □

By a straightforward application of the former lemma and Lemma 1, we get

Proposition 1. *If $(\lim_{\leftarrow i}(G/\Gamma_i, \pi_i), G)$ is a G-subodometer, then there exists a G-odometer which is an extension of this subodometer.*

Proposition 2. *Let $\overleftarrow{G}$ be a G-odometer and (X, G) a dynamical system. If there exists a factor map from $\overleftarrow{G}$ onto X, then there exists a closed subgroup H of $\overleftarrow{G}$ such that the dynamical system $(\overleftarrow{G}/H, G)$ is conjugated to (X, G).*

*Note that this definition is not a profinite completion of the group G because here, we consider only a sequence of decreasing subgroups.

In particular this proposition says that a subodometer is conjugate to the quotient of an odometer by a closed subgroup.

Proof. Let us denote by p the factor map $\overleftarrow{G} \to X$, and by $\mathbf{e}$ the neutral element of $\overleftarrow{G}$. Let H be the closed subset $p^{-1}(p(e))$ of $\overleftarrow{G}$. For $\mathbf{g} = (g_i)_i$ and $\mathbf{h} = (h_i)_i$ in H, we have:

$$p(\mathbf{hg}) = \lim_i p(h_i g_i) = \lim_i h_i.p(g_i) = \lim_i h_i.p(e) = \lim_i p(h_i) = p(e).$$

With the same technique we get:

$$p((\mathbf{g})^{-1}) = \lim p(g_i^{-1}e) = \lim_i g_i^{-1}.p(e) = \lim_i g_i^{-1}.p(g_i) = p(e).$$

So $\mathbf{gh}$ and $\mathbf{g}^{-1}$ belong to H, and H is a group.
Now let us see that $p^{-1}(p(\mathbf{g})) = \mathbf{g}H$ for any $\mathbf{g} \in \overleftarrow{G}$. Let $\mathbf{h}$ be in H, we have:

$$p(\mathbf{gh}) = \lim_i p(g_i h_i) = \lim_i g_i.p(h_i) = \lim_i g_i.p(e) = p(\mathbf{g}).$$

Then $\mathbf{g}H \subset p^{-1}(p(\mathbf{g}))$.
Let $\mathbf{h} \in \overleftarrow{G}$ be such that $p(\mathbf{h}) = p(\mathbf{g})$. Then $\lim_i p(g_i) = \lim_i p(h_i)$ and $p(e) = \lim_i g_i^{-1}.h_i.p(e) = p(\mathbf{g}^{-1}\mathbf{h})$. So $\mathbf{g}^{-1}\mathbf{h}$ belongs to H and $p^{-1}(p(\mathbf{g})) = \mathbf{g}H$. Therefore, the map p factorizes onto a homeomorphism from $\overleftarrow{G}/H$ to X. □

2.2. **Eigenvalues of odometers and subodometers.** Let (X, μ, G) be a measure-theoretic dynamical system with a left action of G. A character χ is a homomorphism from G to the group $\mathbb{S}^1$, the set of complex numbers with modulus 1. Since the group G is equipped with the discrete topology, every character is a continuous map.
A character is an *eigenvalue* of X if there exists $f \in L^2_\mu(X) \setminus \{0\}$ such that $f(g.x) = \chi(g)f(x)$ for all $x \in X$ and $g \in G$. We call f an *eigenfunction* associated to χ. We say that an eigenvalue is a *continuous eigenvalue* if it has an associated continuous eigenfunction.

Since a G-odometer $\overleftarrow{G}$ is a compact group, the normalized left invariant Haar measure λ of $\overleftarrow{G}$ is the only probability measure of $\overleftarrow{G}$ invariant under the action of G. Thus the system $(\overleftarrow{G}, G)$ is uniquely ergodic. Any G-subodometer, as a factor of some G-odometer, is alo uniquely ergodic. Thus when we speak about a subodometer $\overleftarrow{G}$ as a measure-theoretic dynamical system, we mean $\overleftarrow{G}$ equipped with the unique invariant probability measure λ for the action of G.

Proposition 3. *Let $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ be a subodometer. The set of eigenvalues of $\overleftarrow{G}$ is given by $E_G = \bigcup_{n\geq 0}\{$character $\chi : G \to \mathbb{S}^1,\ \chi(\gamma) = 1$ for all $\gamma \in \Gamma_n\}$. Moreover, every eigenvalue of $\overleftarrow{G}$ is a continuous eigenvalue.*

Proof. For $n \geq 0$ we call $C_n = [n; e]$. Since $v, w \in G$ satisfy $v.C_n = w.C_n$ if and only if w and v belong to the same class in G/Γ_n, it makes sense to write $v.C_n$ for $v \in G/\Gamma_n$. Notice that the collection $\mathcal{P}_n = \{v.C_n : v \in G/\Gamma_n\}$ is a clopen partition of G.
Let $\chi \in E_G$ and let $n \geq 0$ be such that $\chi(\gamma) = 1$ for all $\gamma \in \Gamma_n$. This means that χ is constant on each class of G/Γ_n, which implies that $f = \sum_{v\in G/\Gamma_n} \chi(v)1_{v.C_n}$ is a well defined continuous function that verifies $f(h.\mathbf{g}) = \chi(h)f(\mathbf{g})$ for all $\mathbf{g} \in \overleftarrow{G}$ and $h \in G$.
Let χ be an eigenvalue of $\overleftarrow{G}$ and let $f \in L^2_\lambda(\overleftarrow{G}) \setminus \{0\}$ be an associated eigenfunction. For $g \in G$ we have that

$$\chi(g)\left(\int_{C_n} f d\lambda\right) = \int_{g.C_n} f d\lambda.$$

Since $C_n = \gamma.C_n$ for all $\gamma \in \Gamma_n$, it holds that

$$\chi(g)\left(\int_{C_n} f d\lambda\right) = \int_{C_n} f d\lambda \quad \text{for all } g \in \Gamma_n. \tag{1}$$

Observe that

$$\mathbb{E}(f|\mathcal{P}_n) = \sum_{g \in K_n} \frac{\chi(g)}{\lambda(C_n)} \left(\int_{C_n} f d\lambda\right) 1_{g.C_n},$$

for a finite set $K_n \subset G$ containing at least one element of each class of G/Γ_n. Since $\mathcal{B}(\mathcal{P}_n) \uparrow \mathcal{B}(\overleftarrow{G})$, by the increasing Martingale Theorem, we have that $\mathbb{E}(f|\mathcal{P}_n)$ converges to f in $L^2_\lambda(\overleftarrow{G})$. Because $f \neq 0$, this implies there exists $m \geq 0$ such that $\int_{C_m} f d\lambda \neq 0$ and, by (1), we conclude that $\chi(\gamma) = 1$ for all $\gamma \in \Gamma_m$, which means that $\chi \in E_G$. □

3. Characterization of minimal almost 1-1 extensions of odometers

Let (X, G) and (Y, G) be two topological dynamical systems. (Y, G) is said to be the *maximal equicontinuos factor* of (X, G) if it is an equicontinuos factor of (X, G) such that for any other equicontinuous factor (Y', G) of (X, G) there exists a factor map $\pi : Y \to Y'$ that satisfies $\pi \circ f = f'$, with $f : X \to Y$ and $f' : X \to Y'$ factor maps.
It is well known that every topological dynamical system has a maximal equicontinuous factor and if (X, G) is a minimal almost 1-1 extension of a minimal equicontinuous system (Y, G), then (Y, G) is the maximal equicontinuous factor of (X, G) (for more details see [Au]).

3.1. **Regularly recurrent systems.** A subset S of G is said to be *syndetic* if there exists a compact subset K of G such that $G = K.S = \{k.s : s \in S, k \in K\}$. Because we consider a discrete group G, a subset S of G is syndetic if and only there exists a finite subset K of G such that $G = K.S$. It is important to note that a subgroup Γ of G is syndetic if and only if G/Γ is finite.
Let (X, G) be a topological dynamical system and let $x \in X$. The point x is *uniformly recurrent* if for every open neighborhood V of x the set $T_V(x)$ is syndetic. It is well known that $(\Omega_G(x), G)$ is minimal if and only if x is uniformly recurrent.
A point $x \in X$ is *regularly recurrent* if for every open neighborhood V of x there is a syndetic subgroup Γ of G such that $\Gamma \subseteq T_V(x)$. We say that a system is *regularly recurrent* if it is the orbit closure of a regularly recurrent point.
Similarly, we say that a point $x \in X$ is *strongly regularly recurrent* if for every open neighborhood V of x there is a closed neighborhood $W \subset V$ of x such that $T_W(x)$ is a syndetic normal subgroup of G. We say that a system is *strongly regularly recurrent* if it is the orbit closure of a strongly regularly recurrent point. Obviously, a strongly regularly recurrent point is a regularly recurrent point. Regularly recurrent systems are minimal.
The subodometers are examples of regularly recurrent systems, whose any point is regularly recurrent. In the same way, the odometers are strongly regularly recurrent systems whose any point is strongly regularly recurrent.
In this section, we will show that regularly recurrent systems are exactly the minimal almost 1-1 extensions of the subodometers and strongly regularly recurrent systems are the minimal almost 1-1 extensions of the odometers. From that we will conclude that a group G admits an action that is both strongly regularly recurrent and free if and only if G is residually finite.

