C^*-ALGEBRAS OF PENROSE HYPERBOLIC TILINGS

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Abstract. For a quasi-periodic tiling of the hyperbolic plane made with a finite number of tiles, up to affine transformation, the C^*-algebra of the associated hull has no trace. We give here a complete description of the C^*-algebras and the K-theory for a family of such tilings.

1. Introduction

The non-commutative geometry of a quasi-periodic tiling studies an appropriate C^*-algebra of a dynamical system (X, G), for a compact metric space X, called *the Hull*, endowed with a continuous Lie group G action. This C^*-algebra is of relevance to study the space of leaves which is pathological in any topological sense. The Hull owns also a geometrical structure of *Lamination* or *foliated space*, the transverse structure being just metric [8]. The C^*-algebras and the non-commutative tools provide then topological and geometrical invariants for the tiling or the lamination. Moreover, some K-theoretical invariants of Euclidean tilings have a physical interpretation. In particular, when the tiling represents a quasi-crystal, the image of the K-theory under the canonical trace labels the gaps in the spectrum of the Schrödinger operator associated with the quasi-crystal [2].

For an Euclidean tiling, the group G is $\mathbb{R}^d$ and $\mathbb{R}^d$-invariant ergodic probability measures on the Hull are in one-to-one correspondence with ergodic transversal invariant measures and also with extremal traces on the C^*-algebra [3]. These algebras are well studied and this leads, for instance, to give distinct proofs of the *gap labeling conjecture* [3, 4, 9], i.e. for minimal $\mathbb{R}^d$-action, the image of the K-theory under a trace is the countable subgroup of $\mathbb{R}$ generated by the images under the corresponding transversal invariant measure of the compact-open subsets of the (Cantor) canonical transversal.

For an hyperbolic quasi-periodic tiling, the situation is quite distinct. The affine transformations act on the Hull and since this group is not unimodular, there is no transversally invariant measure [15]. This shows up a new phenomena for the C^*-algebra of the tiling: it has no trace. Nevertheless, the affine group is amenable, so the Hull admits at least one invariant probability measure. These measures are actually in one-to-one correspondence with harmonic currents [13], and they provide 3-cyclic cocycles on the smooth algebra of the tiling.

The present paper is devoted to give a complete description of the C^*-algebra and the K-theory of a specific family of hyperbolic tilings derivated from the example given by Penrose in [14]. The dynamic of the Hulls under investigation, have a structure of double suspension (this make sens in term of groupoids as we shall see in section 5.2) which enables to make explicit computations. The K_1-group of the smooth algebra contains as a summand a copy of the K_1-group of the C^*-algebra of the tiling, and the restriction to this summand of the pairing with the 3-cocycle gives the one-dimension gap-labelling for the colouring associated with the tiling. But the right setting to state an analogue of the gap-labelling seems to be Frechet algebras and a natural question is whether this bring in new computable invariants.

In the second section, we construct basic examples of hyperbolic quasi-periodic tilings. Background on the tiling spaces is given in the next section and a description of the considered Hulls is given in Section 4. In Section 5, we recall the background on the groupoids and their C^*-algebras. Section 6 and 7 are devoted to a description of the C^*-algebras of the examples. Their K-theory is given in the section 8. In the section 8, we construct 3-cyclic cocycles associated to these tilings and we discuss an odd version of the gap-labelling.

2. On tilings

Let $\mathbb{H}_2$ be the real hyperbolic 2-space, identified with the upper half complex plane: $\{(x, y) \in \mathbb{R}^2 \ |y > 0\}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. We denote by G the group of *affine maps* of this space: i.e. the isometries of $\mathbb{H}_2$ of the kind $z \mapsto az + b$ with a, b reals and $a > 0$.

A tiling $T = \{t_1, \dots, t_n, \dots\}$ of $\mathbb{H}_2$, is a collection of convex compact polygons t_i with geodesic borders, called *tiles*, such that their union is the whole space $\mathbb{H}_2$, their interiors are pairwise disjoint and they meet full edge to full edge. For a group G of isometries of $\mathbb{H}_2$, a tiling is said of G-*finite type* if there exists a finite number of polygons $\{p_1, \dots, p_n\}$ called *prototiles* such that each t_i is the image of one of these polygons by an element of G. The G finite type tiling will be called also *finite affine type* tilings.
For instance, when F is a fundamental domain of a discrete co-compact group Γ of isometries of $\mathbb{H}_2$, then $\{\gamma(F),\ \gamma \in \Gamma\}$ is a tiling of $\mathbb{H}_2$. However the set of finite type tilings is much richer than the one given by discrete co-compact groups. Besides its famous Euclidean tiling, Penrose in [14] constructs a G-finite type tiling made with a single prototile which is not stable for any Fuchsian group. The construction goes as follows.

2.1. **Hyperbolic Penrose's tiling.** Let P be the convex polygon with vertices A_p with affix $(p-1)/2 + \imath$ for $1 \le p \le 3$ and $A_4 : 2\imath + 1$ and $A_5 : 2\imath$ (see figure **??**): P is a polygon with 5 geodesic edges. Consider the two maps:

$$R : z \mapsto 2z \text{ and } S : z \mapsto z + 1.$$

The hyperbolic Penrose's tiling is defined by $\mathcal{P} = \{R^k \circ S^n P |\ n, k \in \mathbb{Z}\}$ (see figure **??**). This is an example of finite affine type tiling of $\mathbb{H}_2$.

This tiling is stable under no co-compact group of hyperbolic isometries. The proof is homological: we associate with the edge A_4A_5 a positive charge and two negative charges with edges A_1A_2, A_2A_3. If $\mathcal{P}$ was stable for a Fuchsian group, then P would tile a compact surface. Since the edge A_4A_5 can meet only the edges A_1A_2 or A_2A_3, the surface has a neutral charge. This is in contradiction with the fact P is negatively charged.
G. Margulis and S. Mozes [11] have generalized this construction to build a family of prototiles which cannot be used to tile a compact surface. Notice the group of isometries which preserves $\mathcal{P}$ is not trivial and is generated by the transformation R. In order to break this symmetry, it is possible to decorate prototiles to get a new finite type tiling which is not stable for any non trivial isometry (we say in this case that the tiling is *aperiodic*).

3. Background on tiling spaces

In this section, we recall some basic definitions and properties on dynamical systems associated with tilings. We refer to [5], [18] and [10] for the proofs. We give then a description of the dynamical system associated to the hyperbolic Penrose's tiling.

3.1. **Action on tilings space.** First, note that the group G acts transitively, freely (without a fixed point) and preserving the orientation of the surface $\mathbb{H}_2$, thus G is a Lie group homeomorphic to $\mathbb{H}_2$. The metric on $\mathbb{H}_2$ gives a left multiplicative invariant metric on G. We fix the point O in $\mathbb{H}_2$ with affix i that we call *origin*.
For a tiling T of G finite type and an isometry p in G, the image of T by p^{-1} is again a tiling of $\mathbb{H}_2$ of G finite type. We denote by $T.G$ the set of tilings which are image of T by isometries in G. The group G acts on this set by the right action:

$$\begin{array}{ccc} G \times T.G & \longrightarrow & T.G \\ (p, T') & \longmapsto & T'.p = p^{-1}(T'). \end{array}$$

We equip $T.G$ with a metrizable topology, finer as one induced by the metric on $\mathbb{H}_2$. A base of neighborhoods is defined as follows: two tilings are close one of the other if they agree, on a big ball of $\mathbb{H}_2$ centered at the origin, up to an isometry in G close to the identity. This topology can be generated by the metric δ on $T.G$ defined by (see [5]):
For T and T' be two tilings of $T.G$, let

$$A = \{\epsilon \in (0, \frac{1}{\sqrt{2}}] \mid \exists\ g \in B_\epsilon(Id) \subset G \text{ s.t. } T.g \cap B_{1/\epsilon} = T'\}$$

where $B_{1/\epsilon}$ is the set of points $x \in \mathbb{H}_2$ such that $d(x, O) < 1/\epsilon$.
We define:

$$\delta(T, T') = \inf A \text{ if } A \neq \emptyset$$

$$\delta(T, T') = \frac{1}{\sqrt{2}} \text{ else.}$$

The *continuous hull* of the tiling T, is the metric completion of $T.G$ for the metric δ. We denote it by X_T^G. Actually this space is a set of tilings of $\mathbb{H}_2$ of G-finite type.
A *patch* of a tiling T is a finite set of tiles of T. It is straightforward to show that patches of tilings in X_T^G are copies of patches of T. The set X_T^G is then a compact metric set and the action of G can be extended to a continuous right action on this space. The dynamical system (X_T^G, G) has a dense orbit: the orbit of T.
Some combinatorial properties can be interpreted in a dynamical way like, for instance, the following property.

Definition 3.1. *A tiling T satisfies the repetitivity condition if for each patch P, there exists a real $R(P)$ such that every ball of $\mathbb{H}_2$ with radius $R(P)$ intersected with the tiling T contains a translated by an element G of the patch P.*

This definition can be interpreted from a dynamical point of view (for a proof see for instance [18]).

Proposition 3.2 (Gottschalk)**.** *The dynamical system (X_T^G, G) is minimal (any orbit is dense) if and only if the tiling T satisfies the repetitivity condition.*

We call a tiling *aperiodic* if the action of G on X_T^G is free: for all $p \neq Id$ of G and all tilings T' of X_T^G we have $T'.p \neq T'$.
As we saw in the former section the hyperbolic Penrose's tiling is not aperiodic, however, using this example, we shall construct in Section 4 uncountably many examples of repetitive and non-periodic affine finite type tilings.

When the tiling T is non-periodic and repetitive, the hull X_T^G has also a geometric structure of a specific lamination called a *G-solenoid* (see [5]). Locally at any point x, there exists a *vertical germ* which is a Cantor set included in X_T^G, transverse to the local G-action and which is defined independently of the neighborhood of the point x. This implies that X_T^G is locally homeomorphic to the Cartesian product of a Cantor set with an open subset (called a *slice*) of the Lie group G. The connected component of the slices that intersect is called a *leaf* and has a manifold structure. Globally, X_T^G is a disjoint union of uncountably many leaves, and it turns out that each leaf is a G-orbit. Since the action is free, each leaf is homeomorphic to $\mathbb{H}_2$.

In the aperiodic case, the G-action is *expansive*: There exists a positive real ϵ such that for every points T_1 and T_2 in the same vertical in X_T^G, if $\delta(T_1.g, T_2.g) < \epsilon$ for every $g \in G$, then $T_1 = T_2$.
Furthermore this action has locally constant return times: if an orbit (or a leaf) intersects two verticals V and V' at points v and $v.g$ where $g \in G$, then for any point w of V close enough to v, $w.g$ belongs to V'.

3.2. **Structure of the hull of the Penrose Hyperbolic tilings.** First recall the notion of suspension action for X a compact metric space and $f : X \to X$ an homeomorphim. The group $\mathbb{Z}$ acts diagonally on the product space $X \times \mathbb{R}$ by the following homeomorphism denoted $\mathcal{A}_f$

$$\begin{aligned} \mathcal{A}_f :\ & X \times \mathbb{R} \to X \times \mathbb{R} \\ & (x,t) \mapsto (f(x), t-1) \end{aligned}$$

The quotient space of $(X \times \mathbb{R})/\mathcal{A}_f$, where two points are identified if they belong to the same orbit , is a compact set for the product topology and is called *the suspension* of (X, f). The group $\mathbb{R}$ acts also diagonally on $X \times \mathbb{R}$: trivially on X and by translation on $\mathbb{R}$. Since this action commutes with $\mathcal{A}_f$, this induces a continuous $\mathbb{R}$ action on the suspension space $(X \times \mathbb{R})/\mathcal{A}_f$. We call this action: the *suspension action* of the system (X, f) and we denote it by $\big((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{R}\big)$.

We recall here, the construction of the dyadic completion of the integers. On the set of integers $\mathbb{Z}$, we consider the dyadic norm defined by

$$|n|_2 = 2^{-\sup\{p \in \mathbb{N},\ 2^p \text{ divides } |n|\}} \qquad n \in \mathbb{Z}.$$

Let Ω be the completion of the set $\mathbb{Z}$ for the metric given by $|.|_2$. The set Ω has a commutative group structure where $\mathbb{Z}$ is a dense subgroup, and Ω is a Cantor set. The continuous action given by the map $o :\ x \mapsto x+1$ on Ω is called *adding-machine* or *odometer* and is known to be minimal and equicontinuous. We denote by $((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R})$ the suspension action of this homeomorphism.

Recall that a conjugacy map between two dynamical systems is a homeomophism which commutes with the actions. Let $\mathcal{N}$ be the group of maps $\{z \mapsto z+t,\ t \in \mathbb{R}\}$ isomorphic to $\mathbb{R}$.

Proposition 3.3. *Let $X_{\mathcal{P}}^{\mathcal{N}}$ be the completion (for the tiling topology) of the orbit $\mathcal{P}.\mathcal{N}$.*

Then the dynamical system $(X_{\mathcal{P}}^{\mathcal{N}}, \mathcal{N})$ *is conjugate to the suspension action of the odometer* $((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R})$.

Proof. Let $\phi : \mathcal{P}.\mathcal{N} \to (\Omega \times \mathbb{R})/\mathcal{A}_o$ be the map defined by $\phi(\mathcal{P} + t) = [E(t), t]$ where $E(t)$ denotes the integer part of the real t and where $[E(t), t]$ is the class in $(\Omega \times \mathbb{R})/\mathcal{A}_o$ of $[E(t), t] \in \Omega \times \mathbb{R}$. Since the tiling invariant under no translations, the application ϕ is well defined. It is straightforward to check that ϕ is continuous for the tiling topology and for the dyadic topology on Ω. So the map ϕ extends by continuity to $X_{\mathcal{P}}^{\mathcal{N}}$ and factorizes to a a map $\overline{\phi}$ from $X_{\mathcal{P}}^{T}$ to the suspension $(\Omega \times \mathbb{R})/\mathcal{A}_o$.
Conversely, let $\psi : \mathbb{Z} \times \mathbb{R} \to X_{\mathcal{P}}^{\mathcal{N}}$ defined by $\psi(n, t) = \mathcal{P} + n + t$. This application is continuous for the dyadic topology, so it extends by continuity to $\Omega \times \mathbb{R}$. We have $\psi(n+1, t-1) = \psi(n, t)$ for any $n \in \mathbb{Z}$ and any $t \in \mathbb{R}$, so $\psi(n, t)$ is constant along the orbits of the $\mathbb{Z}$ action on $\mathbb{Z} \times \mathbb{R}$ which is dense in $\Omega \times \mathbb{R}$. Thus ψ is constant along the $\mathbb{Z}$-orbits in $\Omega \times \mathbb{R}$ and therefore ψ factorizes to a map $\overline{\psi}$ from the suspension $(\Omega \times \mathbb{R})/\mathcal{A}_o$ to $X_{\mathcal{P}}^{\mathcal{N}}$. It is plain to check that $\overline{\psi} \circ \overline{\phi} = Id$ on the set $\mathcal{P}.\mathcal{N}$ and that $\overline{\phi} \circ \overline{\psi} = Id$ on the set $\pi(\mathbb{N} \times \mathbb{R})$ where $\pi : \Omega \times \mathbb{R} \to (\Omega \times \mathbb{R})/\mathcal{A}_o$ denotes the canonical projection. By density of these sets and by continuity of the maps, the egalities are true on the whole space and so $\overline{\phi}$ is an homeomorphism from $X_{\mathcal{P}}^{\mathcal{N}}$ onto $(\Omega \times \mathbb{R})/\mathcal{A}_o$. It is straightforward to show that $\overline{\phi}$ commutes with the $\mathbb{R}$-actions. □

4. Examples

We construct in this section a family of tilings of $\mathbb{H}_2$ of finite affine type, indexed by sequences on a finite alphabet. For uncountably many of them, the tilings will be aperiodic and repetitive, the action on the associated hull will be free and minimal. A description of these actions in terms of double-suspension is given.

