Université de Picardie Jules Verne

Habilitation à diriger des recherches

Spécialité : Mathématiques

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Actions sur un ensemble de Cantor

De la dynamique symbolique aux quasi-cristaux

Soutenue le 30 novembre 2015 devant le jury composé de :

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Remerciement
Je remercie ma famille, mes parents et Camille qui m'ont aidé et accompagné tout au long de mon travail.

Je tiens également à remercier les membres du LAMFA pour leurs aides, conseils et pour la bonne ambiance qui règne. Tout particulièrement dans l'équipe Padique, je souhaite remercier chaque membre pour leurs discussions et ouverture d'esprit. Un grand merci au personnel du LAMFA pour son dévouement et son efficacité.

Mes remerciements vont également à mes collaborateurs qui m'ont enrichi de leur connaissances et de leur vision de la recherche mathématique. Je leur suis également reconnaissant de la patience qu'ils ont eu à mon égard.

Je remercie également les rapporteurs et les membres du jury pour leur présence et le temps qu'ils m'ont accordé.

RÉSUMÉ

Ce travail est une étude sur des systèmes dynamiques liés à des dynamiques minimales sur un ensemble de Cantor. La première partie concerne la dynamique topologique des systèmes minimaux de Cantor et plus spécifiquement, le groupe d'automorphismes de $\mathbb{Z}$-sous shifts de faible complexité, les problèmes d'équivalences orbitales entre $\mathbb{Z}$ et $\mathbb{Z}^d$ actions et leurs relations avec la spectre continu. La seconde partie est dédiée à la notion plus géométrique d'ensemble de Delone qui sert de modèle aux quasi-cristaux. Nous les étudions dans les géométries euclidienne et hyperbolique. Nous nous intéressons tout particulièrement à leurs propriétés ergodiques et géométriques ainsi qu'à l'équivalence orbitale dans ce contexte. Finalement, la dernière partie traite des propriétés des configurations minimisantes du modèle de Frenkel-Kontorova associé à un environnement de type quasi-cristal.

ABSTRACT

This work is a study of the dynamical systems related to minimal actions on a Cantor set. The first chapter concerns the topological dynamic of minimal Cantor systems, with a focus on the automorphisms groups of $\mathbb{Z}$ subshift, the problem of topological orbit equivalence between $\mathbb{Z}$ and $\mathbb{Z}^d$-actions and their relations with their continuous spectrum. The second chapter is related to the more geometrical notion of Delone set, which is a model for quasicrystals. We study them in the Euclidean and Hyperbolic geometries. We give some of their ergodic and geometric properties and we explore the orbital equivalence class in this context. The final chapter provides properties on the minimizing configurations of the Frenkel-Kontorova model associated with an environment of quasicrystal type.

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Introduction

Ce travail est une étude sur des systèmes dynamiques liés plus ou moins directement à une dynamique minimale sur un ensemble de Cantor (*système minimal de Cantor*). En plus de nous intéresser à la dynamique topologique de ces systèmes minimaux de Cantor, nous regarderons la notion plus géométrique d'*ensemble de Delone* dans différentes géométries. Ces ensembles servent de modèle aux *quasi-cristaux*. Finalement nous étudierons le *modèle de Frenkel-Kontorova* associé à un quasi-cristal. Ces axes de recherche sont motivés par diverses raisons.

Citons tout d'abord la découverte des quasi-cristaux en 1982 par l'équipe de D. Shechtman [SBGC84] qui a remis en cause un paradigme de la cristallographie. Une modélisation de la structure d'un quasi-cristal consiste à considérer chaque atome comme un point d'un espace euclidien. L'ensemble de ces points forme alors un *ensemble de Delone*. Une autre modélisation est possible par l'intermédiaire des pavages où les atomes sont situés à l'intérieur des pavés. Par exemple le quasi-cristal découvert par Schechtman *et al* correspond à une version tridimensionnelle du fameux pavage de Penrose [Mac82], obtenu par une *substitution de type Pisot*. En fait, ces deux modélisations par des pavages ou des ensembles de points, sont équivalentes [BBG06]. Nous privilégions celle des ensembles de Delone car elle évite de tenir compte de la géométrie des pavés qui s'avère inutile pour nos propos.

La découverte des quasi-cristaux soulève de nombreuses questions, notamment sur leur possible classification. Il faut, au préalable, déterminer des invariants calculables à partir de leur structure. Pour se faire, nous allons leur associer un système dynamique, appelé *système de Delone* dont les propriétés sont reliées à leurs propriétés combinatoires. Inversement, la construction d'ensembles de Delone, nous donnera une famille de systèmes dynamiques aux comportements riches et variés. Cette relation entre la dynamique et la combinatoire est, en fait, très classique et remonte aux origines de la théorie ergodique pour déboucher, par exemple, à une preuve du théorème de Szemerédi par Furstenberg [Fur81]. De ce fait, l'étude des systèmes de Delone peut être considérée comme une généralisation de la dynamique symbolique. Comme nous le verrons dans le chapitre 2, il se trouve de plus, que les systèmes de Delone sont intrinsèquement liés à des dynamiques sur l'ensemble de Cantor. La structure du quasi-cristal de Schechtman incite à se concentrer tout particulièrement sur les systèmes d'entropie nulle, comme les *systèmes substitutifs*.

D'autre part, le modèle de Frenkel-Kontorova est un modèle précis et notoirement connu de la physique pour représenter la dislocation d'un cristal. Ce modèle à l'avantage d'être *simple* à expliciter et *universel*, dans le sens où il peut modéliser une grande variété de phénomènes physiques différents [BK04, FBGG05]. Il décrit comment une chaîne infinie d'atomes "minimise l'énergie totale d'un système" lorsque cette énergie prend en compte les interactions entre les proches voisins et un environnement extérieur. Quand l'environnement est périodique, pour un cristal par exemple, les théories dites KAM faible discrète et d'Aubry-

Mather permettent d'étudier ces configurations minimisantes. On peut donc se demander quelles propriétés de ces configurations sont préservées dans le contexte quasi-périodique.

Ajoutons en dernier lieu, que les systèmes de Cantor sont également intéressants du point de vue de l'équivalence orbitale où, rappelons le, deux actions de groupe $G_1 \curvearrowright X_1$ et $G_2 \curvearrowright X_2$ sont dites *orbitalement équivalentes* (topologiquement) s'il existe un homéomorphisme entre les espaces X_1 et X_2 qui envoie les G_1-orbites sur les G_2-orbites, les deux groupes n'étant pas forcément isomorphes. Lorsque les espaces sont connexes, un argument utilisant les théorèmes de Baire, nous montre que cette notion n'est pas très différente de celle de la conjugation. À l'opposé, lorsque les espaces sont des ensembles de Cantor, ces notions deviennent distinctes. De plus, il y a un nombre indénombrable de classe d'équivalence orbitale [HPS92, GMPS10]. Ce phénomène diffère énormément de l'équivalence orbitale mesurée (où l'on ne considère que des applications mesurées) puisque il n'existe qu'une seule classe d'équivalence orbitale parmi toutes les actions de groupes moyennables dénombrables [Dye59, OW80, CFW81].

En relation avec ces problématiques, nous présentons des résultats que nous avons obtenus dans différents articles. Ceux-ci sont cités par un numéro entre crochets (ex. [1]) alors que les travaux d'autres auteurs sont cités par un acronyme entre crochet (ex. [AO95]). L'ordre de présentation des résultats suit l'ordre croissant du nombre notions qu'il faut introduire pour les énoncer.

Ainsi le chapitre 1 concerne l'étude de systèmes minimaux de Cantor. La première section regroupe les résultats sur les $\mathbb{Z}$-actions et la seconde sur des actions de groupes *résiduellement finis*. Nous regardons différents types d'invariants des systèmes minimaux de Cantor. L'entropie étant largement étudiée, nous nous focalisons sur d'autres invariants qui ont été particulièrement peu examinés, notamment pour les systèmes d'entropie nulle. En vue de généralisation aux ensembles de Delone, nos résultats concernent essentiellement la dynamique symbolique.

Le groupe des automorphismes d'un sous-shift unidimensionnel de faible complexité est étudié dans la section 1.1.2. Nous montrons en particulier dans [7] que ce groupe a une faible croissance. Plus précisément, le quotient de ce groupe par celui engendré par le shift est fini pour des sous-shifts minimaux ayant une complexité non super-linéaire, comme les sous-shifts substitutifs. Inversement, nous montrons que n'importe quel groupe fini peut être réalisé de cette manière. Nous construisons également un système symbolique d'entropie nulle dont le groupe d'automorphismes est isomorphe à $\mathbb{Z}^d$ ainsi qu'un sous-shift de complexité polynomiale arbitrairement grande avec un groupe d'automorphismes virtuellement isomorphe à $\mathbb{Z}$.

Un second exemple d'invariant d'un $\mathbb{Z}$-système topologique (X, T) est *le spectre continu*, *i.e.* l'ensemble des valeurs $\lambda \in \mathbb{S}^1$ telles qu'il existe une fonction continue $f \colon X \to \mathbb{S}^1$ vérifiant $f \circ T = \lambda f$. Les exemples typiques de systèmes substitutifs avec un spectre continu non trivial, *i.e.* non faiblement mélangeant, sont donnés par les substitutions de type *Pisot*. Dans la sous-section 1.1.3, nous expliquons pourquoi ces systèmes sont conjugués en mesure à un échange de domaines euclidiens. Ce résultat, présenté dans [8], étend à n'importe quelle substitution de type Pisot unimodulaire (sans condition combinatoire) un résultat de Arnoux-Ito [AI01].

La sous-section 1.1.4, se rapporte aux relations entre le spectre continu d'un $\mathbb{Z}$-système de Cantor (X, T) et sa classe d'équivalence orbitale. Nous obtenons dans [4], des restrictions sur les possibles spectres au sein d'une même classe d'équivalence orbitale forte. Il se trouve que les arguments des valeurs propres $E(X, T)$ forme un sous-groupe de l'intersection des images du groupe de dimension de (X, T) par toutes ses traces (noté $I(X, T)$). Nous montrons que le groupe quotient $I(X,T)/E(X,T)$ est sans torsion lorsque le groupe des infinitésimaux est trivial. Ces hypothèses sont optimales pour ce résultat.

Nous considérons des sous-shifts minimaux pour des groupes résiduellement finis, éventuellement non commutatifs, dans la seconde section 1.2, via une généralisation de la notion de suite Toeplitz obtenue dans [5]. Ceci nous servira de base pour comprendre les propriétés possibles des ensembles de Delone dans différentes géométries. Si le groupe est moyennable, nous réalisons grâce à ces exemples, n'importe quel simplexe de Choquet de mesures de probabilité invariantes par l'action [6]. En particulier, pour le groupe $\mathbb{Z}^d$, $d \geq 1$, nous montrons que tout $\mathbb{Z}$-système Toeplitz est orbitalement équivalent à un $\mathbb{Z}^d$-sous-shift, également de type Toeplitz.

Le chapitre 2 est dédié aux ensembles de Delone. La première section présente quelques définitions et propriétés générales sur les systèmes associés. Nous nous intéressons à ces ensembles en géométrie euclidienne (section 2.2) et en géométrie hyperbolique (section 2.3).

La notion d'ensemble de Delone linéairement répétitif généralise celle d'ensemble substitutif. Ces ensembles linéairement répétitifs possèdent de nombreuses propriétés géométriques et dynamiques rigides (voir le survol [1]). Nous expliquons dans la sous-section 2.2.1, des résultats obtenus dans [3] : un tel système n'a qu'un nombre fini de systèmes de Delone apériodiques non conjugués comme facteur. De plus, chacun de ces facteurs est également linéairement répétitif.

La sous-section 2.2.2 relève de l'étude de l'équivalence orbitale pour les systèmes de Delone euclidiens. Un invariant est alors le groupe des homéomorphismes de l'enveloppe qui sont homotopes à l'identité. Dans [2], nous prouvons que ce groupe est simple et ouvert dans l'ensemble de tous les homéomorphismes. Par un profond résultat de Ben Ami et Rubin [BAR10], nous obtenons que ce groupe est un invariant complet d'équivalence orbitale.

La notion d'ensemble de Delone trouve également une extension dans le cadre de la géométrie hyperbolique [Pen80, MM98, Moz97]. Après avoir présenté divers exemples dans la sous-section 2.3.1, nous expliquons des différences avec le cas euclidien. Par exemple, d'un point de vue ergodique, nous verrons dans la sous-section 2.3.2 que les mesures invariantes pour l'action ne sont plus associées aux mesures transverses invariantes, mais plutôt à la notion géométrique de *mesures harmoniques*. Nous construisons des systèmes minimaux avec un nombre arbitraire de mesures de probabilités harmoniques et ergodiques. Ces résultats sont issus de mes travaux de thèses et publiés dans [13]. Du point de vue des C^*-algèbres, qui donnent des invariants topologiques et géométriques, il n'existe pas de traces. Chaque mesure harmonique donne alors un cocycle 3 cyclique sur la C^*-algèbre associée au système. Nous donnons dans la sous-section 2.3.3 la K-théorie et la cohomologie de Čech pour une famille d'exemples de systèmes de Delone hyperboliques que nous avons traitée dans [12].

Nos travaux sur le modèle de Frenkel-Kontorova sont décris dans le dernier chapitre. Nos preuves utilisent les notions rappelées dans le chapitre 2. La section 3.1 traite des résultats de [10] et se restreint au cas de la dimension 1 sous des hypothèses standard sur l'énergie d'interaction et un environnement induit par un quasi-cristal. Nous montrons que chaque configuration minimisante admet un nombre de rotation, que ce nombre dépend continûment de la configuration et que tout réel positif est un tel nombre de rotation. Ceci généralise, en partie, des résultats de [ALD83].

La dernière section est plus générale et concerne le modèle de Frenkel-Kontorova en dimension quelconque avec un environnement presque périodique. Ceci revient à considérer une famille d'énergie qui est stationnaire par rapport à un système dynamique minimal. Nous introduisons alors dans [11], la notion de configuration calibrée (plus forte que celle de configuration minimisante) et nous prouvons son existence pour certains environnements du système dynamique. En dimension 1, sous des conditions incluant le modèle de la section 3.1, nous montrons l'existence de ces configurations calibrées pour tout environnement dans ce contexte.

Chapitre 1

Étude d'actions minimales sur un ensemble de Cantor

Tout au long de ce chapitre, nous regarderons des actions continues de divers groupes sur un ensemble de Cantor X.

Dans la première section, nous nous restreindrons aux $\mathbb{Z}$-actions minimales. Après avoir rappelé quelques définitions de base, nous étudierons les automorphismes de sous-shifts de faible complexité dans la sous-section 1.1.2. Nous expliquerons nos travaux sur la conjecture Pisot dans la sous-section 1.1.3. Enfin, la sous-section 1.1.4, sera dédiée aux relations entre le spectre continu d'un système de Cantor et sa classe d'équivalence orbitale.

La seconde partie (section 1.2) concernera principalement l'étude d'actions minimales et libres de groupes résiduellement finis. Nous proposerons des exemples de sous-shifts, via une généralisation des notions d'odomètre et de suite Toeplitz décrites dans les sous-sections 1.2.1 et 1.2.2 avec diverses propriétés ergodiques. Finalement la dernière sous-section traitera de l'apport de ces constructions dans la théorie de l'équivalence orbitale.

1.1 Actions minimales de $\mathbb{Z}$ sur un ensemble de Cantor

1.1.1 Notations

Nous nous intéresserons dans cette partie à l'action minimale (toutes les orbites sont denses) d'un homéomorphisme T sur un ensemble de Cantor X. Rappelons qu'un système (Y, S) est un *facteur* de (X, T) s'il existe une surjection continue $\pi\colon X \to Y$ qui commute avec les actions. Le système (X, T) est alors une *extension* de (Y, S). Cette extension est dite *presque injective* si l'application facteur π est injective sur un ensemble G_δ dense (au sens de Baire).

Nous considérerons en particulier des sous-shifts. Ainsi Σ désignera un alphabet fini et $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ l'application *shift* définie par $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. Un sous-shift est alors le système dynamique (X, σ) donné par un sous-ensemble fermé $X \subset \Sigma^{\mathbb{Z}}$, σ-invariant ($\sigma(X) = X$).

Un *mot* w est un élément du monoïde libre Σ^* engendré par Σ avec l'opération de la concaténation, *i.e.* $w = x_1 \dots x_\ell$ pour $x_i \in \Sigma$. L'entier $\ell \geq 1$ est alors la *longueur* du mot w, notée $|w|$. Le langage $\mathcal{L}(X)$ d'un sous-shift X est la collection des mots w (ou *facteurs*) de la forme $w = x_j \dots x_{j+\ell-1}$ pour une suite $x = (x_i)_i \in X$ et un indice $j \in \mathbb{Z}$. Nous dirons que j est une *occurrence* du mot w dans la suite x. Nous utiliserons le même terme pour une suite finie. La *complexité* du sous-shift $p_X(\ell)$ est le nombre de mots du langage $\mathcal{L}(X)$ de longueur ℓ.

Les sous-shifts minimaux sont caractérisés par le fait que pour toute suite x du sous-shift et pour tout mot w du langage, l'ensemble des occurrences de ce mot est non vide est est relativement dense, ou, autrement dit, le mot w apparait une infinité de fois dans x et la distance entre deux occurrences consécutives est uniformément bornée. Des exemples classiques de sous-shifts minimaux sont donnés par des substitutions (voir [Que87] pour une introduction détaillée sur ces systèmes). Rappelons qu'une *substitution* ξ est un morphisme sur Σ^* défini par une application $\Sigma \to \Sigma^*$, tel qu'il existe une lettre $a \in \Sigma$ vérifiant $\lim_k |\xi^k(a)| = +\infty$. Il lui est associé une *matrice d'incidence* $M_\xi = (m_{a,b})_{a,b\in\Sigma}$ où $m_{a,b}$ est le nombre d'occurences de la lettre a dans le mot $\xi(b)$. Une substitution est dite *primitive* si sa matrice d'incidence est primitive, *i.e.* une puissance de la matrice a tous ses coefficients strictement positifs. Le système dynamique associé à ξ est alors le sous-shift Ω_ξ défini par :

$$\{x \in \Sigma^{\mathbb{Z}};\ \text{tout facteur de } x \text{ est un facteur de d'un certain mot } \xi^n(a) \text{ avec } n \geq 1, a \in \Sigma\}.$$

Lorsque la substitution est primitive, il est connu que le système (Ω_ξ, σ) est *linéairement récurrent*, *i.e.* : il existe une constante $L > 1$ telle que pour tout entier $\ell \geq 1$, tout mot de longueur $L\ell$ du langage du sous-shift contient tous les mots de longueur ℓ du sous-shift. Ces systèmes sont minimaux et uniquement ergodiques (voir [Que87, Dur00]). Les propriétés des systèmes linéairement récurrent sont décris dans le chapitre 2 sous-section 2.2.1.

Un autre exemple classique de système symbolique est le sous-shift *sturmien* Il s'agit de l'ensemble des suites qui codent une orbite de la rotation d'angle irrationnel $\alpha \in [0,1] \cap (\mathbb{R}\setminus\mathbb{Q})$ sur le cercle identifié à l'intervalle $[0,1[\mod 1$ avec la partition $\{[0, 1-\alpha[, [1-\alpha, 1[\}$. En d'autres termes, il s'agit de l'ensemble des suites de la forme $(\lfloor (n+1)\alpha+\beta \rfloor - \lfloor n\alpha+\beta \rfloor)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ où $\beta \in [0,1[$ et $\lfloor \cdot \rfloor$ désigne la partie entière. Ce sous-shift a été introduit et étudié en détail dans [MH40] où il est montré qu'il est est minimal, uniquement ergodique et est une extension presque injective de la rotation d'angle α sur le cercle. Seuls les points de l'orbite de 0 ont plusieurs (deux) pré-images. En dehors de cette orbite, l'extension est une conjugaison mesurée [Kur03].

1.1.2 Automorphismes de systèmes symboliques de faible complexité

Un *automorphisme* d'un système dynamique topologique (X, T) est un homéomorphisme de X qui commute avec la transformation T. Le groupe engendré par ces transformations est noté $\mathrm{Aut}(X, T)$. Il est non trivial car il contient celui engendré par l'homéomorphisme T lui même $\langle T \rangle \subset \mathrm{Aut}(X, T)$. Une définition similaire existe dans le cadre mesurable : pour une mesure invariante μ fixée, les automorphismes sont alors bi-mesurables et préservent cette mesure. Le groupe qu'ils engendrent, noté $C(T)$ est appelé *centralisateur* de (X, T, μ).

L'étude du centralisateur est classique et a été largement étudié en théorie ergodique. C'est par exemple le cas pour les systèmes mélangeant de rang finis (voir [Fer97] pour un survol). Mentionnons quelques résultats dans ce contexte. D. Ornstein montre dans [Orn72] que pour un système mélangeant de rang un, son centraliseur est réduit au groupe $\langle T \rangle$. Plus tard del Junco [dJ78] montra la même propriété pour le sous-shift de Chacon qui est de rang un mais seulement faiblement mélangeant. Finalement pour les systèmes mélangeant de rangs finis, King et Thouvenot [KT91] prouvèrent que le groupe quotient $C(T)/\langle T \rangle$ est fini, ou autrement dit, les groupes $C(T)$ et $\langle T \rangle$ sont virtuellement isomorphes.

Dans le cadre topologique et symbolique, il semble que Hedlund [Hed69] fut le premier à étudier groupe des automorphismes du full shift. Cette étude fut étendue aux sous-shifts de types finis mélangeant (ou même seulement d'entropie positive) par notamment Boyle, Lind et Rudolph [BLR88], Kim et Roush [KR90] et Hochmann [Hoc10]. Ils montrent que ces groupes sont gros dans le sens où ils contiennent :

— n'importe quelle somme directe dénombrable de groupe $\mathbb{Z}$ ou de groupes finis,

— le produit libre sur un nombre dénombrable de générateurs,

— le groupe $\mathrm{Aut}(\{1,\dots,n\}^{\mathbb{Z}},\sigma)$ pour tout entier $n \geq 1$.

Un argument essentiel est alors le théorème de Curtis-Hedlund-Lyndon qui stipule que chaque automorphisme d'un sous-shift (X,σ) est donné par une *fonction de bloc glissant* (sliding block codes), ou *automate cellulaire*. Cela implique en particulier que le groupe $\mathrm{Aut}(X,\sigma)$ est dénombrable. La réalisation de groupes passe alors par la notion de *marqueurs*.

De nombreuses questions restent ouvertes sur ces groupes pour un sous-shift en général. Ainsi on se pose les questions de base sur les groupes, sont ils : moyennables ? sofiques ? finiement engendrés ? quel est leur centre ? leurs quotients ?.... et leur relation avec la dynamique du sous-shift. Par exemple la question suivante est toujours ouverte.

Question 1.1.1 *Les groupes Aut*$(\{1,2,3\}^{\mathbb{Z}},\sigma)$ *et Aut*$(\{1,2\}^{\mathbb{Z}},\sigma)$ *sont ils algébriquement isomorphes ?*

Dans le cadre de l'entropie nulle peu de chose sont connues. Host et Parreau [HP89] ont cependant donné des résultats remarquables pour une famille de substitutions de longueur constante (dite *bijective*). Ils montrent un résultat de rigidité : les groupes $C(\sigma)$ et $\mathrm{Aut}(X,\sigma)$ sont identiques et sont virtuellement le groupe $\langle\sigma\rangle$. Durant la même période, Lemańczyk et Mentzen [LM88] réalisent n'importe quel groupe fini comme groupe quotient $C(\sigma)/\langle\sigma\rangle$ pour des systèmes substitutifs dans la classe données par Host et Parreau.

Très récemment, Cyr et Kra [KC15] ont montré que pour un sous-shift transitif avec une complexité sous-quadratique, le groupe $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ est de torsion, signifiant que n'importe quel élément est d'ordre fini. Leur preuve passe par un problème de coloriage de $\mathbb{Z}^2$ et utilise une version affaiblie de la conjecture de Nivat démontrée par Quas et Zamboni [QZ04].

Dans un travail en commun avec S. Donoso, F. Durand et A. Maass [7], nous montrons que la complexité contraint le groupe des automorphismes à être petit.