Lemma 3. *Let (X, G) be a minimal topological dynamical system and let $x \in X$. If $\Gamma \subseteq G$ is a syndetic subgroup of G then $(\Omega_\Gamma(x), \Gamma)$ is minimal.*

Proof. Let H be a syndetic normal subgroup of G contained in Γ (Lemma 1). The group G acts by the natural product action on the compact space $X \times G/H$. Pick a minimal set M

in $X \times G/H$ for this action. The canonical projection on the first coordinate is a factor map that maps M onto a minimal subset of X hence onto X. Thus for every $x \in X$ there exists a point $(x, [a]) \in M$, where $[a]$ denotes the H-class of an element $a \in G$. By minimality of M, this point is uniformly recurrent. The right multiplication by $[a^{-1}]$ on the second axis is a conjugacy that sends the minimal set M onto a minimal set M' that contains $(x, [e])$. The canonical projection $G/H \to G/\Gamma$ induces a factor map from $X \times G/H$ onto $X \times G/\Gamma$ for the product action of G. So it maps M' onto a minimal set of $X \times G/\Gamma$ that contains the point $(x, [e]_\Gamma)$ where $[e]_\Gamma$ denotes the Γ-class of the neutral element e. This implies that for any neighborhood $V \subseteq X$ of x, the set $T_{V\times\{[e]_\Gamma\}}(x, [e]_\Gamma) = \{g \in G : g.x \in V,\, g \in \Gamma\}$ is syndetic. □

Lemma 4. *Let (X, G) be a topological dynamical system and let $x \in X$ be a regularly recurrent point. For every closed neighborhood V of x there exists a syndetic subgroup Γ of G such that $\Gamma \subseteq T_V(x)$ and $\{w(\Omega_\Gamma(x))\}_{w\in G/\Gamma}$ is a clopen partition of X.*
Moreover, if x is strongly regularly recurrent, the former group Γ is normal.

Proof. Let $V \subseteq X$ be a closed neighborhood of a regularly recurrent point x and let $\Gamma' \subseteq G$ be a subgroup with finite index such that $\Gamma' \subseteq T_V(x)$. Let us consider the normal subgroup $H \subset \Gamma'$ given by Lemma 1. By Lemma 3, the set $\Omega_H(x)$ is closed and minimal for the action of H. Since H is normal, for any $g \in G$, the set $g.\Omega_H(x)$, which equals $\Omega_H(g.x)$, is also closed, invariant and minimal for the H-action. Therefore if $w.\Omega_H(x) \cap u.\Omega_H(x) \neq \emptyset$ for $u, w \in G$, we have $w.\Omega_H(x) = u.\Omega_H(x)$.
Furthermore, if u and $w \in G$ are in the same H-class, then we have also $w.\Omega_H(x) = u.\Omega_H(x)$. Since H is syndetic and the G-orbit of x is dense, we have $X = \bigsqcup_{u\in K} u.\Omega_H(x)$, for some finite set $K \subset G$.
Let Γ be the group

$$\Gamma = \{g \in G :\ g.\Omega_H(x) = \Omega_H(x)\}.$$

We have $H \subset \Gamma$, so Γ is syndetic. Since $\Omega_\Gamma(x) = \Omega_H(x)$, we have $\Gamma \subset T_V(x)$ and for any $g \in G$ $g.\Omega_\Gamma(x)$ and $\Omega_\Gamma(x)$ are disjoint or equal because they are minimal closed H-invariant sets. Thus we get :

(1) $g.\Omega_\Gamma(x) = g'.\Omega_\Gamma(x)$ if and only if $g \in g'\Gamma$.
(2) $T_{g.\Omega_\Gamma(x)}(y) = g\Gamma g^{-1}$ for every $y \in g.\Omega_\Gamma(x)$.

It holds that for $w \in G/\Gamma$, $w.\Omega_\Gamma(x)$ is well defined and $\{w.\Omega_\Gamma(x)\}_{w\in G/\Gamma}$ is a clopen partition of X.

When x is a strongly regularly recurrent point of X, we follow the same proof with H being the normal subgroup $T_W(x)$ given by a clopen neighborhood $W \subset V$ of x. Due to this strong property, we have that the group Γ equals H and thus Γ is a normal subgroup of G. □

Corollary 1. *Let (X, G) be a topological dynamical system and let $x \in X$. The point x is regularly recurrent if and only if there exists a fundamental system $(C_i)_{i\geq 0}$ of clopen neighborhoods of x ($\cap_i C_i = \{x\}$), such that for all $y \in C_i$ the set of return times of y to C_i is a syndetic subgroup Γ_i of G, for every $i \geq 0$.*
Moreover, x is strongly regularly recurrent if and only if the groups Γ_i are normal.

Proof. If $x \in X$ has a fundamental system of neighborhoods as written above, it is a (resp. strongly) regularly recurrent point.

The sequences $(C_i)_i$ and $(\Gamma_i)_i$ are defined by induction. If x is a (resp. strongly) regularly recurrent point, let C_1 be the space X and Γ_1 be the group G.
So, given C_i and Γ_i, we take an open neighborhood V_{i+1} of x, whose the closure is strictly contained in C_i. By Lemma 4, we obtain a syndetic (resp. normal) group Γ_{i+1} with $\Gamma_{i+1} \subseteq$

$T_{\overline{V}_{i+1}}(x)$ and $\{w(\Omega_{\Gamma_{i+1}}(x))\}_{w\in G/\Gamma_{i+1}}$ is a clopen partition of X. Clearly, we have $\Gamma_{i+1}\subset\Gamma_i$. We set $C_{i+1}=\Omega_{\Gamma_{i+1}}(x)$ which is a clopen set with $T_{C_{i+1}}(y)=\Gamma_{i+1}$ for all $y\in\Gamma_{i+1}$.
Since $\lim_{i\to\infty}\operatorname{diam}(V_i)=0$, we obtain that $(C_i)_{i\geq 0}$ is a fundamental system of clopen neighborhoods of x. □

Theorem 2. *A minimal topological dynamical system (X,G) is an almost 1-1 extension of a subodometer $\overleftarrow{G}$ by π if and only if (X,G) is a regularly recurrent system.*
A minimal topological dynamical system (X,G) is an almost 1-1 extension of an odometer $\overleftarrow{G}$ by π if and only if (X,G) is a strongly regularly recurrent system.
Moreover, the set of regularly recurrent or strongly regularly recurrent points of X is exactly the pre-image of the set of points in $\overleftarrow{G}$ which have only one pre-image by π.