4.1. **Construction of the examples.** To construct such tilings we will use the hyperbolic Penrose's tiling described in section 2.1, so we will keep the notations of this section. Recall that its stabilizer group under the action of G, is the group $\langle\ R\ \rangle$ generated by the affine map R. The main idea is to "decorate" this tiling in order to break its symmetry, the decoration will be coded by a sequence on a finite alphabet. By a decoration, we mean that we will substitute to each tile t the same polygon t equipped with a color. We take the convention that two colored polygons are the same if and only if the polygons are the same up to an affine map and they have the same color. By substituting each tile by a colored tile, we obtain a new tiling of finite affine type with a bigger number of prototiles.
Notice that the coloration is not canonical. It also possible to do the same by substituting to a tile t, an unique finite family of convex tiles $\{t_i\}_i$, like triangles, such that the union of the t_i is t and the tiles t_i overlaps only on their borders. We choose the coloration only for presentation reasons.

Let r be an integer bigger than 1 and Σ be the set $\{1, \ldots, r\}$. We associate to each element of Σ an unique color. Let P be the polygon defined in Section 2.1 to construct the Penrose's tiling. For an element i of Σ, we denote by P_i the prototile P colored in the color i. To a sequence $w = (w_k)_{k \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$, we associate the G-finite-type tiling $\mathcal{P}(w)$ built with the prototiles P_i for i in Σ and defined by:

$$\mathcal{P}(w) = \{R^q \circ S^n(P_{w_q}),\ n, q \in \mathbb{Z}\}.$$

Notice that the stabilizer of this tiling is a subgroup of $\langle\ R\ \rangle$.

The set of sequences on Σ is the product space $\Sigma^{\mathbb{Z}}$ which is a Cantor set for the product topology. There exists a natural homeomorphism on it called the *shift*. To a sequence $(w_n)_{n\in\mathbb{Z}}$ the shift σ associates the sequence $(w'_n)_{n\in\mathbb{Z}}$ where $w'_n = w_{n+1}$ for any $n \in \mathbb{Z}$. Let Z_w denote the closure of the orbit of w by the action of the shift σ: $Z_w = \overline{\{\sigma^n(w),\ n \in \mathbb{Z}\}}$. The set Z_w is a compact metric space stable under the action of σ. Since $\mathcal{P}(w).R$ denotes the tiling image of $\mathcal{P}(w)$ by R^{-1}, we get the relation

$$\mathcal{P}(w).R = \mathcal{P}(\sigma(w)). \tag{4.1}$$

Notice that this implies that for any $w' \in Z_w$, then $\mathcal{P}(w')$ belongs to $X^G_{\mathcal{P}(w)}$. Thanks to equation 4.1, we obtain the following property:

Lemma 4.1.

- *The sequence w is aperiodic for the shift-action, if and only if $\mathcal{P}(w)$ is stable under no non-trivial affine map.*
- *The dynamical system (Z_w, σ) is minimal, if and only if $(X^G_{\mathcal{P}(w)}, G)$ is minimal.*

Proof. The first point comes from the relation 4.1 and from the fact that the stabilizer of $\mathcal{P}(w)$ is a subgroup of $\langle\ R\ \rangle$. The last point comes from the characterization of minimal sequences: (Z_w, σ) is minimal if and only if each words in w appears infinitely many times with uniformly bounded gap [17]. This condition is equivalent to the repetitivity of $\mathcal{P}(w)$. Since $\mathcal{P}(w)$ is repetitive if and only if $(X^G_{\mathcal{P}(w)}, G)$ is minimal, we get the result. □

Remark 4.2. *The two systems $(X^G_{\mathcal{P}}, G)$ and $(X^G_{\mathcal{P}((1)_{n\in\mathbb{Z}})}, G)$, are conjugate. The map $\phi:\ \mathcal{P}.G \to \mathcal{P}((1)_{n\in\mathbb{Z}}).G$ defined by $\phi(\mathcal{P}.g) = \mathcal{P}((1)_n).g$ is well defined and bijective. The maps ϕ and ϕ^{-1} are continuous for the tiling topology. So ϕ extends by continuity to an homeomorphism, denoted again ϕ, from $X^G_{\mathcal{P}}$ to $X^G_{\mathcal{P}((1)_n)}$. It is straightforward to check that ϕ is a conjugacy map.*

Cette remarque est-elle absolument nécessaire?

Recall that we have defined the group $\mathcal{N} = \{z \mapsto z + t,\ t \in \mathbb{R}\}$ and that $X^{\mathcal{N}}_{\mathcal{P}}$ stands for the closure (for the tiling topology) of the $\mathcal{N}$ orbit $\mathcal{P}.\mathcal{N}$ of the uncolored tiling $\mathcal{P}$. Notice that the continuous action of R on $X^G_{\mathcal{P}}$ preserves the orbit $\mathcal{P}.\mathcal{N}$ so it defines an homeomorphism of $X^{\mathcal{N}}_{\mathcal{P}}$ that we denote also by R. We consider on the space $X^{\mathcal{N}}_{\mathcal{P}} \times Z_w \times \mathbb{R}^*_+$ equipped with the product topology, the homeomorphism $\mathcal{R}$ defined by $\mathcal{R}(\mathcal{T}, w', t) = (\mathcal{T}.R, \sigma(w'), t/2)$. Since the action of $\mathcal{R}$ on the $\mathbb{R}^*_+$ factor is co-compact, the quotient space $(X^{\mathcal{N}}_{\mathcal{P}} \times Z_w \times \mathbb{R}^*_+)/\mathcal{R}$, where the points in the same $\mathcal{R}$ orbit are identified, is a compact space.

The affine group G also acts on the right on $X^{\mathcal{N}}_{\mathcal{P}} \times Z_w \times \mathbb{R}^*_+$; where the action of an element $g : z \mapsto az + b$ is given by the homeomorphism

$$(\mathcal{T}, w', t) \mapsto (\mathcal{T} - bt, w', at) = (\mathcal{T}, w', t).g.$$

It is straightforward to check that the application $\mathcal{R}$ commutes with this action, so this defines a G continuous action on the quotient space $(X^{\mathcal{N}}_{\mathcal{P}} \times Z_w \times \mathbb{R}^*_+)/\mathcal{R}$.

Proposition 4.3. *Let w be an element of $\Sigma^{\mathbb{Z}}$ and let $X_{\mathcal{P}}^{\mathcal{N}}$ and $X_{\mathcal{P}(w)}^{G}$ be respectively the closures (for the tiling topology) of the orbits $\mathcal{P}.\mathcal{N}$ and $\mathcal{P}(w).G$.*
Then the map

$$\begin{array}{rcl}\mathcal{P}(w).G & \to & (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R} \\ \mathcal{P}(w).g & \mapsto & [(\mathcal{P}, w, 1).g]\end{array}$$

where $[x]$ denotes the $\mathcal{R}$-class of x, extends to a conjugacy map between $(X_{\mathcal{P}(w)}^{G}, G)$ and $\big((X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R},\ G\big)$.

Proof. Let Φ be the map $\mathcal{P}.\mathcal{N} \times Z_w \times \mathbb{R}_+^* \to X_{\mathcal{P}(w)}^{G}$ defined by $\Phi(\mathcal{P}+\tau, w', t) = (\mathcal{P}(w')+\tau).R_t$ where R_t denotes the map $z \mapsto tz$. The application Φ is continuous for the tiling topology on $\mathcal{P}.\mathcal{N}$, so it extends by continuity to a continuous map from $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ to $X_{\mathcal{P}(w)}^{G}$. It is plain to check that $\Phi \circ \mathcal{R} = \Phi$ on $\mathcal{P}.\mathcal{N} \times Z_w \times \mathbb{R}_+^*$, thanks to relation 4.1. By the density of the set $\mathcal{P}.\mathcal{N}$ in $X_{\mathcal{P}}^{\mathcal{N}}$ and by continuity of the maps, the relation is true on the whole space $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$. Therefore the map Φ factorizes on a continuous map $\overline{\Phi} : (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R} \to X_{\mathcal{P}(w)}^{G}$. Since the stabilizer of the tiling $\mathcal{P}(w)$ is a subgroup of the one generated by the map R, and by reation 4.1, the map $\Psi : \mathcal{P}(w).G \to \big(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*\big)/\mathcal{R}$ where $\Psi(\mathcal{P}(w).g) = [(\mathcal{P}, w, 1).g]$ for $g \in G$ is well defined. It is straightforward to check that Ψ is continuous, so it extends to a continuous map from $X_{\mathcal{P}(w)}^{G}$ to $\big(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*\big)/\mathcal{R}$ that we denote again Ψ. Furthermore we have $\overline{\Phi} \circ \Psi = Id$ on $\mathcal{P}(w).G$ and $\Psi \circ \overline{\Phi} = Id$ on the dense set $\pi(\mathcal{P}.\mathcal{N} \times Z_w \times \mathbb{R}_+^*)$ where π denotes the canonical projection $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \to \big(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*\big)/\mathcal{R}$. By continuity of the maps, the relations are true on the whole space and so $\overline{\Phi}$ is an homeomorphism. It is straightforward to check that $\overline{\Phi}$ commutes with the action. □

Notice that, $X_{\mathcal{P}}^{\mathcal{N}}$ s locally the Cartesian product of a Cantor set by an interval of $\mathbb{R}$. For $w \in \Sigma^{\mathbb{Z}}$, $X_{\mathcal{P}(w)}^{G}$ is locally homeomorphic the product of a Cantor set by an open subset (a slice) of $\mathbb{R}_+^* \times \mathbb{R}$ since the Cartesian product of two Cantor sets is again a Cantor set. The G action preserves each slice.

4.2. **Ergodic properties of Penrose's tilings.** For a metric space X and a continuous action of a group Γ on it, a Γ-*invariant measure* is a measure μ defined on the Borel σ-algebra of X which is invariant under the action of Γ i.e.: For any measurable set $B \subset X$ and any $g \in \Gamma$, $\mu(B.g) = \mu(B)$. For instance, any group Γ acts on itself by right multiplication, there exists only (up to a scalar) one measure invariant for this action: it is called the *Haar* measure.

Any action of an amenable group Γ (like $\mathbb{Z}$, $\mathbb{R}$ and all their extensions) on a compact metric space X admits a finite invariant measure and in particular, any homeomorphism f of X preserves a probability measure. An *ergodic* invariant measure μ is such that every measurable functions constant along the orbits are μ almost surely constant. Every invariant measure is the sum of ergodic invariant measures [17]. A conjugacy map sends the invariant measure to invariant measure and the ergodic measures to the ergodic measures.

In our case, the group of affine maps G, is the extension of two groups isomorphic to $\mathbb{R}$, hence is amenable. It is well known that the only invariant measures for the suspension action $\big((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{R}\big)$ are locally the images through the canonical projection $\pi : X \times \mathbb{R} \to X \times\ R/\mathcal{A}_f$ of the measures $\mu \otimes \lambda$ where μ is a f-invariant measure on X and λ denotes the Lebesgue measure of $\mathbb{R}$. The proof is actually contained in Property 4.4.

It is well known also that the map $o : x \mapsto x+1$ on the dyadic set of integers Ω, admits only one invariant probability measure: the Haar probability measure on Ω. Hence the suspension of this action $\big((\Omega\times\mathbb{R})/\mathcal{A}_o, \mathbb{R}\big)$ admits only one invariant probability measure. By Proposition 3.3, $X_{\mathcal{P}}^{\mathcal{N}}$ has only one invariant probability measure ν. Notice that the map R preserves $X_{\mathcal{P}}^{\mathcal{N}}$, and since $R\mathcal{N}R^{-1} = \mathcal{N}$, the map R preserves all the $\mathcal{N}$-invariant measures of $X_{\mathcal{P}}^{\mathcal{N}}$ hence preserves ν. Nevertheless, the local product decomposition of ν is not invariant by R, because R divides by 2 the length of the intervals of the $\mathcal{N}$-orbit. So R has to inflate the Haar measure on Ω by a factor 2.