Théorème 1.1.2 ([7]) *Soit* (X,σ) *un sous-shift minimal infini. Si sa complexité vérifie*

$$\liminf_{n\to\infty} \frac{p_X(n)}{n} < \infty,$$

alors le groupe $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ *est fini et le cardinal de* $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ *divise le nombre d'orbites asymptotiques.*

Rappelons que les orbites des éléments $x,y \in X$ du sous-shift, munit d'une distance *dist*, sont *asymptotiques* si pour un certain entier p, $dist(\sigma^n x, \sigma^{n+p} y)$ tend vers 0 quand n tend vers l'infini. Combinatoirement, cela signifie que, quitte à "shifter" une des suites x,y, elles coïncident à partir d'un certain rang. Un argument classique de Morse et Hedlund montre qu'un sous-shift infini (apériodique) contient toujours une paire asymptotique (voir par exemple le chapitre 1 de [Aus88]). La condition sur la complexité implique qu'il n'y a qu'un nombre fini d'orbites asymptotiques [7]. Cette condition était déjà apparue dans une preuve de l'excellent livre de Queffélec [Que87]. Malheureusement lors de la dernière éditon, cette preuve disparut.

Le théorème 1.1.2 concerne tous les sous-shifts sturmiens et les sous-shifts linéairement répétitifs (substitutifs par exemple) car ils ont des complexités linéaires. Précisons dans le cas sturmien, qu'il est bien connu qu'il n'existe qu'une paire asymptotique (e.g. [Dur00]). Leur groupe d'automorphismes est donc trivial. Nous retrouvons également les résultats de Salo et Törmä [ST16] sur les groupes d'automorphismes de substitutions de type Pisot et nous prouvons leur conjecture.

La condition sur la complexité impose au sous-shift d'être d'entropie nulle. Pour autant il est possible d'avoir des complexités avec une croissance sous-exponentielle arbitraire [7]. Citons, qu'au même moment, et indépendamment de nous, Cyr et Kra [CK15] ont obtenu des résultats analogues sur la relation entre la complexité et le groupe d'automorphismes dans [CK15]. Cependant leur preuve est plus combinatoire que la notre. Note preuve, plus topologique, nous permet d'avoir quelques résultats pour des systèmes topologiques plus généraux que les sous-shifts.

L'idée de notre preuve est basée sur la remarque suivante : un automorphisme préserve l'ensemble des paires asymptotiques. Dans le cas minimal, si l'automorphisme fixe une paire, c'est une puissance du shift. Ainsi le groupe des automorphismes modulo le shift $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$, agit librement sur l'ensemble des orbites des paires asymptotiques.

Il en ressort qu'une étude de ces paires asymptotiques nous permet d'étudier le groupe des automorphismes et même de réaliser des exemples, qui montrent que le théorème 1.1.2 est optimal.

Théorème 1.1.3 *Pour tout groupe fini G, il existe un sous-shift minimal substitutif (X,σ) tel que $Aut(X,\sigma)$ est isomorphe à $G \oplus \mathbb{Z}$.*

Ce résultat généralise un peu ceux obtenus par Host-Parreau et Lemańczyk-Mentzen. En outre, nous en donnons une preuve directe. Ajoutons que dans le cas substitutif, il existe un algorithme pour déterminer les paires asymptotiques [BD01] et donc on peut espérer un algorithme pour déterminer le groupe $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ pour un sous-shift substitutif.

Peu de résultats sont connus quant à la réalisation de groupes d'automorphismes pour des sous-shifts de faible complexité. Nous laissons donc la question suivante ouverte.

Question 1.1.4 *Étant donné un groupe dénombrable G (pas nécessairement finement engendré). Existe-t-il un sous-shift qui admette ce groupe comme groupe d'automorphismes ?*

Au delà du cas du théorème 1.1.3, nous ne savons réaliser que le groupe $\mathbb{Z}^d$, pour tout $d \geq 1$, comme groupe d'automorphismes [7]. La difficulté n'étant pas de donner un exemple contenant $\mathbb{Z}^d$ mais dont le groupe est exactement isomorphe à $\mathbb{Z}^d$. À l'inverse, si le système a une récurrence polynomiale, nous savons que le groupe des automorphismes a également une croissance polynomiale (théorème 3.8 [7]). Il s'agit donc d'un groupe virtuellement nilpotent par un résultat célèbre de Gromov. Ces exemples nous poussent à proposer la question suivante qui généraliserait le théorème 1.1.2.

Question 1.1.5 *Soit (X,σ) un sous-shift minimal tel que*

$$d = \inf\{\delta \in \mathbb{N};\ \liminf_n p_X(n)/n^\delta < +\infty\} > 0.$$

Est il vrai que $Aut(X,\sigma)$ est virtuellement isomorphe à $\mathbb{Z}^k$ pour $0 < k \leq d$?

La dynamique et en particulier les facteurs des systèmes contraignent également le groupe des automorphismes. Pour un groupe G, on note par $[f,g] = fgf^{-1}g^{-1}$ le commutateur de $f, g \in G$. On définit, par récurrence, la suite de sous-groupe de commutateurs $G_1 = G$ et

G_{i+1} engendré par les commutateurs $[G_i, G]$, $i \geq 1$. Un groupe est alors dit *nilpotent* de classe d si G_d est trivial. Lorsque G est un groupe nilpotent de classe d, un *nilsystème* de classe d est donné par un sous-groupe discret $\Gamma \subset G$ cocompact et une translation à droite, $R_g \colon x \mapsto xg$ telle que le système $(G/\Gamma, R_g)$ soit minimal. Nous obtenons, dans ce contexte, le résultat suivant.

Théorème 1.1.6 ([7]) *Soit (X, σ) une extension presque injective d'un nilsystème de classe d $(G/\Gamma, R_g)$. Alors $Aut(X, \sigma)$ est un groupe nilpotent de classe au plus d et est un sous-groupe de $Aut(G/\Gamma, R_g)$.*

Ainsi pour un système Toeplitz (cf section 1.2.2), qui est une extension presque injective d'un odomètre, *i.e.* une translation sur un groupe abélien (nilpotent de classe 1), son groupe d'automorphismes est abélien et est un sous-groupe des automorphismes de son facteur équicontinu maximal.

Ajoutons, qu'en construisant des extensions symboliques presque injectives de nilsystèmes, nous obtenons à partir de ce théorème des systèmes symboliques de complexité polynomiales arbitrairement grande avec un groupe d'automorphismes virtuellement isomorphe à $\mathbb{Z}$ [7].

Le théorème 1.1.6 se prouve en utilisant une remarque similaire sur les paires asymptotiques : le groupe des automorphismes préserve l'ensemble des paires *proximales*, *i.e.* les paires de points $x, y \in X$ tels que $\liminf_{n\to\infty} dist(\sigma^n x, \sigma^n y) = 0$. Rappelons qu'un résultat classique (voir par exemple lemme 2.1 dans [7]) assure que les fibres d'extensions presque injectives de systèmes minimaux donnent des paires proximales. Ceci nous permet de montrer que chaque automorphisme se factorise via l'application facteur. La conclusion vient ensuite de l'étude des automorphismes de nilsystèmes.

1.1.3 Sur la conjecture Pisot

Un approche classique, initiée par par Hadamard [Had98] et Morse [Mor21], pour étudier une dynamique géométrique, comme l'étude d'un difféomorphisme d'une variété, consiste à coder sa dynamique par un sous-shift via une partition de Markov. Cette dynamique symbolique est en général plus simple à étudier. Dans son papier fondateur, Rauzy [Rau82] prend le problème sous la vision inverse : étant donné un sous-shift, peut-on lui associer une représentation géométrique ? Comme un ensemble de Cantor n'est pas homéomorphe à une variété, la "représentation géométrique" s'entend par une conjugaison en mesure. Par exemple Rauzy considére dans [Rau82] le sous-shift associé à la substitution de Tribonacci

$$\tau \colon 1 \mapsto 12, 2 \mapsto 13 \text{ et } 3 \mapsto 1.$$

Il montre que ce système est conjugué en mesure à une rotation minimale sur le tore $\mathbb{T}^2$. Plus tard, Arnoux et Rauzy [AR91] montrèrent qu'un sous-shift dont la complexité est $2n+1$ et satisfait une condition combinatoire appelée Condition (*) (condition satisfaite par le sous-shift associé à τ) est mesurablement conjugué à un échange de 3 intervalles.

La propriété fondamentale utilisée par Rauzy dans [Rau82] est que la substitution de Tribonacci τ est une substitution de type *Pisot*.

Définition 1.1.7 *Soit ξ une substitution primitive et P_ξ le polynôme caractéristique de sa matrice d'incidence M_ξ. La substitution est dite de type* Pisot *si le polynôme P_ξ admet a une racine dominante $\beta > 1$ et les autres racines β' vérifient $0 < |\beta'| < 1$.*

Rappelons qu'une substitution ξ est dite *unimodulaire* lorsque $\det M_\xi = \pm 1$.

Ainsi pour une substitution de type Pisot, la valeur propre de Perron de M_ξ est alors un nombre Pisot-Vijayaraghan, i.e. : tous ses conjugués algébriques sont de module strictement plus petit que 1.

Remarquons que le fait d'être Pisot ne dépend pas de la combinatoire de la substitution, mais uniquement de son abélianisé M_ξ et que toute puissance d'une substitution de type Pisot est encore de type Pisot. Ainsi à partir d'une substitution de type Pisot, comme τ, il est possible de créer d'autres substitutions de type Pisot en prenant des puissances de τ puis en permutant les lettres, comme par exemple pour la substitution $1 \mapsto 1123$, $2 \mapsto 211$ et $3 \mapsto 21$.

Les résultats de Rauzy ont donné lieu à la conjecture (encore ouverte) suivante :

Conjecture 1.1.8 (Conjecture Pisot) *Le sous-shift engendré par une substitution de type Pisot a un spectre purement discret, i.e., est conjugué en mesure à une translation sur un groupe.*

Ajoutons également que cette conjecture implique un résultat de rigidité : tous les systèmes substitutifs de type Pisot, avec la même matrice d'incidence, sont mesurablement conjugués. Beaucoup de travaux se sont focalisés sur cette conjecture. La stratégie classique reprend celle initiée par Rauzy dans [Rau82] : on essaye de montrer que le système substitutif est conjugué en mesure à un échange de domaines (voir définition 1.1.9). Puis on prouve que cet échange de domaines, qui a une structure géométrique, est conjugué en mesure à une translation sur un groupe. Précisons tout d'abord ce que l'on entend par échange de domaines.

Définition 1.1.9 *Nous appelons échange de domaines le système dynamique mesurable* $(E, \mathcal{B}, \mathrm{Leb}, T)$ *où* E *est un compact régulier*[*] *d'un espace euclidien,* Leb *désigne la mesure de Lebesgue normalisée sur* E *et* $\mathcal{B}$ *la tribu borélienne de* E *telle que*

— *Il existe des compacts réguliers* $E_1, \ldots, E_n$ *tels que* $E = E_1 \cup \cdots \cup E_n$.
— *Les ensembles* E_i *sont disjoints en mesure.*
— $\mathrm{Leb}(T(E)) = \mathrm{Leb}(E)$.
— *Pour chaque indice* i*, l'application* T *restreint à* E_i *est une translation telle que* $T(E_i) \subset E$.

Des résultats classiques en théorie de la mesure assurent que pour un échange de domaines, l'application T est injective en dehors d'un ensemble de mesure nulle et que l'application T^{-1} est mesurable Un échange de domaines est dit *auto-affine*, s'il existe un nombre fini d'applications affines $f_1, \ldots, f_\ell$, partageant la même partie linéaire, telles que $E = \bigcup_{i=1}^{\ell} f_i(E)$.

Un premier résultat important dans le sens de la conjecture, est dû à Host [Hos86a] qui démontre que n'importe quelle fonction propre d'un système substitutif primitif est continue. Il montre également dans un papier non publié mais très cité, que la conjecture Pisot est vraie pour les substitutions unimodulaires sur un alphabet à deux lettres lorsqu'une condition combinatoire, appelée *condition de forte coïncidence*, est satisfaite. Cette condition apparaissait déjà dans [Dek78]. Barge et Diamond [BD02] montrèrent ensuite que n'importe quelle substitution de type Pisot sur un alphabet à deux lettres vérifiant cette condition. La conjecture Pisot est donc vraie dans ce cas [HS03]. En suivant la stratégie de Rauzy, mais par une approche différente de celle de Host, Arnoux et Ito dans [AI01], associent à chaque substitution unimodulaire, un échange de domaine auto-affine appelé *fractal de Rauzy*. Ils prouvèrent que cet échange de domaine est mesurablement conjugué au système substitutif à condition que la substitution satisfasse une condition combinatoire. Peu de

*. un compact est dit *régulier* s'il est égal à la fermeture de son intérieur.

temps après, Canterini et Siegel généralisent les résultats de Host à toutes les substitutions Pisot unimodulaires [CS01], et non unimodulaires dans [Sie03, Sie04], mais sans supprimer la condition de forte coïncidence. Ces résultats ont conduit à une énorme quantité de travaux pour étudier les fractals de Rauzy (voir [Fog02] et ses références).

Dans un travail en commun avec F. Durand [8], nous généralisons des résultats de [AI01] et [CS01], en supprimant la condition combinatoire : n'importe quel système substitutif de type Pisot unimodulaire est conjugué en mesure à un échange de domaine auto-affine.

Théorème 1.1.10 ([8]) *Soit ξ une substitution unimodulaire de type Pisot sur d lettres et (Ω, S) le sous-shift associé. Alors il existe une transformation d'échange de domaines auto-affine $(E, \mathcal{B}, \mathrm{Leb}, T)$ dans $\mathbb{R}^{d-1}$ et une application continue $F\colon \Omega \to E$ qui est une conjugaison mesurable entre les deux systèmes.*
De plus, en notant $\pi\colon \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$ la projection canonique :
- *il existe une application linéaire N telle que $F \circ \xi = N \circ F$.*
- *L'application $\pi \circ F$ définit un facteur entre le système (Ω, S) est une translation minimale sur le tore $\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$.*
- *Il existe un entier $r \geq 1$ tel que presque toute fibre de $\pi \circ F$ est de cardinal fini r.*

La translation minimale est explicitement décrite dans [CS01, Fog02]. Pour montrer la conjecture Pisot, il reste encore à montrer que cet échange de domaine est conjugué à la translation minimale sur le tore, ou encore que la constante $r = 1$. Précisons que l'échange de domaine n'est pas, *a priori*, le fractal de Rauzy classique associé à ξ car, à la différence de [AI01, Hos92], la matrice N n'est pas forcément identique à une restriction de la matrice d'incidence M_ξ. De plus notre construction diffère assez nettement de la construction géométrique de [AI01].

Pour éviter les problèmes classiques de combinatoire, notre stratégie a été, tout d'abord, de modifier la substitution : en utilisant la notion de *mots de retours*, la combinaison des résultats de [Dur98a, Dur98b, DHS99] montrent que le système substitutif est conjugué à un système avec une substitution *propre*, *i.e.* ayant des propriétés combinatoires sympathiques, comme celle de forte coïncidence. Malheureusement cette nouvelle substitution peut posséder, en plus des valeurs propres originales, les valeurs 0 et 1. Nous prouvons ensuite qu'un système substitutif avec un spectre dynamique non trivial se factorise sur un échange de domaines, grâce à une approximation des fonctions propres donnée dans [BDM05]. Une troisième étape consiste finalement à montrer la conjugaison mesurée lorsque le système substitutif a suffisament de valeurs propres. Ces résultats combinés nous donnent alors le théorème 1.1.10. Cette troisième étape suit le même schéma de preuve que dans [CS01] mais sans utiliser les propriétés standard d'irreducibilité de la substitution. Nous utilisons de façon essentielle que le nombre de valeurs propres dynamiques multiplicativement indépendantes est égal à $\sum_{0<|\lambda|<1} \mathrm{dim} E_\lambda$ où E_λ désigne l'espace caractéristique associé à la valeur propre λ de la matrice d'incidence de la substitution.

Cette condition nous permet d'espérer de pouvoir étendre ces résultats à une classe plus grande de substitutions, comme celle de type *Salem*, où la valeur propre de Perron est un nombre de Salem et les autres valeurs propres ont un module inférieur à 1. Malheureusement nous ne connaissons pas de substitution de ce type ayant suffisamment de valeurs propres, même si *a priori* les restrictions arithmétiques de [FeMN96b] ne l'empêchent pas.

Une autre possibilité d'extension de ces résultats serait également dans le cadre linéairement répétitif. Une première classe d'exemples dans ce sens a été réalisée dans [BJS12].

1.1.4 Équivalence orbitale forte et valeurs propres

Des systèmes dynamiques sont *orbitalement équivalents* s'il existe une bijection entre leurs espaces de phases préservant leur structure (mesurée, topologique, ...) et envoyant chaque orbite sur une orbite. Cette notion est apparue tout d'abord dans le contexte des actions de groupes préservant une mesure de probabilité (on parle dans ce cas d'équivalence orbitale mesurable) lors de l'étude des algèbres de Von Neuman [MVN36]. Un résultat notable de cette théorie est qu'il n'existe qu'une seule classe d'équivalence orbitale parmi toutes les actions de groupes moyennables [Dye59, OW80]. Motivés par ces résultats et la caractérisation de l'équivalence orbitale en termes d'algèbres de Von Neumann, Giordano Putnam et Skau ont obtenu un des résultats les plus importants dans ce contexte : les classes d'équivalences orbitales topologiques de $\mathbb{Z}$-action minimale sur un ensemble de Cantor sont caractérisées par le groupe K_0 de la C^*-algèbre associée [GPS95] .

Plus précisément, deux $\mathbb{Z}$-systèmes minimaux de Cantor (X,T) et (Y,S) sont *orbitalement* équivalents (topologiquement) ou OE, s'il existe un homéomorphisme $F\colon X \to Y$ envoyant les orbites de T sur celles de S. Cela signifie qu'il existe des fonctions $\alpha\colon X \to \mathbb{Z}$ et $\beta\colon X \to \mathbb{Z}$ telles que

$$F(Tx) = S^{\alpha(x)}F(x) \text{ et } F(T^{\beta(x)}x) = SF(x), \qquad \forall x \in X.$$

Lorsque ces fonctions α et β sont continues, Boyle dans [Boy83] démontre que les systèmes sont alors *flip conjugués*, *i.e.* le système (X,T) est conjugué soit à (Y,S) soit à (Y,S^{-1}). Si les fonctions α et β admettent au plus un point de discontinuité, on dit alors que les systèmes (X,T) et (Y,S) sont *fortement orbitalement équivalents* (ou Strongly Orbit Equivalent SOE). Cette notion est en fait une relation d'équivalence sur les systèmes minimaux de Cantor.

Rappelons comment est défini l'invariant total d'équivalence orbitale : le groupe K^0. Notons par $H(X,T)$ le groupe quotient $C(X,\mathbb{Z})/\langle f - f\circ T, f \in C(X,T)\rangle$, où $C(X,\mathbb{Z})$ désigne l'ensemble des fonctions continues sur X à valeurs dans $\mathbb{Z}$ et $\langle f - f\circ T, f \in C(X,T)\rangle$ son sous-groupe engendré par les cobords du sytèmes (X,T). La classe d'une fonction f sera notée $[f]$. Nous définissons le cône positif $H^+(X,T)$ par l'ensemble $\{[f]; f \in C(X,\mathbb{N})\}$. Le triplet

$$K^0(X,T) = (H(X,T), H^+(X,T), [1])$$

est un groupe ordonné ($[f] \geq [g]$ ssi $[f-g] \in H^+(X,T)$) unitaire, d'unitée $[1]$: la classe de la fonction constante égale à 1. Algébriquement, $K^0(X,T)$ est un groupe de dimension (voir [Eff81]). Nous dirons que deux groupes de dimension sont isomorphes s'ils sont isomorphes et l'isomorphie préserve les cônes positifs et les unités.

Le groupe des *infinitésimaux* est alors le sous-groupe $\mathrm{Inf}(K^0(X,T)) = \{[f]; \int f d\mu = 0$ pour toute mesure de probabilité T-invariante$\}$. Le triplet quotient $K^0(X,T)/\mathrm{Inf}(K^0(X,T))$ est :

$$(H(X,T)/\mathrm{Inf}(K^0(X,T)), H(X,T)^+/\mathrm{Inf}(K^0(X,T)), [1] \bmod \mathrm{Inf}(K^0(X,T))).$$

Il forme également un groupe de dimension.

Théorème 1.1.11 ([GPS95]) *Soient (X,T) et (Y,S) deux systèmes minimaux de Cantor. Alors, les systèmes (X,T) et (Y,S) sont SOE si et seulement si leur groupe de dimension $K^0(X,T)$ et $K^0(Y,S)$ sont isomorphes en tant que groupe de dimension.*
Les systèmes (X,T) et (Y,S) sont OE si et seulement si les groupes $K^0(X,T)/\mathrm{Inf}(K^0(X,T))$ et $K^0(Y,S)/\mathrm{Inf}(K^0(Y,S))$ sont isomorphes en tant que groupe de dimension.

Ajoutons que le groupe $K^0(X,T)$ est totalement explicite lorsque l'on peut décrire le système minimal en terme de diagramme de Bratteli [HPS92, DHS99], comme par exemple pour un système substitutif. On obtient ainsi le fait contre-intuitif, qu'un sous-shift sturmien (X,σ) est OE au système minimal (X,σ^2).

Il en ressort également que l'on peut construire une infinité indénombrable de groupes de dimension $K^0(X,T)$, et donc une infinité de classes d'équivalence orbitale. Ceci est très différent du cadre mesurable où tous les $\mathbb{Z}$-systèmes sont équivalents entre eux.

Il est alors naturel de se demander quelles sont les propriétés dynamiques préservées au sein d'une même classe d'équivalence orbitale (forte ou pas). Par exemple dans [HPS92], un isomorphisme affine est construit entre l'ensemble des traces du groupe de dimension $K^0(X,T)$ et l'ensemble $\mathcal{M}(X,T)$ des mesures de probabilités invariantes par l'action. Ainsi le simplexe des mesures de probabilité invariantes d'un système est un invariant de SOE. À l'inverse, l'entropie n'est pas un invariant de SOE, car à l'intérieur d'une même classe, il est possible de construire un exemple de système avec une entropie arbitraire (finie ou non) [BH94, Orm97, Sug03].

Concernant le spectre continu du système dynamique, moins de choses sont connues. Ormes, dans [Orm97] démontre notamment, que le groupe des valeurs propres racines de l'unité est un invariant de SOE, mais pas de OE. Il montre, de plus, qu'à l'intérieur d'une classe d'OE fixée, il est possible de réaliser n'importe quel groupe dénombrable du cercle (éventuellement trivial) comme groupe de valeurs propres mesurables.

Dans [IO07, CDHM03], est apparue une restriction sur le groupe additif des valeurs propres continues $E(X,T) = \{\alpha \in \mathbb{R}; \exp(2i\pi\alpha)$ est une valeur propre continue de $(X,T)\}$. C'est un sous-ensemble de l'image du groupe $K^0(X,T)$ par ses traces :

$$E(X,T) \subset I(X,T) := \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \int f d\mu; f \in C(X,\mathbb{Z}) \right\}.$$

Ce résultat peut être vu comme une version similaire des résultats de Schwartzman sur les cycles asymptotiques [Sch57], mais transposés dans le cas d'ensembles de Cantor (voir également [Pac86, Exe87]).

Dans un travail en commun avec M.I. Cortez et F. Durand, nous montrons la restriction suivante sur le spectre continu.

Théorème 1.1.12 ([4]) *Soit (X,T) un système minimal de Cantor sans infinitésimaux,* i.e. $\mathrm{Inf} K^0(X,T) = \{[0]\}$. *Alors le groupe quotient $I(X,T)/E(X,T)$ est sans torsion.*

Pour illustrer ce résultat, prenons le cas où $K^0(X,T) = \mathbb{Z} + \alpha\mathbb{Z} = I(X,T)$ avec α un nombre irrationnel, comme c'est le cas, par exemple, pour un sous-shift sturmien. Le seul sous-groupe propre de $\mathbb{Z} + \alpha\mathbb{Z}$ sans torsion étant le groupe $\mathbb{Z}$, le théorème implique que n'importe quel autre système (Y,S) SOE à (X,T), admet comme groupe additif de valeur propre $E(Y,S)$ soit le groupe $\mathbb{Z} + \alpha\mathbb{Z}$ (c'est le cas pour un système sturmien), soit admet le groupe $\mathbb{Z}$. Il est alors topologiquement faiblement mélangeant (c'est le cas donné par les réalisations de Ormes [Orm97]). Mentionnons, que des résultats similaires ont été obtenus par Giordano, Handelman et Hosseini [GHH].