Proof. Let (X,G) be a minimal almost 1-1 extension of a subodometer $\overleftarrow{G}=\lim_{\leftarrow i}(G/\Gamma_i,\pi_i)$. Let $\pi: X\to\overleftarrow{G}$ be the almost 1-1 factor map and let $x\in X$ be such that $\{x\}=\pi^{-1}(\{\pi(x)\})$. Since π is continuous, if $\pi(x)=(a_i)_{i\geq 0}\in\overleftarrow{G}$ then $(\pi^{-1}([i;a_i]))_i$ is a decreasing sequence of clopen neighborhoods of x that satisfies

$$\bigcap_{i\geq 0}\pi^{-1}([i;a_i])=\{x\}.$$

We know that for every $\mathbf{g}\in[i;a_i]$, the set $T_{[i;a_i]}(\mathbf{g})$ is a group conjugated to Γ_i, therefore for all y in $\pi^{-1}([i;a_i])$, we have $T_{\pi^{-1}([i;a_i])}(y)$ is a group conjugated to Γ_i. So, by Corollary 1 we conclude that x is a regularly recurrent point of X. When $\overleftarrow{G}$ is an odometer, the groups Γ_i are normal and the point x is then a strongly regularly recurrent point of X.
Let (X,G) be a regularly recurrent system and let $x\in X$ be a regularly recurrent point. By Corollary 1 there exists a decreasing sequence $(C_i)_{i\geq 0}$ of clopen neighborhoods of x such that $\bigcap_{i\geq 0}C_i=\{x\}$, and there is a syndetic (resp. normal) subgroup Γ_i such that $T_{C_i}(y)=\Gamma_i$ for all $y\in C_i$, $i\geq 0$. Since $C_{i+1}\subseteq C_i$, we have that $\Gamma_{i+1}\subseteq\Gamma_i$, $i\geq 0$. So, we can define the subodometer $\overleftarrow{G}=\lim_{\leftarrow i}(G/\Gamma_i,\pi_i)$. We define $\pi: X\to\overleftarrow{G}$ by $\pi=(f_i)_{i\geq 0}$ where f_i is the continuous map $f_i: X\to G/\Gamma_i$ given by $f_i(y)=[z]$, where $[z]$ denotes the Γ_i-class of $z\in G$, if and only if $y\in z.C_i$ for $y\in X$, $z\in G$ and $i\geq 0$. The function π is a factor map, and, since $\bigcap_{i\geq 0}C_i=\{x\}$, we have that $\pi^{-1}\{\mathbf{e}\}=\{x\}$. So, π is an almost 1-1 extension. When x is strongly regularly recurrent, the groups Γ_i are normal and $\overleftarrow{G}$ is an odometer.
If $\pi': X\to\overleftarrow{G'}$ is another almost 1-1 factor map and $\overleftarrow{G'}$ an subodometer or an odometer, $\overleftarrow{G}$ and $\overleftarrow{G'}$ are maximal equicontinuous factors of (X,G), therefore, they are conjugate. Thus there exists a factor map $\pi'':\overleftarrow{G'}\to\overleftarrow{G}$ such that $\pi''\circ\pi'=\pi$, which implies that $\pi'^{-1}\{x\}=\pi^{-1}\{\pi''(x)\}$ for any x of $\overleftarrow{G'}$. We conclude that the set of regularly recurrent or strongly regularly recurrent points is exactly the pre-image of the points in $\overleftarrow{G}$ which have only one pre-image. □

By a straightforward application of Theorem 2 we get the following corollaries.

Corollary 2. *Every point of a system (X,G) is regularly recurrent if and only if (X,G) is conjugate to a subodometer.*
Similarly, every point of (X,G) is strongly regularly recurrent if and only if (X,G) is conjugate to an odometer.

Corollary 3. *A discrete finitely generated group G admits a strongly regularly recurrent free action on a compact metric space if and only if G is residually finite.*

Corollary 4. *Let* (X,G) *be a regularly recurrent system and let* $\overleftarrow{G}$ *be its maximal equicontinuous factor. The set of continuous eigenvalues of* X *is* E_G.

Proof. It is clear that E_G is contained in the set of continuous eigenvalues of X. Conversely, if χ is a continuous eigenvalue of X we can take $f : X \to \mathbb{S}^1$ an associated continuous eigenfunction which is a factor map between (X,G) and the dynamical system $(f(X),G)$, where the action of $g \in G$ on $\exp(2i\pi x) \in f(X)$ is given by $g.\exp(2i\pi x) = \chi(g)\exp(2i\pi x)$, which is an isometry. Thus the system $(f(X),G)$ is equicontinuous and therefore there exists a factor map $\pi : \overleftarrow{G} \to f(X)$. Since π is an eigenfunction associated to χ we conclude that $\chi \in E_G$. □

4. Regularly recurrent Cantor systems with G amenable.

We say that a topological dynamical system (X,G) is a *regularly recurrent Cantor system* if it is regularly recurrent and X is a Cantor set. In this section we suppose that (X,G) is a regularly recurrent Cantor system.

Proposition 4. *Let* (X,G) *be a regularly recurrent Cantor system. There exists a sequence*

$$(\mathcal{P}_n = \{w.C_{n,k} : w \in D_n,\, 1 \leq k \leq k_n\})_{n\geq 0},$$

of finite clopen partitions of X*, where* $D_n \subseteq G$ *and* $C_{n,k} \subseteq X$ *is a clopen set, satisfying, for every* $n \geq 0$*, the following:*

1. $C_{n+1} \subseteq C_n = \bigcup_{k=1}^{k_n} C_{n,k} \subset X$.
2. *There exists a syndetic subgroup* Γ_n *of* G *such that* D_n *is a subset of* G *containing exactly one representing element of each class in* G/Γ_n *and such that* $T_{C_n}(x) = \Gamma_n$*, for all* $x \in C_n$.
3. $\mathcal{P}_{n+1}$ *is finer than* $\mathcal{P}_n$.
4. *The family of sets* $\{\mathcal{P}_n,\ n \geq 0\}$ *spans the topology of* X.

Proof. The idea of the proof (the same as used in [HPS] and [Pu]) is to show that any minimal Cantor $\mathbb{Z}$-system has a nested sequence of clopen Kakutani-Rohlin partitions.
We recall the algorithm introduced in [Pu] to generate a Kakutani-Rohlin partition finer than a given one. Let $\mathcal{R}$ be a finite clopen partition of X. Suppose that

$$\mathcal{Q} = \{w.C_j : w \in D,\, 1 \leq j \leq k\},$$

is another finite clopen partition of X, and that there exists a syndetic subgroup Γ of G such that $D = \{w_1, \cdots, w_l\}$ is a subset of G containing exactly one representing element of each class in G/Γ, and that the set of return times of any point in $C = \bigcup_{j=1}^{k} C_j$ to C is equal to Γ. The following algorithm produces a partition $\mathcal{R} \wedge \mathcal{Q} = \{w.B_j : w \in D,\, 1 \leq j \leq d\}$ verifying

- $\mathcal{R} \wedge \mathcal{Q}$ is finer than $\mathcal{R}$ and $\mathcal{Q}$.
- $C = \bigcup_{j=1}^{d} B_j$

Step 1: let $1 \leq j \leq k$. Consider $A_{1,j,i_1}, \cdots, A_{1,j,i_{l_{1,j}}}$, the sets in $\mathcal{R}$ such that

$$w_1^{-1}.A_{1,j,i_s} \cap C_j \neq \emptyset, \text{ for every } 1 \leq s \leq l_{1,j}.$$

We denote by $B_{1,1}, \cdots, B_{1,k_1}$, with $k_1 = \sum_{j=1}^{k} l_{1,j}$, the elements of the collection

$$\{w_1^{-1}.A_{1,j,i_s} \cap C_j : 1 \leq s \leq l_{1,j}, 1 \leq j \leq k\}.$$

We have that $\mathcal{Q}_1 = \{w.B_{1,j} : w \in D,\, 1 \leq j \leq k_1\}$ is a clopen finite partition of X. In addition, for every $1 \leq i \leq k_1$ there exist $1 \leq j \leq k$ and $1 \leq s \leq l_{1,j}$ such that $w_1.B_{1,i} \subseteq A_{1,j,i_s}$, $w_1.B_{1,i} \subseteq w_1.C_j$ and $\bigcup_{s=1}^{l_j} B_{1,i} = C_j$. In other words, we have obtained a clopen partition $\mathcal{Q}_1 = \{w.B_{1,j} : w \in D,\, 1 \leq j \leq k_1\}$, satisfying

- For every $1 \leq j \leq k_1$, there exist A in $\mathcal{R}$ and B in $\mathcal{Q}$ such that $w_1.B_{1,j}$ is contained in $A \cap B$.
- $\bigcup_{j=1}^{k_1} B_{1,j} = C$.

Now, for $2 \leq n \leq l$, we suppose that the step $n-1$ has produced a finite clopen partition $\mathcal{Q}_{n-1} = \{w.B_{n-1,j} : w \in D,\, 1 \leq j \leq k_{n-1}\}$ such that

- For every $1 \leq j \leq k_{n-1}$ and every $1 \leq i \leq n-1$, there exists A in $\mathcal{R}$ and $B \in \mathcal{Q}$ such that $w_i.B_{n-1,j}$ is contained in $A \cap B$.
- $\bigcup_{j=1}^{k_{n-1}} B_{n-1,j} = C$.