Proposition 4.4. *Let $w \in \Sigma^{\mathbb{Z}}$, and let $X_{\mathcal{P}}^{\mathcal{N}}$ be the closure of the orbit $\mathcal{P}.\mathcal{N}$.*
A finite invariant measure of $\big((X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R},\ G\big)$ is locally the image through the projection $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \to (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$ of a measure $\nu\otimes\mu\otimes m$ where

- *ν is the only invariant probability measure of $(X_{\mathcal{P}}^{\mathcal{N}}, \mathbb{R})$;*
- *m is the Haar measure of $(\mathbb{R}_+^*, \times)$;*
- *μ is a finite invariant measure of (Z_w, σ).*

Proof. It is enough to prove this for an ergodic finite G-invariant measure $\overline{\theta}$ on the suspension $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$. Since $\mathcal{R}$ acts cocompactly on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$, $\overline{\theta}$ defines a finite measure on a fundamental domain of $\mathcal{R}$, and the sum of all the images of this measure by $\mathcal{R}$ defines a σ-finite measure θ on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ which is G and $\mathcal{R}$-invariant.
Let $\pi_2 : X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \to Z_w$ be the projection to the second coordinate. This map being equivariant with respect to the actions of $\mathbb{Z}$ respectivelly given by $\mathcal{R}$ on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ and the shift action on Z_w, we get that $\pi_{2*}\theta$ is shift invariant measure on Z_w. The measure θ can be disintegrated over $\pi_{2*}\theta = \mu$ by a family of measures $\{\lambda_{w'}\}_{w'}$ defined for μ-almost every $w' \in Z_w$ on $X_{\mathcal{P}}^{\mathcal{N}} \times \{w'\} \times \mathbb{R}_+^*$ such that

$$\theta(B\times C) = \int_C \lambda_{w'}(B)d\mu(w'),$$

for any Borel sets $B \subset X_{\mathcal{P}}^{\mathcal{N}} \times \mathbb{R}_+^*$ and $C \subset Z_w$.
The G-invariance of θ implies that the measures $\lambda_{w'}$ are almost G-invariant. The projection to the first coordinate $\pi_1 : X_{\mathcal{P}}^{\mathcal{N}} \times \{w'\} \times \mathbb{R}_+^* \to X_{\mathcal{P}}^{\mathcal{N}}$ is $\mathcal{N}$-equivariant. The measures $\pi_{2*}\lambda_{w'}$ are then $\mathcal{N}$-invariant measures, hence are proportional to ν. Each measure $\lambda_{w'}$ can be disintegrated over ν by a family of measures $\{m_{x,w'}\}_{x,w'}$ on $\mathbb{R}_+^*$ defined for ν almost every $x \in X_{\mathcal{P}}^{\mathcal{N}}$ such that

$$\lambda_{w'}(B\times\{w'\}\times I) = \int_B\int_I m_{x,w'}d\nu(x),$$

for any Borel sets $B \subset X_{\mathcal{P}}^{\mathcal{N}}$ and $I \subset \mathbb{R}_+^*$. Each measure $\lambda_{w'}$ is invariant under the action of maps of the kind $z \mapsto az$ for $a \in \mathbb{R}_+^*$. It is then straightforward to check that the measures $m_{x,w'}$ are multiplication-invariant. By unicity of the Haar measure, there exists a measurable positive function $(x, w') \mapsto h(x, w')$ defined almost everywhere so that $m_{x,w'} = h(x, w')m$. The $\mathcal{N}$-invariance of the measures $\lambda_{w'}$ implies that the map h is almost surely constant along the $\mathcal{N}$-orbits, and the $\mathcal{R}$-invariance of θ implies that h is almost surely constant along the $\mathcal{R}$-orbits. This defines then a measurable map on the quotient space by $\mathcal{R}$ which is G-invariant, the ergodicity of $\overline{\theta}$ implies that this map is almost surely constant. □

Notice that an invariant measure on $X_{\mathcal{P}(w)}^G$ can be decomposed locally into the product of a measure on a Cantor set by a measure along the leaves. Since the map R does not preserves the transversal measure on Ω in $X_{\mathcal{P}}^{\mathcal{N}}$, the holonomy groupoid of $X_{\mathcal{P}(w)}^G$ does not preserves the transversal measure on the Cantor set.
The G-action is locally free and acts transitively on each leave, so each orbits inherits a hyperbolic 2-manifolds structure. Actually $X_{\mathcal{P}(w)}^G$, can be equipped with a continuous metric with a constant curvature equals to -1 in restriction to each leave. This metric induces a measure on each leaf which can be locally written as $dxdy/y^2$. A finite invariant measure on $X_{\mathcal{P}(w)}^G$ in restriction on the leaves can be written $dxdy/y$ (since the Haar measure m for $(\mathbb{R}_+^*, \times)$ is dy/y). So the density of these measures with respect to the Riemannian measure is an harmonic function. The invariant measures of $X_{\mathcal{P}(w)}^G$ have then also a geometric interpretation in terms of *harmonic* measures, a notion introduced by L. Garnett in [7].

Definition 4.5. *A probability measure μ on M is harmonic if*

$$\int_M \Delta f d\mu = 0$$

for any continuous function f $\mathcal{C}^2$ in restriction on the leaves, where Δ denotes the Laplace-Beltrami operator in restriction on each leave.

Actually, it is shown in [13] , that in $X_{\mathcal{P}(w)}^G$ the notions of harmonic and invariant measures are the same and such measures can be described in terms of inverse limit of vectoriel cones.

5. Transformation groupoids

We gather this section with results on groupoids and their C^*-algebras. Good material on this topic can be found in [**?**, **?**]. Let $\mathcal{G}$ be a locally compact groupoid, with base space X, range and source maps respectively $r : \mathcal{G} \to X$ and $s : \mathcal{G} \to X$. For any element x of X, we set

$$\mathcal{G}^x = \{\gamma \in \mathcal{G} \text{ such that } r(\gamma) = x\}$$

and

$$\mathcal{G}_x = \{\gamma \in \mathcal{G} \text{ such that } s(\gamma) = x\}.$$

Thourought this section, all the groupoids will be assumed locally compact and second countable.

5.1. **Semi-direct product groupoid.** Let H be a locally compact group acting on a locally compact space X. The semi-direct product groupoid $X \rtimes H$ of X by H is defined by

- $X \times H$ as a topological space ;
- the base space is X and the structure maps are $r : X \rtimes H \to X$; $(x, h) \mapsto x$ and $s : X \rtimes H \to X$; $(x, h) \mapsto h^{-1}x$;
- the product is $(x, h) \cdot (h^{-1}x, h') = (x, hh')$ for x in X and h and h' in H.

Let μ be a left Haar mesure on H. Then the groupoid $X \rtimes H$ is equipped with a Haar system $\lambda^\mu = (\lambda^\mu_x)_{x\in X}$ given for any f in $C_c(X \times H)$ and any x in X by $\lambda^\mu_x(f) = \int_H f(x,h)d\mu(h)$. Then $C^*(X \times H, \lambda^\mu)$ is the usual crossed-product $C_0(X) \rtimes H$. In this paper, will equip $\mathbb{R}$ with its standard Haar measure (i.e the Lebesgue measure).

5.2. **Suspension of a groupoid.**

Definition 5.1. *Let $\mathcal{G}$ be a groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x\in X}$. A groupoid automorphism $\alpha : \mathcal{G} \to \mathcal{G}$ is said to preserve the Haar System λ if there exists a continuous function $\rho_\alpha : \mathcal{G} \to \mathbb{R}^+$ such that for any x in X the measures $\alpha_*\lambda^x$ and $\lambda^{\alpha(x)}$ on $\mathcal{G}^{\alpha(x)}$ are in the same class and ρ_α restricts on $\mathcal{G}^{\alpha(x)}$ to $\frac{d\alpha_*\lambda^x}{d\lambda^{\alpha(x)}}$. The map ρ_α is called the density of α.*

Remark 5.2. *Let $\mathcal{G}$ be a groupoid with base space X and Haar system $\lambda = (\lambda^x)_{x\in X}$ and let $\alpha : \mathcal{G} \to \mathcal{G}$ be an automorphism of groupoid preserving the Haar system λ.*

(1) *Let us denote for any γ in $\mathcal{G}$ by $L_\gamma : \mathcal{G}^{s(\gamma)} \to \mathcal{G}^{r(\gamma)}$ the left translation by γ. Since $L_\gamma \circ \alpha = \alpha \circ L_{\alpha^{-1}(\gamma)}$ for any γ in $\mathcal{G}$, we get that*
$$L_{\gamma,*}\alpha_*\lambda^{\alpha^{-1}(s(\gamma))} = \alpha_* L_{\alpha^{-1}(\gamma),*}\lambda^{s(\alpha^{-1}(\gamma))} = \alpha_*\lambda^{r(\alpha^{-1}(\gamma))}.$$
Then $L_{\gamma,*}\alpha_*\lambda^{\alpha^{-1}(s(\gamma))}$ is a measure on $\mathcal{G}^{r(\gamma)}$ absolutly continuous with respect to $L_{\gamma,*}\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ with density $\rho_\alpha \circ L_\gamma$ and thus $\rho_\alpha \circ L_{\gamma^{-1}} = \rho_\alpha$. In particular ρ_α is constant on $\mathcal{G}_x$ for any x in X.

(2) *The automorphism of groupoid $\alpha^{-1} : \mathcal{G} \to \mathcal{G}$ also preserves the Haar system λ and $\rho_{\alpha^{-1}} = 1/\rho_\alpha \circ \alpha$.*

Definition 5.3. *Let $\mathcal{G}$ be a groupoid with base space X, range and source map r and s and let $\alpha : \mathcal{G} \to \mathcal{G}$ be a groupoid automorphism. Using the notations of section 3.2 the suspension of the groupoid $\mathcal{G}$ respectively to α is the groupoid $\mathcal{G}_\alpha \overset{def}{=\!=} (\mathcal{G} \times \mathbb{R})/\mathcal{A}_\alpha$ with basis $(X \times \mathbb{R})/\mathcal{A}_\alpha$ (recall that α induces a homeomorphism on X). For any γ in $\mathcal{G}$ and t in $\mathbb{R}$, let us denote by $[\gamma, t]$ the class of (γ, t) in $\mathcal{G}_\alpha$.*

- *The range map r_α and the source map s_α are defined in the following way:*
  - *$r_\alpha([\gamma, t]) = [r(\gamma), t]$ for every γ in $\mathcal{G}$ and t in $\mathbb{R}$;*
  - *$s_\alpha([\gamma, t]) = [s(\gamma), t]$ for every γ in $\mathcal{G}$ and t in $\mathbb{R}$;*
- *Let γ and γ' be elements of $\mathcal{G}$ such that $s(\gamma) = r(\gamma')$ and let t be in $\mathbb{R}$, then $[\gamma, t] \circ [\gamma', t] = [\gamma \circ \gamma', t]$;*
- *$[\gamma, t]^{-1} = [\gamma^{-1}, t]$.*

There is an action of $\mathbb{R}$ on $\mathcal{G}_\alpha$ by automorphisms given for s in $\mathbb{R}$ and $[\gamma, t]$ in $\mathcal{G}_\alpha$ by $s \cdot [\gamma, t] = [\gamma, t + s]$.

Lemma 5.4. *Let $\mathcal{G}$ be a groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x\in X}$ and let $\alpha : \mathcal{G} \to \mathcal{G}$ be an automorphism preserving the Haar System λ. Let us assume that $\rho_\alpha = \rho_\alpha \circ \alpha$. Then $\mathcal{G}_\alpha$ admits a Haar system $\lambda_\alpha = \left(\lambda_\alpha^{[x,t]}\right)_{[x,t]\in(X\times\mathbb{R})/\mathbb{Z}}$ given for any $[x, t]$ in $(X \times \mathbb{R})/\mathcal{A}_\alpha$ and any continuous fonction f in $C_c\left(\mathcal{G}_\alpha^{[x,t]}\right)$ by*
$$\lambda_\alpha^{[x,t]}(f) = \int_{\mathcal{G}^x} \rho_\alpha(\gamma)^{-t} f([\gamma, t])d\lambda^x(\gamma).$$

Proof.

- Let us prove first that the definition of $\lambda_\alpha^{[x,t]}(f)$ for $[x,t]$ in $(X\times\mathbb{R})/\mathcal{A}_\alpha$ and f in $C_c\left(\mathcal{G}_\alpha^{[x,t]}\right)$ makes sense.

$$\begin{aligned}\int_{\mathcal{G}^x}\rho_\alpha(\gamma)^{-t}f([\gamma,t])d\lambda^x(\gamma) &= \int_{\mathcal{G}^x}\rho_\alpha(\alpha(\gamma))^{-t}f([\alpha^{-1}(\alpha(\gamma),t])d\lambda^x(\gamma)\\ &= \int_{\mathcal{G}^{\alpha(x)}}\rho_\alpha(\gamma)^{-t+1}f([\alpha^{-1}(\gamma),t])d\lambda^{\alpha(x)}(\gamma)\\ &= \int_{\mathcal{G}^{\alpha(x)}}\rho_\alpha(\gamma)^{-t+1}f([\gamma,t-1])d\lambda^{\alpha(x)}(\gamma)\end{aligned}$$

- Let us show then that $(\lambda^{[x,t]})_{[x,t]\in(X\times\mathbb{R})/\mathbb{Z}}$ is a Haar system. Let γ' be an element of $\mathcal{G}$, let t be a real number and let f be a function in $C_c\left(\mathcal{G}_\alpha^{[r(\gamma'),t]}\right)$. Then

$$\begin{aligned}\lambda_\alpha^{[r(\gamma'),t]}(f) &= \int_{\mathcal{G}^{r(\gamma')}}\rho_\alpha(\gamma)^{-t}f([\gamma,t])d\lambda^{r(\gamma')}(\gamma)\\ &= \int_{\mathcal{G}^{s(\gamma')}}\rho_\alpha(\gamma'\cdot\gamma)^{-t}f([\gamma'\cdot\gamma,t])d\lambda^{s(\gamma')}(\gamma)\\ &= \int_{\mathcal{G}^{s(\gamma')}}\rho_\alpha(\gamma)^{-t}f([\gamma'\cdot\gamma,t])d\lambda^{s(\gamma')}(\gamma)\\ &= \lambda_\alpha^{[s(\gamma'),t]}(f\circ L_{[\gamma',t]}),\end{aligned}$$

where the third equality holds in view of remark 5.2.