Précisons également que l'hypothèse sur les infinitésimaux est optimale. En effet, si (X,σ) désigne un sous-shift sturmien, le système (X,σ^2) est également minimal, admet la même mesure de probabilité invariante que (X,σ), admet des infinitésimaux (donné par des corbords pour σ qui ne sont pas des cobords pour σ^2), et il est standard de vérifier que $I(X,\sigma^2) = I(X,\sigma) = \mathbb{Z} + \alpha\mathbb{Z}$ alors que $E(X,\sigma^2) = \mathbb{Z} + 2\alpha\mathbb{Z}$.

Nous donnons en outre, dans [4], un exemple où le groupe quotient $I(X,T)/E(X,T)$ a de la torsion et qui montre que le groupe d'automorphismes $\mathrm{Aut}(X,T)$ (voir la section 1.1.2) n'est pas invariant dans la classe d'OE.

La preuve du théorème 1.1.12 passe par les partitions de Kakutani-Rohlin et leurs descriptions en terme de diagrammme de Bratteli-Vershik. Comme dans la section 1.1.3, une description fine des fonctions propres continues à partir du diagramme nous permet de donner des conditions arithmétiques sur les valeurs propres additives au sein du groupe $E(X,T)$.

En paraphrasant ce théorème, nous obtenons, pour (X,T) un système minimal de Cantor sans infinitésimaux, que l'ensemble $\{E(Y,S); (YS)$ système minimal de Cantor SOE à $(X,T)\}$ est inclu dans

$$\{\Gamma \text{ sous-groupe dénombrable de } I(X,T); \mathbb{Z} \subset \Gamma, I(X,T)/\Gamma \text{ est sans torsion}\}.$$

On se demande alors si, en fait, ceci caractérise complètement les sous-groupes de valeurs propres au sein d'une même classe d'équivalence orbitale. Nous obtenons seulement une réponse partielle, grâce à un résultat de réalisation de Sugisaki [Sug11].

Proposition 1.1.13 ([4]) *L'inclusion précédente est une égalité si et seulement si : pour tout sous-groupe dénombrable dense $\Gamma \subset \mathbb{R}$ contenant $\mathbb{Z}$, il existe un système de Cantor minimal (X,T) tel que $E(X,T) = \Gamma$ et $K^0(X,T) \simeq (\Gamma, \Gamma \cap \mathbb{R}^+, 1)$.*

Techniquement, pour réaliser la condition de cette proposition avec des diagrammes de Bratteli, il nous est nécessaire d'avoir une bonne vitesse d'approximation (de type sommable) de vecteurs définis par Γ par une suite de matrices à coefficients entiers. Malheureusement, nous n'avons pas trouvé de résultats d'approximation satisfaisant ces conditions.

1.2 Actions minimales de groupes résiduellement finis

Peu d'exemples et de résultats sont connus pour les actions continues et libre de groupes non commutatifs sur des ensembles de Cantor. Il se trouve que cette première base est nécessaire pour se forger une intuition pour l'étude d'ensembles de Delone dans différentes géométries. Une classe d'exemples de sous-shifts qui puisse s'étendre assez facilement à des groupes plus généraux que $\mathbb{Z}$ sont les sous-shifts Toeplitz. Nous verrons qu'ils fournissent une riche classe de dynamiques, tout particulièrement pour l'équivalence orbitale.

Les $\mathbb{Z}$-sous-shifts Toeplitz ont été introduits par Jacobs et Keane [JK69] en adaptant une technique de Toeplitz [Toe28] pour créer des fonctions presque périodiques (au sens de H. Bohr) explicites. Du fait de leur nature arithmetico-combinatoire, cette famille de $\mathbb{Z}$-sous-shifts a fourni de nombreux exemples de $\mathbb{Z}$-actions minimales avec des propriétés ergodiques et dynamiques intéressantes et a été largement étudiée [GJ00, Dow05]. Citons notamment Williams [Wil84], qui utilise ces sous-shifts pour créer une $\mathbb{Z}$-action minimale mais possédant une infinité non dénombrable de mesures de probabilité ergodiques invariantes. Généralisant ce résultat, Downarowicz [Dow91] montre l'existence d'un tel sous-shift dont le simplexe des mesures de probabilité invariantes est affinement homéomorphe à un simplex de Choquet arbitraire. Le spectre continu des sous-shifts Toeplitz est bien compris car on sait décrire explicitement leur facteur équicontinu maximal : c'est un odomètre [MP79] (mais voir [DL98] pour une preuve). Plus précisément, Downarowicz et Lacroix caractérisent les sous-shifts Toeplitz comme étant les systèmes symboliques qui sont des extensions presque injectives d'odomètres [DL98]. De plus n'importe quel entropie finie peut être réalisée par un sous-shift Toeplitz [Kur03]. Il est également possible de réaliser topologiquement diverses autres propriétés ergodiques par ces systèmes Toeplitz [Dow97].

Dans cette section, nous généraliserons la notion de suite Toeplitz à une classe plus large de groupe et nous étendrons quelques propriétés connues à ces actions. Pour cela, il nous faudra tout d'abord généraliser la notion d'odomètre aux groupes résiduellement finis (sous-section 1.2.1), puis voir les propriétés relatives des suites Toeplitz dans la sous-section 1.2.2. Nous verrons finalement l'apport de ces exemples dans la théorie de l'équivalence orbitale dans la sous section 1.2.3.

1.2.1 Groupes résiduellement finis et odomètres

Un groupe au plus dénombrable G est dit *résiduellement fini* s'il existe une suite $\Gamma_1 \supset \cdots \supset \Gamma_n \supset \Gamma_{n+1} \supset \cdots$ de sous-groupes de G d'indices finis telle que leur intersection soit triviale :

$$\bigcap_{n\geq 1} \Gamma_n = \{e\}, \tag{1.2.1}$$

où e désigne l'élément neutre de G.

Bien évidement un groupe fini est résiduellement fini. Un premier exemple infini est le groupe des entiers $\mathbb{Z}$, avec la suite de groupes $\Gamma_n = n!\mathbb{Z}$. Plus généralement, un résultat classique de Mal'cev [Mal39], assure que pour tout corps $\mathbb{K}$ de caractéristique nulle, n'importe quel groupe finiment engendré de $GL_n(\mathbb{K})$ est résiduellement fini. En particulier, le groupe libre F_n et le groupe des tresses B_n engendrés par n éléments sont résiduellement finis. Dans la suite, nous supposerons que G est un groupe dénombrable, finiment engendré et résiduellement fini.

Pour un tel groupe, il existe ainsi des projections canoniques $\pi_n \colon G/\Gamma_{n+1} \to G/\Gamma_n$ sur les ensembles de classes à droites. Nous appelons *G-sous-odomètre* ou *adding machine* associé à la suite $(\Gamma_n)_n$, la limite projective suivante :

$$\overleftarrow{G} = \varprojlim (G/\Gamma_n, \pi_n) = \{(x_n)_n \in \prod_{n=1}^{\infty} G/\Gamma_n;\ \pi_n(x_{n+1}) = x_n \ \forall n \geq 1\}.$$

On munit chaque ensemble fini G/Γ_n de la topologie discrète, et $\overleftarrow{G}$ est un sous-ensemble compact de $\prod_{n=1}^{\infty} G/\Gamma_n$ pour la topologie produit. Le groupe G agit continûment par multiplication à gauche sur $\overleftarrow{G}$: pour $h \in G$ et $(g_n)_n \in \overleftarrow{G}$, $h.(g_n)_n := (h._n g_n)_n$ où $h._n\cdot$ désigne la multiplication à gauche par l'élément h dans G/Γ_n. Cette action généralise l'odomètre usuel (pour le groupe $\mathbb{Z}$) puisqu'on peut montrer qu'une telle action est équicontinue et minimale sur un ensemble de Cantor [5]. Ajoutons que deux suites de sous-groupes emboîtés $(\Gamma_n^{(1)})_n$ $(\Gamma_n^{(2)})_n$, donnent des sous-odomètres aux dynamiques différentes dès que $\Gamma_1^{(2)}$ n'est inclus dans aucun groupe $\Gamma_n^{(1)}$. Il est donc facile d'en créer un nombre indénombrable.

Une subtilité apparait dans le contexte non commutatif. En effet, lorsque les groupes Γ_n sont normaux, les quotients G/Γ_n ont une structure de groupe et $\overleftarrow{G}$ également. Nous appelons alors $\overleftarrow{G}$ un *odomètre*. Du fait la relation (1.2.1), G est un sous-groupe de $\overleftarrow{G}$ et son action correspond à la multiplication sur $\overleftarrow{G}$, elle est donc libre : le stabilisateur de tout point est trivial. Réciproquement, n'importe quel groupe topologique homéomorphe à un ensemble de Cantor est un odomètre [Ser63, Proposition 0]. Par des résultats classiques, on peut aussi montrer que n'importe quel sous-odomètre est le quotient d'un odomètre par un sous-groupe fermé [5].

Ajoutons que les valeurs propres mesurables d'un sous-domètre $\overleftarrow{G} = \varprojlim(G/\Gamma_n, \pi_n)$, sont les caractères $\chi\colon G \to \mathbb{S}^1$ telles que $\chi(\gamma) = 1$, $\forall \gamma \in \Gamma_n$ [5].

1.2.2 Sous-shift Toeplitz

Pour un alphabet fini Σ, l'action à gauche du *shift* $\sigma\colon G \curvearrowright \Sigma^G$ sur l'ensemble des fonctions $x\colon G \to \Sigma$, est défini pour $g \in G$ par $\sigma^g(x)(h) = x(g^{-1}h)$, pour tout $h \in G$. Lorsque Σ est muni de la topologie discrète, Σ^G est un ensemble de Cantor et l'action du shift est continue. Comme pour le cas de $\mathbb{Z}$, un sous-ensemble fermé $X \subset \Sigma^G$ invariant par l'action du shift est appelé *sous-shift*.

Définition 1.2.1 *Une suite $x \in \Sigma^G$ est dite* Toeplitz *si pour tout indice $g \in G$, il existe un sous-groupe d'indice fini $\Gamma \subset G$ tel que $\sigma^\gamma(x)(g) = x(\gamma^{-1}g) = x(g)$, pour tout $\gamma \in \Gamma$.*

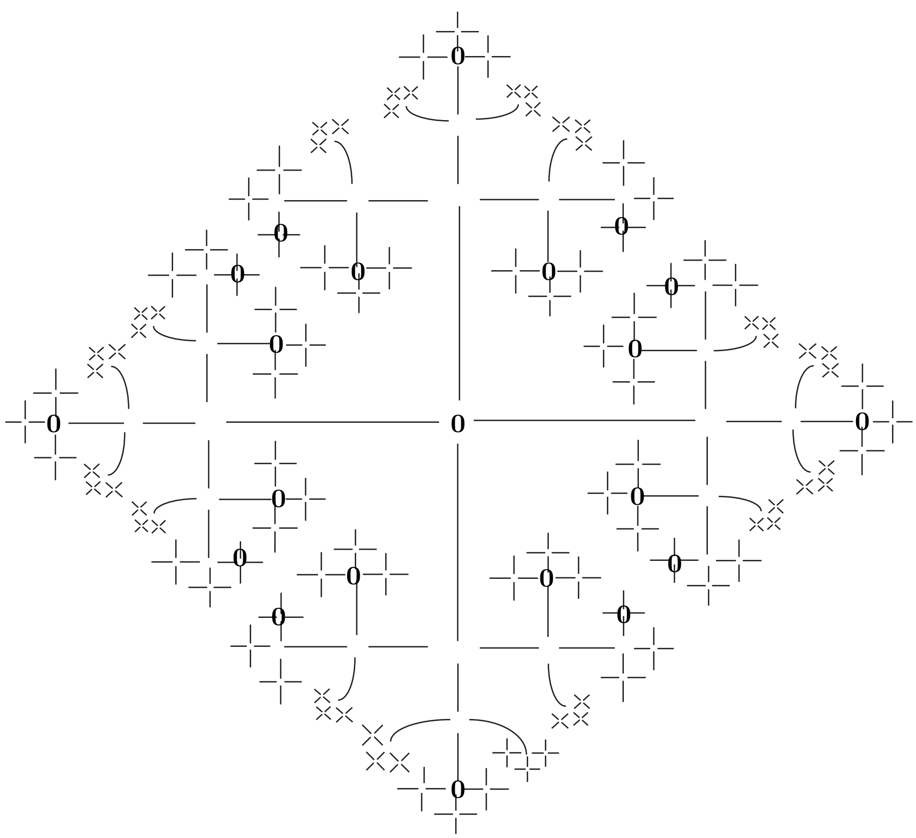

FIGURE 1.1 – Étape 1. On colorie e par $\mathbf{0}$ puis on prolonge par Γ_1-périodicité où $\Gamma_1 = \langle a^3, b^3, aba^{-1}, a^{-1}ba, bab^{-1}, b^{-1}ab \rangle$ et a, b désignent des générateurs du groupe libre F_2.

Le lecteur attentif remarquera que si l'on permute l'ordre des quantificateurs, on obtient la définition de suite périodique pour un sous-groupe d'indice fini. Étant donné une suite

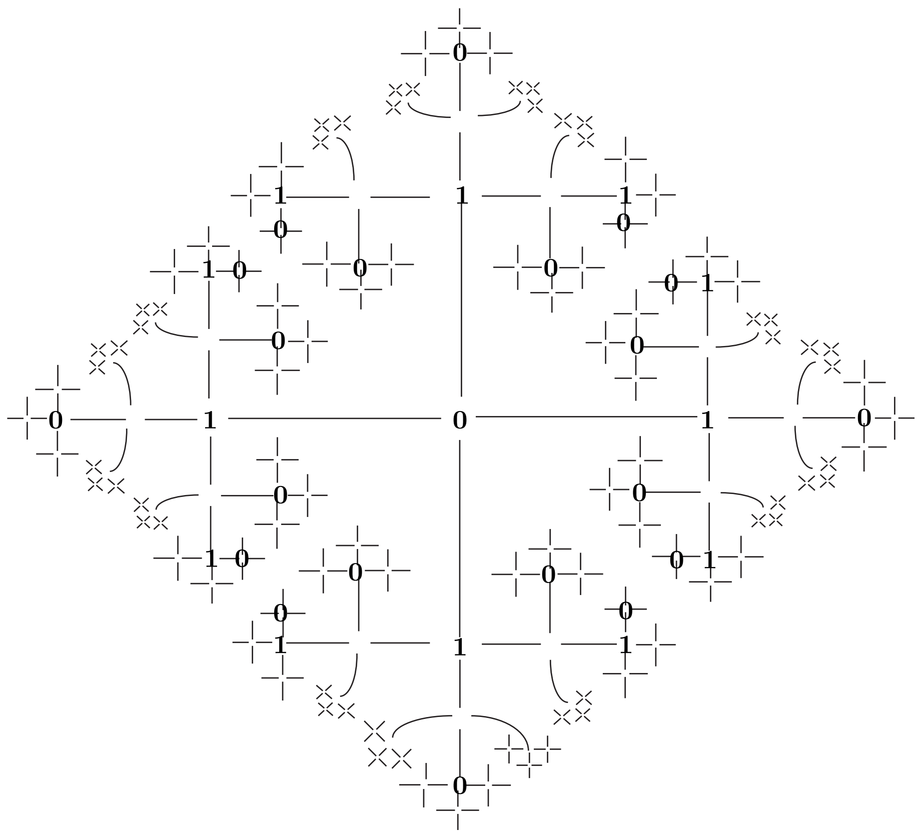

FIGURE 1.2 – Étape 2. On complète par **1** la coloration du domaine fondamental de Γ_1 puis on prolonge par Γ_2-périodicité pour un certain groupe $\Gamma_2 \subset \Gamma_1$.

décroissante de sous-groupe $(\Gamma_n)_n$ d'indices finis et d'intersection triviale, il est possible de construire une suite croissante de domaines fondamentaux finis D_n pour G/Γ_n de sorte que $\bigcup_n D_n = G$ et chaque D_{n+1} est une union de translatés de D_n. On construit alors par récurrence une suite Toeplitz de la façon suivante : à l'étape n, on choisit arbitrairement $x_{|D_n}$ (pour les indices non encore définis) puis on prolonge par Γ_{n+1}-périodicité : pour tout $\gamma \in \Gamma_{n+1}$, $h \in D_n$, on pose $x(\gamma h) := x(h)$ (voir les figures 1.1 et 1.2).

Un *G-sous-shift Toeplitz* X est l'adhérence d'une orbite, par l'action du shift, d'une suite Toeplitz $\overline{\{\sigma^g(x); g \in G\}}$. La G-action restreinte à ce sous-shift est alors *régulièrement récurrent*, dans le sens où pour tout voisinage V de x, il existe un sous-groupe $\Gamma \subset G$ d'indice fini tel que $\sigma^\gamma(v) \in V$ pour tout $\gamma \in \Gamma$ [Kri06, 5]. Cette condition est plus forte que la presque périodicité et assure donc que l'action de G est minimale sur ce sous-shift [Aus88]. Nous dirons que la restriction de l'action du shift sur le sous-shift X est *fortement régulièrement récurrent* s'il existe un voisinage W arbitrairement petit de x tel que $\{\gamma \in G; \sigma^\gamma(x) \in W\}$ est un sous-groupe normal de G d'indice fini. Bien évidemment cette condition implique la récurrence régulière.

Dans un travail en commun avec M. I. Cortez, nous caractérisons les sous-shifts qui sont des extensions presque injectives de sous-odomètres en étendant au contexte non commutatif les résultats de [DL98, Cor06]. Rappelons que le système (X, G) est une *extension presque injective* du système $(\overleftarrow{G}, G)$ s'il existe une application continue $\pi\colon X \to \overleftarrow{G}$ surjective, commutant avec les actions, telle que π est injective sur un ensemble G_δ-dense (au sens de Baire).

Théorème 1.2.2 ([5]) *Un sous-shift minimal (X, G) est une extension presque injective d'un sous-odomètre (resp. un odomètre) $(\overleftarrow{G}, G)$ par π si et seulement si X est un G-sous-shift Toeplitz (resp. fortement régulièrement récurrent).*
De plus l'ensemble des points d'injectivité de π est l'ensemble des suites Toeplitz de X.

Ce théorème implique en particulier que le facteur équicontinu maximal d'un sous-shift Toeplitz est un sous-odomètre $\overleftarrow{G}$ [Aus88]. Ces systèmes partagent donc le même spectre continu. En fait, nous pouvons être beaucoup plus précis, et caractériser exactement l'odomètre en fonction de la combinatoire d'une suite Toeplitz. Il en ressort le théorème d'existence suivant :

Théorème 1.2.3 ([5]) *Pour tout sous-odomètre $\overleftarrow{G}$, il existe une suite Toeplitz $x \in \{0,1\}^G$ telle que $(\overleftarrow{G}, G)$ est le facteur équicontinu maximal du G-sous-shift Toeplitz associé à x.*

Citons également le travail de Krieger [Kri06] qui, lorsque le groupe G est moyennable, réalise n'importe quelle entropie par un sous-shift Toeplitz.

1.2.3 Équivalence orbitale pour les sous-shifts Toeplitz

Nous considérons ici le cas d'un groupe G résiduellement fini, finiment engendré et moyennable. Ainsi chaque action continue de G sur un espace métrique compact admet une mesure de probabilité invariante. Le théorème 1.2.2 peut faire penser que les sous-shifts Toeplitz et les odomètres ont des dynamiques très proches. Nous montrons, en fait, qu'il n'en est rien dans le contexte mesuré. Il est possible de réaliser n'importe simplexe (raisonnable) de mesures de probabilité invariantes par l'action du shift sur un sous-shift Toeplitz.

Théorème 1.2.4 ([6]) *Pour tout simplexe de Choquet K et n'importe quel odomètre $\overleftarrow{G}$, il existe un G-sous-shift Toeplitz X qui est une extension presque injective de $(\overleftarrow{G}, G)$ et dont l'ensemble des mesures de probabilité invariantes est affinement homéomorphe à K.*

Ce résultat généralise celui de Downarowicz [Dow91] au cadre non commutatif moyennable. Une grande différence est que notre preuve est constructive. Elle repose sur une description de la dynamique par une partition à la Kakutani-Rohlin à l'aide d'une suite de Følner $(F_n)_n$. Nous construisons une telle suite avec une propriété combinatoire : il en existe une de sorte qu'à chaque étape, F_{n+1} est pavable par des translatés de F_n et chaque F_n est un domaine fondamental pour le groupe Γ_n (voir également [Wei01]). Nous en déduisons que pour une suite de matrices d'incidences $(M_n)_n$ vérifiant des conditions arithmétiques données par la suite de nombres $(\text{card } F_{n+1}/\text{card } F_n)_n$, il existe une suite Toeplitz dont le remplissage à l'étape n est "contrôlé" par la matrice M_n. Nous obtenons le résultat en décrivant le simplexe K par une limite inverse de simplexes de dimensions finies, ce qui nous donne les matrices d'incidences.

Une autre conséquence de notre construction arrive dans le contexte de l'équivalence orbitale. Suite à de nombreux travaux Giordano, Matui, Putnam et Skau [GMPS10] montrent que pour toute $\mathbb{Z}^d$-action $(X, \mathbb{Z}^d)$ minimale sur un ensemble de Cantor, il existe une $\mathbb{Z}$-action (Y, S) minimale sur un Cantor qui lui est orbitalement équivalente, *i.e.* il existe un homéomorphisme de $X \to Y$ préservant les orbites des systèmes. Dans ce contexte, le groupe de dimension modulo les infinitésimaux (définition similaire à celle de la section 1.1.4) est également un invariant total de l'équivalence orbitale [GMPS10].

Ce résultat est similaire à ceux de [OW80, CFW81] dans le contexte mesuré. Cependant, on ne sait toujours pas s'il peut s'étendre à n'importe quelle action minimale sur un Cantor d'un groupe dénombrable moyennable. À l'opposé, on sait que le même résultat est faux pour des groupes libres puisque Gaboriau [Gab00] a montré que si des actions des groupes libres F_n et F_p sont orbitalement équivalentes (même seulement mesurablement) alors le rang des groupes est préservé, *i.e.* $n = p$. De plus, nous ne savons toujours pas quelles sont les $\mathbb{Z}$-actions qui peuvent être réalisées comme des classes de $\mathbb{Z}^d$-actions. Nous donnons dans [6] une réponse partielle qui apparait comme un corollaire de notre preuve du théorème 1.2.4.

Théorème 1.2.5 ([6]) *Soit $(X, \mathbb{Z})$ un $\mathbb{Z}$ sous-shift Toeplitz. Alors pour tout entier $d \geq 1$, il existe un $\mathbb{Z}^d$ sous-shift Toeplitz qui est orbitalement équivalent à $(X, \mathbb{Z})$.*

Pour le démontrer, nous utilisons une description en terme de partition de Kakutani-Rohlin pour caractériser le groupe de dimension modulo les infinitésimaux associés à la $\mathbb{Z}^d$-action. En appliquant directement le résultat de [GMPS10], il suffit alors de décrire le groupe de dimension d'un $\mathbb{Z}$-sous-shift Toeplitz par une limite inverse de groupes pour donner une suite de matrices d'incidences permettant de définir le $\mathbb{Z}^d$-Toeplitz.

Chapitre 2

Systèmes de Delone

Les concepts de base reliés aux ensembles de Delone euclidiens ou hyperboliques sont rappelés dans la première section, notamment la notion d'enveloppe topologique (sous-section 2.1.1) et ses propriétés géométriques (sous-section 2.1.2).

La section 2.2 ne traite que du cas euclidien. Nous décrivons d'abord quelques propriétés des ensembles linéairement répétitifs dans la sous-section 2.2.1, puis nous traitons des propriétés des homéomorphismes de l'enveloppe dans la sous-section 2.2.2.

La dernière section est relative aux ensembles de Delone en géométrie hyperbolique. Nous rappelons quelques constructions, qui ne sont pas toujours simple à trouver dans la littérature, dans la sous-section 2.3.1. La sous-section 2.3.2 concerne le lien entre les mesures harmoniques et les mesures invariantes des systèmes associés. La dernière sous-section traite de la K-théorie d'une famille d'exemple d'ensembles de Delone hyperboliques.

2.1 Généralités

2.1.1 Ensemble de Delone et enveloppe topologique

Nous considérerons à la fois la géométrie euclidienne et hyperbolique. Ainsi, en suivant les notions issues de [BG03], $\mathbb{G}$ désignera dans cette section

- soit le groupe $\mathbb{R}^d$ des translations de l'espace euclidien de dimension $d \geq 1$,
- soit le groupe des transformations affines $\{z \mapsto az + b; a > 0, b \in \mathbb{R}\}$ vu comme sous-groupe d'isométries du demi plan hyperbolique $\mathbb{H}^2 = \{z \in \mathbb{C}; \Im(z) > 0\}$ munit de la métrique $ds^2 = \frac{dx^2+dy^2}{y^2}$.