Step n: let $1 \leq j \leq k_{n-1}$. Consider $A_{n,j,i_1}, \cdots, A_{n,j,i_{l_{n,j}}}$, the sets in $\mathcal{R}$ such that

$$w_n^{-1}.A_{n,j,i_s} \cap B_{n-1,j} \neq \emptyset, \text{ for every } 1 \leq s \leq l_{n,j}.$$

We denote by $B_{n,1}, \cdots, B_{n,k_n}$, with $k_n = \sum_{j=1}^{k} l_{n,j}$, the elements in the collection

$$\{w_n^{-1}.A_{j,i_s} \cap B_{n-1,j} : 1 \leq s \leq l_{n,j}, 1 \leq j \leq k_{n-1}\}.$$

We have $\mathcal{Q}_n = \{w.B_{n,j} : w \in D,\, 1 \leq j \leq k_n\}$ is a clopen finite partition of X. In addition, for every $1 \leq l \leq k_n$ there exist $1 \leq j \leq k_{n-1}$ and $1 \leq s \leq l_{n,j}$ such that $B_{n,l} \subseteq B_{n-1,j}$ and $B_{n,l} \subseteq w_n^{-1}.A_{n,j,s}$. This implies that for every $1 \leq i \leq n-1$, $w_i.B_{n,l} \subseteq w_i.B_{n-1,j}$ and by hypothesis, $w_i.B_{n,l}$ is contained in a subset A_i in $\mathcal{R}$. Since $\bigcup_{l=1}^{k_n} B_{n,l} = \bigcup_{i=1}^{k_{n-1}} B_{n-1,i}$, the partition $\mathcal{Q}_n$ satisfies

- For every $1 \leq j \leq k_n$ and $1 \leq i \leq n$, there exist A in $\mathcal{R}$ and B in $\mathcal{Q}$ such that $w_i.B_{n,j}$ is contained in $A \cap B$.
- $\bigcup_{j=1}^{k_n} B_{n,j} = C$.

This implies that after the step l, we obtain a partition

$$\mathcal{R} \wedge \mathcal{Q} = \mathcal{Q}_l = \{w.B_{l,j} : w \in D,\, 1 \leq j \leq k_l\},$$

which is finer than $\mathcal{R}$ and $\mathcal{Q}$, and which satisfies $\bigcup_{j=1}^{k_l} B_{n,j} = C$.

Now we use this algorithm to prove the Proposition 4 . From Corollary 1, there exists a decreasing sequence $(C_n)_{n\geq 0}$ of clopen subsets of X and a decreasing sequence $(\Gamma_n)_{n\geq 0}$ of syndetic subgroups of G such that $|\bigcap_{n\geq 0} C_n| = 1$ and $T_{C_n}(x) = \Gamma_n$ for all $x \in C_n$.
For every $n \geq 0$, we take a subset D_n of G containing exactly one representing element in each class of G/Γ_n, and we define

$$\mathcal{Q}_n = \{w.C_n : w \in D_n\}.$$

The collection $\mathcal{Q}_n$ is a finite clopen partition of X.
Since X is a Cantor set, it is always possible to take a sequence $(\mathcal{R}_n)_{n\geq 0}$ of finite clopen partitions of X which spans its topology.
We construct the desired sequence $(\mathcal{P}_n)_{n\geq 0}$ as follows:

- We set $\mathcal{P}_0 = \mathcal{R}_0 \wedge \mathcal{Q}_0$.
- For $n > 0$. First, we set $\mathcal{P}'_n = \mathcal{R}_n \wedge \mathcal{Q}_n$, and then $\mathcal{P}_n = \mathcal{P}_{n-1} \wedge \mathcal{P}'_n$.

From this construction we get

$$(\mathcal{P}_n = \{w.C_{n,j} : w \in D_n,\, 1 \leq j \leq k_n\})_{n\geq 0},$$

a sequence of finite clopen partition of X satisfying, for every $n \geq 0$:

(i) $\mathcal{P}_n$ is finer than $\mathcal{P}_{n-1}$ and $\mathcal{R}_n$.
(ii) $\bigcup_{j=1}^{k_n} C_{n,j} = C_n$.

The condition (i) implies $(\mathcal{P}_n)_{n\geq 0}$ is a nested sequence and that it spans the topology of X. The condition (ii) implies that this sequence verifies conditions 1. and 2. of Proposition 4. $\square$

Let us recall that a group G is amenable if and only if any continuous G-action on a compact metric space admits an invariant probability measure. We have the following characterization: The group G is amenable if and only if it has a Følner sequence, that is, a sequence $(F_n)_{n\geq 0}$ of finite subsets of G such that for every $g \in G$

$$\lim_{n\to\infty} \frac{|gF_n \bigtriangleup F_n|}{|F_n|} = 0.$$

Let (X, G) be a regularly recurrent Cantor system with G amenable, so this action admits an invariant probability measure. Consider the sequence of finite clopen partitions of X as in Proposition 4:

$$(\mathcal{P}_n = \{w.C_{n,k} : w \in D_n,\ 1 \leq k \leq k_n\})_{n\geq 0}.$$

Let $n \geq 0$. The incidence matrix between $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ is $A_n \in \mathcal{M}_{k_n\times k_{n+1}}(\mathbb{Z}^+)$ defined by

$$A_n(i, j) = |\{w \in D_{n+1} : w.C_{n+1,j} \subseteq C_{n,i}\}|.$$

Notice that $\sum_{i=1}^{k_n} A_n(i, j) = q_{n,j}$ is the number of $w \in D_{n+1}$ such that $w.C_{n+1,j} \subseteq C_n$. Since the set of return times of the points in C_n to C_n is equal to Γ_n, the number $q_{n,j}$ does not depend on j and it is equal to the number of $w \in D_{n+1}$ which are in Γ_n. So, $q_{n,j} = \frac{|D_{n+1}|}{|D_n|}$ for every $1 \leq j \leq k_{n+1}$. Consider the set

$$\triangle_n = \{(x_1, \cdots, x_{k_n}) \in (\mathbb{R}^+)^{k_n} : \sum_{i=1}^{k_n} x_i = \frac{1}{|D_n|}\}.$$

Since, for every $1 \leq j \leq k_{n+1}$, $\sum_{i=1}^{k_n} A_n(i, j) = \frac{|D_{n+1}|}{|D_n|}$, the map $A_n : \triangle_{n+1} \to \triangle_n$ is well defined by ordinary matrix multiplication.

Because $(\mathcal{P}_n)_{n\geq 0}$ is a countable collection of clopen sets that spans the topology of X, any invariant measure defined on this family of sets extends to a unique invariant measure on the Borel σ-algebra of X. So, any invariant measure μ on $(\mathcal{P}_n)_{n\geq 0}$ must verify

$$\mu(C_{n,i}) = \sum_{j=1}^{k_{n+1}} A_n(i, j)\mu(C_{n+1,j}),\ \text{for every } 1 \leq i \leq k_n \text{ and } n \geq 0\ ,$$

and it is completely determined by this relation. In other words, we can identify an invariant measure with an element in the inverse limit $\lim_{\leftarrow n}(\triangle_n, A_n)$. In the next Proposition we provide a sufficient condition for the reversed identification.

Remark 1. From [We], and more explicitely in [Kr], we have the following theorem.

Theorem 3 (Weiss)**.** *Let G be a numerable and amenable group and $(\Gamma_n)_{n\in\mathbb{N}}$ a nested sequence of normal subgroups s.t. $\bigcap_n \Gamma_n$ is trivial. Then there exist a Følner sequence $(D_n)_{n\in\mathbb{N}}$ of G and a subsequence $(\Gamma_{\varphi(n)})_n$ of $(\Gamma_n)_n$ s.t.:*

- *Each D_n contains exactly one representing element in each class of $G/\Gamma_{\varphi(n)}$*
- *$D_n \subset D_{n+1}$.*
- *$\bigcup_n D_n = G$.*

We deduce that in the case of an amenable group G with $\bigcap_n \Gamma_n$ trivial, up to take a subsequence, it is possible to take the sequence $(D_n)_{n\geq 0}$, defined as in Proposition 4, as a Følner sequence.

Theorem 4. *If G is amenable and the sequence $(D_n)_{n\geq 0}$ is Følner then $\mathcal{M}_G(X)$ is affinely-homeomorphic to $\lim_{\leftarrow n}(\triangle_n, A_n)$.*

Proof. Let $((x_{n,1},\cdots,x_{n,k_n}))_{n\geq 0}$ be an element in $\lim_{\leftarrow n}(\triangle_n, A_n)$. It defines a probability measure on X by setting

$$\mu(u.C_{n,i}) = x_{n,i}, \text{ for every } 1\leq i\leq k_n,\ u\in D_n \text{ and } n\geq 0.$$

To show this measure is invariant it is sufficient to show that for every $n\geq 0$, $1\leq k\leq k_n$ and $v\in G$, $\mu(v.C_{n,k}) = \mu(C_{n,k}) = x_{n,k}$.