□

5.3. **C^*-algebra of a suspension groupoid.** Let us recall first the construction of the reduced C^*-algebra $C_r^*(\mathcal{G},\lambda)$ associated to a groupoid $\mathcal{G}$ with base X and Haar system $\lambda=(\lambda^x)_{x\in X}$. Let $\mathcal{L}^2(\mathcal{G},\lambda)$ be the $C_0(X)$-Hilbert completion of $C_c(\mathcal{G})$ equipped with the $C_0(X)$-valued scalar product

$$\langle\phi,\phi'\rangle(x)=\int_{\mathcal{G}^x}\bar{\phi}(\gamma^{-1})\phi'(\gamma^{-1})d\lambda^x(\gamma)$$

for ϕ and ϕ' in $C_c(\mathcal{G})$ and x in X, i.e the completion of $C_c(\mathcal{G})$ with respect to the norm $\|\phi\|=\sup_{x\in X}\langle\phi,\phi\rangle^{1/2}$. The $C_0(X)$-module structure on $C_c(\mathcal{G})$ extends to $\mathcal{L}^2(\mathcal{G},\lambda)$ and $\langle\bullet,\bullet\rangle$ extends to a $C_0(X)$-valued scalar product on $\mathcal{L}^2(\mathcal{G},\lambda)$. Recall that an operator $T:\mathcal{L}^2(\mathcal{G},\lambda)\to\mathcal{L}^2(\mathcal{G},\lambda)$ is called adjointable if there exists an operator $T^*:\mathcal{L}^2(\mathcal{G},\lambda)\to\mathcal{L}^2(\mathcal{G},\lambda)$, called the adjoint of T such that $\langle T^*\phi,\phi'\rangle=\langle\phi,T\phi'\rangle$ for all ϕ and ϕ' in $\mathcal{L}^2(\mathcal{G},\lambda)$. Notice that the adjoint, when it exists is unique and that operator that admits an adjoint are automatically $C_0(X)$-linear and continuous. The set of adjointable operators on $\mathcal{L}^2(\mathcal{G},\lambda)$ is then a C^*-algebra with respect to the operator norm. Then any f in $C_c(\mathcal{G})$ acts as an adjointable operator on $\mathcal{L}^2(\mathcal{G},\lambda)$ by convolution

$$f\cdot\phi(\gamma)=\int_{\mathcal{G}^{r(\gamma)}}f(\gamma')\phi(\gamma'^{-1}\gamma)d\lambda^{r(\gamma)}(\gamma')$$

where ϕ is in $C_c(\mathcal{G})$, the adjoint of this operator being given by the action of $f^*:\gamma\mapsto\bar{f}(\gamma^{-1})$. The convolution product provides a structure of involutive algebra on $C_c(\mathcal{G})$ and using the action defined above, this algebra can be viewed as a subalgebra of the C^*-algebra of adjointable operators of $\mathcal{L}^2(\mathcal{G},\lambda)$. The reduced

C^*-algebra $C_r^*(\mathcal{G},\lambda)$ is then the closure of $C_c(\mathcal{G})$ in the C^*-algebra of adjointable operators of $\mathcal{L}^2(\mathcal{G},\lambda)$. Namely, if we define for x in X the measure on $\mathcal{G}_x$ by $\lambda_x(\phi)=\int_{\mathcal{G}^x}\phi(\gamma^{-1})d\lambda^x(\gamma)$ for any ϕ in $C_c(\mathcal{G}_x)$, then $\mathcal{L}^2(\mathcal{G},\lambda)$ is a continuous field of Hilbert spaces over X with fiber $\mathcal{L}^2(\mathcal{G}_x,\lambda_x)$ at x in X. The corresponding $C_0(X)$-structure on $C_r^*(\mathcal{G},\lambda)$ is then given for h in $C_0(X)$ by the multiplication by $h\circ s$. Let us denote for any x in X by ν_x the representation of $C_r^*(\mathcal{G},\lambda)$ on the fiber $\mathcal{L}^2(\mathcal{G}_x,\lambda_x)$. Then for any f in $C_r^*(\mathcal{G},\lambda)$, we get that $\|f\|_{C_r^*(\mathcal{G},\lambda)}=\sup_{x\in X}\|\nu_x(f)\|$.

Lemma 5.5. *Let $\mathcal{G}$ be a locally compact groupoid with base space X equipped with a Haar system $\lambda=(\lambda^x)_{x\in X}$ and let $\alpha:\mathcal{G}\to\mathcal{G}$ be an automorphism preserving the Haar System λ. Let us define the continuous map $\rho'_\alpha:\mathcal{G}\to\mathbb{R};\,\gamma\mapsto\rho_\alpha(\gamma^{-1})$. Then α induces a unique automorphism $\tilde{\alpha}$ of the C^*-algebra $C_r^*(\mathcal{G},\lambda)$ such that for every f in $C_c(\mathcal{G})$ we have $\tilde{\alpha}(f)=(\rho'_\alpha\rho_\alpha)^{1/2}f\circ\alpha^{-1}$.*

Proof. The map $C_c(\mathcal{G})\to C_c(\mathcal{G});\,\phi\mapsto\rho_\alpha'^{1/2}\phi\circ\alpha^{-1}$ extends to a continuous linear and invertible map $W:\mathcal{L}^2(\mathcal{G},\lambda)\to\mathcal{L}^2(\mathcal{G},\lambda)$ such that

$$\langle W\cdot\phi,W\cdot\phi\rangle(x)=\langle\phi,\phi\rangle(\alpha^{-1}(x)),$$

for all x in X. Its inverse W^{-1} is defined by $W^{-1}(\phi)=({\rho'}_\alpha\circ\alpha)^{-1/2}\phi\circ\alpha$ for all ϕ in $C_c(\mathcal{G})$. Let us define

$$\tilde{\alpha}:C_r^*(\mathcal{G},\lambda)\to C_r^*(\mathcal{G},\lambda);\,x\mapsto W\cdot x\cdot W^{-1}.$$

Then $W\cdot f\cdot W^{-1}=(\rho'_\alpha\rho_\alpha)^{1/2}f\circ\alpha^{-1}$ for all f in $C_c(\mathcal{G})$. □

Recall that if A is a C^*-algebra and if β is an automorphism of A then the mapping torus of A is the C^*-algebra

$$A_\beta=\{f\in C([0,1],A)\text{ such that }\beta(f(1))=f(0)\}.$$

Namely, the mapping torus A_β can be viewed as the algebra of continuous function $h:\mathbb{R}\to A$ such that $h(t)=\beta(h(t+1))$ for all t in $\mathbb{R}$. In this picture, there is an action of $\mathbb{R}$ on A_β by translations defined for t in $\mathbb{R}$ and f in A_β by $t\cdot f(s)=f(t-s)$ for any s in $\mathbb{R}$. Translations then define a strongly continuous action by automorphisms $\widehat{\beta}$ of $\mathbb{R}$ on A_β. By the mapping torus isomorphism, we have a natural Morita equivalence between $A\rtimes_\beta\mathbb{Z}$ and $A\rtimes_{\widehat{\beta}}\mathbb{R}$.

Let α be an automorphism of a groupoid $\mathcal{G}$ preserving a Haar system λ and with density ρ_α. For a function f in $C_c(\mathcal{G}_\alpha)$, we define $\hat{f}$ in $C_c([0,1]\times\mathcal{G})\subset C([0,1],C_r^*(\mathcal{G},\lambda))$ by $\hat{f}(t,\gamma)=\rho_\alpha^{-t/2}(\gamma){\rho'}_\alpha^{-t/2}(\gamma)f([\gamma,t])$. We can check easily that $\hat{f}$ belongs to the mapping torus $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$.

Proposition 5.6. *Let $\mathcal{G}$ be a locally compact groupoid with basis space X equipped with a Haar system $\lambda=(\lambda^x)_{x\in X}$ and let $\alpha:\mathcal{G}\to\mathcal{G}$ be an automorphism preserving the Haar System λ such that $\rho_\alpha\circ\alpha=\rho_\alpha$. Then there is a unique automorphism of C^*-algebras*

$$\Lambda_\alpha:C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)\to C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$$

such that $\Lambda_\alpha(f)=\hat{f}$ for any f in $C_c(\mathcal{G}_\alpha)$.

Proof. Let f be a function of $C_c(\mathcal{G}_\alpha)$. Then

$$\begin{aligned}\|\hat{f}\|_{C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}} &= \sup_{t\in[0,1]}\|\hat{f}(t,\bullet))\|_{C_r^*(\mathcal{G},\lambda)}\\ &= \sup_{t\in[0,1],\,x\in X}\|\nu_x(\hat{f}(t,\bullet))\|\end{aligned}$$

On the other hand, $\|f\|_{C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)} = \sup_{t\in[0,1],\, x\in X} \|\nu_{[x,t]}(f)\|$, where $\nu_{[x,t]}$ is the representation of $C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)$ on the fiber $\mathcal{L}^2(\mathcal{G}_{\alpha,[x,t]},\lambda_{\alpha,[x,t]})$ at the fiber $[x,t]\in(X\times\mathbb{R})/\mathcal{A}_\alpha$. If we define for t in $[0,1]$ the map $\pi_t:\mathcal{G}\to\mathcal{G}_\alpha:\gamma\mapsto[\gamma,t]$, then

$$C_c(\mathcal{G}_{[x,t]})\to C_c(\mathcal{G}_x):\phi\mapsto {\rho'}_\alpha^{-t/2}\phi\circ\pi_t$$

extends to an isometry $W_t:\mathcal{L}^2(\mathcal{G}_{\alpha,[x,t]},\lambda_{\alpha,[x,t]})\to\mathcal{L}^2(\mathcal{G}_x,\lambda_x)$ and W_t conjugate $\nu_{[x,t]}(f)$ and $\nu_x(\hat{f}(t,\bullet))$. Thus $\|\hat{f}\|_{C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}}=\|f\|_{C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)}$ and

$$C_c(\mathcal{G}_\alpha)\to C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}};\ f\mapsto\hat{f}$$

extends to a monomorphism $\Lambda_\alpha:C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)\to C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$. The set

$$\mathcal{A}_\alpha=\{h\in C_c([0,1]\times\mathcal{G})\text{ such that }h(1,\alpha(\gamma))={\rho'}_\alpha^{t/2}\rho_\alpha^{t/2}h(0,\gamma)\text{ for all }\gamma\in\Gamma\}$$

is dense in $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$. Let us define for an element h of $\mathcal{A}_\alpha$ the map $\tilde{h}:\mathcal{G}\times\mathbb{R}\to\mathbb{C}$ as the unique map such that

- $\tilde{h}(\gamma,t)={\rho'}_\alpha^{-t/2}\rho_\alpha^{-t/2}h(t,\gamma)$ for all γ in $\mathcal{G}$ and t in $[0,1]$;
- $h(\alpha(\gamma),t)=h(\gamma,t+1)$ for all γ in $\mathcal{G}$ and t in $\mathbb{R}$.

Then $\tilde{h}$ defines a continuous map of $C_c(\mathcal{G}_\alpha)$ whose image under Λ_α is h. Hence Λ_α has dense range in $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$ and thus is surjective. □

Remark 5.7. *With the notations of the above proposition, let us define for a real s the automorphism of groupoid $\theta_s:\mathcal{G}_\alpha\to\mathcal{G}_\alpha$; $[\gamma,t]\mapsto[\gamma,s+t]$. Then θ_s is preserving the Haar system $\lambda_\alpha=(\lambda^{[x,t]})_{[x,t]\in X\times\mathbb{R}/\mathbb{Z}}$ with density $\mathcal{G}_\alpha\to\mathbb{R}$; $[\gamma,t]\mapsto\rho_\alpha(\gamma)^s$. We obtain from lemma 5.5 an automorphism $\tilde{\theta}_s$ of $C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)$ which gives rise to a strongly continous action of $\mathbb{R}$ on $C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)$ by automorphism. On the other hand, the isomorphism $\Lambda_\alpha:C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)\to C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$ of proposition 5.6 is then $\mathbb{R}$-equivariant, where the action of $\mathbb{R}$ on $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$ is the action $\widehat{\tilde{\alpha}}$ associated to a mapping torus.*

6. The dynamic of the uncolored Penrose tiling under translations

As we have seen before, the continuous hull $X_{\mathcal{P}}^{\mathbb{R}}$ of the action of the uncolored Penrose tiling $\mathcal{P}$ under the actions of $\mathbb{R}$ by translations is the suspension $\frac{\Omega\times\mathbb{R}}{\mathcal{A}_o}$ of the odometer homeomorphism $o:\Omega\to\Omega$; $x\mapsto x+1$, where Ω is the dyadic completion of the integers. The $\mathbb{R}$-algebra $C(\frac{\Omega\times\mathbb{R}}{\mathcal{A}_o})$ is then the mapping torus algebra of $C(\Omega)$ with respect to automorphism induced by o. In consequence, the crossed-product algebras $C(X_{\mathcal{P}}^{\mathbb{R}})\rtimes\mathbb{R}$ and $C(\Omega)\rtimes\mathbb{Z}$ are Morita equivalent. The purpose of this section is to recall the explicit description of the isomorphism $C(\Omega)\rtimes\mathbb{Z}\xrightarrow{\cong}C(X_{\mathcal{P}}^{\mathbb{R}})\rtimes\mathbb{R}$ arising from this Morita equivalence.

For this, let us define on $C_c(\Omega\times\mathbb{R})$ the $C(\Omega)\rtimes\mathbb{Z}$-valued inner product

$$\langle\xi,\,\xi'\rangle(\omega,k)=\int_{\mathbb{R}}\bar{\xi}(\omega,s)\xi'(\omega-k,s+k)ds$$

for ξ and ξ' in $C_c(\Omega\times\mathbb{R})$ and (ω,k) in $(\Omega\times\mathbb{R})$. This inner product is positive and gives rise to a right $C(\Omega)\rtimes\mathbb{Z}$-Hilbert module $\mathcal{E}$, the action of $C(\Omega)\rtimes\mathbb{Z}$ being given for h in $C_c(\Omega\times\mathbb{Z})$ and ξ in $C_c(\Omega\times\mathbb{R})$ by

$$\xi\cdot h(\omega,t)=\sum_{n\in\mathbb{Z}}\xi(n+\omega,t-n)h(n+\omega,n)$$

for (ω,k) in $\Omega\times\mathbb{R}$. The right $C(\Omega)\rtimes\mathbb{Z}$-Hilbert module $\mathcal{E}$ is also equipped with a left action of $C\left(\frac{\Omega\times\mathbb{R}}{\mathcal{A}_o}\right)\rtimes\mathbb{R}$ given for f in $C_c\left(\frac{\Omega\times\mathbb{R}}{\mathcal{A}_o}\times\mathbb{R}\right)$ and ξ in $C_c(\Omega\times\mathbb{R})$ by

$$f\cdot\xi(\omega,t)=\int_{\mathbb{R}}f([\omega,t],s)\xi(\omega,t-s)ds$$

for (ω,k) in $\Omega\times\mathbb{R}$. We get in this way a $C\left(\frac{\Omega\times\mathbb{R}}{\mathcal{A}_o}\right)\rtimes\mathbb{R}-C(\Omega)\rtimes\mathbb{Z}$ imprimitivity bimodule which implements the Morita equivalence we are looking for. Actually, there is an isomorphism of right $C(\Omega)\rtimes\mathbb{Z}$-Hilbert module

$$\Psi:\mathcal{E}\to L^2([0,1])\otimes C(\Omega)\rtimes\mathbb{Z}$$

defined in a unique way by $\Psi(g)=g\otimes\delta_0$ for g in $C_c(\mathbb{R})$ supported in $(0,1)$ (here δ_0 is the Dirac function at $0\in\mathbb{Z}$). Using the right $C\left(\frac{\Omega\times\mathbb{R}}{\mathcal{A}_o}\right)\rtimes\mathbb{R}$-module structure of the $C\left(\frac{\Omega\times\mathbb{R}}{\mathcal{A}_o}\right)\rtimes\mathbb{R}-C(\Omega)\rtimes\mathbb{Z}$ imprimitivity bimodule $\mathcal{E}$ and the isomorphism Ψ, we get an isomorphism

$$C\left(\frac{\Omega\times\mathbb{R}}{\mathcal{A}_o}\right)\rtimes\mathbb{R}\stackrel{\cong}{\to}\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}. \tag{6.1}$$