Remarquons que les deux groupes agissent par isométries transitivement et librement sur leur espace homogène associé de sorte que l'on peut identifier ces groupes d'isométries et leurs espaces. Le groupe $\mathbb{G}$ est munit d'une distance invariante par multiplication à gauche. Nous noterons par $B_R(x)$ la boule de rayon $R > 0$ centrée en un point $x \in \mathbb{G}$. Nous renvoyons le lecteur à [LP03, KP00] pour une présentation détaillée des ensembles de Delone euclidiens et à [Rud89, Rob96, Sol97] pour une introduction aux propriétés des systèmes de Delone associés. Tout ce qui est présenté dans cette section peut s'étendre aux pavages et à des groupes de Lie connexes [BG03].

Définition 2.1.1 *Un* (r_X, R_X)-ensemble de Delone $X \subset \mathbb{G}$ *est un sous-ensemble*

- r_X-uniformément discret, i.e. *pour tout* $x \in X$, *card* $B_{r_X}(x) \cap X \leq 1$ *;*

— R_X-relativement dense, i.e. *pour tout point* $y \in \mathbb{G}$*, card* $B_{R_X}(y) \cap X \geq 1$.

Pour alléger les notations, nous parlerons d'ensemble de Delone, sans préciser les constantes r_X et R_X. Un *R-patch* $\mathtt{P}$ est un sous-ensemble fini de X de la forme $B_R(x) \cap X$ pour un certain $x \in X$. Deux patchs $\mathtt{P}_1$ et $\mathtt{P}_2$ sont dits $\mathbb{G}$-*équivalents* s'il existe une isométrie $g \in \mathbb{G}$ telle que $g(\mathtt{P}_1) = \mathtt{P}_2$. Un ensemble de Delone X est dit *de* $\mathbb{G}$*-type fini* si pour tout $R > 0$, il n'existe qu'un nombre fini de classes de $\mathbb{G}$-équivalence de R-patchs. Un point $y \in X$ est une *occurrence* d'un R-patch $\mathtt{P}$ si les patchs $B_R(y) \cap X$ et $\mathtt{P}$ sont $\mathbb{G}$-équivalents.

Nous allons donner à présent une topologie sur l'ensemble des ensembles de Delone de $\mathbb{G}$-type fini. Remarquons tout d'abord qu'à chaque ensemble de Delone X, nous pouvons associer une mesure de Radon $\nu_X := \sum_{x \in X} \delta_x$ où δ_x désigne la mesure de Dirac en x. Comme l'ensemble des mesures $\mathcal{M}(\mathbb{G}, r, R)$ obtenus à partir de (r, R)-ensembles de Delone est un sous-ensemble du dual des fonctions continues à support compact $\mathcal{C}_c(\mathbb{G}, \mathbb{R})$. La topologie faible-* induit une topologie métrique sur cet ensemble fermé, appelée topologie de *Gromov-Hausdorff*, où une suite $(X_n)_{n \in \mathbb{N}}$ d'ensembles de Delone converge vers X si et seulement si pour tout ouvert borné U de $\mathbb{G}$, la suite d'ensembles $(U \cap X_n)_{n \in \mathbb{N}}$ converge vers $U \cap X$ pour la topologie de Hausdorff. De façon plus combinatoire, pour cette topologie, deux ensembles de Delone sont proches s'ils coïncident, par une isométrie de $\mathbb{G}$ proche de l'identité, sur une grande boule centrée à l'origine.

Le groupe des transformations $\mathbb{G}$ agit continûment par translation à droite* sur l'ensemble des ensembles de Delone de $\mathbb{G}$-type fini $X.g := g^{-1}(X) = \{g^{-1}(x);\ x \in X\}$. Cette action peut également être vue comme la restriction de l'action par translation à droite de $\mathbb{G}$ sur $\mathcal{M}(\mathbb{G}, r, R)$ donnée par $\nu.g(f) := \nu(f(g^{-1}\cdot))$ pour $\nu \in \mathcal{M}(\mathbb{G})$, $f \in \mathcal{C}_c(\mathbb{G}, \mathbb{R})$ et $g \in \mathbb{G}$.

L'*enveloppe topologique* $\Omega(X)$ de l'ensemble de Delone X est la fermeture pour la topologie de Gromov-Hausdorff de l'orbite de X par l'action de $\mathbb{G}$. Le système dynamique $(\Omega(X), \mathbb{G})$ est alors appelé *système de Delone*. Il est direct de montrer que $\Omega(X)$ est un ensemble métrique compact (cf [KP00]). Le fait de se restreindre aux ensembles de Delone de $\mathbb{G}$-type fini implique une certaine rigidité : chaque élément est un ensemble de Delone X' dont chaque patch est équivalent à un patch de X [KP00, BG03]. De plus, l'action de $\mathbb{G}$ est continue et le système dynamique $(\Omega(X), \mathbb{G})$ possède, par construction, une orbite dense (celle de X). Un résultat classique donne la caractérisation combinatoire suivante.

Proposition 2.1.2 ([Aus88]) *Soit* X *un ensemble de Delone de* $\mathbb{G}$*-type fini. Le système de Delone* $(\Omega(X), \mathbb{G})$ *est minimal si et seulement si* X *est* répétitif *:* i.e. *pour tout rayon* $R > 0$ *et pour tout R-patch* $\mathtt{P}$ *de* X*, il existe une constante* $M(R) > 0$ *telle que n'importe quelle boule de rayon* M *intersectée avec* X *contient une occurence de* $\mathtt{P}$*.*

Nous dirons qu'un ensemble de Delone X est *apériodique* si pour tout $g \in \mathbb{G} \setminus \{e\}$ $X.g \neq X$. Cet ensemble est dit *totalement apériodique* si pour tout ensemble $Y \in \Omega(X)$ et pour tout $g \in \mathbb{G} \setminus \{e\}$ $Y.g \neq Y$ ou, autrement dit, l'action de $\mathbb{G}$ sur $\Omega(X)$ est libre. Dans le cas commutatif $\mathbb{G} = \mathbb{R}^d$, si X est répétitif et apériodique, alors il est totalement apériodique. Ce n'est pas le cas dans le cadre non commutatif. Les exemples classiques d'ensembles de Delone apériodiques répétitifs en géométrie euclidienne sont données par les méthodes de *coupée-projection* [dB81, KD86] et de substitution [Gar77, GS89]. Rappelons que le pavage de Penrose peut être obtenu par ces deux méthodes. Nous renvoyons à la sous-section 2.3.1 pour des exemples d'ensembles de Delone apériodiques et répétitifs en géométrie hyperbolique. Pour des exemples sur des groupes de Lie plus généraux, nous réferrons par exemple à [Moz97].

*. Le choix de l'action à droite deviendra clair lors de la présentation de la structure géométrique de l'enveloppe.

Pour des raisons techniques, il est utile d'introduire également le sous-ensemble de l'enveloppe Ξ appelé *transversale canonique* défini comme

$$\Xi(X) = \{Y \in \Omega(X);\ \text{l'origine } e \in Y\}.$$

Topologiquement, il est assez simple de voir que l'hypothèse de $\mathbb{G}$-type fini implique que cet ensemble est totalement discontinu. Lorsque l'ensemble de Delone X est répétitif et totalement apériodique, c'est un ensemble de Cantor. Il est transverse à la dynamique dans le sens où pour tout $g \in \mathbb{G} \setminus \{e\}$ suffisamment proche de l'identité, les ensembles $\Xi(X).g$ et $\Xi(X)$ sont disjoints.

2.1.2 Structure géométrique de l'enveloppe

Nous rappelons ici la structure géométrique de lamination de ces enveloppes. Nous faisons référence à [BBG06, BG03] pour une présentation plus détaillée.

Soit (Ω, dist) un espace métrique compact, et supposons qu'il existe un recouvrement par des ouverts U_i et des homéomorphismes, appelés *carte*, $h_i \colon U_i \to V_i \times \Xi_i$ où V_i est un ensemble ouvert du groupe de Lie $\mathbb{G}$ et Ξ_i est un espace métrique compact totalement discontinu. La collection d'ouverts et d'homéomorphismes (U_i, h_i) est appelé *atlas d'une lamination plate* si pour chaque *application de transition* $h_{i,j} = h_i \circ h_j^{-1}$, il existe un $g_{i,j} \in \mathbb{G}$ et un homéomorphisme $f_{i,j} \colon \Xi'_j \subset \Xi_j \to \Xi'_i \subset \Xi_i$ de sorte que $h_{i,j}$ s'écrive dans son domaine de définition

$$h_{i,j}(g, \xi) = (g_{i,j}.g, f_{i,j}(\xi)), \tag{2.1.1}$$

où $g_{i,j}.g$ désigne la multiplication à gauche de $x \in \mathbb{G}$ par $g_{i,j}$. Il est important de noter que l'élément $g_{i,j}$ et l'homéomorphisme $f_{i,j}$ sont indépendants des coordonnées (g, ξ). Deux atlas sont dits *équivalents* si leur union est également un atlas de lamination plate. Une *lamination plate* est la donnée d'un espace métrique compact Ω et d'une classe d'équivalence d'atlas de lamination plate. Un atlas est alors dit *maximal* s'il contient n'importe quel atlas équivalent. Une *boîte* est un domaine d'une carte dans l'atlas maximal.

Ainsi un tel espace est une lamination géométrique [CGSY99, chapitre 2]. Plus précisément une telle lamination est une *variété boîte d'allumettes*, ou *matchbox manifold* [AO95] car son espace transverse est totalement discontinu. Encore plus spécifiquement, il admet une transversale globale. En résumé, nous avons :

1. des *plaques* : une plaque est un ensemble de la forme $h_i^{-1}(V_i \times \{\xi\})$ dans une carte.
2. Des *verticales* : une verticale est un ensemble de la forme $h_i^{-1}(\{g\} \times \Xi_i)$ dans une carte.

Comme les applications de transition envoient les plaques (resp. les verticales) sur des plaques (resp. des verticales), ces notions sont bien définies, indépendamment du choix de la carte, et leur union forme un ensemble globalement bien défini. Ainsi une *feuille* est la composante connexe des plaques qui s'intersectent. L'espace Ω est une union disjointe de feuilles et chaque feuille a une structure de variété différentiable. Une lamination plate Ω est dite *minimale* si toutes ses feuilles sont denses dans Ω. De la même façon, une union disjointe de verticales est bien définie et est appelée une *transversale* de Ω.

De la forme spéciale des applications de transition, on peut définir (indépendamment du choix de la carte) une action par multiplication à droite par un élément de $\mathbb{G}$ proche de l'identité. Suivant l'appellation originelle [BG03], nous appellerons $\mathbb{G}$-*solénoïde*† toute

†. Pour éviter une certaine ambiguité avec le terme solénoïde, réservé à des actions équicontinues, certains auteurs préfèrent utiliser le terme de *lamination pavable*.

lamination plate Ω dont chaque feuille est isométrique à $\mathbb{G}$. Dans ce cas, l'action locale de $\mathbb{G}$ s'étend en une action globale continue de $\mathbb{G}$ sur Ω. Celle-ci est libre (le stabilisateur de tout point est trivial) et les feuilles correspondent aux orbites de l'action.

Une propriété importante de cette action est qu'elle envoie une verticale sur une verticale. Ceci lui donne une prorpiété que nous appellerons un peu abusivement *temps de retour localement constant* : *i.e.* pour tout $g \in \mathbb{G}$ et pour tout point $x \in \Omega$ dans une verticale $\tilde{\Xi}$ d'une boîte B, si $x.g$ appartient à une verticale $\tilde{\Xi}_2$ d'une boite B_2, alors pour tout $y \in \tilde{\Xi}$ suffisamment proche de x, nous avons $y.g \in \tilde{\Xi}_2$. Il en ressort que chaque verticale est *transverse* à l'action puisque pour tout $g \in \mathbb{G}$ suffisamment proche de l'identité et pour toute verticale $\tilde{\Xi}$ suffisamment petite, les ensembles $\tilde{\Xi}.g$ et $\tilde{\Xi}$ sont disjoints.

Des exemples de $\mathbb{G}$-solénoïdes minimaux sont alors obtenus par des suspensions d'actions minimales libres d'un réseau, ou d'un semi-réseau cocompact de $\mathbb{G}$ (ex : $\mathbb{Z}^d$ pour $\mathbb{G} = \mathbb{R}^d$, ou le semi-groupe de Baumslag-Solitar pour le groupe $\mathbb{G}$ des transformations affines) sur un ensemble de Cantor Ξ, en prenant des applications *temps* localement constantes. Voir la section 1.2 pour de tels exemples. La suspension d'un odomètre (cf sous-section 1.2.1), nous donne ainsi un $\mathbb{G}$-solénoïde avec une action de $\mathbb{G}$ équicontinue.

À l'opposé des systèmes équicontinus, la notion suivante est caractéristique des systèmes associés aux ensembles de Delone [FS14].

Définition 2.1.3 *Le $\mathbb{G}$-solénoïde Ω est dit* expansif *s'il existe un $\eta > 0$ vérifiant pour tout homéomorphisme $h\colon \mathbb{G} \to \mathbb{G}$ tel que $h(e) = e$, si $x, y \in \Omega$ sont tels que $\sup_{g \in \mathbb{G}} \operatorname{dist}(x.g, y.h(g)) < \eta$ alors il existe un g_0 η-proche de e tel que $x.g_0 = y$.*

La relation entre les ensembles de Delone et les solénoïdes est donnée par le théorème suivant.

Théorème 2.1.4 ([BG03]) *Soit $X \subset \mathbb{G}$ un ensemble de Delone totalement apériodique de $\mathbb{G}$-type fini. Alors l'enveloppe continue de $\Omega(X)$ a une structure de $\mathbb{G}$-solénoïde expansif où l'action de $\mathbb{G}$ sur $\Omega(X)$ coïncide avec celle de $\mathbb{G}$ sur le solénoïde.*
Réciproquement, si Ω est un $\mathbb{G}$-solénoïde expansif minimal, alors il existe un ensemble de Delone $X \subset \mathbb{G}$ totalement apériodique de $\mathbb{G}$-type fini et répétitif tel que les systèmes dynamiques $(\Omega, \mathbb{G})$ et $(\Omega(X), \mathbb{G})$ sont conjugués.

La structure de lamination des espaces de Delone avait déjà été remarquée dans [CGSY99, chapitre 2]. Les boîtes s'obtiennent assez naturellement dans le contexte des pavages. En effet, pour un R-patch $\mathtt{P}$ que l'on peut supposer, quitte à le translater, centré en l'origine, nous noterons par conv $\mathtt{P} \subset \mathbb{G}$ son enveloppe convexe. Soit $i_{\mathtt{P}}$ l'application

$$\begin{array}{rcl} i_{\mathtt{P}}\colon \operatorname{conv} \mathtt{P} \times \Xi_{\mathtt{P}} & \to & \Omega(X) \\ (g, Y) & \mapsto & Y.g = g^{-1}(Y) \end{array}$$

où $\Xi_{\mathtt{P}}$ désigne l'ensemble $\{Y \in \Xi(X), B_R(0) \cap Y = \mathtt{P}\}$. Il est alors simple de vérifier que cette application est un homéomorphisme sur son ensemble image. De plus, la collection de ces applications forme un atlas de lamination plate. Ainsi l'enveloppe d'un ensemble de Delone répétititf totalement apériodique est localement homéomorphe au produit cartésien d'un ensemble de Cantor par un ouvert de $\mathbb{G}$ et l'ensemble $\Xi(X)$ est une transversale globale.

Une *décomposition en boîtes* est une collection de boîtes ouvertes deux à deux disjointes $\mathcal{B} = \{B^{(1)}, \ldots, B^{(t)}\}$ de Ω telle que l'union de leur fermeture recouvre l'espace Ω. En identifiant les points appartenant à une même verticale de la fermeture d'une boîte, nous obtenons un simplexe B. Comme la lamination est plate, ce simplexe hérite d'une structure supplémentaire de *variété branchée* [Wil74, BG03], *i.e.* est localement le recollement de variétés tangentes entre elles. Cette variété possède donc un espace tangeant en chacun de ses points. En prenant une suite de décompositions en boîtes $(\mathcal{B}_n)_{n\geq 0}$ "bien emboîtées", dans le sens où chaque verticale d'une boîte de $\mathcal{B}_n$ est incluse dans une verticale d'une boîte de $\mathcal{B}_{n+1}$, plus quelques conditions techniques, Benedetti et Gambaudo obtiennent qu'un $\mathbb{G}$-solénoïde est conjugué à une limite projective de variétés branchées

$$\varprojlim(B_n, f_n) = \left\{ (x_n)_{n\geq 0} \in \prod_{n\geq 0} B_n;\ f_n(x_{n+1}) = x_n \right\},$$

où chaque $f_n\colon B_{n+1} \to B_n$ st une application continue surjective, envoyant le lieu singulier sur le lieu singulier [BG03, Sad03].

2.2 Ensembles de Delone euclidiens

2.2.1 Ensembles linéairement répétitifs

Rappelons que d'après la Proposition 2.1.2, un ensemble de Delone $X \subset \mathbb{R}^d$ est répétitif si tout R-patch apparait dans n'importe quelle boule de rayon M. La plus petite valeur M possible pour un R fixé sera appelée *fonction de répétitivité* et sera désignée par $M_X(R)$.

Définition 2.2.1 *Un ensemble de Delone $X \subset \mathbb{R}^d$ est dit* linéairement répétitif *si sa fonction de répétitivité $M_X(R)$ croit au plus linéairement,* i.e. *il existe une constante $L > 1$ telle que $M_X(R) \leq LR$ pour tout $R > 0$.*

Dans la suite, nous dirons que X est *linéairement répétitif (de constante L)*. Cette notion est apparue dans [LP02]. De façon indépendante, la notion similaire de $\mathbb{Z}$-sous-shift *linéairement récurrent* avait été introduite, peu de temps auparavant dans [Dur96, DHS99] pour étudier les relations entre les systèmes substitutifs et les groupes de dimension stationnaires.

Selon le théorème suivant, il s'agit de la croissance la plus lente possible de fonction de répétitivié pour un ensemble de Delone apériodique.

Théorème 2.2.2 ([LP02], Thm 2.3) *Soit $d \geq 1$. Il existe une constante $c(d) > 1$ telle que tout ensemble de Delone $X \subset \mathbb{R}^d$ vérifiant*

$$M_X(R) \leq c(d)R \qquad \textit{pour un certain } R > 0,$$

a une période non nulle : i.e. *il existe $\vec{v} \in \mathbb{R}^d \setminus \{0\}$ tel que $X + \vec{v} = X$.*

Citons également un autre résultat de [LP02] qui stipule que si pour un $R > 0$, $M_X(R) < \frac{4}{3}R$, alors l'ensemble de Delone X a d périodes indépendantes.

Les exemples classiques d'ensembles de Delone apériodiques, c'est-à-dire ceux issus des substitutions primitives, sont linéairement répétitifs [Sol98]. Pourtant, de différents points de vue, la collection des ensembles de Delone linéairement répétitifs est petite dans la famille des ensembles de Delone de $\mathbb{R}^d$. Par exemple, dans [MH40], Morse et Hedlund caractérisent les sous-shifts sturmiens linéairement récurrents (ou linéairement répétitifs) comme étant ceux qui codent une rotation d'angle $\alpha \in [0, 1[$ dont le développement en fraction continue est borné. Ces angles sont alors mal approchés par les nombres rationnels. Ils forment un ensemble maigre au sens de Baire et de mesure nulle pour la mesure de Lebesgue. Cependant, cet ensemble est de dimension de Hausdorff maximal 1. Mentionnons que tous ces résultats ont été étendus aux dimensions supérieures aux ensembles de Delone obtenus par coupé-projection dans [HKW].

Un ensemble de Delone $X \subset \mathbb{R}^d$ linéairement répétitif apériodique possède de nombreuses propriétés rigides. Par exemple d'un point de vue combinatoire, deux occurrences d'un même R-patch sont à une distance au moins linéaire en R. Ceci implique que la complexité de X est la plus basse possible [Len04, LP03] : si $N_X(R)$ désigne le nombre de R-patch, à translation près, alors $N_X(R) \in \Theta(R^d)$ signifiant qu'il existe des constantes $cste_1, cste_2 > 0$ telles que pour tout R suffisamment grand

$$cste_1 R^d \leq N_X(R) \leq cste_2 R^d.$$

Sa structure hiérarchique est également très contrainte. Étant donnée n'importe quelle taille $R > 0$, il est possible de décomposer X en de gros patchs (chacun contenant au moins un R-patch) de sorte que le nombre de ces gros patchs, à translation près, est uniformément borné en R [3]. D'un point de vue structurel, il est possible de décrire le $\mathbb{R}^d$-solénoïde $\Omega(X)$ par une limite projective de variétés branchées $\varprojlim(B_n, f_n)$ avec un nombre fini de variétés branchées à homéomorphisme près et où la suite des applications $(f_n^*)_n$ en homologie est uniformément bornée [APC11].

D'un point de vue ergodique, les systèmes associés aux ensembles de Delone linéairement répétitifs sont minimaux et uniquement ergodiques. Il est même possible de déterminer des vitesses de convergences du nombre moyen d'occurrences d'un patch [LP03, APC11]. Précisons que ces systèmes sont d'entropie nulle et ne sont jamais mesurablement fortement mélangeants [1]. Ces ensembles de Delone possèdent bien d'autres propriétés géométriques et nous renvoyons le lecteur à [1] et ses références pour un survol des différentes propriétés et caractérisations connues des ensembles de Delone linéairement répétitifs.

D'un point de vue dynamique, nous montrons dans [3] que la famille des ensembles de Delone linéairement répétitifs est "stable" dans la famille des systèmes de Delone apériodiques.

Proposition 2.2.3 ([3]) *Soit $X \subset \mathbb{R}^d$ un ensemble de Delone linéairement répétitif de constante $L > 1$. Alors il existe une constante $C(L, d)$ (ne dépendant que de L et de d) telle que pour toute application facteur $\pi\colon (\Omega(X), \mathbb{R}^d) \to (\Omega(Y), \mathbb{R}^d)$ sur un système de Delone apériodique,*

- *L'ensemble de Delone Y est linéairement répétitif, et*
- *chaque fibre de π contient au plus C éléments.*

Ainsi tout système de Delone apériodique facteur d'un ensemble de Delone linéairement répétitif X est lui aussi linéairement répétitif. De plus, chaque application facteur a des fibres de cardinal uniformément borné. Concernant la preuve, le premier point se déduit assez directement des définitions. Quant au second point, il repose sur la propriété de répulsion des occurrences de patchs. Si un facteur possède beaucoup de pré-images d'un même point, chacune de ces pré-images définit un R-patch différent en l'origine. Tous ces patchs ont donc une occurence dans une boule de rayon LR de X. Nous obtenons par l'image de π un patch de Y avec des occurrences beaucoup trop proches.

Dans [3], nous en déduisons qu'il ne peut y avoir qu'un nombre fini de tels facteurs non conjugués.

Théorème 2.2.4 ([3]) *Soient un réel $L > 1$ et un entier $d \geq 1$. Il existe une constante $N(L, d)$ telle que tout ensemble de Delone $X \subset \mathbb{R}^d$ linéairement répétitif de constante L, a au plus N systèmes de Delone apériodiques, non conjugués, facteurs du système $(\Omega(X), \mathbb{R}^d)$.*

Ce résultat étend aux systèmes de Delone un résultat de [Dur00] pour les sous-shifts linéairement récurrent.

La preuve repose de façon essentielle sur les contraintes de structure hiérarchique d'un ensemble de Delone linéairement répétitif. Pour simplifier, considérons les cas où chaque facteur $\pi \colon \Omega(X) \to \Omega(Y)$ est localement constant le long de chaque verticale d'une décomposition en boîtes de $\Omega(X) \simeq \varprojlim(B_n, f_n)$. Il s'ensuit que chaque facteur π se factorise en une application $\tilde{\pi}$ d'une variété branchée B_n vers $\Omega(Y)$. Comme il n'y a qu'un nombre fini de variétés branchées non isomorphes, il n'y a qu'un nombre fini d'identification possible des faces d-dimensionnelles de ces variétés branchées par des applications de type $\tilde{\pi}$. Il suffit alors de montrer que des facteurs identifiant les mêmes d-faces d'un variété branchée ont les mêmes fibres impliquant ainsi que les systèmes facteurs sont conjugués. Le cas général se déduit de celui-ci en prouvant qu'une application facteur est une perturbation d'une application localement transversalement constante.