Fix $v\in G$ and $m>n\geq 0$. Consider the sets

$$J(m,n,k,l) = \{w\in D_m : w.C_{m,l}\subseteq C_{n,k}\},$$

$J_1(m,n,k,l) = \{w\in J(m,n,k,l) : vw\in D_m\}$ and $J_2(m,n,k,l) = J(m,n,k,l)\setminus J_1(m,n,k,l)$. We have

$$v.C_{n,k} = \bigcup_{l=1}^{k_m}\bigcup_{w\in J(m,n,k,l)} vw.C_{m,l},$$

and then

$$\mu(v.C_{n,k}) = \sum_{l=1}^{k_m}\sum_{w\in J(m,n,k,l)}\mu(vw.C_{m,l}) = \sum_{l=1}^{k_m}\sum_{w\in J_1(m,n,k,l)}\mu(vw.C_{m,l}) + \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(vw.C_{m,l}).$$

Since $\mu(u.C_{m,l}) = \mu(C_{m,l})$ for $u\in D_m$, we get

$$\mu(v.C_{n,k}) = \sum_{l=1}^{k_m}|J_1(m,n,k,l)|\mu(C_{m,l}) + \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(vw.C_{m,l})$$

$$= \mu(C_{n,k}) - \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(C_{m,l}) + \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(vw.C_{m,l}).$$

Thus we have

$$|\mu(v.C_{n,k}) - \mu(C_{n,k})| \leq \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(C_{m,l}) + \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(vw.C_{m,l}).$$

Because $J_2(m,n,k,l)\subset\{w\in D_m : vw\notin D_m\}$, we have

$$|\mu(v.C_{n,k}) - \mu(C_{n,k})| \leq \sum_{\{w\in D_m : vw\notin D_m\}}\sum_{l=1}^{k_m}\mu(C_{m,l}) + \sum_{\{w\in D_m : vw\notin D_m\}}\sum_{l=1}^{k_m}\mu(vw.C_{m,l})$$

$$= \sum_{\{w\in D_m : vw\notin D_m\}}\mu\left(\bigcup_{l=1}^{k_m}C_{m,l}\right) + \sum_{\{w\in D_m : vw\notin D_m\}}\mu\left(\bigcup_{l=1}^{k_m}vw.C_{m,l}\right).$$

Since $|\{w\in D_m : vw\notin D_m\}|\leq |v.D_m\triangle D_m|$ and $\mu\left(\bigcup_{l=1}^{k_m}C_{m,l}\right) = \mu\left(\bigcup_{l=1}^{k_m}vw.C_{m,l}\right) = \frac{1}{|D_m|}$, we have

$$|\mu(v.C_{n,k}) - \mu(C_{n,k})| \leq \frac{2|v.D_m\triangle D_m|}{|D_m|}.$$

So, because $(D_n)_{n\geq 0}$ is Følner, we get $\mu(v.C_{n,k}) = \mu(C_{n,k})$.

□

5. Semicocycles

The notion of a semicocycle has been extensively used in the theory of one-dimensional Toeplitz flows (see [Do]). In this section it is not used but we develop it for actions of a residually finite discrete group G for further utility.
Recall that for a residually finite group G and a decreasing sequence $(\Gamma_i)_{i\geq 0}$ of syndetic subgroups of G with $\bigcap_{i\geq 0}\Gamma_i = \{e\}$, the stabilizer of $\mathbf{e} = (e_i)_{i\geq 0}$ in the free G-subodometer $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ is trivial. This defines an immersion τ of G into $\overleftarrow{G}$.

Definition 3. *Let $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ be a G-subodometer with $\bigcap_{i\geq 0}\Gamma_i = \{e\}$ and let K be a compact metric space. A function $f : G \to K$ is a* semicocycle on $\overleftarrow{G}$ *if it is continuous with respect $\Theta_{\overleftarrow{G}}$, where $\Theta_{\overleftarrow{G}}$ is the topology on G inherited from $\overleftarrow{G}$ (we identify $\tau(G)$ with G).*

The functions $f : G \to K$ may be seen as elements of the topological dynamical system (K^G, G), where K^G is endowed with the metrizable product topology, and the left-action of $\gamma \in G$ on $f = (f(g))_{g\in G} \in K^G$ is the shift action, defined by $\gamma.f \in K^G$, where $\gamma.f(g) = f(g\gamma)$ for every $g \in G$.
The proofs of Theorems 5 and 6 below follow the same ideas as used in [Do] for $G = \mathbb{Z}$.

Theorem 5. *If $f \in K^G$ is a semicocycle on some subodometer $\overleftarrow{G}$ then f is a regularly recurrent point of (K^G, G).*

Proof. Fix $\epsilon > 0$ and a finite set C in G. The pair (ϵ, C) determines a basic open set V in the Tychonov topology. Since f is continuous on G for the topology induced by the odometer $\overleftarrow{G}$, there exists $\delta > 0$ such that for every $g \in C$ and $g' \in G$, $\mathrm{dist}(g, g') < \delta$ (for the metric inherited from $\overleftarrow{G}$) implies $\mathrm{d}(f(g), f(g')) < \epsilon$ in K. By definition of a subodometer, there exist a finite index subgroup Γ of G and a factor map $\pi : \overleftarrow{G} \to G/\Gamma$ such that for any element w of G/Γ, $\pi^{-1}(w)$ is a clopen subset of $\overleftarrow{G}$ with diameter smaller than δ. Furthermore, for any $y \in \pi^{-1}(w)$, $T_{\pi^{-1}(w)}(y)$ is a group conjugated to Γ. Let us consider now the finite index normal subgroup $H = \cap_{g\in G} g\Gamma g^{-1}$. Since Γ is of finite index in G, there is just a finite number of groups conjugated to Γ and the former intersection is a finite intersection. The group H is a subgroup of any group of the kind $T_{\pi^{-1}(w)}(y)$ with $w \in G/\Gamma, y \in \pi^{-1}(w)$. Thus, $\mathrm{dist}(n'.g, n') < \delta$ for any $g \in H$, $n' \in G$, by the normality of H. Hence $d(f(n'.g), f(n')) < \epsilon$ for any $g \in H$ and $n' \in G$. We have proved that the H-orbit of f is contained in V and then f is a regularly recurrent point of K^G. □

Proposition 2 and Theorem 5 imply that $(\Omega_G(f), G)$ is a minimal almost 1-1 extension of some free subodometer, where $\Omega_G(f)$ represents the orbit closure of a semicocycle f in K^G with a trivial stabilizer under the action of G. Notice that $\overleftarrow{G}$ needs not to be the maximal equicontinuous factor of $(\Omega_G(f), G)$, as we will see later.

Let $f \in K^G$ be a semicocycle on a G-subodometer $\overleftarrow{G}$. Since we have identified the group G with G embedded in $\overleftarrow{G}$, it makes sense to define F to be the closure of the graph of f in $\overleftarrow{G} \times K$ endowed with the product topology, $F = \overline{\{(g, f(g)) : g \in G\}} \subseteq \overleftarrow{G} \times K$. Let $F(\mathbf{g})$ be the set $\{k \in K : (\mathbf{g}, k) \in F\}$ for $\mathbf{g} \in \overleftarrow{G}$.
We call C_f the set of $\mathbf{g} \in \overleftarrow{G}$ such that $|F(\mathbf{g})| = 1$ and $D_f = \overleftarrow{G} \setminus C_f$. Since f is continuous we have that $F(g) = \{g\}$ for all $g \in G$. Thus C_f is the subset where f can be continuously extended by $f(\mathbf{g}) = F(\mathbf{g})$.

The semicocycle f is said to be *invariant under no rotation* if for every $\mathbf{h}_1 \neq \mathbf{h}_2 \in \overleftarrow{G}$ there exists a $g \in G$ such that $F(g.\mathbf{h}_1) \neq F(g.\mathbf{h}_2)$.