This isomorphism can be described as follows. Let us define for f and g in $L^2([0,1])$ the rank one operator

$$\Theta_{f,g}:L^2([0,1])\to L^2([0,1]);\ h\mapsto f\langle g,h\rangle.$$

We define for ξ and ξ' in $C_c(\Omega\times\mathbb{R})$ the continuous function of $C_c\left(\frac{\Omega\times\mathbb{R}}{\mathcal{A}_o}\times\mathbb{R}\right)$

$$\Theta^{\Omega}_{\xi,\xi'}([\omega,s],t)=\sum_{k\in\mathbb{Z}}\xi(\omega+k,s-k)\bar{\xi}'(\omega+k,s-t-k)$$

for all ω in Ω and s and t in $\mathbb{R}$. It is straightforward to check that $\Theta^{\Omega}_{\xi,\xi'}$ is well defined and that

$$\Theta^{\Omega}_{\xi,\xi'}\cdot\eta=\xi\langle\xi',\eta\rangle$$

for all η in $C_c(\Omega\times\mathbb{R})$. If we set for f and g in $C_c(\mathbb{R})$ with support in $(0,1)$ and for ϕ in $C(\Omega)$, $\xi=1\otimes f$, $\xi'=\phi\otimes g$ and $\xi'':\Omega\times\mathbb{R}\to\mathbb{R};(\omega,t)\mapsto g(t+1)$, then the image of $\Theta^{\Omega}_{\xi,\xi'}$ under the isomorphism of equation 6.1 is $\Theta_{f,g}\otimes\phi\in\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$ and moreover,

$$\Theta^{\Omega}_{\xi,\xi'}([\omega,s],t)=\sum_{k\in\mathbb{Z}}f(s-k)\bar{\phi}(\omega+k)\bar{g}(s-t-k). \tag{6.2}$$

Let us denote by u the unitary of $C(\Omega)\rtimes\mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$. Then the image of $\Theta^{\Omega}_{\xi,\xi''}$ under the ismorphism of equation 6.1 is $\Theta_{f,g}\otimes u\in\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$ and moreover,

$$\Theta^{\Omega}_{\xi,\xi''}([\omega,s],t)=\sum_{k\in\mathbb{Z}}f(s-k)\bar{g}(s+1-t-k). \tag{6.3}$$

Let us define by α the automorphism of the groupoid $\frac{\Omega\times\mathbb{R}}{\mathbb{Z}}\rtimes\mathbb{R}$ by

- $\alpha([\omega,s],t)=([\omega/2,s/2],t/2)$ if ω is even;
- $\alpha([\omega,s],t)=([(\omega+1)/2,(s+1)/2],t/2)$ if ω is odd.

Notice that $\alpha^{-1}([\omega,s],t)=([2\omega,2s],2t)$ for all ω in Ω and s and t in $\mathbb{R}$. Then α preserves the Haar system of $\frac{\Omega\times\mathbb{R}}{\mathbb{Z}}\rtimes\mathbb{R}$ arising from the Haar mesure on $\mathbb{R}$ and has constant density $\rho_\alpha=2$. Hence according to lemma 5.5, the automorphism of groupoid α induces an automorphism $\tilde{\alpha}$ of C^*-algebra $C\left(\frac{\Omega\times\mathbb{R}}{\mathbb{Z}}\right)\rtimes\mathbb{R}$ such that $\tilde{\alpha}(h)=2h\circ\alpha^{-1}$ for all h in $C\left(\frac{\Omega\times\mathbb{R}}{\mathbb{Z}}\times\mathbb{R}\right)$. We are now in position to describe how $\tilde{\alpha}$ is transported under the isomorphism of Equation (6.1) to an automorphism Υ of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$. With ξ, ξ' and ξ'' as defined above,

$$\begin{aligned}\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi'})([\omega,s],t) &= 2\Theta^{\Omega}_{\xi,\xi'}([2\omega,2s],2t)\\ &= 2\sum_{k\in\mathbb{Z}} f(2s-k)\bar{\phi}(2\omega+k)\bar{g}(2s-2t-k)\\ &= 2\sum_{k\in\mathbb{Z}} f(2s-2k)\bar{\phi}(2\omega+2k)\bar{g}(2s-2t-2k)+\\ &\quad 2\sum_{k\in\mathbb{Z}} f(2s-2k-1)\bar{\phi}(2\omega+2k-1)\bar{g}(2s-2t-2k-1)\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi''})([\omega,s],t) &= 2\Theta^{\Omega}_{\xi,\xi''}([2\omega,2s],2t)\\ &= 2\sum_{k\in\mathbb{Z}} f(2s-k)\bar{g}(2s+1-2t-k)\\ &= 2\sum_{k\in\mathbb{Z}} f(2s-2k)\bar{g}(2s+1-2t-2k)+\\ &\quad 2\sum_{k\in\mathbb{Z}} f(2s-2k-1)\bar{g}(2s-2t-2k).\end{aligned} \tag{6.5}$$

To complete the description of the automorphism of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$ corresponding to $\tilde{\alpha}$, we will need to introduce some extra notations. We define the partial isometries U_0,U_1 and V of $L^2([0,1])$ by

- $U_0f(t)=\sqrt{2}f(2t)$ if $t\in[0,1/2]$ and $U_0f(t)=0$ otherwise;
- $U_1f(t)=\sqrt{2}f(2t-1)$ if $t\in[1/2,1]$ and $U_1f(t)=0$ otherwise;
- $Vf(t)=f(t+1/2)$ if $t\in[0,1/2]$ and $Vf(t)=0$ otherwise,

for f in $C([0,1])$. Let use define also the endomorphisms W_0 and W_1 of the C^*-algebra $C(\Omega)$ by $W_0\phi(\omega)=\phi(2\omega)$ and $W_1\phi(\omega)=\phi(2\omega+1)$, for ϕ in $C(\Omega)$ and ω in Ω. Using this notations, equations 6.4 and 6.5 can be rewriten

$$\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi'})([\omega,s],t)=\sum_{k\in\mathbb{Z}}U_0f(s-k)V_0\bar{\phi}(\omega+k)U_0\bar{g}(s-t-k)+\sum_{k\in\mathbb{Z}}U_1f(s-k)V_1\bar{\phi}(\omega+k)U_1\bar{g}(s-t-k)$$

and

$$\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi''})([\omega,s],t)=\sum_{k\in\mathbb{Z}}U_0f(s-k)U_1\bar{g}(s-t-k+1)+\sum_{k\in\mathbb{Z}}U_1f(s-k)U_0\bar{g}(s-t-k).$$

Thus, in view of equations 6.4 and 6.5, we get that

$$\Upsilon(\Theta_{f,g}\otimes\phi)=\Theta_{U_0f,U_0g}\otimes V_0\phi+\Theta_{U_1f,U_1g}\otimes V_1\phi$$

and

$$\Upsilon(\Theta_{f,g}\otimes u)=\Theta_{U_0f,U_1g}\otimes u+\Theta_{U_1f,U_0g}\otimes 1.$$

From this we deduce

$$\Upsilon(k\otimes\phi)=U_0\cdot k\cdot U_0^*\otimes V_0\phi+U_1\cdot k\cdot U_1^*\otimes V_1\phi$$

and

$$\begin{aligned}\Upsilon(k\otimes u) &= U_0\cdot k\cdot U_1^*\otimes u+U_1\cdot k\cdot U_0^*\otimes 1\\ &= U_0\cdot k\cdot U_0^*\cdot V\otimes u+U_1\cdot k\cdot U_1^*\cdot V^*\otimes 1\\ &= (U_0\cdot k\cdot U_0^*+U_1\cdot k\cdot U_1^*)\cdot(V\otimes u+\cdot V^*\otimes 1)\end{aligned}$$

where the second equality holds since $V^*\cdot U_0=U_1$ and $V\cdot U_1=U_0$ and the third holds since $V^*U_1=VU_0=0$. In consequence, if we extends Υ to the multiplier algebra of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$, we finally obtain that the automorphism Υ is the unique morphism of C^*-algebra such that

$$\Upsilon(k\otimes\phi)=U_0^*\cdot k\cdot U_0\otimes V_0\phi+U_1^*\cdot k\cdot U_1\otimes V_1\phi$$

and

$$\Upsilon(1\otimes u)=V\otimes u+V^*\otimes 1, \tag{6.6}$$

where k is in $\mathcal{K}(L^2([0,1]))$, ϕ is in $C(\Omega)$ and $1\otimes u$ and $V\otimes u+V^*\otimes 1$ are viewed as multipliers of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$.

The following lemma will be useful to compute the K-theory of the C^*-algebra of the Penrose hyperbolic tiling. For short, we will denote from now $\mathcal{K}(L^2([0,1]))$ by $\mathcal{K}$.

Lemma 6.1. *Let A be the unitarisation of $\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$ and let f be a norm one function of $L^2([0,1])$. Then the unitaries*

$$(1-\Theta_{f,f}\otimes 1)+\Theta_{f,f}\otimes u$$

and

$$\Theta_{U_0f,U_1f}\otimes u+\Theta_{U_1f,U_0f}\otimes 1+1-\Theta_{U_0f,U_0f}\otimes 1-\Theta_{U_1f,U_1f}\otimes 1 \tag{6.7}$$

of A are homotopic.

Proof. If we set $f_0=f$ and complete to a Hilbertian basis $f_0,\dots,f_n,\dots$ of $L^2([0,1])$, then $U_0f_0,\dots,U_0f_n,\dots;U_1f_0,\dots,U_1f_n,\dots$ is a Hilbertian basis of $L^2([0,1])$. In this basis the unitary of equation 6.7 can be written down

$$\left(\begin{array}{ccc|ccc} 0 & & & u & & \\ & 1 & & & 0 & \\ & & \ddots & & & \ddots \\ \hline 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & \ddots & & & \ddots \end{array}\right)$$

which is homotopic to

$$\left(\begin{array}{ccc|ccc} u & & & 0 & & \\ & 1 & & & 0 & \\ & & \ddots & & & \ddots \\ \hline 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & \ddots & & & \ddots \end{array}\right).$$

All unitaries that can be writen down in such way in some hilbertian basis of $L^2([0,1])$ are homotopic and since this is the case for $1-\Theta_{f,f}\otimes 1+\Theta_{f,f}\otimes u$, we get the result. □

7. The C^*-algebra of a Penrose hyperbolic tiling

Let us consider the semi-direct product groupoid $\mathcal{G}=(X^{\mathbb{R}}_{\mathcal{P}}\times Z_w)\rtimes\mathbb{R}$ corresponding to the diagonal action of $\mathbb{R}$ on $X^{\mathbb{R}}_{\mathcal{P}}\times Z_w$, by translation on $X^{\mathbb{R}}_{\mathcal{P}}$ and trivial on Z_ω. Let us denote by $\lambda=(\lambda_{(\mathcal{P}',\omega)})_{(\mathcal{P}',\omega)\in X^{\mathbb{R}}_{\mathcal{P}}\times Z_w}$ the Haar system provided by the left Haar mesure on $\mathbb{R}$. Let us define the groupoid automorphism $\alpha_\omega:\mathcal{G}\to\mathcal{G};\,(\mathcal{P}',\omega',t)\mapsto(R\cdot\mathcal{P}',\sigma(\omega'),2t)$. Then α_ω preserves the Haar system λ with constant density $\rho_{\alpha_\omega}=1/2$ and thus according to lemma 5.4 the mapping torus groupoid $\mathcal{G}_{\alpha_\omega}$ admits a Haar system λ_{α_ω}. The semi-direct product groupoid $X^G_{\mathcal{P}(w)}\rtimes\mathbb{R}$, where $\mathbb{R}$ acts on $X^G_{\mathcal{P}(w)}$ by translations, is equipped with an action of $\mathbb{R}$ by automorphisms $\beta_t:X^G_{\mathcal{P}(w)}\rtimes\mathbb{R}\to X^G_{\mathcal{P}(w)}\rtimes\mathbb{R};\,(\mathcal{T},s)\mapsto(2^t\cdot\mathcal{T},2^ts)$ for any t in $\mathbb{R}$. The automorphism β_t preserves the Haar system with constant density 2^t and thus in view of proposition 5.6 induced a strongly continuous action of $\mathbb{R}$ on the C^*-algebra of the semi-direct groupoid $\mathcal{G}$.

Lemma 7.1. *Let ω be an element of $\{1,\dots,n\}^{\mathbb{Z}}$. Then there is a unique isomorphism of groupoids $\Psi_\omega:\mathcal{G}_{\alpha_\omega}\longrightarrow X^G_{\mathcal{P}(w)}\rtimes\mathbb{R}$ such that:*

1. *$\Psi_\omega([\mathcal{P}+x,\omega,y,0])=(\mathcal{P}(\omega)+x,y)$ for all x and y in $\mathbb{R}$;*
2. *Ψ_ω is equivariant with respect to the actions of $\mathbb{R}$;*
3. *$\Psi_{\omega,*}\lambda_{\alpha_\omega}$ is the Haar system on $X^G_{\mathcal{P}(w)}\rtimes\mathbb{R}$ provided by the Haar measure on $\mathbb{R}$;*
4. *The map*
$$C_c(X^G_{\mathcal{P}(w)}\rtimes\mathbb{R})\to C_c(\mathcal{G}_{\alpha_\omega});\,f\mapsto f\circ\Psi_\omega$$
induced a $\mathbb{R}$-equivariant isomorphism
$$\widetilde{\Psi}_\omega:C_0(X^G_{\mathcal{P}(w)})\rtimes\mathbb{R}\to C^*(\mathcal{G}_{\alpha_\omega},\lambda_{\alpha_\omega}).$$

Proof. For $\mathcal{T}$ in $X^{\mathbb{R}}_{\mathcal{P}}$ and ω' in $\{1,...n\}^{\mathbb{Z}}$, let us define the Penrose hyperbolic tiling $\mathcal{T}(\omega')$ by coloring $\mathcal{T}$ with ω'. Let us denote by R^t the action of 2^t on $X^G_{\mathcal{P}(w)}$. If ω' belongs to Z_ω, then $\mathcal{T}(\omega')$ belongs to $X^G_{\mathcal{P}(w)}$ and $X^{\mathbb{R}}_{\mathcal{P}}\times Z_\omega\to X^G_{\mathcal{P}(w)};\,(\mathcal{T},\omega')\mapsto\mathcal{T}(\omega')$ is continuous. Since $(\mathcal{T}\cdot R)(\sigma(\omega'))=\mathcal{T}(\omega')\cdot R$ for all $\mathcal{T}$ in $X^{\mathbb{R}}_{\mathcal{P}}$ and ω' in Z_ω, the continuous map
$$\mathcal{G}\times\mathbb{R}\to X^G_{\mathcal{P}(w)}\rtimes\mathbb{R};\,(\mathcal{T},\omega',x,y)\mapsto(\mathcal{T}(\omega')R^y,2^yx)$$
induces a continuous morphism of groupoids
$$\Psi_\omega:\mathcal{G}_{\alpha_\omega}\to X^G_{\mathcal{P}(w)}\rtimes\mathbb{R}.$$
This map is clearly injective since the equality $\mathcal{T}(\omega')R^t=\mathcal{T}'(\omega'')$ for t in $\mathbb{R}$, $\mathcal{T}$ and $\mathcal{T}'$ in $X^{\mathbb{R}}_{\mathcal{P}}$ and ω' and ω'' in Z_ω holds if and only if t is integer, $\omega'=\sigma^t(\omega'')$ and $\mathcal{T}=\mathcal{T}'R^t$. To prove surjectivity, let us remark that any element of $X^G_{\mathcal{P}(w)}$ can be written as $\mathcal{T}(\omega')R^a$, with a in $\mathbb{R}$ and $\mathcal{T}$ in $X^{\mathbb{R}}_{\mathcal{P}}$. We get then
$$\Psi_\omega([\mathcal{T},\omega',2^{-a}t,a])=(\mathcal{T}(\omega')R^a,t)$$
for all t in $\mathbb{R}$.