2.2.2 Sur les homéomorphismes de l'enveloppe

Rappelons que deux systèmes de Delone $(\Omega(X), \mathbb{R}^d)$, $(\Omega(Y), \mathbb{R}^d)$ sont dits *orbitalement équivalents* ou *flot-équivalents*, s'il existe un homémomorphisme $F \colon \Omega(X) \to \Omega(Y)$ tel que pour tout $x \in \Omega(X)$

$$F(\mathrm{Orb}_{\mathbb{R}^d}(x)) = \mathrm{Orb}_{\mathbb{R}^d}(F(x)).$$

Dans le cadre des ensembles de Delone apériodiques, les orbites correspondent aux feuilles de la lamination, c'est-à-dire aux composantes connexes par arc de cet espace. Ainsi tout homéomorphisme envoie une feuille sur une feuille. Dire que deux systèmes de Delone sont flot-équivalents revient donc à dire que leur enveloppe sont homéomorphes. Un invariant naturel de flot-équivalence est alors le groupe des homéomorphismes $\mathrm{Homeo}(\Omega(X))$ de l'enveloppe $\Omega(X)$. Nous nous intéresserons dans cette section à ce groupe et, plus particulièrement à sa composante connexe de l'identité dans le cadre un peu plus général de $\mathbb{R}^d$-solénoïde Ω. Nous montrons en particulier que ce groupe est simple, *i.e.* il n'admet pas de sous-groupe normal propre non trivial.

Rappelons que le groupe des homéomorphismes $\mathrm{Homeo}(M)$ d'une variété M a déjà bien été étudié. Dans les années 60, Fisher [Fis60] puis Anderson [And62] ont montré que ce groupe est simple lorsque la variété M est compacte sans bord. Un résultat important dans ce domaine est dû à D. Epstein [Eps70] qui a établi une condition suffisante sur un groupe d'homémomorphismes pour que son sous-groupe dérivé (engendré par les commutateurs) soit simple. Ainsi, un groupe vérifiant la condition d'Epstein est simple si et seulement s'il est *parfait* (*i.e.* tout élément du groupe s'écrit comme un produit de commutateurs). Plus tard cette problématique fut étendue aux sous-groupes de difféomorphismes. Mentionnons alors les travaux de Herman [Her73], Thurston [Thu74] et Mather [Mat74] qui ont fourni une classification presque complète des groupes simples de difféomorphismes de variétés. Plus récemment ces problèmes de simplicité, ont été étudiés dans le contexte des feuilletages. Par exemple, étant donné un feuilletage $\mathcal{F}$ d'une variété M, Rybicki [Ryb95] et Tsuboi [Tsu06] ont caractérisé la simplicité et la perfection de la composante connexe de l'identité du groupe des difféomorphismes de M préservant le feuilletage $\mathcal{F}$. Précisons que ces groupes ne vérifient pas la condition d'Epstein.

Pour donner nos résultats dans ce contexte, nous devons introduire quelques notations. Pour un $\mathbb{R}^d$-solénoïde Ω, nous désignerons par $D(\Omega)$ le groupe des *déformations*, *i.e.* la composante connexe par arc de l'identité dans $\mathrm{Homeo}(\Omega)$. En d'autres termes, pour chaque déformation, il existe un chemin continu d'homéomorphismes (pour la topologie C^0) la reliant à l'identité. De façon générale, pour un groupe topologique G nous noterons par G^0 la composante connexe de l'identité dans G.

Dans le cas des systèmes de Delone, certaines subtilités dans les définitions disparaissent puisque la composante connexe de l'identité est le groupe des déformations.

Proposition 2.2.5 ([2]) *Soit Ω un $\mathbb{R}^d$-solénoïde expansif. Alors $Homeo^0(\Omega)$ est égal à $D(\Omega)$ et est ouvert dans* $\mathrm{Homeo}(\Omega)$.

Il s'ensuit par le Théorème 2.1.4 que n'importe quel homéomorphisme d'une enveloppe $\Omega(X)$, suffisamment proche de l'identité, est une déformation. De plus, comme l'identité préserve chaque feuille, remarquons que chaque déformation préserve également chaque feuille. Nous noterons alors par $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ le groupe des homéomorphismes de Ω, qui préservent chaque feuille de Ω. Ce phénomène est très différent du cas équicontinue car, dans ce cas, le solénoïde a une structure de groupe topologique abélien où $\mathbb{R}^d$ est sous-groupe dense. Ainsi n'importe quelle translation par un élément x proche de l'élément neutre de Ω, définit un homéomorphisme arbitrairement proche de l'identité. Si x n'est pas dans la feuille de l'élément neutre, cette translation ne préserve aucune feuille.

Un autre sous-groupe apparaissant naturellement dans le cadre des systèmes de Delone est celui des homéomorphismes préservant des verticales.

Définition 2.2.6 *Un homéomorphisme* préserve la structure verticale *d'un $\mathbb{R}^d$-solénoïde Ω si, pour tout point $x \in \Omega$ dans une verticale Ξ d'une boîte B, si $f(x)$ appartient à une verticale Ξ_2 d'une boite B_2, alors pour tout $y \in \Xi$ suffisamment proche de x, on a $f(x) \in \Xi_2$.*

Par exemple, du fait de la propriété de temps de retour localement constant, chaque homéomorphisme associé à la $\mathbb{R}^d$-action $x \mapsto x + \vec{v}$, $\vec{v} \in \mathbb{R}^d$, préserve la verticale. Nous noterons par $\mathrm{Homeo}_{pv}(\Omega)$ l'ensemble des homéomorphismes de $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ préservant la verticale. De même $D_{pv}(\Omega)$ désignera la composante connexe par arc de l'identité dans le groupe $\mathrm{Homeo}_{pv}(\Omega)$.

Nous obtenons, dans le cas général le théorème suivant.

Théorème 2.2.7 ([2]) *Soit Ω un $\mathbb{R}^d$-solénoïde. Notons par G soit le groupe* $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ *soit le groupe* $\mathrm{Homeo}_{pv}(\Omega)$. *Alors*

1. $\mathrm{Homeo}_{\mathcal{L}}(\Omega)^0 = D(\Omega)$ *et* $\mathrm{Homeo}_{pv}(\Omega)^0 = D_{pv}(\Omega)$ *;*
2. *G^0 est ouvert dans G ;*
3. *G^0 est simple.*

Ce théorème concerne aussi bien les enveloppes d'ensembles de Delone apériodiques que les suspensions de $\mathbb{Z}^d$-actions équicontinues, distales, ...

La preuve du Théorème 2.2.7 suit la même stratégie que dans [Fis60] pour des variétés triangulées. En utilisant une généralisation du théorème de Schoenflies dûe à Edwards et Kirby [EK71], nous montrons que les groupes considérés sont *factorisables* (ou *fragmentables*), *i.e* tout homéomorphisme se décompose en un produit de déformations à supports arbitrairement petits. Ceci permet de démontrer la Proposition 2.2.5 et les items (1) et (2) du Théorème 2.2.7. Comme le critère d'Epstein ne s'applique pas ici, nous donnons dans [2] une condition suffisante pour qu'un sous-groupe de commutateurs de $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ soit simple. Nous montrons ensuite que les groupes $\mathrm{Homeo}_{\mathcal{L}}(\Omega)^0$ et $\mathrm{Homeo}_{pv}(\Omega)^0$ sont parfaits et vérifient le critère précédent de simplicité.

Dans le cas particulier de la dimension 1, nous montrons en outre que ces groupes sont *uniformément parfaits* : tout élément s'écrit comme un produit de 2 commutateurs [2]. Ce dernier résultat empêche notamment l'existence de quasi-morphisme sur ces groupes.

Du point de vue de la flot équivalence, grâce à un théorème difficile de Ben Ami et Rubin [BAR10], nous obtenons également en corollaire de la preuve du Théorème 2.2.7 que les groupes $D(\Omega)$ et $D_{pv}(\Omega)$ sont des invariants complets de la classe d'équivalence orbitale du système $(\Omega, \mathbb{R}^d)$.

Proposition 2.2.8 ([2]) *Soient Ω_1, Ω_2 deux $\mathbb{R}^d$-solénoïdes. Pour $i = 1, 2$, désignons par G_i soit le groupe $\mathrm{Homeo}_{\mathcal{L}}(\Omega_i)^0$ soit le groupe $\mathrm{Homeo}_{pv}(\Omega_i)^0$.*

i) Si Ω_1 et Ω_2 sont homéomorphes, alors les groupes $\mathrm{Homeo}_{\mathcal{L}}(\Omega_1)^0$ et $\mathrm{Homeo}_{\mathcal{L}}(\Omega_2)^0$ sont isomorphes.

ii) Réciproquement pour tout isomorphisme $\varphi\colon G_1 \to G_2$, il existe un homéomorphisme $h\colon \Omega_1 \to \Omega_2$ tel que $\varphi(g) = h \circ g \circ h^{-1}$ pour tout $g \in G_1$.

Cette propriété montre ainsi que l'on peut interpréter chaque invariant de flot-équivalence en des termes algébriques. Par exemple, il est montré dans [Jul12] que les croissances asymptotiques des fonctions complexité et répétitivité sont des invariants de flot-équivalence. Il reste à interpréter ces croissances de façon algébrique sur le groupe des homéomorphismes de l'enveloppe.

2.3 Ensembles de Delone hyperboliques

Dans cette section, nous nous intéresserons aux ensembles de Delone X de l'espace hyperbolique $\mathbb{H}^2$ qui sont de type fini pour le groupe affine $GA = \{z \mapsto az + b; a > 0, b \in \mathbb{R}\}$

2.3.1 Exemples

Dans l'article [Pen80], Penrose rappelle les principales propriétés de son célèbre pavage éponyme, et donne en plus un exemple de pavage de $\mathbb{H}^2$ de GA-type fini (voir figure 2.1).

Il considère le polygone convexe P_a de sommets, les points A_n d'affixe $a(p-1)/2 + i$ pour $1 \leq n \leq 3$, A_4 d'affixe dans le plan complexe $\mathbb{C}$, $2i + a$ et A_5 : $2i$, pour un paramètre $a > 0$ fixé. Le polygone P_a possède ainsi 5 arêtes géodésiques. Considérons les isométries

$$R\colon z \mapsto 2z \text{ et } S\colon z \mapsto z + a.$$

Le pavage hyperbolique de Penrose est alors défini par $\mathcal{P}_a = \{R^k \circ S^n P_a; n, k \in \mathbb{Z}\}$ (voir figure 2.1). Penrose montre dans [Pen80] qu'il est *faiblement apériodique* dans le sens où, aucun groupe cocompact Γ d'isométries de $\mathbb{H}^2$ ne préserve ce pavage. L'argument homologique de Penrose est le suivant : si cela était le cas, le pavé P_a pourrait paver la surface compacte $\mathbb{H}^2/\Gamma$. Mettons une charge positive sur les arêtes A_4A_5 et une charge négative sur chaque arête A_1A_2 et A_2A_3. Comme une arête de type A_4A_5 ne peut rencontrer qu'une arête de type A_1A_2 ou A_2A_3, la surface compacte $\mathbb{H}^2/\Gamma$ possède une charge globale nulle. Pourtant chaque pavé P_a a une charge négative et la surface doit avoir une charge globale strictement négative.

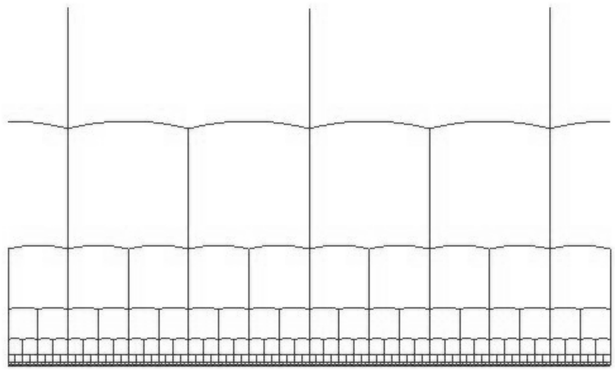

FIGURE 2.1 – Le pavage hyperbolique de Penrose

L'argument homologique de Penrose a été généralisé de différentes manières, notamment dans [BW92] pour d'autres espaces non moyennables. De façon plus géométrique, G. Margulis et S. Mozes font remarquer dans [MM98] que la formule de Gauss-Bonnet implique que l'aire d'une surface hyperbolique de courbure constante égale à -1, est un multiple entier de π. Comme l'aire de P_a, qui ne dépend que de ses angles, varie continûment avec le paramètre a, le pavé P_a a une aire non rationnellement liée avec π pour un nombre indénombrable de paramètres a. Il ne peut donc pas paver une surface compacte.

Le pavage de la figure 2.1 n'est donc pas stable pour un réseau cocompact, cependant on peut montrer que son stabilisateur est le groupe $\langle R \rangle$ engendré par la transformation R. Cette isométrie préserve une géodésique verticale (dans le modèle du demi-plan supérieur). En décorant les pavés le long de cette géodésique de façon à casser cette symétrie, nous construisons un pavage non périodique. Dans [GS05], Goodmann-Strauss utilise cette idée. Il code la dynamique d'une application de l'intervalle sans point périodique, puis il décore les pavés P_a selon ce codage de façon à ce que la décoration des pavés le long d'une géodésique verticale code la dynamique de l'application. Les pavages obtenus avec ces pavés ne sont invariants par aucune isométrie [GS05]. Ils sont donc totalement apériodiques (voir section 2.1.1). Pour des groupes de Lie semi-simples de rang supérieur à 3, Mozes construit dans [Moz97] une famille finie de pavés ne pouvant engendrer que des pavages totalement apériodiques. Pour être plus complet sur ce thème, ajoutons qu'un résultat récent de Aubrun, Barbieri et Thomassé [ABT15] démontrent que n'importe quel groupe dénombrable admet un pavage totalement apériodique.

Mentionnons également qu'il est possible de construire des ensembles de Delone hyperboliques de type fini par la la méthode de coupé-projection. En effet, il suffit pour cela de considérer l'espace produit $\mathbb{H}^2 \times \mathbb{H}^2$ muni de la distance produit. Le groupe $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ agit isométriquement sur l'espace produit. Considérons alors un sous-groupe discret Γ, cocompact et *irréductible* : *i.e.* tel que les groupes $\Gamma \cap (\mathrm{PSL}_2(\mathbb{R}) \times \{\mathrm{Id}\})$ et $\Gamma \cap (\{\mathrm{Id}\} \times \mathrm{PSL}_2(\mathbb{R}))$ n'engendrent pas un sous-groupe d'indice fini dans Γ. Notons par π_1, π_2 les projections $\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2$ sur, respectivement, les premières et secondes coordonnées. Pour une origine

$x \in \mathbb{H}^2 \times \mathbb{H}^2$ fixée, soit $B \subset \mathbb{H}^2$ la boule centrée en $\pi_2(x)$ de rayon supérieur au diamètre d'un domaine fondamental de Γ, nous considérons alors l'ensemble

$$X = \pi_1\{\gamma(x); \gamma \in \Gamma, \pi_2(\gamma(x)) \in B\} \subset \mathbb{H}^2.$$

Comme le groupe Γ est cocompact, il est finiment engendré et l'ensemble X est uniformément discret. De plus, il n'existe qu'un nombre fini de R-patchs à isométrie prés. Le choix de la fenêtre B assure que X est relativement dense. Ainsi X forme un ensemble de Delone.

Malheureusement, même si les résultats de théorie des groupes nous assure l'existence de réseaux Γ cocompacts et irréductibles, il est difficile d'en construire un de façon effective pour en faire un dessin explicite. Si l'on relaxe l'hypothèse de cocompacité par celle de covolume fini ($\mathbb{H}^2 \times \mathbb{H}^2/\Gamma$ a un volume fini pour la mesure de Haar invariante à droite), il est plus simple d'en décrire un. Par exemple, considérons l'anneau $\mathbb{Z}[\sqrt{2}]$, ou $\mathbb{Z}[\sqrt{d}]$ pour un entier d différent d'un carré. Il admet deux plongements dans $\mathbb{R}$ définis par $a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}] \mapsto a \pm b\sqrt{2} \in \mathbb{R}$. CEux-ci s'étendent en deux plongements denses du groupe $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}])$ dans $\mathrm{PSL}_2(\mathbb{R})$. Le produit de ces deux plongements donne alors un plongement de $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}])$ dans $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ dont l'image est un sous-groupe discret de covolume fini et irréductible [Mar91, Zim84]. Il est possible de réaliser des plongements similaires à partir de corps de nombres quaternioniques pour obtenir des groupes cocompacts irréductibles [MR03].

L'ensemble de Delone obtenu par cette méthode de coupé-projection n'est pas nécessairement totalement apériodique. Il faut alors modifier les patchs en les décorant. Une façon de l'obtenir est de considérer un sous-shift $X \subset \Sigma^\Gamma$ sur un alphabet fini Σ, dont l'action du shift est libre (voir section 1.2.2 pour des exemples) et de considérer l'action diagonale de Γ sur $\mathbb{H}^2 \times \mathbb{H}^2 \times X$.

Remarquons qu'il est également possible de faire de même sur $\mathbb{H}^2 \times \mathbb{Q}_p$ où $\mathbb{Q}_p$ désigne l'ensemble des entiers p-adique pour un nombre premier p avec le groupe d'isométries $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{Q}_p)$. Nous utilisons une technique similaire dans la section 2.3.3.

2.3.2 Mesures invariantes et mesures harmoniques

À la différence du contexte euclidien, nous verrons qu'il n'est pas possible de définir de fréquence d'apparition d'un motif pour un ensemble de Delone X de GA type fini.

Ainsi, dans le contexte euclidien, pour un système de Delone $(\Omega(Y), \mathbb{R}^d)$ uniquement ergodique, *i.e.* admettant une unique mesure de probabilité μ invariante par la $\mathbb{R}^d$ action (c'est le cas pour un ensemble de Delone substitutif), le théorème ergodique de Birkhoff assure que la limite suivante existe

$$\lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R(0))} \int_{B_R(0)} f(Y + \vec{v}) d\mathrm{Leb}(\vec{v}) = \int_{\Omega(Y)} f(x) d\mu(x),$$

pour toute fonction intégrable $f \colon \Omega(Y) \to \mathbb{R}$, où Leb désigne la mesure de Lebesgue de $\mathbb{R}^d$. En prenant comme fonction f, la fonction indicatrice d'une boîte associée à un $R_{\mathtt{P}}$-patch $\mathtt{P}$ centré en l'origine (cf section 2.1.2), nous obtenons en divisant la limite précédente par le volume de conv $\mathtt{P}$, que la limite suivante existe

$$\lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R(0))} \mathrm{card}\ \{\vec{v} \in B_R(0); (Y + \vec{v}) \cap B_{R_{\mathtt{P}}}(0) = \mathtt{P}\};$$

ou, autrement dit, que le patch $\mathtt{P}$ apparait dans Y avec une certaine fréquence. Cette limite correspond en fait, avec les notations de la section 2.1.2, à la valeur $\nu(\Xi_{\mathtt{P}})$ pour une certaine mesure ν sur la transversale Ξ obtenue comme limite $\mu(i_{\mathtt{P}}(B_\epsilon(0) \times \Xi_{\mathtt{P}}))/\mathrm{vol}(B_\epsilon(0))$ lorsque ϵ tend vers 0 [BBG06]. Géométriquement cette mesure correspond à une *mesure transverse invariante*.

Définition 2.3.1 *Une* mesure transverse invariante ν *pour une* $\mathbb{G}$*-lamination est la donnée d'une mesure positive* ν_i *sur chaque espace transverse* Ξ_i *de la lamination, de sorte que si* $B \subset \Xi_i$ *est un borélien contenu dans le domaine de définition de* $f_{i,j}$*, alors* $\mu_i(B) = \mu_j(f_{i,j}(B))$.

L'hypothèse de l'existence d'une telle mesure est une hypothèse forte.

Proposition 2.3.2 ([13]) *Un GA-solénoïde n'admet pas de mesure transverse invariante.*

Ceci provient essentiellement du fait que le groupe GA n'est pas *unimodulaire*, *i.e.* il admet une mesure de Haar invariante par multiplication à gauche λ_L qui est différente de la mesure de Haar invariante par multiplication à droite λ_R, ou plus précisément, λ_L n'est pas invariante par la multiplication à droite. Ainsi, étant donnée une mesure transverse invariante $\{\nu_i\}$, il est possible de construire une mesure finie sur le solénoïde qui localement est le produit de ν_i par λ_L. Nous obtenons alors une contradiction en étudiant l'image de cette mesure par la GA-action par multiplication à droite. La propriété 2.3.2 peut s'étendre aisément à n'importe quelle lamination dont les feuilles sont isométriques à un groupe de Lie non unimodulaire.

Pour passer outre ce problème et étudier les fréquences d'apparition des motifs, nous devons passer par une autre notion géométrique de mesure : les *mesures harmoniques*. Pour cela, nous sommes amenés à introduire les formes différentielles sur un $\mathbb{G}$-solénoide Ω, ou plus généralement sur une lamination [CGSY99, chapitre 2].

Dans une boîte homéomorphe à $V \times \Xi$, nous appellerons *k-forme différentielle*, $k = 0, 1, 2$, une famille de k-formes différentielles réelles (de classe C^∞) dans les plaques $V \times \{\xi\}$ qui dépend continûment du paramètre ξ pour la topologie C^∞. Une k-forme sur Ω est alors donnée par des k-formes différentielles dans chaque boîte compatible sur les intersections, dans un sens évident. L'espace $A^k(\Omega)$ désignera l'espace vectoriel topologique des k-formes sur Ω. L'opérateur de différentiation le long des feuilles définit un opérateur $d\colon A^k(\Omega) \to A^{k+1}(\Omega)$.

Les *cycles feuilletés*, introduits par Sullivan [Sul76], sont des formes linéaires continues $A^2(\Omega) \to \mathbb{R}$ strictement positives sur les formes strictement positives et qui s'annulent sur les formes exactes. Un résultat important de [Sul76] est que les cycles feuilletés correspondent aux mesures transverses invariantes. Ainsi, une autre formulation de la proposition 2.3.2 est : un GA-solénoïde n'admet pas de cycle feuilleté.

Nous sommes alors amenés à étudier la notion plus générale de *courant harmonique* introduit par Garnett dans [Gar83]. L'opérateur laplacien le long des feuilles Δ donne un opérateur réel $A^0(\Omega) \to A^2(\Omega)$ dont l'image $\mathrm{Im}\Delta$ est inclu dans l'ensemble des formes exactes. Un *courant harmonique* est une forme linéaire continue $A^2(\Omega) \to \mathbb{R}$ strictement positive sur les formes strictement positives et nulles sur $\mathrm{Im}\Delta$. Les cycles feuilletés sont donc des exemples de courants harmoniques mais n'importe quelle lamination, en particulier n'importe quel $\mathbb{G}$-solénoïde, admet un courant harmonique [Gar83, Can03].

Comme pour les cycles feuilletés, en identifiant l'espace des fonctions $C^\infty(\Omega)$ et $A^2(\Omega)$ par le choix d'une métrique le long des feuilles, il est possible d'associer à un courant harmonique une mesure positive et finie sur Ω. Ces mesures, appelées *mesures harmoniques*, sont alors caractérisées de la façon suivante : pour n'importe quelle fonction $f\colon \Omega \to \mathbb{R}$ mesurable bornée et lisse le long des feuilles, l'intégrale $\int \Delta f d\mu = 0$ [‡] si et seulement si la mesure μ est

‡. où Δ désigne le laplacien le long des feuilles pour la métrique choisie

harmonique [Gar83, Can03].

Une telle mesure se désintègre localement, au travers des cartes, dans $V \times \Xi$ en le produit d'une mesure ν sur l'espace transverse Ξ par une mesure μ_ξ harmonique définie sur ν- presque toutes les plaques $V \times \{\xi\}$. Il se trouve que chaque mesure μ_ξ est absolument continue par rapport à la mesure de Riemann dz le long des feuilles et dont la densité $z \mapsto f(z, \xi)$ est, pour ν-presque tout ξ, une fonction harmonique positive. Ainsi pour tout borélien $B \subset V \times \Xi$, la mesure de B s'écrit

$$\iint_B f(z, \xi) dz d\nu(\xi).$$

Bien que cette décomposition ne soit pas unique, la fonction $\phi_\mu \colon z \mapsto \int f(z, \xi) d\nu(\xi)$ est intrinsèque à la mesure harmonique.