Theorem 6. *Let (X, G) be a minimal topological dynamical system and $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ be a G-subodometer with $\bigcap_{i\geq 0}\Gamma_i = \{e\}$. There exists an almost 1-1 factor π of (X, G) onto $(\overleftarrow{G}, G)$ with $|\pi^{-1}(\mathbf{e})| = 1$ if and only if (X, G) is conjugated to $(\Omega_G(f), G)$, where f is a semicocycle on $\overleftarrow{G}$, invariant under no rotation.*

Proof. Consider the system $(\Omega_G(f), G)$. By definition, for every $x \in \Omega_G(f)$, there exists a sequence $(g_i)_i \subset G$ such that for each $h \in G$, $\lim_i f(hg_i) = x(h)$. Let $\mathbf{j} \in \overleftarrow{G}$ be an accumulation point of the sequence $(g_i)_i$. We have $x(h) \in F(h.\mathbf{j})$. By a straightforward calculation, we check that for each such $\mathbf{j}$, the set $\{(h.\mathbf{j}, x(h))|\ h \in G\}$ is a dense subset of F. Since f is invariant under no rotation, $\mathbf{j}$ is determined for any x in an unique way. So we have proved that if $g_i.f \to x$ then $g_i \to \mathbf{j}$. The map $\pi : x \in \Omega_G(f) \mapsto \mathbf{j} \in \overleftarrow{G}$ is a continuous extension onto $\Omega_G(f)$ of the application $g.f \mapsto g$. It is straightforward to check that π is a factor map that sends f to $\mathbf{e}$. If $\pi(x) = \mathbf{e}$ then $x(h) \in F(h) = \{f(h)\}$ and $x(h) = f(h)$ for any $h \in G$. Since the system $(\Omega_G(f), G)$ is minimal, π is an almost 1 to 1 factor map.
Conversely, consider a minimal almost 1-1 extension (X, G) of a G-subodometer and $\pi : X \to \overleftarrow{G}$ the associated factor map. Consider $x \in X$ such that $\pi(x)$ has a singleton fiber by π. The same holds for all the elements of its G-orbit. The map $f : g \in G \mapsto \pi^{-1}(g.\pi(x)) = g.\pi^{-1}(x) \in X$ is continuous for the induced topology on G, it is then a semicocycle. This is straightforward to check that $F(\mathbf{j}) = \pi^{-1}(\mathbf{k})$ where $\mathbf{k} \in \overleftarrow{G}$ is the limit point of the sequence $(g_i.\pi(x))_i$ with (g_i) a sequence of G that converges to $\mathbf{j}$. The set $\pi^{-1}(\mathbf{k})$ does not depend of the choice of the sequence (g_i). It is then straightforward to show that f is invariant under no rotation. The conjugating map from $(\Omega_G(f), G)$ onto (X, G) is the projection onto the neutral element coordinate: $\phi \mapsto \phi(\mathbf{e})$. By a standard way, we check this application is a homeomorphism which commutes with the G-action. □

Corollary 5. *A topological dynamical system (X, G) is a minimal almost 1-1 extension of a free odometer $(\overleftarrow{G}, G)$ if and only if it is conjugated to $(\Omega_G(f), G)$, where f is a semicocycle on G, invariant under no rotation.*

Proof. For a factor map $p : X \to \overleftarrow{G}$ and any point $x \in X$, by a right multiplication by $p(x)^{-1}$, we obtain again a factor map that sends the point x to $\mathbf{e}$. The result follows from Theorem 6. □

6. G-Toeplitz Arrays

In this section we assume that G is a discrete finitely generated group. For a finite alphabet Σ equipped with the discrete topology, we consider the left action of G on Σ^G, continuous with respect to the product topology, defined for $x = (x(g))_{g\in G} \in \Sigma^G$ and $\gamma \in G$ by :

$$\gamma.x(g) = x(g\gamma) \text{ for any } g \in G.$$

Recall that we denote by $\Omega_G(x)$ the closure of the G-orbit of x for this action. In this section, we consider regularly recurrent systems of the kind $(\omega_G(x), G)$.

For a syndetic group $\Gamma \subseteq G$ and $x = (x(g))_{g\in G} \in \Sigma^G$ we define:

$$Per(x, \Gamma, \sigma) = \{g \in G : x(g\gamma) = \sigma \text{ for all } \gamma \in \Gamma\}, \quad \sigma \in \Sigma,$$

$$Per(x, \Gamma) = \bigcup_{\sigma\in\Sigma} Per(x, \Gamma, \sigma).$$

Clearly for two subgroups Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2$, we have $Per(x, \Gamma_2, \sigma) \subset Per(x, \Gamma_1, \sigma)$. When $Per(x, \Gamma) \neq \emptyset$ we say that Γ is a *group of periods of* x. Furthermore, $Per(x, \Gamma)$ is invariant under the right multiplication by an element of Γ. We say that x is a *G-Toeplitz array* (or simply a Toeplitz array) if for all $g \in G$ there exists a syndetic subgroup $\Gamma \subseteq G$ such that $g \in Per(x, \Gamma)$.

Proposition 5. *The following statements concerning $x \in \Sigma^G$ are equivalent:*

(1) *x is a Toeplitz array.*
(2) *There exists a sequence of syndetic subgroups $(\Gamma_n)_{n\geq 0}$, such that $\Gamma_{n+1} \subset \Gamma_n$ for all $n \geq 0$ and $G = \cup_n Per(x, \Gamma_n)$.*
(3) *x is regularly recurrent.*

Proof. Let D_n be the ball of radius n in G centered at the neutral element.
Suppose that x is a Toeplitz array. Since for any two groups Z_1, Z_2 of periods of x we have $Per(x, Z_1) \subset Per(x, Z_1 \cap Z_2)$, for any $n \geq 0$, there exists a syndetic subgroup Z_n such that $D_n \subset Per(x, Z_n)$. Let $\Gamma_0 = Z_0$ and $\Gamma_{n+1} = \Gamma_n \cap Z_n$. The sequence $(\Gamma_n)_n$ satisfies the statement (2).
Let $(\Gamma_n)_n$ be a sequence as in statement (2). Let C_n be the set $\{y \in \Sigma^G : y(D_n) = x(D_n)\}$ for all $n \geq 0$, $(C_n)_{n\geq 0}$ is a fundamental system of clopen neighborhoods of x. Since D_n is contained in $Per(x, \Gamma_n)$, the set of return times of x to C_n contains Γ_n which implies that x is regularly recurrent.
Suppose that x is regularly recurrent. For $n \geq 0$ we take Γ_n a syndetic subgroup of G such that $\Gamma_n \subseteq T_{C_n}(x)$. It holds that G is equal to $\bigcup_{n\geq 0} Per(x, \Gamma_n)$, which means that x is a Toeplitz array. □

A subshift (X, G) is a *G-Toeplitz system* (or simply a Toeplitz system) if there exists a Toeplitz array x such that $X = \Omega_G(x)$. From Theorem 2 and Proposition 5 we conclude that the family of minimal subshifts which are almost 1-1 extensions of subodometers coincides with the family of Toeplitz systems.
In order to know the maximal equicontinuous factor of a given Toeplitz system, we will introduce the concepts of essential group of periods and period structure.

Definition 4. *Let $x \in \Sigma^G$. A syndetic group $\Gamma \subset G$ is called an* essential group of periods of x *if $Per(x, \Gamma, \sigma) \subseteq Per(g.x, \Gamma, \sigma)$ for every $\sigma \in \Sigma$ implies that $g \in \Gamma$.*

Lemma 5. *If Γ is an essential group of periods of x then every group of periods Γ' satisfying $Per(x, \Gamma) \subseteq Per(x, \Gamma')$ is contained in Γ.*

Proof. Let Γ be an essential group of periods of x. Suppose that Γ' is a group of periods such that $Per(x, \Gamma) \subseteq Per(x, \Gamma')$. For $w \in Per(x, \Gamma, \sigma)$ and $g \in \Gamma'$ we have $w\gamma g \in Per(x, \Gamma', \sigma)$ for every $\gamma \in \Gamma$. This implies that $x(w\gamma g) = g.x(w\gamma) = \sigma$ for every $\gamma \in \Gamma$, which means that $w \in Per(g.x, \Gamma', \sigma)$. Because Γ is essential, we conclude that $g \in \Gamma$ and then $\Gamma' \subseteq \Gamma$. □

Remark 2. From Lemma 5 we deduce that the family of the essential groups of periods is contained in the family of the groups generated by essential periods introduced in [Co] for the case $G = \mathbb{Z}^d$.
Notice that for any $x \in \Sigma^G$, $g \in G$ and any group $\Gamma \subset G$, we have the relation $Per(g.x, \Gamma, \sigma) = Per(x, g^{-1}\Gamma g, \sigma)g^{-1}$ for any $\sigma \in \Sigma$. This relation will be useful in the following to characterize the essential groups of periods.