It is then straightforward to check that condition (3) of the lemma is satisfied. The uniqueness of Ψ_ω is a consequence on one hand of the equivariance and on the other hand of the density of the $\mathbb{R}$-orbit of $\mathcal{P}$ in $X_{\mathcal{P}}^{\mathbb{R}}$. Condition (4) follows then from condition (3).

□

Proposition 7.2. *Using the notations of lemmas 5.5 and 7.1, the C^*-algebras $C(X^G_{\mathcal{P}(w)})\rtimes G$ and $C_r^*(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_\omega}}\mathbb{Z}$ are Morita equivalent.*

Proof. Recall that $G=\mathbb{R}\rtimes\mathbb{R}_+^*$, where the group $(\mathbb{R}_+^*,\cdot)$ acts on $(\mathbb{R},+)$ by multiplication. Iterate crossed-products leads to an isomorphism

$$C(X^G_{\mathcal{P}(w)})\rtimes G\cong (C(X^G_{\mathcal{P}(w)})\rtimes\mathbb{R})\rtimes\mathbb{R}_+^*.$$

If we identify the groups $(\mathbb{R},+)$ and $(\mathbb{R}_+^*,\cdot)$ using the isomorphism

$$\mathbb{R}\to\mathbb{R}_+^*;\, t\mapsto 2^t,$$

this provides the action under consideration in lemma 7.1 of $\mathbb{R}$ on $C(X^G_{\mathcal{P}(w)})\rtimes\mathbb{R}$ and hence, the algebras $C(X^G_{\mathcal{P}(w)})\rtimes G$ and $C^*(\mathcal{G}_{\alpha_\omega},\lambda_{\alpha_\omega})$ are isomorphic. In view of lemma 5.6, the C^*-algebra $C(X^G_{\mathcal{P}(w)})\rtimes G$ is isomorphic to $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}\rtimes\mathbb{R}$. But since $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$ is the mapping torus algebra with respect to the automorphism $\tilde{\alpha}:C_r^*(\mathcal{G},\lambda)\to C_r^*(\mathcal{G},\lambda)$, the crossed product C^*-algebra $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}\rtimes\mathbb{R}$ is Morita equivalent to $C_r^*(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha}}\mathbb{Z}$ and hence we get the result. □

8. The K-theory of the C^*-algebra of a Penrose hyperbolic tiling

Let us consider the semi-direct groupoid $\mathcal{G}=(X_{\mathcal{P}}^{\mathbb{R}}\times Z_w)\rtimes\mathbb{R}$ corresponding to the diagonal action of $\mathbb{R}$ on $X_{\mathcal{P}}^{\mathbb{R}}\times Z_w$, by translation on $X_{\mathcal{P}}^{\mathbb{R}}$ and trivial on Z_ω. According to proposition 7.2 we have an isomorphism

$$K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)\stackrel{\cong}{\to}K_*(C_r^*(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_\omega}}\mathbb{Z})$$

induced by the Morita equivalence. In order to compute this K-theory group. We will need to recall some basic facts concerning the K-theory group of a crossed product of a C^*-algebra A by an action of $\mathbb{Z}$ provided by an automorphism θ of A. This K-theory can be computed by using the Pimnsner-Voiculescu exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\theta_*-Id} & K_0(A) & \xrightarrow{\iota_*} & K_0(A\rtimes_\theta\mathbb{Z})\\ \uparrow & & & & \downarrow\\ K_1(A\rtimes_\theta\mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\theta_*-Id} & K_1(A)\end{array},$$

where ι_* is the morphism induced in K-theory by the inclusion $\iota:A\hookrightarrow A\rtimes_\theta\mathbb{Z}$ and θ_* is the morphism in K-theory induced by θ. The vertical maps $K_*(A\rtimes_\theta\mathbb{Z})\longrightarrow K_{*+1}(A)$ are given by the composition

$$K_*(A\rtimes_\theta\mathbb{Z})\stackrel{\cong}{\longrightarrow}K_*(A_\theta\rtimes_{\widehat{\theta}}\mathbb{R})\stackrel{\cong}{\longrightarrow}K_{*+1}(A_\theta)\stackrel{\mathrm{ev}_*}{\longrightarrow}K_{*+1}(A),$$

where

- A_θ is the mapping torus of A with respect to the action θ endowed, with its associated action $\widehat{\theta}$ of $\mathbb{R}$;
- the first map is induced by the Morita equivalence between $A\rtimes_\theta\mathbb{Z}$ and $A_\theta\rtimes_{\widehat{\theta}}\mathbb{R}$;
- the second map is the Thom-Connes isomorphism;

- the third map is induced in K-theory by the evaluation map $\mathrm{ev} : A_\theta \to A;\ f \mapsto f(0)$.

This yields short exact sequences

$$0 \to \operatorname{Coinv} K_0(A) \to K_0(A \rtimes_\theta \mathbb{Z}) \to \operatorname{Inv} K_1(A) \to 0 \tag{8.1}$$

and

$$0 \to \operatorname{Coinv} K_1(A) \to K_1(A \rtimes_\theta \mathbb{Z}) \to \operatorname{Inv} K_0(A) \to 0, \tag{8.2}$$

where $\operatorname{Inv} K_*(A)$ and $\operatorname{Coinv} K_*(A)$ stand respectively for the invariants and the coinvariants group of $K_*(A)$ with respect to the action of $\mathbb{Z}$ provided by the aumorphism of group θ_* (recall that for an abelian group M equipped with an automorphism Ψ then the group of coinvariants of M is the quotient group of M by the subgroup $\{x - \Psi(x), x \in M\}$). Moreover the inclusions in these exact sequence are induced by ι_*. In order to compute $K_*(C^*_r(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_\omega}} \mathbb{Z})$, we will need the following lemma which is straightforward to prove.

Lemma 8.1. *Let Z be a Cantor set and let us denote by $C(Z,\mathbb{Z})$ the algebra of continuous and integer valued functions on Z. For a compact open subset E of Z we denote by χ_E the characteristic function of E.*

1. *we have an isomorphism $C(Z,\mathbb{Z}) \to K_0(C(Z));\ \chi_E \mapsto [\chi_E]$.*
2. *$K_1(C(Z)) = \{0\}$.*

Plugging $C^*_r(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_\omega}} \mathbb{Z}$ into the short exact sequences (8.1) and (8.2), we get

$$0 \to \operatorname{Coinv} K_0(C^*_r(\mathcal{G},\lambda)) \to K_0(C^*_r(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_\omega}} \mathbb{Z}) \to \operatorname{Inv} K_1(C^*_r(\mathcal{G},\lambda)) \to 0 \tag{8.3}$$

and

$$0 \to \operatorname{Coinv} K_1(C^*_r(\mathcal{G},\lambda)) \to K_1(C^*_r(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_\omega}} \mathbb{Z}) \to \operatorname{Inv} K_0(C^*_r(\mathcal{G},\lambda)) \to 0. \tag{8.4}$$

According to equation 6.1, the C^*-algebra $C^*_r(\mathcal{G},\lambda)$ is isomorphic to $C(Z_\omega) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$. The K-theory of $C^*_r(\mathcal{G},\lambda)$ can be the computed by using the Künneth formula: In view of lemma 8.1, $K_0(C(Z_\omega)) \cong C(Z_\omega,\mathbb{Z})$ is torsion free and $K_1(C(Z_\omega)) = \{0\}$ and by Morita equivalence, we get that

$$K_0(C^*_r(\mathcal{G},\lambda)) \cong C(Z_\omega,\mathbb{Z}) \otimes K_0(C(\Omega) \rtimes \mathbb{Z})$$

and

$$K_1(C^*_r(\mathcal{G},\lambda)) \cong C(Z_\omega,\mathbb{Z}) \otimes K_1(C(\Omega) \rtimes \mathbb{Z}).$$

This isomorphism, up to the Morita equivalence and to the isomorphism of equation (6.1) are implemented by the external product in K-theory and will be precisely described later on. Once again, $K_*(C(\Omega) \rtimes \mathbb{Z})$ can be computed from the short exact sequences (8.1) and (8.2), and we get, using lemma 8.1 that

$$K_0(C(\Omega) \rtimes \mathbb{Z}) \cong \operatorname{Coinv} C(\Omega,\mathbb{Z}) \tag{8.5}$$

and

$$K_1(C(\Omega) \rtimes \mathbb{Z}) \cong \operatorname{Inv} C(\Omega,\mathbb{Z}) \cong \mathbb{Z}. \tag{8.6}$$

The isomorphism of equation (8.5) is induced by the composition

$$C(\Omega, Z) \overset{\cong}{\hookrightarrow} K_0(C(\Omega)) \to K_0(C(\Omega) \rtimes \mathbb{Z}),$$

which factorizes through $\operatorname{Coinv} C(\Omega, Z)$, where the first map is described in lemma 8.1, and the second map is induced on K-theory by the inclusion $C(\Omega) \hookrightarrow C(\Omega) \rtimes \mathbb{Z}$. In the first isomorphism of equation 8.6 the class of $[u]$ in $K_1(C(\Omega) \rtimes \mathbb{Z})$ of the unitary u of $C(\Omega) \rtimes \mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$ is mapped to the constant function 1 of $C(\Omega, \mathbb{Z})$.

Lemma 8.2. *Let ν be the Haar measure on Ω. Then*

(1) *$\int f d\nu$ is in $\mathbb{Z}[1/2]$ for all f in $C(\Omega, \mathbb{Z})$;*
(2) *$C(\Omega, \mathbb{Z}) \to \mathbb{Z}[1/2]; f \mapsto \int f d\nu$ factorizes through an isomorphism*

$$\operatorname{Coinv} C(\Omega, \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}[1/2].$$

Proof. It is enought to check the first point for characteristic function of compact-open subset of Ω. For an integer n and k in $\{0, \ldots, 2^{n-1}\}$, we set $F_{n,k} = 2^n\Omega + k$. Then $(F_{n,k})_{n\in\mathbb{N},\, 0\leq k\leq 2^{n-1}}$ is a basis of compact open neighborhoods for Ω and thereby, every compact open subset of Ω is a disjoint union of some $F_{n,k}$. Since $\nu(F_{n,k}) = 2^{-n}$, we get the first point.

The measure μ being invariant by translation, the map

$$C(\Omega, \mathbb{Z}) \to \mathbb{Z}[1/2]; f \mapsto \int f d\nu$$

factorizes through a group morphism $\operatorname{Coinv} C(\Omega, \mathbb{Z}) \to \mathbb{Z}[1/2]$. This morphism admits a cross-section

$$\mathbb{Z}[1/2] \to \operatorname{Coinv} C(\Omega, \mathbb{Z});\ 2^{-n} \mapsto [\chi_{F_{n,0}}]. \tag{8.7}$$

This map is well defined since $F_{n,0} = F_{n+1,0} \coprod (2^n + F_{n+1,0})$ and thus $[\chi_{F_{n,0}}] = [\chi_{F_{n+1,0}}] + [\chi_{2^n + F_{n+1,0}}] = 2[\chi_{F_{n+1,0}}]$ in $\operatorname{Coinv} C(\Omega, \mathbb{Z})$. Since the $(\chi_{F_{n,k}})_{n\in\mathbb{N},\, 0\leq k\leq 2^{n-1}}$ generates $C(\Omega, \mathbb{Z})$ as an abelian group, it is enought to check that the cross-section of equation 8.7 is a left inverse on $\chi_{F_{n,k}}$, which is true since $[\chi_{F_{n,k}}] = [\chi_{k+F_{n,0}}] = [\chi_{F_{n,0}}]$ in $\operatorname{Coinv} C(\Omega, \mathbb{Z})$. □

Proposition 8.3. *Let $C(Z_\omega, \mathbb{Z}[1/2]) \cong C(Z_\omega, \mathbb{Z}) \otimes \mathbb{Z}[1/2]$ be the algebra of continuous function on Z_ω, valued in $\mathbb{Z}[1/2]$ (equipped with the discrete topology). Then with the notations of the proof of lemma 8.2, we have isomorphisms*

(1)

$$\begin{array}{ccc} C(Z_\omega, \mathbb{Z}[1/2]) & \stackrel{\cong}{\longrightarrow} & K_0(C(Z_\omega) \otimes C(\Omega) \rtimes \mathbb{Z}) \\ \dfrac{\chi_E}{2^n} & \mapsto & [\chi_E \otimes \chi_{F_{n,0}}], \end{array}$$

where E is a compact open subset of Z_ω and χ_E is its characteristic function.

(2)

$$\begin{array}{ccc} C(Z_\omega, \mathbb{Z}) & \stackrel{\cong}{\longrightarrow} & K_1(C(Z_\omega) \otimes C(\Omega) \rtimes \mathbb{Z}) \\ \chi_E & \mapsto & [\chi_E \otimes u + (1 - \chi_E) \otimes 1], \end{array}$$

where u is the unitary of $C(\Omega) \rtimes \mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$.