Les feuilles d'un $\mathbb{G}$-solénoïde étant simplement connexes, il est possible, en recollant les fonctions harmoniques $f(z, \xi)$ définies sur chaque plaque, de définir une fonction globale $f \colon \mathbb{G} \to \mathbb{R}_+$ harmonique. Dans le cas d'un $\mathbb{R}^2$-solénoïde, comme il n'existe pas de fonction harmonique positive non constante sur le plan, chaque densité locale f est constante. Ainsi chaque mesure harmonique se désintégre en un produit d'une mesure ν sur la transversale par la mesure de Riemann et ν définit une mesure transverse invariante.

Garnett donne également dans [Gar83] un théorème ergodique remarquable associé aux mesures harmoniques. Pour cela, il est nécessaire de considérer pour x un point base d'une lamination Ω, la *mesure de Wiener* $\mathbb{P}_x$ sur l'ensemble des chemins continus $\gamma \colon \mathbb{R}_+ \to \Omega$ tels que $\gamma(0) = x$. Cette mesure est obtenue par la théorie du mouvement brownien en utilisant le noyau de la chaleur.

Théorème 2.3.3 ([Gar83]) *Soit μ une mesure harmonique d'une lamination Ω. Pour n'importe quelle fonction bornée $f \colon \Omega \to \mathbb{R}$, la limite $\ell(x, \gamma) = \lim_{n\to\infty} 1/n \sum_{k=0}^{n-1} f(\gamma(i))$ existe pour μ-presque tout point $x \in \Omega$ et $\mathbb{P}_x$-presque tout chemin γ.*
Cette limite est constante le long des feuilles de Ω et $\ell(x, \gamma)$ est constante pour $\mathbb{P}_x$-presque tout chemin γ. De plus $\int \ell(x) d\mu(x) = \int f(x) d\mu(x)$.

De la même façon que pour les mesures invariantes, on pourra parler de mesure harmonique *ergodique* si la mesure de tout borélien saturé de feuilles est de mesure nulle ou totale. Ainsi dans le cas d'enveloppe d'un ensemble de Delone de GA-type fini totalement apériodique, nous pourrons parler de fréquence de passage dans un motif le long d'un chemin brownien pour une mesure harmonique ergodique.

Théorème 2.3.4 ([13]) *Une mesure finie sur un GA-solénoïde est harmonique si et seulement si elle est invariante par l'action du groupe affine.*

La preuve vient essentiellement du fait bien connu suivant : en notant λ_L la mesure de Haar sur GA invariante par multiplication à gauche et $R_g * \lambda_L$ la mesure image de λ par la multiplication à droite par g, la fonction $\varphi \colon g \mapsto d\lambda / dR_g * \lambda$ est une fonction harmonique *minimale*, *i.e.* toute fonction harmonique positive dont le rapport avec φ est bornée, est proportionnelle à φ. Ainsi, pour $g_*\mu$ la mesure image d'une mesure harmonique μ par l'action de $g \in GA$, un calcul direct nous permet de conclure que l'intégrale $\int f dg_*\mu$ est indépendante de g pour toute fonction test f.

Ce résultat peut être étendu au cas des laminations définies par des actions localement libres du groupe affine. Par exemple, pour une lamination $\mathcal{L}$ par surfaces hyperboliques, son fibré unitaire tangent $T^1\mathcal{L}$ hérite également d'une structure de lamination. Cette lamination $T^1\mathcal{L}$ est, de plus, munie de l'action des flots géodésique et horocyclique stable (le long des feuilles). Ces deux flots conjoints engendrent une action continue du groupe affine GA. La même preuve du Théorème 2.3.4 donne que les mesures harmoniques de cette lamination sont invariantes par l'action conjointe des flots géodésique et horocyclique. Il est possible alors de montrer que ces mesures se projettent de façon surjective en des mesures hamoniques sur $\mathcal{L}$. Ceci est fait dans [Mar06, BM08].

En utilisant de plus la strucure de limite projective $\varprojlim_n (B_n, f_n)$ de variétés branchées d'un GA-solénoïde, nous obtenons la caractérisation suivante :

Théorème 2.3.5 ([13]) *Il existe une suite de morphismes linéaires $(A_n)_{n\geq 0}$ tel que l'ensemble des mesures harmoniques est isomorphe à la limite projective de cônes dans l'espace des 2-chaînes des variétés branchées $\varprojlim_n (\mathcal{C}_2(B_n, \mathbb{R})^+, A_n)$.*

Une caractérisation similaire est donnée pour les mesures transverses invariantes de $\mathbb{G}$-solénoïde pour $\mathbb{G}$ unimodulaire dans [BG03]. La différence est qu'il n'y a pas ici de mesure transverse invariante. Le Théorème 2.3.4 nous permet de décrire la forme des fonctions locales ϕ_μ, pour une mesure harmonique μ. Ce sont des fonctions harmoniques minimales. La caractérisation s'obtient alors en considérant la suite des intégrales $\int_V \phi_\mu(z)dz$ pour des boîtes de la forme $V \times \Xi$.

Nous obtenons en corollaire que le nombre de mesures de probabilité harmoniques et ergodiques est borné par le nombre maximal de faces des variétés branchées. Grâce à cela nous donnons des exemples explicites avec un nombre arbitraire de mesures harmoniques.

Théorème 2.3.6 ([13]) *Pour tout entier $r \geq 1$, il existe un ensemble de Delone $X \subset \mathbb{H}^2$ de GA-type fini, totalement apériodique et répétitif, dont l'enveloppe $\Omega(X)$ a exactement r mesures de probabilité hamoniques ergodiques.*

La réalisation de ces exemples est, par essence, similaire à celle du Théorème 1.2.4. Ce résultat illustre la grande variété de dynamique que l'on peut obtenir à l'aide des ensembles de Delone. À l'opposé, si l'on considère des laminations hyperboliques dont la dynamique transverse est conforme, il y a, dans le cas minimal, une unique mesure harmonique [DK07]. Par la suite, ces exemples de laminations minimales non uniquement ergodiques ont été étendus au cas de feuilletages lisses d'une variété de dimension 5 dans [Der09] et dans le cas de la codimension 1 dans [DV11].

2.3.3 C^*-algèbres

La géométrie non-commutative des systèmes de Delone étudie les C^*-algèbres de leur enveloppe. Le lecteur pourra consulter [Ren80, Ren09] pour des introductions lumineuses sur les C^*-algèbres associées aux systèmes dynamiques. Une telle C^*-algèbre permet alors de donner des invariants topologiques et géométriques de la lamination. De plus, certains invariants K-théoriques de systèmes de Delone euclidiens ont une interprétation physique. En particulier lorsque l'on modélise un quasi-cristal par un ensemble de Delone, l'image de la K-théorie de l'enveloppe par la trace canonique indice les trous dans le spectre de l'opérateur de Schrödinger associé au quasi-cristal [Bel92].

Pour un ensemble de Delone euclidien, les traces de la C^*-algèbre sont en bijection avec les mesures transverses invariantes de l'enveloppe [BBG06]. Ces algèbres ont été depuis bien étudiées et ont conduit à diverses preuves de la conjecture du *gap-labelling* [BBG06, BOO03, KP03] : pour une action de $\mathbb{R}^d$ minimale, l'image de la K-théorie par une trace est le sous-groupe dénombrable de $\mathbb{R}$ engendré par les mesures de clopens (ensembles ouverts et fermés) de la transversale canonique pour une mesure transversalement invariante.

La situation est différente lorsque que l'ensemble de Delone est hyperbolique de GA-type fini, puisqu'il n'existe pas de mesure transversalement invariante (Proposition 2.3.2). Ainsi, la C^*-algèbre ne possède pas de trace. Cependant chaque mesure harmonique (ou GA-invariante par le Théorème 2.3.4) donne un cocyle 3-cyclique ([ENN88]) sur la C^*-algèbre du système. Avec Oyono-Oyono, nous avons étudié dans [12] ces C^*-algèbres pour une famille spécifique d'exemples. Nous en donnons une description explicite et nous calculons leur K-théorie ainsi que leur cohomologie de Čech.

La famille d'exemples est obtenue en décorant les pavés du pavage hyperbolique donné par Penrose dans [Pen80] (voir section 2.3.1). Ainsi pour le pavé P_1 défini dans la section 2.3.1 et un entier $i \in \{1, \dots, r\}$, $r \geq 1$, nous noterons par $P^{(i)}$ un ensemble de $i + 5$ points, dont 5 points sont aux sommets du pavé P_1 et i points (la *décoration*) sont à l'intérieur d'une boule de rayon 1/6 centrée au barycentre du pavé P_1. Cette décoration est faite de sorte que, pour n'importe quel pavage fait de transaltés (pour le groupe affine) de P_1 et pour tout choix de décoration i pour chaque pavé, correspond un unique ensemble de Delone union des translatés des $P^{(i)}$. §

Pour une suite $w = (w_k)_{k\in\mathbb{Z}} \in \{1, \dots, r\}^{\mathbb{Z}}$, l'ensemble de Delone de GA-type fini $\mathcal{P}(w)$ est défini par $\mathcal{P}(w) = \{R^p \circ S^n(P^{(w_{-q})}), n, q \in \mathbb{Z}\}$, avec les notations de la section 2.3.1. Son stabilisateur est un sous-groupe de $\langle R\rangle$. Désignons par $Z_w = \overline{\{\sigma^n(w), n \in \mathbb{Z}\}}$ le sous-shift engendré par w. Il est assez simple de vérifier que l'ensemble de Delone $\mathcal{P}(w)$ est totalement apériodique si la suite w est non périodique pour le shift et qu'il est répétititf si et seulement si le système (Z_w, σ) est minimal.

L'enveloppe $\Omega(\mathcal{P}(w))$ de l'ensemble de Delone $\mathcal{P}(w)$ a une structure de suspension. Pour voir cela, considérons $GA(\mathbb{Z}[1/2])$ le groupe des transformations affines $a_{n,b}\colon z \mapsto 2^n z + b$ où $n \in \mathbb{Z}$ et b est un rationnel de la forme $p/2^q$ $p, q \in \mathbb{Z}$. Ce groupe agit naturellement à gauche sur le produit $\mathbb{H}^2 \times \mathbb{Q}_2$ où $\mathbb{Q}_2$ désigne l'ensemble des rationnels 2-adiques. Il agit également sur Z_w par $a_{n,b}.w' = \sigma^{-n}(w')$. Le produit de ces deux actions donne une action à gauche de $GA(\mathbb{Z}[1/2])$ sur $\mathbb{H}^2 \times \mathbb{Q}_2 \times Z_w$ qui est continue propre et sans point fixe. L'enveloppe $\Omega(\mathcal{P}(w))$ est alors conjuguée au quotient de $\mathbb{H}^2 \times \mathbb{Q}_2 \times Z_w$ par cette action. En décomposant encore plus cette suspension, il est possible de la décrire en termes de double suspension (suspension de suspension). Ceci nous permet dans [12] d'expliciter sa C^*-algèbre à partir de celle donnée par Z_w.

Ainsi si $C(Z_w, A)$ désigne l'ensemble des fonctions continues sur Z_w à valeurs dans l'anneau A, nous noterons les groupes quotients par

$$\begin{aligned}
\text{inv } C(Z_w, \mathbb{Z}) &= C(Z_w, \mathbb{Z})/\langle f = f \circ \sigma; f \in C(Z_w, \mathbb{Z})\rangle, \\
\text{coinv } C(Z_w, \mathbb{Z}) &= C(Z_w, \mathbb{Z})/\langle f - f \circ \sigma^{-1}; f \in C(Z_w, \mathbb{Z})\rangle, \\
\text{coinv } C(Z_w, \mathbb{Z}[1/2]) &= C(Z_w, \mathbb{Z}[1/2])/\langle f - 2f \circ \sigma^{-1}; f \in C(Z_w, \mathbb{Z}[1/2])\rangle.
\end{aligned}$$

Théorème 2.3.7 ([12]) *Nous avons les isomorphismes suivants :*

$$\begin{aligned}
K^0(\Omega(\mathcal{P}(w)) &\simeq \text{inv } C(Z_w, \mathbb{Z}) \oplus \text{coinv } C(Z_w, \mathbb{Z}[1/2]), \textit{et} \\
K^1(\Omega(\mathcal{P}(w)) &\simeq \text{coinv } C(Z_w, \mathbb{Z}).
\end{aligned}$$

§. Dans [12] nous considérons la décoration d'un pavage. Le choix de la décoration est fait ici pour se placer dans le contexte équivalent d'ensemble de Delone.

Concernant la cohomologie de Čech à coefficients entiers, nous avons

$$\begin{array}{rcl} \check{H}^0(\Omega(\mathcal{P}(w)),\mathbb{Z}) & \simeq & \mathrm{inv}\ C(Z_w,\mathbb{Z}), \\ \check{H}^1(\Omega(\mathcal{P}(w)),\mathbb{Z}) & \simeq & \mathrm{coinv}\ C(Z_w,\mathbb{Z}), \\ \check{H}^2(\Omega(\mathcal{P}(w)),\mathbb{Z}) & \simeq & \mathrm{coinv}\ C(Z_w,\mathbb{Z}[1/2]). \end{array}$$

Le lecteur remarquera que ces invariants topologiques sont similaires aux groupes de dimension de la dynamique sur la transversale (cf section 1.1.4). Ajoutons que les isomorphismes sont décrits de façon explicite dans [12].

Chapitre 3

Modèle de Frenkel-Kontorova associé à un quasi-cristal

Introduction

Le modèle de Frenkel-Kontorova (FK) [BK04, FBGG05] décrit comment une chaîne infinie d'atomes "minimise l'énergie totale d'un système" lorsque cette énergie prend en compte les interactions entre les proches voisins et un environnement extérieur. Les configurations sont modélisées par une suite $(x_n)_{n\in\mathbb{Z}} \subset \mathbb{R}^d$ où x_n représente la position de l'atome indicé par n. Une configuration $(x_n)_{n\in\mathbb{Z}}$ est dite *minimisante* pour une *énergie d'interaction* $E\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ si l'énergie de chaque segment $(x_m, x_{m+1}, \ldots, x_n)$ ne peut être abaissée en changeant la configuration à l'intérieur du segment sans changer les deux points du bord. Plus formellement, en notant

$$E(x_m, x_{m+1}, \ldots, x_n) := \sum_{k=m}^{n-1} E(x_k, x_{k+1}),$$

une configuration $(x_n)_{n\in\mathbb{Z}}$ est dite minimisante si pour tous les entiers $m < n$ et pour tous les points $y_m, y_{m+1}, \ldots, y_n \in \mathbb{R}^d$ vérifiant $y_m = x_m$ et $y_n = x_n$, nous avons

$$E(x_m, x_{m+1}, \ldots, x_n) \leq E(y_m, y_{m+1}, \ldots, y_n).$$

En dimension $d = 1$, un exemple classique d'énergie d'interaction E est de la forme $E(x, y) = V(x) + U(y - x)$ où $V\colon \mathbb{R} \to \mathbb{R}$ est potentiel 1-*périodique* ($V(x+1) = V(x)$ pour tout réel x) représentant l'interaction avec un environnement à structure périodique, et $U\colon \mathbb{R} \to \mathbb{R}$ représentant l'interaction entre les atomes. En particulier, le modèle original de Frenkel-Kontorova [FK38] considère l'énergie d'interaction $E(x, y) = 1/2(1 - \cos(\pi x)) + 1/2C(y - x - \mu)^2$, pour des paramètres C et μ, Les hypothèses suivantes sur les potentiels sont dites *classiques* et étendent le modèle original :

— *régularité* : U et $V\colon \mathbb{R} \to \mathbb{R}$ sont C^2 ;
— *twist*, ou *stricte convexité* : $U^{''}(x) > 0$, $\forall x \in \mathbb{R}$;
— *super-linéarité* : $\lim_{x\to\pm\infty} \frac{U(x)}{|x|} = +\infty$.

Il est alors simple de montrer que des configurations minimisantes existent dans ce contexte et que l'ensemble de ces configurations est fermé pour la topologie produit sur

$\mathbb{R}^{\mathbb{Z}}$. Ces configurations ont été initialement décrites par S. Aubry -P.Y Le Daeron [ALD83]. Leurs travaux précurseurs ainsi que ceux, indépendants, de J. Mather [Mat82, Mat91] et de nombreux autres (voir les survols [MF94, CI99, Fatar]), ont débouchés à la *théorie d'Aubry-Mather* qui donne une assez bonne compréhension de ces configurations minimisantes. Ajoutons que les propriétés du modèle FK ont été étendues aux dimensions supérieures par E. Garibaldi et P. Thieullen dans [GT11].

À la vue de ces résultats et de ceux sur les ensembles de Delone, il est naturel d'espérer pouvoir comprendre également les propriétés des configurations minimisantes dans un contexte quasi-périodique. À notre connaissance, ce problème fut initalement étudié dans [vE99]. La première section de ce chapitre concerne quelques propriétés de ces configurations dans le cas de la dimension 1, sous des hypothèses classiques, et avec un potentiel V équivariant relativement à un quasi-cristal. La seconde section traite d'une énergie d'interaction presque périodique en dimension quelconque. Cela revient à considérer une famille d'énergies stationnaire par rapport à un système dynamique minimal. Ce contexte englobe celui de la première section.

3.1 FK associé à un quasi-cristal de dimension un

Pour une énergie de la forme $E(x, y) = V(x) + U(y - x)$ où $U, V : \mathbb{R} \to \mathbb{R}$ satisfont les hypothèses classiques et V est périodique, Aubry et Le Daeron, montrent dans [ALD83], que chaque configuration minimisante $(x_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ admet un *nombre de rotation* ρ, *i.e.* la limite suivante existe :

$$\lim_{n\to\pm\infty} \frac{x_n}{n} = \rho \geq 0.$$

Remarquons que l'inverse de cette limite peut s'interpréter en termes de densité de particules de la configuration minimisante. Aubry et Le Daeron montrent également que le nombre de rotation ρ dépend continûment de la configuration minimisante $(x_n)_{n\in\mathbb{Z}}$ (pour la topologie produit). De plus, ces nombres de rotations ne possèdent aucune propriété arithmétique particulière puisque tout réel positif est le nombre de rotation d'une configuration minimisante. Ce dernier point implique aussi qu'il existe un nombre indénombrable de configurations minimisantes. Précisons qu'Aubry et Le Daeron démontrent, en fait, des résultats beaucoup plus précis concernant la combinatoire de ces configurations. Nous nous restreindrons cependant à leurs derniers résultats mentionnés.

Avec J.M. Gambaudo et P. Guiraud [10], nous étendons ces résultats au contexte d'un potentiel V quasi-périodique associé à un *quasi-cristal* $X \subset \mathbb{R}$. Pour la suite, nous dirons que $X \subset \mathbb{R}$ est un quasi-cristal si

1. X est un ensemble de Delone de $\mathbb{R}$-type fini (voir section 2.1.1) ;
2. X est répétitif (voir proposition 2.1.2) ;
3. Pour tout R-patch $\mathtt{P}$ de X et tout réel $x \in \mathbb{R}$, la limite suivante existe

$$\lim_{M\to+\infty} \frac{\operatorname{card}\{v \in [x - M, x + M]; v \text{ est une occurrence de } \mathtt{P}\}}{2M} = \nu(\mathtt{P}) > 0,$$

uniformément et indépendamment de x.

Rappelons que la condition (2), par la proposition 2.1.2, est équivalente à la minimalité de la $\mathbb{R}$-action sur l'enveloppe de X. La condition (3) est équivalente à l'unique ergodicité de cette dernière action (voir la section 2.3.2). Par exemple, les ensembles de Delone linéairement répétitifs forment des quasi-cristaux (voir la section 2.2.1).

Pour un quasi-cristal fixé X, nous considérerons des fonctions *fortement X-équivariantes* (ou *strongly equivariant functions*) dans le sens de [Kel03]. Un potentiel $V\colon \mathbb{R} \to \mathbb{R}$ est dit fortement X-équivariant s'il existe une constante $R_V > 0$ telle que

$$V(x) = V(y), \qquad \forall x, y \in \mathbb{R} \text{ tels que } (B_{R_V}(x) \cap X) - x = (B_{R_V}(y) \cap X) - y.$$

Ainsi, si x et de y sont deux occurrences d'un même R_V-patch, la valeur du potentiel est la même en ces deux points.

Bien évidemment un potentiel périodique est fortement équivariant pour un quasi-cristal périodique. De façon général, si $\delta := \sum_{x\in X} \delta_x$ désigne la mesure de Radon associée à un quasi-cristal X, et si $g\colon \mathbb{R} \to \mathbb{R}$ est une fonction lisse (resp. C^p) à support compact. Il est alors simple de vérifier que la convolution $g{*}\delta$ donne une fonction fortement équivariante lisse (resp. C^p). Il est simple également de voir que toute fonction fortement équivariante s'étend de façon unique en une fonction continue sur l'enveloppe de X, $\hat{V}\colon \Omega(X) \to \mathbb{R}$ [10, Kel03]. Cette fonction vérifie $\hat{V}(X - t) = V(t)$ pour tout réel t et $\hat{V}$ est localement constante sur les verticales Ξ (cf section 2.1.2). L'enveloppe est alors un espace similaire au cercle dans le cas périodique.

Dans ce contexte, nous obtenons le résultat suivant.

Théorème 3.1.1 ([10]) *Pour un quasi-cristal X, un potentiel $V\colon \mathbb{R} \to \mathbb{R}$ C^2 fortement X-équivariant et une fonction $U\colon \mathbb{R} \to \mathbb{R}$, C^2, super-linéaire et strictement convexe, nous avons*

i) toute configuration minimisante $(x_n)_n$ admet un nombre de rotation

$$\rho((x_n)_n) = \lim_{n\to\pm\infty} \frac{x_n}{n}.$$

ii) La fonction nombre de rotation $(x_n)_n \mapsto \rho((x_n)_n)$ est continue pour la topologie produit.

iii) Tout nombre $\rho \geq 0$ est le nombre de rotation d'une configuration minimisante.

La stratégie de la preuve d'Aubry et Le Daeron dans le cas périodique [ALD83] consiste à montrer que les configurations minimisantes ont une combinatoire particulière : si on projette une telle configuration $(x_n)_n$ sur le cercle $\mathbb{R}/\mathbb{Z}$, il existe un homéomorphisme f du cercle, préservant l'orientation tel que $f(x_n \mod \mathbb{Z}) = x_{n+1} \mod \mathbb{Z}$, pour tout entier n. Ainsi, une telle configuration admet un nombre de rotation. Cette propriété combinatoire remarquable repose sur le fameux *lemme de croisement d'Aubry* qui utilise fortement la condition twist.

Nous suivons une stratégie similaire dans le cas quasi-périodique [10], où nous montrons qu'une configuration minimisante admet une combinatoire particulière. Plus précisément, si $x, y \in \mathbb{R}$ sont deux occurrences d'un même R'-patch de X ($(B_{R'}(x) \cap X) - x = (B_{R'}(y) \cap X) - y$, $R' \geq R_V$), alors

$$|\text{card } (\{x_n\}_n \cap B_{R'}(x)) - \text{card } (\{x_n\}_n \cap B_{R'}(y))| \leq 3.$$

Grâce à cette propriété et à la fréquence uniforme des occurrences des patchs de X (condition (3)), nous obtenons l'existence des nombres de rotation et leur continuité. Si l'hypothèse d'unique ergodicité était enlevée, nous obtiendrions que la suite $(x_n/n)_n$ admet plusieurs valeurs d'adhérences.

Remarquons que cette condition combinatoire est un peu plus faible que dans le cas périodique puisqu'elle n'implique pas que chaque configuration minimisante est l'orbite d'un homéomorphisme de l'enveloppe $\Omega(X)$.

Pour construire des configurations minimisantes avec un nombre de rotation prescrit, nous considérons des configurations finies qui minimisent l'énergie totale sur un motif avec un nombre bien choisi d'atomes. Une limite de ces configurations, en prenant des motifs de plus en plus grands, nous donne une configuration minimisante adéquate.