In the following Lemma we show the existence of essential groups of periods.

Lemma 6. *Let $x \in \Sigma^G$. If $\Gamma \subseteq G$ is a group of periods of x then there exists an essential group $K \subseteq G$ of periods of x such that $Per(x, \Gamma) \subseteq Per(x, K)$.*

Proof. Let $\Gamma \subseteq G$ be a group of periods of x and Γ' be a syndetic normal subgroup of Γ. We denote by $\hat{\Gamma}'$ the collection of shifted groups:

$$\bigcup_{g\in G}\{Hg : H \text{ syndetic subgroup of } G \text{ such that } Per(x,\Gamma',\sigma) \subseteq Per(x, g^{-1}Hg, \sigma)g^{-1}, \forall \sigma \in \Sigma\}.$$

Let K be the group generated by the elements of the union of all sets in $\hat{\Gamma}'$. Let $w \in Per(x,\Gamma',\sigma)$. For any $\gamma \in \Gamma'$ and any $Hg \in \hat{\Gamma}'$, $w\gamma$ belongs to $Per(x,\Gamma',\sigma) \subseteq Per(x,g^{-1}Hg,\sigma)g^{-1}$. This implies that for every $hg \in Hg \in \hat{\Gamma}'$ we have $w\gamma hg \in Per(x, g^{-1}Hg, \sigma)$. Since Γ' is a normal subgroup, we get for any $\gamma \in \Gamma'$ and any $hg \in Hg \in \hat{\Gamma}'$, $x(whg\gamma) = \sigma$, which means that $whg \in Per(x,\Gamma',\sigma)$. Thus we obtain that for any $h_1g_1,\dots,h_ng_n$ with h_ig_i belonging to a set in $\hat{\Gamma}'$ and $w \in Per(x,\Gamma',\sigma)$, we have $x(wh_1g_1\dots h_ng_n) = \sigma$. In other words, $Per(x,\Gamma',\sigma)$ is contained in $Per(x,K,\sigma)$. So, we have $Per(x,\Gamma,\sigma) \subseteq Per(x,\Gamma',\sigma) \subseteq Per(x,K,\sigma)$. If $g \in G$ is such that $Per(x,K,\sigma) \subseteq Per(g.x,K,\sigma) = Per(x,g^{-1}Kg,\sigma)g^{-1}, \forall \sigma \in \Sigma$, then Kg belongs to $\hat{\Gamma}'$, which implies that g is in K. □

Corollary 6. *Let $x \in \Sigma^G$ be a Toeplitz array. There exists a sequence $(\Gamma_n)_{n\geq 0}$ of essential group of periods of x such that $\Gamma_{n+1} \subseteq \Gamma_n$ and $\bigcup_{n\geq 0} Per(x,\Gamma_n) = G$.*

Proof. From Proposition 5 (2) we conclude there exists a decreasing sequence $(\Gamma'_n)_{n\geq 0}$ of syndetic groups of periods of x such that $\bigcup_{n\geq 0} Per(x,\Gamma'_n) = G$. We set Γ_0 an essential group of periods of x such that $Per(x,\Gamma'_0) \subseteq Per(x,\Gamma_0)$. For $n > 0$ we set $\Gamma''_n = \Gamma'_n \cap \Gamma_{n-1}$ which is a syndetic subgroup of G, and since $Per(x,\Gamma_{n-1})$ and $Per(x,\Gamma'_n)$ are contained in $Per(x,\Gamma''_n)$, Γ''_n is a group of periods of x. Thus, by Lemma 6, there exists an essential group of periods Γ_n, such that $Per(x,\Gamma_{n-1}) \subseteq Per(x,\Gamma''_n) \subseteq Per(x,\Gamma_n)$. Since Γ_{n-1} is an essential group of periods, from Lemma 5 we get $\Gamma_n \subseteq \Gamma_{n-1}$. Because $\bigcup_{n\geq 0} Per(x,\Gamma'_n) = G$, we deduce $\bigcup_{n\geq 0} Per(x,\Gamma_n) = G$. □

Definition 5. *A sequence of groups as in Corollary 6 is called a* period structure *of x.*

In the sequel, we will show that from a period structure $(\Gamma_n)_{n\geq 0}$ of a G-Toeplitz array x it is possible to construct a sequence of nested finite clopen partitions of $\Omega_G(x)$. From this sequence of partitions it will be easy to define an almost 1-1 factor map between the Toeplitz system $(\Omega_G(x), G)$ and the odometer $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$.
Let $x \in \Sigma^G$ be a Toeplitz array, let $y \in \Omega_G(x)$ and let $\Gamma \subseteq G$ be group of periods of y. We define the set:

$$C_\Gamma(y) = \{x' \in \Omega_G(x) : Per(x',\Gamma,\sigma) = Per(y,\Gamma,\sigma),\ \forall\ \sigma \in \Sigma\}.$$

Lemma 7. *$C_\Gamma(y) = \gamma.C_\Gamma(y)$ for every $\gamma \in \Gamma$.*
For every $x' \in C_\Gamma(y)$, we have $\Omega_\Gamma(x') \subseteq C_\Gamma(y)$.

Proof. Let $x' \in \gamma.C_\Gamma(y)$. There exists $x'' \in C_\Gamma(y)$ such that $x' = \gamma.x''$. If $g \in Per(x',\Gamma,\sigma)$ then $x'(g\gamma') = \sigma$ for every $\gamma' \in \Gamma$. In particular, we have

$$\sigma = x'(g\gamma'\gamma^{-1}) = \gamma^{-1}.x'(g\gamma') = x''(g\gamma'),\ \forall \gamma' \in \Gamma,$$

which implies $Per(x',\Gamma,\sigma) \subseteq Per(x'',\Gamma,\sigma) = Per(y,\Gamma,\sigma)$. On the other hand, if $g \in Per(x'',\Gamma,\sigma)$ then

$$\sigma = x''(g\gamma') = x''(g\gamma'\gamma) = \gamma.x''(g\gamma') = x'(g\gamma'),\ \forall \gamma' \in \Gamma,$$

which implies that $Per(y,\Gamma,\sigma) \subseteq Per(x',\Gamma,\sigma)$. Thus we obtain that $\gamma.C_\Gamma(y) \subseteq C_\Gamma(y)$. Since this is true also for γ^{-1}, we conclude that $\gamma.C_\Gamma(y) = C_\Gamma(y)$.

To show the second point, let us first consider $x' \in \Omega_\Gamma(y)$. It is straightforward to show that for any σ, $Per(y,\Gamma,\sigma) \subseteq Per(x,\Gamma,\sigma)$. Since $\Omega_\Gamma(y)$ is a minimal Γ-invariant closed set (Lemma 3), y belongs to $\Omega_\Gamma(x')$ and therefore $Per(x',\Gamma,\sigma) = Per(y,\Gamma,\sigma)$. So we have $\Omega_\Gamma(y) \subseteq C_\Gamma(y)$. To conclude, notice that $C_\Gamma(y) = C_\Gamma(x')$ for any $x' \in C_\Gamma(y)$, so we get $\Omega_\Gamma(x') \subseteq C_\Gamma(x')$ for any $x' \in C_\Gamma(y)$. □

We will use the following convention: for a Γ-periodic subset C of $\Omega_G(x)$, i.e. such that $w.C = w'.C$ whenever $w^{-1}w' \in \Gamma$, we will write $v.C$ instead of $w.C$, where v is the projection of w to G/Γ.

Proposition 6. *Let $x \in \Sigma^G$ be a Toeplitz array and let $y \in \Omega_G(x)$. If $\Gamma \subseteq G$ is an essential group of periods of y then $\Omega_\Gamma(y) = C_\Gamma(y)$ and $\{w.C_\Gamma(y)\}_{w\in G/\Gamma}$ is a clopen partition of $\Omega_G(x)$.*

Proof. By Lemma 7, we have $\Gamma \subseteq T_{C_\Gamma}(x')$ for every $x' \in C_\Gamma(y)$. In the sequel, we will show that for an essential group of periods Γ, we have $T_{C_\Gamma(y)}(x') = \Gamma$ for every $x' \in C_\Gamma(y)$.