Proof. As we have already mentionned, $K_*(C(Z_\omega))$ is torsion free and the Künneth formula provides isomorphisms

$$\begin{array}{ccc} K_0(C(Z_\omega)) \otimes K_0(C(\Omega) \rtimes \mathbb{Z}) & \stackrel{\cong}{\to} & K_0(C(Z_\omega) \otimes C(\Omega) \rtimes \mathbb{Z}) \\ {[p] \otimes [q]} & \mapsto & [p \otimes q], \end{array}$$

where p and q are some matrix projectors with coefficients respectively in $C(Z_\omega)$ and $C(\Omega) \rtimes \mathbb{Z}$, and

$$\begin{array}{rcl} K_0(C(Z_\omega))\otimes K_1(C(\Omega)\rtimes\mathbb{Z}) & \stackrel{\cong}{\to} & K_1(C(Z_\omega)\otimes C(\Omega)\rtimes\mathbb{Z}) \\ [p]\otimes[v] & \mapsto & [p\otimes v+(I_k-p)\otimes I_l], \end{array}$$

where p is a projector in $M_l(C(Z_\omega))$ and v is a unitary in $M_k(C(\Omega)\rtimes\mathbb{Z})$. The proposition is then consequence of lemmas 8.1, 8.2 and of the discution related to equations (8.5) and (8.6). □

In order to compute the invariants and the coinvariants of

$$K_*(C_r^*(\mathcal{G},\lambda))\cong K_*(C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})),$$

we will need a carefull description of the action induced in K-theory by the automorphism $\sigma^*\otimes\mu$ of $C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})$, where σ^* is the automorphism of $C(\Omega)$ induced by the shift σ and where Υ was defined in section 6.

Lemma 8.4. *If we equip $C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$ with the $\mathbb{Z}$-action provided by $\sigma^*\otimes\Upsilon$ and under the $\mathbb{Z}$-equivariant isomorphism*

$$C_r^*(\mathcal{G},\lambda)\cong C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z},$$

the action induced by α_ω on $K_0(C_r^*(\mathcal{G},\lambda)\cong C(Z_\omega,\mathbb{Z}[1/2])$ and on $K_1(C_r^*(\mathcal{G},\lambda)\cong C(Z_\omega,\mathbb{Z})$ are given by the automorphisms of abelian groups

$$\begin{array}{rcl} \Psi_0: C(Z_\omega,\mathbb{Z}[1/2]) & \to & C(Z_\omega,\mathbb{Z}[1/2]) \\ f & \mapsto & 2f\circ\sigma^{-1} \end{array}$$

and

$$\begin{array}{rcl} \Psi_1: C(Z_\omega,\mathbb{Z}) & \to & C(Z_\omega,\mathbb{Z}) \\ f & \mapsto & f\circ\sigma^{-1}. \end{array}$$

Proof. According to proposition 8.3 and to equation (6.4) of section 6 and using the Morita equivalence between $C(Z_\omega)\otimes C(\Omega)\rtimes\mathbb{Z}$ and $C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$, in order to describe Ψ_0, we have to compute the image under $(\sigma^*\otimes\Upsilon)_*$ of $[\chi_E\otimes\Theta_{f,f}\otimes\chi_{F_{n,0}}]\in K_0(C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})$ where,

- χ_E is the characteristic function of a compact open subset E of Z_ω;
- $\chi_{F_{n,0}}$ is the characteristic function of $F_{n,0}=2^n\Omega$ for $n\geq 1$;
- $\Theta_{f,f}$ is the rank one projector associated to a norm 1 function f of $L^2([0,1])$.

We have

$$\begin{array}{rcl} \sigma^*\otimes\Upsilon(\chi_E\otimes\Theta_{f,f}\otimes\chi_{F_{n,0}}) & = & \chi_{\sigma(E)}\otimes\Theta_{U_0f,U_0f}\otimes V_0\chi_{F_{n,0}}+\chi_{\sigma(E)}\otimes\Theta_{U_1f,U_1f}\otimes V_1\chi_{F_{n,0}} \\ & = & \chi_{\sigma(E)}\otimes\Theta_{U_0f,U_0f}\otimes\chi_{F_{n-1,0}}, \end{array}$$

where the last equality holds since $V_0\chi_{F_{n,0}}=\chi_{F_{n-1,0}}$ and $V_1\chi_{F_{n,0}}=0$. Since Θ_{U_0f,U_0f} is again a rank one projector, then up to the Morita equivalence between $C(Z_\omega)\otimes C(\Omega)\rtimes\mathbb{Z}$ and $C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$, the image of $[\chi_E\otimes\chi_{F_{n,0}}]\in K_0(C(Z_\omega)\otimes C(\Omega)\rtimes\mathbb{Z})$ under $(\sigma^*\otimes\Upsilon)_*$ is $[\chi_{\sigma(E)}\otimes\chi_{F_{n-1,0}}]\in K_0(C(Z_\omega)\otimes C(\Omega)\rtimes\mathbb{Z})$. Using proposition 8.3, this completes the description of Ψ_0. For Ψ_1, notice first that up to the isomorphism

$$K_0(C(Z_\omega))\otimes K_0(\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})\stackrel{\cong}{\to}K_1(C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})$$

provided by the Künneth formula, the action of $(\sigma^*\otimes\Upsilon)_*$ is $\sigma^*_*\otimes\Upsilon_*$ and then the result is a consequence of lemma 6.1 and of proposition 8.3. □

Let us equip $C(Z_\omega,\mathbb{Z}[1/2])$ and $C(Z_\omega,\mathbb{Z})$ with the $\mathbb{Z}$-actions respectively provided by Ψ_0 and Ψ_1. Then since $\|\Psi_0(h)\|=\|2h\|$ for any h in $C(Z_\omega,\mathbb{Z}[1/2])$, we get that $\operatorname{Inv} C(Z_\omega,\mathbb{Z}[1/2])=\{0\}$ We are now in position to get a complete description of the K-theory of $C(X^G_{\mathcal{P}(w)})\rtimes G$. In view of the short exact sequences of equations (8.3) and (8.4), the two following theorems are then consequences of lemma 8.4 and of proposition 8.3.

Theorem 8.5. *We have a short exact sequence*

$$0\to \operatorname{Coinv} C(Z_\omega,\mathbb{Z}[1/2])\stackrel{\iota_0}{\to} K_0(C(X^G_{\mathcal{P}(w)})\rtimes G)\to \operatorname{Inv} C(Z_\omega,\mathbb{Z})\to 0,$$

where up to the Morita equivalence $C(X^G_{\mathcal{P}(w)})\rtimes G\cong C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_\omega}}\mathbb{Z}$, the element $\iota_0[2^{-n}\chi_E]$ is the image of $[\chi_E\otimes\Theta_{f,f}\otimes\chi_{F_{n,0}}]\in K_0(C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})$ under the morphism induced in K-theory by the inclusion

$$C(Z_\omega)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}\cong C^*_r(\mathcal{G},\lambda)\hookrightarrow C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_\omega}}\mathbb{Z},$$

where

- χ_E *is the characteristic function of a compact open subset E of Z_ω;*
- $\chi_{F_{n,0}}$ *is the characteristic function of $F_{n,0}=2^n\Omega$;*
- $\Theta_{f,f}$ *is the rank one projector associated to a norm 1 function f of $L^2([0,1])$.*

Theorem 8.6. *We have an isomorphism*

$$\operatorname{Coinv} C(Z_\omega,\mathbb{Z})\stackrel{\cong}{\to} K_1(C(X^G_{\mathcal{P}(w)})\rtimes G)$$

induced on the coinvariants by the composition

$$C(Z_\omega,\mathbb{Z})\cong K_0(C(Z_\omega))\stackrel{\otimes[u]}{\to} K_1(C(Z_\omega)\otimes C(\Omega)\rtimes\mathbb{Z})\cong K_1(C^*_r(\mathcal{G},\lambda)\to K_1(C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_\omega}}\mathbb{Z}),$$

where

- $\otimes[u]$ *is the external poduct in K-theory by the class in $K_1(C(\Omega)\rtimes\mathbb{Z})$ of the unitary u of $C(\Omega)\rtimes\mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$;*
- *the last map in the composition is the morphism induced in K-theory by the inclusion $C^*_r(\mathcal{G},\lambda)\hookrightarrow C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_\omega}}\mathbb{Z}$.*

The short exact sequence of theorem 8.5, admits an explicit splitting which can be describe in the following way: Assume first that (Z_ω,σ) is minimal. In particular, $\operatorname{Inv} C(Z_\omega,\mathbb{Z})\cong\mathbb{Z}$ is generated by $1\in C(Z_\omega,\mathbb{Z})$. Let us considerer the following diagram, whose left square is commutative

$$\begin{array}{ccccc} K_1(C^*(\mathbb{R})) & \longrightarrow & K_1(C^*(\mathcal{G},\lambda)) & & \\ \downarrow & & \downarrow & & \\ \mathbb{Z}\cong K_0(\mathbb{C}) & \longrightarrow & K_0(C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_\omega)) & \xrightarrow{\mathrm{ev}_*} & K_0(C(\Omega\times Z_\omega)) \end{array},$$

where

- the horizontal map of the left square are induced by the inclusion $\mathbb{C}\hookrightarrow C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_\omega)$.
- vertical maps are the Thom-Connes isomorphism.
- The map $\mathrm{ev}: C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_\omega))\longrightarrow C(\Omega\times Z_\omega)$ is induced by the continuous map $\Omega\to X^{\mathcal{N}}_{\mathcal{P}}\cong(\Omega\times\mathbb{R})/\mathcal{A}_o$; $x\mapsto(x,0)$;

Up to the Morita equivalence between $C^*(\mathcal{G},\lambda)$ and $C(Z_\omega)\otimes C(\Omega)\rtimes\mathbb{Z}$, the left down stair case is the boundary of the Pimnsner-Voiculescu six-term exact sequence. From this, we see that $K_1(C^*(\mathcal{G},\lambda)\cong\mathbb{Z}$ is generated by the image of the canonical generator of $K_1(C^*(\mathbb{R}))\cong K_1(C_0(\mathbb{R}))\cong K_0(\mathbb{C})\cong\mathbb{Z}$. On the other hand, we have a diagram with commutative square

$$\begin{array}{ccccccc}
K_0(C^*(\mathbb{R})\rtimes\mathbb{R}_+^*) & \longrightarrow & K_0(C(X^G_{\mathcal{P}_\omega})\rtimes G) & \xrightarrow[\cong]{} & K_0(C^*(\mathcal{G}_{\alpha_\omega},\lambda_{\alpha_\omega})\rtimes\mathbb{R}) & \xrightarrow[\cong]{} & K_0(C^*(\mathcal{G},\lambda)_{\alpha_\omega}\rtimes\mathbb{R})\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
K_1(C^*(\mathbb{R})) & \longrightarrow & K_1(C(X^G_{\mathcal{P}_\omega})\rtimes\mathbb{R}) & \xrightarrow[\cong]{} & K_1(C^*(\mathcal{G}_{\alpha_\omega},\lambda_{\alpha_\omega})) & \xrightarrow[\cong]{} & K_1(C^*(\mathcal{G},\lambda)_{\alpha_\omega})\\
 & & & & & & \downarrow{\scriptstyle \mathrm{ev}_*}\\
 & & & & & & K_0(C^*(\mathcal{G},\lambda))
\end{array}$$

where,

- the horizontal map of the left square are induced by the inclusion $C^*(\mathbb{R})\hookrightarrow C(X^G_{\mathcal{P}_\omega})\rtimes\mathbb{R}$;
- the horizontal map of the middle square are induced by the isomorphism of lemma 7.1
- the horizontal map of the left square are induced by the isomorphism of proposition 5.6
- the first row of vertical maps are Thom-Connes isomorphisms.

It is then straightforward to check that the downstaircase of the diagram is indeed induced by the inclusion $C^*(\mathbb{R})\hookrightarrow C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_\omega)\rtimes\mathbb{R}=C^*(\mathcal{G},\lambda)$. Notice that $K_0(C^*(\mathbb{R})\rtimes\mathbb{R}_+^*)\cong\mathbb{Z}$ (by Thom-Connes isomorphism). Moreover, the inclusion $\mathcal{K}(L^2(\mathbb{R}_+^*))\cong C_0(\mathbb{R}_+^*)\rtimes\mathbb{R}_+^*\hookrightarrow C_0(\mathbb{R})\rtimes\mathbb{R}_+^*\cong C^*(\mathbb{R})\rtimes\mathbb{R}_+^*$ provides a generator for $K_0(C^*(\mathbb{R})\rtimes\mathbb{R}_+^*)$ whose image under the left vertical map is the canonical generator for $K_1(C^*(\mathbb{R}))\cong K_0(\mathbb{C})\cong\mathbb{Z}$. Using the description of the boundary map of Pimnsner-Voiculescu six-term exact sequence, we see that if e is any rank one projector in $\mathcal{K}(L^2(\mathbb{R}_+^*))\cong C_0(\mathbb{R}_+^*)\rtimes\mathbb{R}_+^*\subset C^*(\mathbb{R})\rtimes\mathbb{R}_+^*$. Then p can be viewed as an element of $C(X^G_{\mathcal{P}_\omega})\rtimes G$ whose class in K-theory provides a section for the short exact sequence of theorem 8.5.

In general, $\operatorname{Inv}C(Z_\omega,\mathbb{Z})$ is generated by characteristic functions of $\mathbb{Z}$-invariant compact-open subsets of Z_ω. According to proposition 4.3, any $\mathbb{Z}$-invariant compact-open subset E of Z_ω provides a $\mathbb{R}$-invariant compact subset $\widetilde{E}$ of $X^G_{\mathcal{P}}$. Hence, with above notations, if $\chi_{\widetilde{E}}$ is the characteristic function for $\widetilde{E}$, then $\chi_{\widetilde{E}}e$ can be viewed as an element of $C(X^G_{\mathcal{P}_\omega})\rtimes G$. Let $s:\operatorname{Inv}C(Z_\omega,\mathbb{Z})\to K_0(C(X^G_{\mathcal{P}_\omega})\rtimes G)$ be the group homomorphism uniquelly defined by $s(\chi_E)=\chi_{\widetilde{E}}e$ for E a $\mathbb{Z}$-invariant compact-open subset of Z_ω. Then s is a section for the short exact sequence of theorem 8.5.

9. The cyclic cocycle associated to a harmonic probabilty

Recall that according to discution ending section **??**, a probability is harmonic if and only if it is G-invariant. In this section, we associate to an harmonic probability a 3-cyclic cocycle on the smooth cross product algebra of $X^G_{\mathcal{P}(\omega)}\rtimes G$. This cyclic cocycle is indeed builded out a 1-cyclic cocycle on the algebra of smooth (along the leaves) functions on $X^G_{\mathcal{P}(\omega)}$ by using the analogue in cyclic cohomology of the

Thom-Connes isomorphism (see [6]). We give a description of this cocycle and we discuss an odd version of the gap-labelling.

9.1. **Review on smooth cross products.** We collect here results from [6] concerning smooth cross products by an action of $\mathbb{R}$ that we will need later on.

Let $\mathcal{A}$ be a Frechet algebra with respect to an increasing family of semi-norms $(\|\bullet\|_k)_{k\in\mathbb{N}}$.