Pour conclure cette section, ajoutons qu'il existe un point de vue plus dynamique des configurations minimisantes. En remarquant qu'un minimum est un point critique, nous obtenons des équations d'Euler-Lagrange discrètes : pour toute configuration minimisante $(x_n)_n$ et pour tout entier n,

$$\frac{\partial E}{\partial y}(x_{n-1}, x_n) + \frac{\partial E}{\partial x}(x_n, x_{n+1}) = 0.$$

Ce qui se traduit, sous les hypothèses standard, par

$$\begin{cases} p_{n+1} = & p_n - V'(x_n) \\ x_{n+1} = & x_n - (U')^{-1}(p_n - V'(x_n)), \end{cases}$$

en posant $p_n = x_{n+1} - x_n$. En conséquence, toute configuration minimisante est une orbite d'un homéomorphisme Φ de $\mathbb{R}^2$ appelé *application standard* (ou *standard map*) qui possède de nombreuses propriétés. Dans le cas périodique, cet homéomorphisme se projette en un homéomorphisme sur le fibré tangent du cercle $\mathbb{S}^1 \times \mathbb{R}$. Dans le cas quasi-périodique, il s'étend en un homéomorphisme du fibré tangent de l'enveloppe $\Omega(X) \times \mathbb{R}$. Il est alors naturel de se demander si les propriétés classiques des homéomorphismes de l'anneau s'étendent dans le contexte quasi-périodique. Une première étude de base a été effectuée dans [AP10].

3.2 FK multidimensionel et stationnaire

Nous étendons le modèle de FK quasi-périodique à un contexte plus général dans [11] avec, cette fois, une approche lagrangienne plus proche du point de vue de Mather. Afin de simplifier la présentation, nous ne donnerons pas ici les hypothèses les plus faibles pour les résultats et nous en référons à [11] pour le lecteur interessé.

Nous considérons une famille d'énergie d'interaction en dimension supérieure $d \geq 1$ $E_\omega \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, dépendant d'un environnement ω. Nous supposons de plus, que la famille de tous les environnements forme un espace métrique compact Ω muni d'une $\mathbb{R}^d$-action continue $\tau \colon \mathbb{R}^d \times \Omega \to \Omega$, telle que le système dynamique $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d})$ soit minimal. Nous nous intéressons aux familles d'énergies $\{E_\omega\}_{\omega\in\Omega}$ possédant *une forme lagrangienne*, *i.e.* dont il existe une fonction continue $L \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$, appelée *lagrangien*, telle que

$$\forall \omega \in \Omega, \forall x, y \in \mathbb{R}^d,\ E_\omega(x, y) = L(\tau_x(\omega), y - x).$$

Le triplet $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ est appelé *modèle d'interaction presque périodique*[∗].

Définition 3.2.1 *Soit $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ un modèle d'interaction presque périodique. Le lagrangien L est dit* super-linéaire *si*

$$\lim_{R\to+\infty} \inf_{\omega\in\Omega} \inf_{\|t\|\geq R} \frac{L(\omega, t)}{\|t\|} = +\infty.$$

∗. Précisons que l'expression "presque périodique" est liée à la minimalité du système et est plus générale que la notion de "presque périodique au sens de H. Bohr"

L est dit faiblement twist *s'il existe une fonction $U : \Omega \to \mathbb{R}$, C^0 telle que pour tout $\omega \in \Omega$, la fonction $\tilde{E}_\omega(x, y) := E_\omega(x, y) + U(\tau_x\omega) - U(\tau_y(\omega)$ est C^2 et*

$$\forall x, y \in \mathbb{R}^d, \qquad \frac{\partial^2 \tilde{E}_\omega}{\partial x \partial y}(x, \cdot) < 0 \ \textit{et} \ \frac{\partial^2 \tilde{E}_\omega}{\partial x \partial y}(\cdot, y) < 0 \qquad \textit{p.p.}$$

Dans ce contexte, nous retrouvons tous les exemples traités dans la section précédente 3.1.

Exemple 3.2.2 (Ex. périodique) *Pour $\Omega = \mathbb{R}/\mathbb{Z}$ et τ_t la translation par $t \in \mathbb{R}$ modulo $\mathbb{Z}$, nous retrouvons l'énergie d'interaction originelle en prenant $L(\omega, t) = V(\omega) + U(t)$ pour $\omega \in \mathbb{R}/\mathbb{Z}, t \in \mathbb{R}$, avec $U : \mathbb{R} \to \mathbb{R}$ et $V : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ continue.*

Exemple 3.2.3 (Ex. quasi-cristal) *De même, en considérant $\Omega(X)$ l'enveloppe d'un quasi-cristal X et une fonction continue $\hat{V} : \Omega(X) \to \mathbb{R}$, le lagrangien $L(\omega, t) = \hat{V}(\omega) + U(t)$ pour $\omega \in \Omega(X)$ et $t \in \mathbb{R}$, nous retrouvons l'énergie d'interaction associée à un quasi-cristal traitée dans la section 3.1.*

Remarquons que dans ces deux cas, les propriétés de super-linéarité et twist de la fonction U impliquent les propriétés de super-linéarité et de faible twist du lagrangien.

Exemple 3.2.4 (Ex. presque périodique au sens de Bohr) *Le lagrangien défini, pour des constantes K_1, K_2, λ, par*

$$L((\omega_1, \omega_2), t) = \frac{1}{2}|t - \lambda|^2 + \frac{K_1}{(2\pi)^2}(1 - \cos 2\pi\omega_1) + \frac{K_2}{(2\pi)^2}(1 - \cos 2\pi\omega_2),$$

où $t \in \mathbb{R}$ et $(\omega_1, \omega_2) \in \mathbb{T}^2$, donne également un modèle d'interaction presque périodique, super-linéaire et faiblement twist pour le flot $\tau_t(\omega) = \omega + t(1, \sqrt{2}) \in \mathbb{T}^2$.

Ainsi le contexte de modèle d'interaction quasi-périodique est beaucoup plus large que celui étudié dans la section précédente puisque il englobe des systèmes expansifs, équicontinus ou avec des régularités intermédiaires (distal, *etc* ...). Ajoutons que ce modèle stationnaire a déjà été étudié pour l'équation d'Hamilton-Jacobi dans le contexte stationnaire ergodique (la dépendence en ω n'est alors que mesurable et l'action de τ est ergodique) [LS03, DS09, DS12] ou alors associé à des fonctions presque périodiques au sens de H. Bohr (par ex. [SdlL12, Ish99]).

Remarquons qu'un modèle d'interaction presque périodique avec un lagrangien L super-linéaire implique les propriétés suivantes pour chaque énergie d'interaction $E_\omega : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\omega \in \Omega$:

- *stationnarité* : $\forall t, x, y \in \mathbb{R}^d \ E_\omega(x + t, y + t) = E_{\tau_t(\omega)}(x, y)$.
- *bornée par translation* : $\forall R > 0$, $\sup_{\|y-x\| \le R} E_\omega(x, y) < +\infty$.
- *uniforme continuité par translation* : $\forall R > 0 \ E_\omega(x, y)$ est uniformément continue en $\|y - x\| \le R$.
- *super-linéarité* : $\lim_{R \to +\infty} \inf_{\|y-x\| \ge R} \frac{E_\omega(x,y)}{\|y-x\|} = +\infty$.

Sous les seules hypothèses de super-linéarité, de borné par translation et d'uniforme continuité par translation de l'énergie, l'existence de configuration bi-infinie minimisante n'est pas clair. Nous savons prouver seulement l'existence de configuration minimisante semi-infini $(x_n)_{n \ge 0}$ (voir proposition 3.2.6). Nous obtenons tout de même un peu plus, car nous montrons que ces configurations sont *calibrées*. C'est une notion clef dans la théorie KAM faible.

Définition 3.2.5 *Nous disons qu'une configuration $(x_n)_n$ est* c-calibrée *pour une énergie d'interaction E (au niveau $c \in \mathbb{R}$) si pour tous les entiers $m < n$,*

$$E(x_m, \ldots, x_n) - (m-n)c \leq \inf_{k \geq 1} \inf_{y_0 = x_m, \ldots, y_k = x_n} [E(y_0, \ldots, y_k) - kc].$$

Il est simple de voir qu'une configuration calibrée $(x_n)_n$ est une configuration minimisante, mais la réciproque n'est pas vraie. Mentionnons également, qu'en ces termes, Aubry et Le Daeron prouvent dans [ALD83] que pour une énergie d'interaction périodique en dimension $d = 1$, toute configuration minimisante est en faite calibrée pour un certain c.

Sous des hypothèses assez faible, nous obtenons l'existence de configuration semi-infinie calibrée.

Proposition 3.2.6 ([11]) *Pour une énergie d'interaction $E \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, C^0, super-linéaire, bornée par translation et uniformément continue par translation, il existe une calibration $\bar{c} \in \mathbb{R}$ et une configuration $(x_n)_{n \geq 0}$ $\bar{c}$-calibrée.*

Nous avons donc pour un modèle d'interaction presque périodique $(\Omega, \{\tau_t\}_{t \in \mathbb{R}^d}, L)$, avec un lagrangien L C^0 et super-linéaire, l'existence de configurations calibrées semi-infinies pour tout environnement $\omega \in \Omega$. La preuve de la proposition 3.2.6 provient d'une généralisation directe de la stratégie moderne de la théorie d'Aubry-Mather et est similaire à celle de [Zav12]. Le modèle FK apparait naturellement comme une discrétisation en temps de l'équation d'Hamilton Jacobi. Le pendant discret des solutions de viscosité ou solution KAM faible de ces équations, sont les fonctions propres d'un opérateur dit *de Lax-Oleinik*. L'étude de cet opérateur, nous permet dans l'annexe de [11] de démontrer la proposition 3.2.6.

Pour un modèle d'interaction quasi-périodique, une calibration importante est donnée par l'*énergie minimale*.

Définition 3.2.7 *Pour une famille d'interaction $\{E_\omega\}_{\omega \in \Omega}$ de forme lagrangienne, l'*énergie minimale *est la quantitée*

$$\bar{E} = \lim_{n \to +\infty} \inf_{\omega \in \Omega} \inf_{x_0, \ldots, x_n \in \mathbb{R}^d} \frac{1}{n} E_\omega(x_0, \ldots, x_n).$$

Cette limite est en fait un supremum par super-additivité et est finie pour L super-linéaire. Cette constante $\bar{E}$ joue le rôle d'un drift et $E_\omega(x,y) - \bar{E}$ est similaire à une "distance signée".

Nous montrons dans [11] l'existence de configurations $\bar{E}$-calibrées bi-infinies pour les environnements ω qui sont dans le projeté de l'*ensemble de Mather* : $Mather(L) \subset \Omega \times \mathbb{R}^d$, défini un peu plus loin (définition 3.2.10). Pour être plus précis, notons $pr \colon \Omega \times \mathbb{R}^d \to \Omega$ la projection sur la première coordonnée, nous avons :

Théorème 3.2.8 ([11]) *Soit $(\Omega, \{\tau_t\}_{t \in \mathbb{R}^d}, L)$ un modèle d'interaction presque périodique, avec L un lagrangien superlinéaire C^0. Alors pour tout $\omega \in pr(Mather(L))$, il existe une configuration $\bar{E}$-calibrée $(x_n)_{n \in \mathbb{Z}}$ telle que $x_0 = 0$ et $\sup_n \|x_{n+1} - x_n\| < +\infty$.*

Pour définir l'ensemble de Mather, il faut tout d'abord considérer la notion de *mesure holonomique*. Une telle mesure μ est une mesure de probabilité sur $\Omega \times \mathbb{R}^d$ telle que

$$\forall f \in C^0(\Omega), \qquad \int f(\omega) \mu(d\omega, dt) = \int f(\tau_t \omega) \mu(d\omega, dt).$$

La mesure de Dirac $\delta_{(\omega,0)}$ est un exemple de mesure holonomique. Plus généralement, n'importe quelle mesure de probabilité invariante par l'application standard est holonomique.

La proposition suivante est fondamentale pour les configurations calibrées, car elle relie leur existence avec un problème de minimisation ergodique plus simple à traiter.

Proposition 3.2.9 ([11]) *Si L est C^0 super-linéaire, alors $\bar{E} = \inf\{\int L d\mu; \mu$ est holonomique$\}$ et l'infimum est atteint pour une certaine mesure.*

Une mesure qui réalise l'infimum précédent est dite *minimisante*.

Définition 3.2.10 *L'ensemble de Mather de L est l'ensemble*

$$Mather(L) := \bigcup_{\mu \ minimisante} supp(\mu) \subset \Omega \times \mathbb{R}^d.$$

L'ensemble de Mather est non vide (Proposition 3.2.9) et est en fait compact lorsque le lagrangien L est super-linéaire [11]. Il est cependant possible que son projeté $pr(Mather(L))$ ne rencontre pas toutes les orbites du flot τ. Dans l'exemple 3.2.4, en prenant $\lambda = 0$, il est simple de voir que $\bar{E} = 0$ et que les mesures minimisantes sont des combinaison convexes de mesures de Dirac. De plus la projection de l'ensemble de Mather ne rencontre qu'un nombre fini de τ-orbite.

Dans les exemples associés aux quasi-cristaux, nous montrons, en dimension $d = 1$, que cette pathologie disparait. Dans ce cas, le projeté de l'ensemble de Mather rencontre toutes les orbites.

Plus précisément, pour Ω un $\mathbb{R}$-solénoïde (voir section 2.1.2), nous dirons qu'un lagrangien $L\colon \Omega \times \mathbb{R} \to \mathbb{R}$ est *transversalement constant* si pour toute verticale Ξ, $L(\cdot, t)_{|\Xi}$ est localement constant pour tout t fixé. Dans la suite, nous appellerons *modèle quasi-cristallin* un tel modèle d'interaction $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ avec un lagrangien L super-linéaire, faiblement twist et transversalement constant. L'exemple 3.2.3 pour une fonction V fortement X-équivariante donne un exemple d'un tel système.

Théorème 3.2.11 ([11]) *Soit $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ un modèle quasi-cristallin où l'action τ est uniquement ergodique. Alors le projeté de l'ensemble de Mather $pr(Mather(L))$ rencontre toute τ-orbite. En particulier, pour tout $\omega \in \Omega$, il existe une configuration $\bar{E}$-calibrée $(x_{n,\omega})_{n\in\mathbb{Z}}$ pour E_ω avec des sauts et une distance à l'origine uniformément bornés en ω :*

$$\sup_{\omega\in\Omega} \sup_{n\in\mathbb{Z}} |x_{n+1,\omega} - x_{n,\omega}| < +\infty, \ \ et \ \sup_{\omega\in\Omega} |x_{0,\omega}| < +\infty.$$

La démonstration du théorème 3.2.8 n'est pas directe et passe par l'étude des propriétés du *potentiel de Mañé*. La preuve du théorème 3.2.11 provient d'une description des mesures invariantes sur le solénoïde via des tours de Kakutani-Rohlin et d'une analyse de la combinatoire des configurations minimisantes dans ce contexte, via l'étude du potentiel de Mañé et le lemme de croisement d'Aubry.

Remarquons que la théorie d'Aubry-Mather dans le cas périodique, montre que l'ensemble de Mather est inclus dans un graphe de $\Omega \times \mathbb{R}$. Nous ne savons pas démontrer ce dernier point dans notre contexte, même si nous en sommes proche. Des simulations numériques (semi-rigoureuses) tendent à assurer l'existence de ces graphes invariants. De plus, dans le cas particulier de l'équation du pendule quasi-périodique, où l'on remplace le terme usuel en sin par une fonction fortement équivariante par rapport à ensemble de Delone donné par une substitution Pisot, ou par une suite sturmienne, nous obtenons l'existence d'une solution de viscosité et l'ensemble de Mather est supporté dans un graphe donné par cette solution. Ici les calculs sont assez explicites et nous devons utiliser fortement que les suites considérées sont *balancées* ce qui assure une vitesse de convergence rapide pour les moyennes ergodiques du nombre d'apparition d'un motif arbitraire.

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Progress in Mathematics, Vol. 309, 193–220

Linearly Repetitive Delone Sets

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Abstract. Linearly repetitive Delone sets are the simplest aperiodic repetitive Delone sets of the Euclidean space, e.g. any self similar Delone set is linearly repetitive. We present here some combinatorial, ergodic and mixing properties of their associated dynamical systems. We also give a characterization of such sets via the patch frequencies. Finally, we explain why a linearly repetitive Delone set is the image of a lattice by a bi-Lipschitz map of the space.

Mathematics Subject Classification (2010). 37B50.

Keywords. Delone sets, tiling systems, linearly repetitive, Voronoï cell.

1. History and motivations

The notion of *linearly recurrent subshift* has been introduced in [Du, DHS] to study the relations between substitutive dynamical systems and stationary dimension groups. In an independent way, the similar notion of *linearly repetitive Delone sets* of the Euclidean space $\mathbb{R}^d$ appears in [LP1]. For a Delone set X of $\mathbb{R}^d$, the *repetitivity function* $M_X(R)$ is the least M (possibly infinite) such that every closed ball B of radius M intersected with X contains a translated copy of any patch with diameter smaller than $2R$.

A Delone set X is said *linearly repetitive* if there exists a constant L such that $M_X(R) < LR$ for all $R > 0$. Observe that we can assume that the constant L is greater than 1. According to the following theorem, the slowest growth for the repetitivity function of an aperiodic Delone set is linear.

Theorem 1 ([LP1, Thm. 2.3]). *Let $d \geq 1$. There exists a constant $c(d) > 0$ such that for any Delone set X of $\mathbb{R}^d$ such that*

$$M_X(R) < c(d)R \qquad \text{for some } R > 0,$$

then X has a non-zero period.

Financial support from the ANR SUBTILE 0879. This work is part of the program MathAmSud DYSTIL 12Math-02.

Even more, if for some R, $M_X(R) < \frac{4}{3}R$, then the Delone set X is a *crystal*, *i.e.*, has d independent periods ([LP1, Thm. 2.2]).

The classical examples of aperiodic Delone systems, e.g., the ones arising from substitution, are linearly repetitive.

Lemma 2 ([So2, Lem. 2.3]). *A primitive self similar tiling is linearly repetitive.*

In many senses that we will not specify, the family of linearly repetitive Delone sets is small inside the family of all the Delone sets of the Euclidean space $\mathbb{R}^d$. For instance, in the class of Sturmian subshifts, several authors [MH, Du1, Du, LP2] show the following result.

Proposition 3. *The Sturmian subshift associated to an irrational number α is linearly recurrent if and only if the coefficients of the continued fraction of α are bounded.*

Let us recall that for the standard topology, the set of numbers with bounded continued fraction are badly approximable by rational numbers. It is known that they form a Baire meager set, with 0 Lebesgue measure but with Hausdorff dimension 1.

As we shall see, the linearly repetitive Delone sets possess many rigid properties. In the next section we present some combinatorial properties of these sets. For instance, their complexity appears to be the slowest possible among all the aperiodic repetitive Delone sets. Section 3 is devoted to the structure of the hull of an aperiodic linearly repetitive Delone set. A tower system with uniform bound is described. We deduce from this the main properties of the system. We focus in Section 4 on the ergodic properties of dynamical systems associated to linearly repetitive Delone sets. They are strictly ergodic (i.e., each patch appears with a frequency). But they are not wild since they are never measurably mixing. They satisfy also a subbaditive ergodic theorem. We present a characterization of the linear repetitivity by using a bound on the frequencies of the occurrences of the patches. The dynamical factors of these systems are studied in Section 5. They admit as factors just a finite number of non conjugate aperiodic Delone systems. We give also a characterization of their continuous and measurable eigenvalues by studying cohomological equations. The last section concerns the deformation of linearly repetitive Delone sets: each one is the image through a Lipschitz map of a lattice in $\mathbb{R}^d$.

2. Combinatorial properties

In this section we give the basic definitions and combinatorial properties concerning linearly repetitive Delone sets of $\mathbb{R}^d$. Most of these properties are obvious for self-similar tilings. Recall that a set $X \subset \mathbb{R}^d$, with $d \geq 1$, is a (r_X, R_X)-Delone set (or a Delone set for short) if it is a discrete subset of the Euclidean space $\mathbb{R}^d$, with the following properties:

1. *Uniform discreteness*: each open ball of radius $r_X > 0$ in $\mathbb{R}^d$ contains at most one point of X.
2. *Relative density*: each closed ball of radius R_X in $\mathbb{R}^d$ contains at least one point of X.

A classical example is given by the lattice $\mathbb{Z}^d$ of points with integer coefficients is a Delone set. But notice also that the image of any Delone set by a bi-Lipschitz map of $\mathbb{R}^d$ provides a Delone set. We denote by $B_R(x)$ the Euclidean closed ball of radius $R > 0$ centered at the point $x \in \mathbb{R}^d$.

2.1. Return vectors to a patch

Let X be a (r_X, R_X)-Delone set. A *R-patch* is a set of the kind $\mathtt{P} = X \cap B_R(x)$ centered at some point $x \in X$ and for some $R > R_X$[1]. In the rest of this paper we assume that all the Delone sets have *finite local complexity*, that is for any real $R > 0$ there is a finite number of R-patches, up to translations. This is actually equivalent to the fact $X - X$ is a discrete subset of $\mathbb{R}^d$ [La].

For a R-patch $\mathtt{P}$, we define the set

$$\mathcal{R}_{\mathtt{P}}(X) = \{v \in \mathbb{R}^d : \mathtt{P} + v \text{ is a } R\text{-patch of } X\}.$$

It is called the set of *return vectors* to $\mathtt{P}$. For a fixed center $x_{\mathtt{P}}$ of $\mathtt{P}$, any point in $\mathcal{R}_{\mathtt{P}}(X) + x_{\mathtt{P}} =: X_{\mathtt{P}}$ is an *occurrence* of the patch $\mathtt{P}$.

Observe that the null vector 0 always belongs to $\mathcal{R}_{\mathtt{P}}(X)$. It is straightforward to check that $X_{\mathtt{P}}$ is a Delone set when X is linearly repetitive (see definition in the introduction). Furthermore, $X_{\mathtt{P}}$ has finite local complexity because $X_{\mathtt{P}} - X_{\mathtt{P}} \subset X - X$.

When X is aperiodic and linearly repetitive with constant L, there are uniform bounds on the constants $r_{X_{\mathtt{P}}}$ and $R_{X_{\mathtt{P}}}$ associated to the Delone set $X_{\mathtt{P}}$. The following lemma shows that two occurrences of a patch can not be too close. The proof can be found in [Le, Lem. 2.1] and in [So2, Du1].

Lemma 4. *Let X be a linearly repetitive aperiodic Delone set with constant $L > 1$. Then, for every patch $\mathtt{P} = X \cap B_R(x)$ with $x \in X$, $R > 0$, we have*

$$\frac{R}{L+1} \leq r_{X_P} \leq R_{X_{\mathtt{P}}} \leq LR.$$

Proof. By contradiction: let us assume there exist $x \neq y \in X$ with

$$(X \cap B_R(x)) - x = X \cap B_R(y) - y$$

and

$$r_X \leq \|x - y\| < \frac{R}{(L+1)}.$$

Then for any point z' in $B_R(x) \cap X$, we have $z' + (y - x) \in X$. For any $z \in X$, the set $X \cap B_R(x)$ contains a translated copy centered in $z' \in X \cap B_R(x)$ of the patch $B_{\frac{R}{L+1}}(z) \cap X$. Thus $z' + (y - x) \in X \cap B_{\frac{R}{L+1}}(z')$ and finally $z + (y - x) \in X$ and

[1]Note: a given patch may be defined by several centers x and radius R. So when we consider a R-patch $\mathtt{P}$, we choose a center $x_{\mathtt{P}}$ and a radius R.

so $X + (y - x) \subset X$. In a similar way we obtain $X + (x - y) \subset X$, so that finally we get $X + x - y = X$ contradicting the aperiodicity of X. $\square$

This repulsion property on the occurrences of patches has several consequences on the combinatorics of the Delone set X.

First of all on the complexity. Let us denote $N_X(R)$ the number of different R-patches $B_R(x) \cap X$ with $x \in X$, up to translation. Since any ball of radius $M_X(R)$ contains the centers of occurrences of any R-patch, we easily deduce that $N_X(R)^{\frac{1}{d}} = O(M_X(R))$ as $R \to \infty$ (see [LP2]).

Lemma 5 ([Le, Lem. 2.2]). *Let X be an aperiodic linearly repetitive Delone set. Then*

$$\liminf_{R \to +\infty} \frac{N_X(R)}{R^d} > 0.$$

From this, we conclude that for an aperiodic linearly repetitive Delone set $M_X(R) = O(N_X(R)^{\frac{1}{d}})$ as $R \to \infty$.