Suppose that $g \in G$ satisfies $g.y \in C_\Gamma(y)$. This implies $Per(g.y,\Gamma,\sigma) = Per(y,\Gamma,\sigma)$ for every $\sigma \in \Sigma$. Since Γ is an essential group of periods of y, we obtain $g \in \Gamma$ and $T_{C_\Gamma(y)}(y) = \Gamma$. By Lemma 3, for any $x' \in \Omega_G(x)$, the set $\Omega_\Gamma(x')$ is a minimal Γ-invariant set, hence by syndicity of Γ and by minimality of the G-action, $\Omega_G(x)$ is a finite and disjoint union of minimal Γ-invariant sets. So the sets $\Omega_\Gamma(x')$ are clopen sets and by Lemma 7, $C_\Gamma(y)$ is a finite union of clopen sets. By minimality of the G-action, it is straightforward to check that $T_{C_\Gamma(y)}(x') = \Gamma$ for every $x' \in C_\Gamma(y)$. Thus we get that $\{w.C_\Gamma(y)\}_{w\in G/\Gamma}$ is a collection of disjoint sets. Moreover, this collection is a partition of $\Omega_G(x)$ because $w.\Omega_\Gamma(y) \subseteq w.C_\Gamma(y)$ for every $w \in G/\Gamma$ and $\{w.\Omega_\Gamma(y)\}_{w\in G/\Gamma}$ is a covering of $\Omega_G(x)$. This also implies that $\Omega_\Gamma(x) = C_\Gamma(x)$. □

Proposition 7. *Let $x \in \Sigma^G$ be a Toeplitz array. If $(\Gamma_n)_{n\geq 0}$ is a period structure of x then the subodometer $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ is the maximal equicontinuous factor of $(\Omega_G(x), G)$.*

Proof. By Proposition 6, if $(\Gamma_n)_{\geq 0}$ is period structure of the Toeplitz array x, then $(\{C_{g.\Gamma_n}(x) : g \in G/\Gamma_n\})_{n\geq 0}$ is a sequence of nested clopen partitions of $\Omega_G(x)$. This implies that the function $f_n : \Omega_G(x) \to G/\Gamma_n$ given by $f_n(y) = g$ if and only if $y \in g.C_{\Gamma_n}(x)$ is a well defined continuous function, $y \in \Omega_G(x)$, $n \geq 0$. The function $\pi : \Omega_G(x) \to \overleftarrow{G}$ given by $\pi = (f_n)_{n\geq 0}$ is a factor map. Since, by definition $\bigcap_{n\geq 0} C_{\Gamma_n}(x) = \{x\}$, we have that $\pi^{-1}\{\mathbf{e}\} = \{x\}$ and then π is an almost 1-1 factor map. □

Theorem 7. *For every subodometer $\overleftarrow{G}$ there exists a Toeplitz array $x \in \{0,1\}^G$ such that $\overleftarrow{G}$ is the maximal equicontinuous factor of $(\Omega_G(x), G)$.*

Proof. Let $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ be a subodometer with $\Gamma_0 = G$. We distinguish two cases:
Case 1. There exists $m \geq 0$ such that $\Gamma_n = \Gamma_m$ for all $n \geq m$. In this case $\overleftarrow{G}$ is the finite group G/Γ_m and then every minimal almost 1-1 extension will be conjugate to $\overleftarrow{G}$. For example, $x \in \{0,1\}^G$ defined by $x(v) = 0$ for all $v \in \Gamma_m$ and $x(v) = 1$ if not, provides a Toeplitz sequence x such that $\overleftarrow{G}$ is the maximal equicontinuous factor of the system associated to x.

Case 2. For every $m \geq 0$ there exists $n > m$ such that $\Gamma_n \neq \Gamma_m$. In this case we can take a subsequence $(\Gamma_n)_{n\geq 0}$ such that $\Gamma_{n+1} \neq \Gamma_n$ and $[\Gamma_n : \Gamma_{n+1}] \geq 2$ for all $n \geq 0$. By Proposition 2, $\overleftarrow{G}$ is conjugate to the subodometer obtained from this sequence. In order to construct the Toeplitz array x we will consider a sequence $(D_n)_{n\geq 0}$ of compact subsets of G such that:

- for each n, D_n is a fundamental domain of Γ_n (i.e. D_n contains an unique element of each class of G/Γ_n). The set D_0 is the singelton set $\{e\}$.
- For each n, $D_n \subset D_{n+1}$ and $D_{n+1} = \bigsqcup_{k\in K_n} D_n.k$ for some finite set $K_n \subset G$ containing the neutral element e of G. By assumption, the cardinal of K_n is bigger than 2.

- $\bigcup_{g\in\cap_n\Gamma_n}\bigcup_{n\geq 0} D_n.g = G$.

We define now a sequence of subsets of G $(S_n)_{n\geq 0}$ by induction. Let S_0 be the singleton $\{e\}$. Let v_1 be an element of D_1 distinct from e and let $S_1=\{v_1\}$. For $n>1$, let S_n be the set $v_{n-1}.\Gamma_{n-1}\cap(D_n\setminus D_{n-1})$ and let v_n be a point in S_n. We define then $x\in\{0,1\}^G$ by :

$$x(w)=\begin{cases} 0 & \text{if } w \text{ belongs to } \cup_{n\geq 0}S_{2n}.\Gamma_{2n+1}, \\ 1 & \text{else.}\end{cases} \tag{2}$$

Notice that $x(w)=1$ for the element w of $\cup_{n\geq 0}S_{2n+1}.\Gamma_{2n+2}$. Since $\bigcup_{j\in\{j,\ 0\leq 2j+1\leq n\}} S_{2j}\Gamma_{2j+1}$ $\subseteq Per(x,\Gamma_n,0)$ and $(D_{n-1}\setminus \bigcup_{j\in\{j,\ 0\leq 2j+1\leq n\}} S_{2j}\Gamma_{2j+1})\subseteq Per(x,\Gamma_n,1)$ for any $n\geq 1$, it holds that $D_{n-1}\subseteq Per(x,\Gamma_n)$, and for any $g\in\bigcap_n\Gamma_n$, we have also $D_{n-1}.g\subseteq Per(x,\Gamma_n)$. Thus, we get $G=\bigcup_{n\geq 0}Per(x,\Gamma_n)$ and x is a Toeplitz array. To conclude that $\overleftarrow{G}$ is the maximal equicontinuous factor of the system associated to x, by Proposition 7, it is enough to show that $(\Gamma_n)_{n\geq 0}$ is a period structure of x.

Let us prove by induction on n that Γ_n is an essential group of periods of x. For $n=0$, $\Gamma_0=G$ and this is obviously true. Suppose now that $n>0$ and that Γ_{n-1} is an essential group of periods. Let $g\in G$ be such that $Per(x,\Gamma_n,\sigma)\subset Per(g.x,\Gamma_n,\sigma)$, for all σ of $\{0,1\}$. Since $\Gamma_n\subset\Gamma_{n-1}$, we have $Per(x,\Gamma_{n-1},\sigma)\subset Per(x,\Gamma_n,\sigma)$. Let γ_{n-1} in Γ_{n-1}, there exist $\gamma\in D_n$ and $\gamma_n\in\Gamma_n$ such that $\gamma_{n-1}=\gamma\gamma_n$. For w in $Per(x,\Gamma_{n-1},\sigma)$, we have $w\gamma_{n-1}\gamma_n^{-1}=w\gamma$ belongs to $Per(x,\Gamma_{n-1},\sigma)\subset Per(g.x,\Gamma_n,\sigma)$. So we have $\sigma=g.x(w\gamma)=g.x(w\gamma.\gamma_n)=g.x(w.\gamma_{n-1})$ and therefore $w\in Per(g.x,\Gamma_{n-1},\sigma)$ for all $w\in Per(x,\Gamma_{n-1},\sigma)$. By the hypothesis of induction we get that g belongs to Γ_{n-1}.
By the definition of x, the element v_{n-1} belongs to $Per(x,\Gamma_n,\sigma)$ with $\sigma=x(v_{n-1})$, so $x(v_{n-1}.g)=\sigma$. Since $g\in\Gamma_{n-1}$ and by the construction of x, g belongs to Γ_n and so Γ_n is an essential group of periods of x. □

Remark 3. It is interesting to note that when $\overleftarrow{G}$ is a free odometer, the action of G on $\overleftarrow{G}$ is free and minimal. The G-Toeplitz array x, constructed as above, is such that $(\Omega_G(x),G)$ is an almost 1-1 extension of the system $(\overleftarrow{G},G)$, so the action of G on $\Omega_G(x)$ is also free and minimal. All the elements of $\Omega_G(x)$ are not stable for the G-action.

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