Definition 9.1. *A smooth action on $\mathcal{A}$ is a homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}\, \mathcal{A}$ such that*

(1) *For every t in $\mathbb{R}$ and a in $\mathcal{A}$, the function $t \mapsto \alpha_t(a)$ is smooth.*
(2) *For every integers k and m, there exist integers j and n and a real C such that $\left\|\frac{d^k}{dt^k}\alpha_t(a)\right\|_m \leq C(1+t^2)^{j/2}\|a\|_n$ for every a in $\mathcal{A}$.*

If α is a smooth action on $\mathcal{A}$, then the smooth cross product $\mathcal{A}\rtimes_\alpha\mathbb{R}$ is defined as the set of smooth functions $f : \mathbb{R} \to \mathcal{A}$ such that

$$\|f\|_{k,m,n} \overset{\text{def}}{=\!=} \sup_{t\in\mathbb{R}}(1+t^2)^{k/2}\left\|\frac{d^m}{dt^m}f(t)\right\|_n < +\infty$$

for all integers k, m and n. The smooth cross product $\mathcal{A}\rtimes_\alpha\mathbb{R}$ provided with the family of semi-norm $\|\bullet\|_{k,m,n}$ for k, m and n integers together with the convolution product

$$f * g(t) = \int f(s)\alpha_s(g(t-s))dt$$

is then a Frechet algebra. Notice that a smooth action α on a Frechet algebra $\mathcal{A}$ gives rise to a bounded derivation Z_α of $\mathcal{A}\rtimes_\alpha\mathbb{R}$ defined by $Z_\alpha(f)(t) = tf(t)$ for all f in $\mathcal{A}\rtimes_\alpha\mathbb{R}$ and t in $\mathbb{R}$.

Let $\mathcal{A}^G_{\mathcal{P}(\omega)}$ be the algebra of continuous and smooth along the leaves functions on $X^G_{\mathcal{P}(\omega)}$, i.e functions whose restrictions to leaves admit at all order differential which as functions on $X^G_{\mathcal{P}(\omega)}$ are continuous. Let β_0 and β_1 be the two actions of $\mathbb{R}$ on $\mathcal{A}^G_{\mathcal{P}(\omega)}$ respectively induced by

$$\mathbb{R}\times X^G_{\mathcal{P}(\omega)} \to X^G_{\mathcal{P}(\omega)};\, (t,\mathcal{T}) \mapsto \mathcal{T}+t$$

and

$$\mathbb{R}\times X^G_{\mathcal{P}(\omega)} \to X^G_{\mathcal{P}(\omega)};\, (t,\mathcal{T}) \mapsto \mathcal{T}\cdot R^{2^t}.$$

Let X and Y be respectively the vector fields associated to β^0 and β^1. Then $\mathcal{A}^G_{\mathcal{P}(\omega)}$ is a Frechet algebra with respect to the family of semi-norms

$$\|\bullet\|_{k,l} : f \mapsto \sup_{X^G_{\mathcal{P}(\omega)}} |X^kY^l(f)|,$$

where k and l run through integers. It is clear that β^0 is a smooth action on $\mathcal{A}^G_{\mathcal{P}(\omega)}$. Moreover,

$$\mathbb{R}\times\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R} \to \mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R};(t,f)\mapsto[s\mapsto\beta^1(f(2^{-t}s))]$$

is an action of $\mathbb{R}$ on $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}$ by automorphisms. This action is not smooth in the previous sense. Nevertheless, the action β^1 satisfies conditions (1),(2) and (3) of [6, Section 7.2] with respect to the family of functions $\rho_n : \mathbb{R}\to\mathbb{R}; t\mapsto 2^{2n|t|}$, where n runs through integers. In this situation, we can define the smooth cross product $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R}$ of $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}$ by β_1 to be set of smooth functions $f:\mathbb{R}\to\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}$ such that

$$\|f\|_{k,l,m,n}\stackrel{\text{def}}{=\!=}\sup_{t\in\mathbb{R}}\rho_k(t)\left\|\frac{d^l}{dt^l}f(t)\right\|_{m,n}<+\infty$$

for all integers k,l,m and n. Then $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R}$ provided with the family of semi-norm $\|\bullet\|_{k,l,m,n}$ for k,m and n integers together with the convolution product is a Frechet algebra. Moreover, this algebra can be viewed as a dense subalgebra of $C(X^G_{\mathcal{P}(\omega)})\rtimes G$. As for smooth actions, the action β^1 gives rise to a derivation Z_{β^1} of $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R}$ (defined by the same formula).

9.2. **The 3-cyclic cocycle.** Let η be a G-invariant probability on $X^G_{\mathcal{P}(\omega)}$. Define

$$\tau_{\omega,\eta}:\mathcal{A}^G_{\mathcal{P}(\omega)}\times\mathcal{A}^G_{\mathcal{P}(\omega)}\to\mathbb{C};\ (f,g)\mapsto\int Y(f)g\,d\eta.$$

Using the Leibnitz rules and the invariance of G, it is straightforward to check that $\tau_{\omega,\eta}$ is 1-cyclic cocycle. In [6] was constructed for a smooth action α on a Frechet algebra $\mathcal{A}$ a homomorphism $H^n_\lambda(\mathcal{A})\to H^{n+1}_\lambda(\mathcal{A}\rtimes_\alpha\mathbb{R})$, where $H^*_\lambda(\bullet)$ stands for the cyclic cohomology. This homomorphism is indeed induced by a homomorphism at the level of cyclic cocycles $\#_\alpha: Z^n_\lambda(\mathcal{A})\to Z^{n+1}_\lambda(\mathcal{A}\rtimes_\alpha\mathbb{R})$ and commutes with the periodisation operator S. Hence it gives rise to homomorphim in periodic cohomology $HP^*(\mathcal{A})\to HP^{*+1}(\mathcal{A}\rtimes_\alpha\mathbb{R})$ which turns out to be an isomorphism. This isomorphism is for periodic cohomology the analogue of the Thom-Connes isomorphism in K-theory.

We give now the description of $\#_{\beta^0}\tau_{\omega,\eta}$. Let us define first $X_{\beta^0}:\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta_0}\mathbb{R}\to\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}$ and $Y_{\beta^0}:\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\to\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta_0}\mathbb{R}$ respectively by $X_{\beta^0}f(t)=X(f)(t)$ and $Y_{\beta^0}f(t)=Y(f)(t)$, for all f in $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta_0}\mathbb{R}$ and t in $\mathbb{R}$. Using the relation $Y\circ\beta^0_t=\beta^0_t\circ Y-t\ln 2\,\beta^0_t\circ X$ and applying the definition of $\#^0_\beta$, we get:

Proposition 9.2.

(1)

$$\#_{\beta^0}\tau_{\omega,\eta}(f,g,h)=-2\pi i\eta(Y_{\beta^0}f*g*Z_{\beta^0}h(0)+Z_{\beta^0}f*g*Y_{\beta^0}h(0)$$
$$-2\pi i\ln 2(\eta(1/2{Z_{\beta^0}}^2f*g*X_{\beta^0}h(0)+Z_{\beta^0}f*Z_{\beta^0}g*X_{\beta^0}(h)(0)-1/2X_{\beta^0}f*g*{Z_{\beta^0}}^2(0))$$

for all f, g and h in $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\rtimes\mathbb{R}$

(2) *The cocycle $\#_{\beta^0}\tau_{\omega,\eta}$ is β^1-invariant, i.e $\#_{\beta^0}\tau_{\omega,\eta}(\beta^1_tf,\beta^1_tg,\beta^1_th)=\#_{\beta^0}\tau_{\omega,\eta}(f,g,h)$ for all t in $\mathbb{R}$ and f, g and h in $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\rtimes\mathbb{R}$.*

According to [6, Section 7.2], the action β^1 on $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\rtimes\mathbb{R}$ also gives rise to a homomorphism $\#_{\beta^1}: Z^n_\lambda(\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R})\to Z^{n+1}_\lambda(\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R})$ which induces an isomorphim $HP^*(\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R})\stackrel{\cong}{\longrightarrow}HP^{*+1}(\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R})$. A direct application of the definition of $\#_{\beta^1}$ leads to

Lemma 9.3. *Let ϕ be a 3-cyclic cocycle for $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}$. Let us define for any f,g and h in $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R}$.*

$$\widetilde{\phi}(f,g,h)=2\pi\imath\int_{t_0+t_1+t_2=0}f(t_0)\beta^1_{t_0}g(t_1)\beta^1_{-t_2}(t_2).$$

Then

$$\#_{\beta^1}\phi(f_0,f_1,f_2,f_3) = -\widetilde{\phi}(f_0,f_1,f_2*Z_{\beta^1}f_3)+\widetilde{\phi}(Z_{\beta^1}f_0*f_1,f_2,f_3) - \widetilde{\phi}(f_0,Z_{\beta^1}f_1*f_2,f_3)-\widetilde{\phi}(Z_{\beta^1}f_0,f_1*f_2,f_3)$$

Definition 9.4. *With above notations, the* 3*-cyclic cocycle on* $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R}$ *associated to the Penrose hyperbolic tiling coloured by* ω *and to a* G*-invariant probability* η *on* $X^G_{\mathcal{P}(\omega)}$ *is* $\phi_{\omega,\eta}=\#_{\beta^1}\#_{\beta^0}\tau_{\omega,\eta}$.

Notice that if we perform this construction for the tiling of the euclidian space with continuous hull $X^{\mathbb{R}^2}$ with respect to the $\mathbb{R}^2$-action by translations, we get a 3-cyclic cocycle which is indeed equivalent to the 1-cycle cocycle on $C(X^{\mathbb{R}^2})\rtimes\mathbb{R}^2\cong(C(X^{\mathbb{R}^2})\rtimes\mathbb{R})\rtimes\mathbb{R}$ arising from the trace on $C(X^{\mathbb{R}^2})\rtimes\mathbb{R}$ associated to an $\mathbb{R}$-invariant probability on $X^{\mathbb{R}^2}$.

The class of $\phi_{\omega,\eta}$ in $HP^1(\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R})$ is the image of the class of $\tau_{\omega,\eta}$ under the composition of isomorphism

$$HP^1(\mathcal{A}^G_{\mathcal{P}(\omega)})\stackrel{\cong}{\longrightarrow}HP^0(\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R})\stackrel{\cong}{\longrightarrow}HP^1(\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R}).$$

The 3-cyclic cocycle $\phi_{\omega,\eta}$ provides a linear map

$$K^1(\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R})\to\mathbb{C};\ x\mapsto\langle\phi_{\omega,\eta},\bullet\rangle.$$

Recall from proposition 4.4 that G-invariant probabilities on $X^G_{\mathcal{P}(\omega)}$ are in one to one correspondance with $\mathbb{Z}$-invariant probabilities on $\mathbb{Z}_\omega$. Let $\hat{\eta}$ be the probability on $\mathbb{Z}_\omega$ associated to η. On can show indeed that there is a natural choice of generator for $K_1(C(X^G_{\mathcal{P}(\omega)})\rtimes G)$ coming from $K_1(\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R})$ under the morphism induced by the inclusion $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R}\hookrightarrow C(X^G_{\mathcal{P}(\omega)})\rtimes G$ whose range under $\langle\phi_{\omega,\eta},\bullet\rangle$ generates $\mathbb{Z}[\hat{\eta}]\stackrel{\text{def}}{=\!=}\{\hat{\eta}(E),\,E$ compact open subset of $Z_\omega\}$. To make this statement more precise, let us consider the Frechet algebra $\mathcal{S}(\mathbb{R},C(Z_\omega))$ of $C(Z_\omega)$-valued Schwartz functions on $\mathbb{R}$ equipped with the convolution product $f*g(t)=\int f(s)g(t-s)ds$ for any f and g in $\mathcal{S}(\mathbb{R},C(Z_\omega))$ ($\mathcal{S}(\mathbb{R},C(Z_\omega))$ can be viewed in fact as the smooth cross product of $C(Z_\omega)$ by the trivial action of $\mathbb{R}$). Using the groupoids homomorphism $\mathcal{G}=(X^G_{\mathcal{P}(\omega)}\times Z_\omega)\rtimes\mathbb{R}\to\mathbb{Z}_\omega\rtimes\mathbb{R};\ (\mathcal{T},\omega',t)\mapsto(\omega',t)$ (the action of $\mathbb{R}$ on Z_ω being trivial), we see that $\mathcal{S}(\mathbb{R},C(Z_\omega))$ can be viewed as a subalgebra of $C(X^{\mathcal{N}}_{\mathcal{P}})\rtimes\mathbb{R}$ (indeed $\mathcal{S}(\mathbb{R},C(Z_\omega))$ is a Frechet subalgebra of the smooth cross product of the algebra of smooth elements of $C(X^{\mathcal{N}}_{\mathcal{P}})$). Moreover, with notations of section 4, this subalgebra is $\mathcal{R}$-invariant. Let us then define the smooth suspension of $\mathcal{S}(\mathbb{R},C(Z_\omega))$

$$\mathcal{B}_\omega=\{f:\mathbb{R}\to\mathcal{S}(\mathbb{R},C(Z_\omega))\text{ smooth and such that } f(t+1,\omega',s)=f(t,\sigma(\omega'),s/2);\,s,\,t\in\mathbb{R},\omega'\in Z_\omega\}.$$

In view of lemma 7.1, the Frechet algebra $\mathcal{B}_\omega$ can be viewed as a β^1-invariant Frechet subalgebra of $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}$. Hence, $\mathcal{B}_\omega\rtimes^\rho_{\beta^1}\mathbb{R}$ is a Frechet subalgebra of $\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R}$. On can show then the following :

- The composition of inclusions

$$\mathcal{B}_\omega\rtimes^\rho_{\beta^1}\mathbb{R}\stackrel{\iota}{\hookrightarrow}\mathcal{A}^G_{\mathcal{P}(\omega)}\rtimes_{\beta^0}\mathbb{R}\rtimes^\rho_{\beta^1}\mathbb{R}\hookrightarrow C(X^G_{\mathcal{P}(\omega)})\rtimes G$$

induces an isomorphism

$$K_1(\mathcal{B}_\omega \rtimes^\rho_{\beta^1} \mathbb{R}) \xrightarrow{\cong} K_1(C(X^G_{\mathcal{P}(\omega)}) \rtimes G);$$

- $$\{\langle \phi_{\omega,\eta}, \iota_*(x)\rangle;\ x \in K_1(\mathcal{B}_\omega \rtimes^\rho_{\beta^1} \mathbb{R})\} = \mathbb{Z}[\hat{\eta}].$$

This can be view as an odd version of the gap labelling, the then question arising being whether we have

$$\{\langle \phi_{\omega,\eta}, x\rangle;\ x \in K_1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R})\} = \mathbb{Z}[\hat{\eta}]?$$

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