Proof. As X is relatively dense, there exist constants $\lambda_1 > 0$ and $R_1 > 0$ such that

$$\sharp(X \cap B_R(x)) \geq \lambda_1 R^d \qquad \text{for any } x \in X,\ R \geq R_1.$$

By the previous lemma all the patches $(X - x) \cap B_R(0)$ for $x \in X \cap B_{\frac{R}{3(L+1)}}(0)$ are pairwise different. Thus for any $R \geq 3(L+1)R_1$, we have

$$N_X(R) \geq \sharp(X \cap B_{\frac{R}{3(L+1)}}(0)) \geq \lambda_1 \left(\frac{R}{3(L+1)}\right)^d,$$

that gives us the result. $\square$

Another property is on the hierarchical structure of the linearly repetitive Delone sets, that is quite simple: for any size $R > 0$, it is possible to decompose the Delone set into big patches (each one containing a R-patch), so that the number of these patches, up to translations, is independent of the size R. To be more precise, we need the notion of *Voronoï cell of a patch*. For a (r_X, R_X)-Delone set X, the *Voronoï cell* V_x of a point $x \in X$ is the set

$$V_x = \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - x'\|, \forall x' \in X\}.$$

It is then direct to check that any Voronoï cell V_x is a convex polyhedra, its diameter is smaller or equal to $2R_X$ and it contains the ball $B_{\frac{r_X}{2}}(x)$. Moreover when the Delone set X is of finite local complexity, the collection of Voronoï cells $\{V_x\}_{x \in X}$ forms a tiling of $\mathbb{R}^d$ of finite local complexity.

For a patch R-patch $\mathtt{P} = B_R(x_\mathtt{P}) \cap X$ of a repetitive Delone set X, we denote by $V_{\mathtt{P},x}$ the Voronoï cell associated to the Delone set $X_\mathtt{P}$ and an occurrence $x \in X_\mathtt{P}$. Notice that the Voronoï cell associated to the set of return vectors $\mathcal{R}_\mathtt{P}(X)$ and a return vector $v \in \mathcal{R}_\mathtt{P}(X)$, is the Voronoï cell of the occurrence $x_\mathtt{P} + v \in X_\mathtt{P}$ translated by the vector $-x_\mathtt{P}$.

It follows by Lemma 4 that for an aperiodic linearly repetitive Delone set with constant L, for any R-patch $\mathtt{P}$,

$$\operatorname{diam} V_{\mathtt{P},x} \leq 2LR, \qquad B_{\frac{R}{2(L+1)}}(x) \subset V_{\mathtt{P},x}, \qquad \text{for any } x \in X_{\mathtt{P}}. \tag{2.1}$$

Lemma 6 ([CDP, Lem. 11]). *Let X be an aperiodic linearly repetitive Delone set with constant L. There exists an explicit positive constant $c(L)$ such that for every $R > 0$ and every R-patch $\mathtt{P} = X \cap B_R(x)$, the collection $\{X \cap V_{\mathtt{P},x} : x \in X_{\mathtt{P}}\}$ contains at most $c(L)$ elements up to translation.*

Observe here that the bound, explicit in the proof, does not depend on the combinatorics of X but just on the constant of repetitivity.

Proof. Let us consider B the union of Voronoï cells $V_{\mathtt{P},x}$, $x \in X_{\mathtt{P}}$ that intersects the ball $B_{L^2R}(0)$. We have then

$$B_{L^2R}(0) \subset B \subset B_{L^2R+2LR}(0).$$

By linear repetitivity, $B \cap X$ contains a translated copy of any patch of the kind $X \cap V_{\mathtt{P},x}$ with $x \in X_{\mathtt{P}}$. Since any Voronoï cell contains a ball of radius $\frac{R}{2(L+1)}$, the number of patches in $B \cap X$ of the kind $X \cap V_{\mathtt{P},x}$ with $x \in X_{\mathtt{P}}$ is smaller than

$$\frac{\operatorname{vol} B_{RL(L+2)}(0)}{\operatorname{vol} B_{\frac{R}{2(L+1)}}(0)} \leq (2L(L+2)^2)^d = c(L). \qquad \square$$

Even stronger, the next lemma gives for an aperiodic linearly repetitive Delone set, a uniform bound (in R) on the number of occurrences of a patch inside a ball of radius KR.

Lemma 7. *Let X be an aperiodic linearly repetitive Delone set with constant $L \geq 1$, and let $K \geq L$. Then for any R-patch $\mathtt{P}$ of X and any point $y \in \mathbb{R}^d$,*

$$\sharp\{v \in \mathbb{R}^d; \mathtt{P} - v \subset B_{KR}(y) \cap X\} \leq 12^d K^d L^d.$$

Proof. Let B be the union of all the Voronoï cells $V_{\mathtt{P},x}$, $x \in X_{\mathtt{P}}$ that intersect the ball $B_{KR}(y)$. It follows that

$$B \subset B_{KR+2LR}(y).$$

By Lemma 4, the sets $B_{\frac{R}{2(L+1)}}(z)$, where the points $z \in B_{KR}(y) \cap X_{\mathtt{P}}$ are occurrences of $\mathtt{P}$, are pairwise disjoint and are included in B. Then it follows that

$$\sharp\{v \in \mathbb{R}^d; \mathtt{P} - v \subset B\} \leq \frac{\operatorname{vol}(B)}{\operatorname{vol} B_{\frac{R}{2(L+1)}}(0)} \leq 2^d (K + 2L)^d (L+1)^d,$$

that gives us the result. $\square$

Here again, observe that the bound depends just on the repetitivity constant.

3. Structure of the hull of a linearly repetitive Delone set

3.1. Background on solenoids, boxes

In this section, we will see the specific geometrical structure of the associated hull Ω of an aperiodic repetitive Delone set. We recall here, from [BBG, BG], the local structure of this space.

3.1.1. Local transversals and return vectors. Let $(\Omega, \mathbb{R}^d)$ be an aperiodic minimal Delone system. The *canonical transversal* of Ω is the set Ω^0 composed of all Delone sets in Ω that contain the origin 0. This terminology is motivated by the fact that if Y is in Ω^0, then every small translation of Y will not be in Ω^0. A *cylinder* in Ω is a set of the form

$$C_{Y,S} := \{Z \in \Omega \mid Z \cap B_S(0) = Y \cap B_S(0)\},$$

where $Y \in \Omega$ and $S > 0$ are such that $Y \cap B_S(0) \neq \emptyset$. The next lemma is well known.

Proposition 8. *Every cylinder in Ω is a Cantor set. Moreover, a basis for the topology of Ω is given by sets of the form*

$$\{Z - v \mid Z \in C_{Y,S}, v \in B_\varepsilon(0)\}.$$

In particular, the canonical transversal Ω^0 is a Cantor set.

A *local transversal* in Ω is a clopen (both closed and open) subset of some cylinder in Ω. By Proposition 8, a local transversal C is a Cantor set. This implies that the *recognition radius* defined as

$$\operatorname{rec}(C) := \inf\{S > 0 \mid C_{Y,S} \subseteq C \text{ for all } Y \in C\}$$

is finite. The motivation to define $\operatorname{rec}(C)$ is the following: suppose that a Delone set $Y \in \Omega$ is given and we want to check if Y belongs to C. Then it suffices to look whether the patch $Y \cap \overline{B}_{\operatorname{rec}(C)}(0)$ is equivalent to $Y_i \cap \overline{B}_{\operatorname{rec}(C)}(0)$ for some Y_i. Of course, if $C = C_{Y,S}$, then its recognition radius is smaller than S. Proposition 8 implies also that the collection

$$\{C_{Y,S} \mid Y \in C, S > \operatorname{rec}(C)\}$$

forms a basis for its topology. Indeed, since C is a Cantor set, it is easy to find a finite set $\{Y_1, \ldots, Y_m\}$ in C such that

$$C = \bigcup_{i=1}^{m} C_{Y_i, \operatorname{rec}(C)}.$$

Given a local transversal C and $D \subseteq \mathbb{R}^d$, the following notation will be used throughout the paper:

$$C[D] = \{Y - x \mid Y \in C, x \in D\}.$$

As we define a return vector to a patch, one can define the set of return vectors to a local transversal. Given a local transversal C and a Delone set $Y \in \Omega$, we define

$$\mathcal{R}_C(Y) = \{x \in \mathbb{R}^d \mid Y - x \in C\}.$$

When Y belongs to C, we refer to $\mathcal{R}_C(Y)$ as the *set of return vectors* of Y to C. The following lemma is standard (see, e.g., [C])

Lemma 9. *Let C be a local transversal. Then for each $Y \in C$, the set of return vectors $\mathcal{R}_C(Y)$ is a repetitive Delone set. Moreover, the following quantities*

$$r(C) = \frac{1}{2}\inf\{\|x-y\| \; x, y \in \mathcal{R}_C(Y),\, x \neq y\}, \quad \textit{and} \tag{3.1}$$

$$R(C) = \inf\{R > 0 \; \mathcal{R}_C(Y) \cap \overline{B}_R(y) \neq \emptyset \textit{ for all } y \in \mathbb{R}^d\}, \tag{3.2}$$

do not depend on the choice of Y in C.

3.1.2. Solenoids and boxes. In this section, we recall some definitions and results of [BBG, BG] that will be used throughout the paper. The hull Ω is locally homeomorphic to the product of a Cantor set and $\mathbb{R}^d$. Moreover, there exists an open cover $\{U_i\}_{i=1}^n$ of Ω such that for each $i \in \{1, \ldots, n\}$, there are $Y_i \in \Omega$, $S_i > 0$ and open sets $D_i \subseteq \mathbb{R}^d$ such that $U_i = C_{Y_i,S_i}[D_i]$ and the map $h_i : D_i \times C_i \to U_i$ defined by $h_i(t, Z) = Z - t$ is a homeomorphism. Furthermore, there are vectors $v_{i,j} \in \mathbb{R}^d$ (depending *only* on i and j) such that the transition maps $h_i^{-1} \circ h_j$ satisfy

$$h_i^{-1} \circ h_j(t, Z) = (t - v_{i,j}, Z - v_{i,j}) \tag{3.3}$$

at all points (t, Z) where the composition is defined. Following [BG], we call such a cover a $\mathbb{R}^d$*-solenoid's atlas*. It induces, among others structures, a laminated structure as follows. First, *slices* are defined as sets of the form $h_i(D_i \times \{Z\})$. Equation (3.3) implies that slices are mapped onto slices. Thus, the *leaves* of Ω are defined as the smallest connected subsets that contain all the slices they intersect. It is not difficult to check, using (3.3), that the leaves coincide with the orbits of Ω.

A *box* in Ω is a set of the form $B := C[D]$ where C is a local transversal in Ω, and $D \subseteq \mathbb{R}^d$ is an open set such that the map from $D \times C$ to B given by $(x, Y) \mapsto Y - x$ is a homeomorphism. This is true, for instance, if $D \subseteq B_{r(C)}(0)$ (*cf.* (3.1)).

3.2. Tower systems

In this section we review the concepts of box decompositions and tower systems introduced in [BBG, BG]. We focus on linearly repetitive Delone sets. The main results of this section can be found in [AC]. For all this section, Ω denotes the hull of an aperiodic repetitive Delone set X.

3.2.1. Box decompositions and derived tilings. A *box decomposition* is a finite and pairwise-disjoint collection of boxes $\mathcal{B} = \{B_1, \ldots, B_t\}$ in Ω such that the closures of the boxes in $\mathcal{B}$ cover the hull. For simplicity, we always write $B_i = C_i[D_i]$, where C_i and D_i are fixed and C_i is contained in B_i. In particular, the set D_i contains 0. We refer to C_i as the *base* of B_i. In this way, we call the union of all C_i the *base* of $\mathcal{B}$. The reasoning for fixing a local transversal in each B_i comes from the fact that box decompositions can be constructed in a canonical way starting from the set $\mathcal{R}_C(Y)$ of return vectors to a given local transversal C [BBG].

An alternative way of understanding a box decomposition is given by a family of tilings, known as *derived tilings*, which are constructed by intersecting the box decomposition with the orbit of each Delone set in the hull.

Let us start by recalling some basic definitions about tilings. A *tile* T in $\mathbb{R}^d$ is a compact set that is the closure of its interior (not necessarily connected). A *tiling* $\mathcal{T}$ of $\mathbb{R}^d$ is a countable collection of tiles that cover $\mathbb{R}^d$ and have pairwise disjoint interiors. Tiles can be *decorated*: they may have a color and/or be punctured at an interior point. Formally, this means that decorated tiles are tuples (T, i, x), where T is a tile, i lies in a finite set of *colors*, and x belongs to the interior of T. Two tiles have the same type if they differ by a translation. If the tiles are punctured, then the translation must also send one puncture to the other, and when they are colored, they must have the same color.

To construct a derived tiling, the idea is to read the intersection of the boxes in the box decomposition with the orbit of a fixed Delone set in the hull. In the sequel, it will be convenient to make the following construction. Let $\{C_i\}_{i=1}^t$ be a collection of local transversals and $\{D_i\}_{i=1}^t$ be a collection of bounded open subsets of $\mathbb{R}^d$ containing 0. Define $\mathcal{B} = \{C_i[D_i]\}_{i=1}^t$ and observe that the sets in $\mathcal{B}$ are not necessarily boxes of Ω. For each $Y \in \Omega$, define the (decorated) *derived collection* of $\mathcal{B}$ at Y by

$$\mathcal{T}_{\mathcal{B}}(Y) := \{(\overline{D_i} + v, i, v) \mid i \in \{1, \ldots, t\}, v \in \mathcal{R}_{C_i}(Y)\}.$$

The following lemma gives the relation between box decomposition and tilings.

Lemma 10 ([AC, Lem. 3.1]). *Let $\mathcal{B} = \{C_i[D_i]\}_{i=1}^t$, where the C_i's are local transversals and the D_i's are open bounded subsets of $\mathbb{R}^d$ that contain 0. Then, $\mathcal{B}$ is a box decomposition if and only if $\mathcal{T}_{\mathcal{B}}(Y)$ is a tiling of $\mathbb{R}^d$ for every $Y \in \Omega$. In this case, we call $\mathcal{T}_{\mathcal{B}}(Y)$ the* derived tiling *of $\mathcal{B}$ at Y.*

Proof. It is easy to see that if $\mathcal{B}$ is a box decomposition, then $\mathcal{T}_{\mathcal{B}}(Y)$ is a tiling for every $Y \in \Omega$. We now show the converse. For convenience, set $C = \cup_i C_i$. Fix $Y \in \Omega$ and suppose there are $i, j \in \{1, \ldots, t\}$, $Y_1 \in C_i, Y_2 \in C_j$, $x_1 \in D_i$ and $x_2 \in D_j$ such that $Y = Y_1 - x_1 = Y_2 - x_2$. This implies that the tiles $\overline{D_i} - x_1$ and $\overline{D_j} - x_2$ of $\mathcal{T}_{\mathcal{B}}(Y)$ meet an interior point. Since $\mathcal{T}_{\mathcal{B}}(Y)$ is a tiling, these tiles must coincide, and hence $i = j$ and $x_1 = x_2$. We conclude i the maps $h_i : C_i \times D_i \to C_i[D_i]$ given by $(Y, t) \mapsto Y - t$ are one-to-one, and moreover their image are pairwise disjoint. It is then straightforward to check that the maps h_i are homeomorphims. $\square$

3.2.2. Properly nested box decompositions. A box decomposition $\mathcal{B}' = \{C'_i[D'_i]\}_{i=1}^{t'}$ is *zoomed out* of another box decomposition $\mathcal{B} = \{C_j[D_j]\}_{j=1}^t$ if the following properties are satisfied:

(Z.1) If $Y \in C'_i$ is such that $Y - x \in C_j - y$ for some $x \in \overline{D'_i}$ and $y \in \overline{D_j}$, then $C'_i - x \subseteq C_j - y$.

(Z.2) If $x \in \partial D'_i$, then there exist j and $y \in \partial D_j$ such that $C'_i - x \subseteq C_j - y$.

(Z.3) For every box B' in $\mathcal{B}'$, there is a box B in $\mathcal{B}$ such that $B \cap B' \neq \emptyset$ and $\partial B \cap \partial B' = \emptyset$.

For each $i \in \{1, \ldots, t'\}$ and $j \in \{1, \ldots, t\}$ define

$$O_{i,j} = \{x \in D'_i \mid C'_i - x \subseteq C_j\}. \tag{3.4}$$

(Z.4) For each $i \in \{1, \ldots, t'\}$ and $j \in \{1, \ldots, t\}$,

$$\overline{D'_i} = \bigcup_{j=1}^{t} \bigcup_{x \in O_{i,j}} \overline{D_j} + x,$$

where all the sets in the right-hand side of the equation have pairwise disjoint interiors.

Observe that in the case that D_j is connected, then properties (Z.1) and (Z.2) imply (Z.4).

Since we are considering the C'_i's and C_j's as the bases of the boxes, we ask the following additional property to be satisfied:

(Z.5) The base of $\mathcal{B}'$ is included in the base of $\mathcal{B}$, that is, $\cup_i C'_i \subseteq \cup_j C_j$.

By (Z.4), we have that the tiling $\mathcal{T}_{\mathcal{B}'}(Y)$ is a super-tiling of $\mathcal{T}_{\mathcal{B}}(Y)$ in the sense that each tile T in $\mathcal{T}_{\mathcal{B}'}(Y)$ can be decomposed into a finite set of tiles of $\mathcal{T}_{\mathcal{B}}(Y)$. By (Z.3), one of these tiles is included in the interior of T.

Lemma 11. *For every $j \in \{1, \ldots, t\}$ we have*

$$C_j = \bigcup_{i=1}^{t'} \bigcup_{x \in O_{i,j}} C'_i - x.$$

Proof. By the definition of $O_{i,j}$ and (Z.1), it suffices to show that every $Y \in C_j$ belongs to the interior of some box $C'_i[D'_i]$. Suppose not, then $Y \in C'_i - x$ with $x \in \partial D'_i$ for some i since $\mathcal{B}'$ is a box decomposition. Moreover, by (Z.2) we deduce that Y must be in the boundary of some box $B_{j'}$ in $\mathcal{B}$, which gives a contradiction. □

3.3. Tower systems of linearly repetitive Delone system

A *tower system* is a sequence of box decompositions $(\mathcal{B}_n)_{n \in \mathbb{N}}$ such that $\mathcal{B}_{n+1}$ is zoomed out of $\mathcal{B}_n$ for all $n \in \mathbb{N}$. An iteration of the construction of zoomed out box decomposition gives the following result.

Theorem 12 ([BBG]). *The hull of any aperiodic minimal Delone set possesses a tower system.*

We have explained in Section 3.2.1 how to construct a box decomposition and in Section 3.2.2 the notion of zoomed out box decomposition. In this section, we specify the construction of a tower system to the linear repetitive case.

Fot a decreasing sequence $(C_n)_{n \in \mathbb{N}}$ of local transversals with diameter going to 0, and a tower system $(\mathcal{B}_n)_n$, we say that $(\mathcal{B}_n)_n$ is *adapted* to $(C_n)_n$, if for any $n \in \mathbb{N}$ we have $\mathcal{B}_n = \{C_{n,i}[D_{n,i}]\}_{i=1}^{t_n}$ such that $C_n = \cup_i C_{n,i}$ and t_n is a positive integer. In this case, for each $n \in \mathbb{N}^*$ we define, as in (3.4),

$$O_{i,j}^{(n)} = \{x \in D_{n,i} \mid C_{n,i} - x \subseteq C_{n-1,j}\} \tag{3.5}$$

and

$$m_{i,j}^{(n)} = \sharp O_{i,j}^{(n)}$$

for every $i \in \{1, \ldots, t_n\}$ and $j \in \{1, \ldots, t_{n-1}\}$. The *transition matrix* of level n (associated to the tower system $(\mathcal{B}_n)_{n\in\mathbb{N}}$) is then defined as the matrix $M_n = (m_{i,j}^{(n)})_{i,j}$, so M_n has size $t_n \times t_{n-1}$. From (Z.4), we get

$$\mathrm{vol}(D_{n,i}) = \sum_{j=1}^{t_{n-1}} m_{i,j}^{(n)} \, \mathrm{vol}(D_{n-1,j}). \tag{3.6}$$

Given a box decomposition $\mathcal{B} = \{C_i[D_i]\}_{i=1}^{t}$, define its external and internal radius by

$$R_{\mathrm{ext}}(\mathcal{B}) = \max_{i\in\{1,\ldots,t\}} \inf\{R > 0 : B_R(0) \supseteq D_i\};$$
$$r_{\mathrm{int}}(\mathcal{B}) = \min_{i\in\{1,\ldots,t\}} \sup\{r > 0 : B_r(0) \subseteq D_i\},$$

respectively. Define also $\mathrm{rec}(\mathcal{B}) = \max_{i\in\{1,\ldots,t\}} \mathrm{rec}(C_i)$.

With all theses definitions, we can state the following result for aperiodic linearly repetitive Delone systems.

Theorem 13 ([AC, Thm. 3.4]). *Let X be an aperiodic linearly repetitive Delone set with constant $L > 1$ and $0 \in X$. Given $K \geq 6L(L+1)^2$ and $s_0 > 0$, set $s_n := K^n s_0$ and $C_n := C_{X,s_n}$ for all $n \in \mathbb{N}$. Then, there exists a tower system $(\mathcal{B}_n)_n$ of Ω adapted to $(C_n)_{n\in\mathbb{N}}$ that satisfies the following additional properties:*

i) *for every $n \geq 0$, $C_{n+1} \subseteq C_{n,1}$;*

ii) *there exist constants*

$$K_1 := \frac{1}{2(L+1)} - \frac{L}{K-1} \quad \textit{and} \quad K_2 := \frac{LK}{K-1},$$

which satisfy $0 < K_1 < 1 < K_2$, such that for every $n \in \mathbb{N}$ we have

$$K_1 s_n \leq r_{\mathrm{int}}(\mathcal{B}_n) < R_{\mathrm{ext}}(\mathcal{B}_n) \leq K_2 s_n; \tag{3.7}$$

iii) *for every $n \in \mathbb{N}$,*

$$\mathrm{rec}(\mathcal{B}_n) \leq (2L+1)s_n. \tag{3.8}$$

As an application of this result, we have the nice following structure.

Theorem 14. *Let X be an aperiodic linearly repetitive Delone set. Then, the tower system of Ω obtained in Theorem 13 satisfies the following:*

1. *For every $n \in \mathbb{N}^*$, the matrix $M_n = (m_{i,j}^{(n)})_{i,j}$ has strictly positive coefficients;*
2. *The matrices $\{M_n\}_{n\in\mathbb{N}^*}$ are uniformly bounded in size and norm.*

In the self-similar case, the family of matrices $\{M_n\}$ can be reduced to only one element.

Proof. Take the notations of Theorem 13. Indeed, by the definition of linearl repetitivity, we have $M_X(\mathrm{rec}(\mathcal{B}_n)) \le L\,\mathrm{rec}(\mathcal{B}_n)$ for all $n \in \mathbb{N}^*$. Combining this with (3.8), the left-hand inequality of (3.7) and the definition of s_n we get

$$M_X(\mathrm{rec}(\mathcal{B}_n)) \le \frac{L(2L+1)}{KK_1} r_{\mathrm{int}}(\mathcal{B}_{n+1}).$$

Since $K \ge 6L(L+1)^2$, it follows that $L(2L+1) \le K_1 K$ and we obtain for all $n \ge 0$

$$M_X(\mathrm{rec}(\mathcal{B}_n)) \le r_{\mathrm{int}}(\mathcal{B}_{n+1}).$$

Thus any $\mathrm{rec}(\mathcal{B}_n)$-patch occurs in a set $D_{n+1,i} \cap Y$ for any $Y \in C_{n+1,i}$, and the coefficients $m_{i,j}^{(n)}$ are positive. Moreover, since $D_{n,i}$ is included in a ball of radius $R_{\mathrm{ext}}(\mathcal{B}_{n-1})$ and each $D_{n-1,j}$ contains a ball of radius $r_{\mathrm{int}}(\mathcal{B}_{n-1})$, we deduce from 3.6 that

$$\sum_{j=1}^{t_{n-1}} m_{i,j}^{(n)} \le \left(\frac{R_{\mathrm{ext}}(\mathcal{B}_{n-1})}{r_{\mathrm{int}}(\mathcal{B}_{n-1})}\right)^d \le \left(K\frac{K_2}{K_1}\right)^d.$$

So we get that the matrices $\{M_n\}_n$ are uniformly bounded. □

4. Ergodic properties of linearly repetitive system

4.1. Background on transverse invariant measure

A Borel measure μ on the hull Ω of a repetitive Delone set is *translation invariant* if $\mu(B - v) = \mu(B)$ for every Borel set B and $v \in \mathbb{R}^d$. It is well known that any continuous $\mathbb{R}^d$ action on a compact space admits an invariant measure.

Let C be a local transversal and $0 < r < r(C)$. Each translation invariant measure μ induces a measure ν on C (see [Gh] for the general construction): given a Borel subset V of C, its *transverse measure* is defined by

$$\nu(V) = \frac{\mu(V[B_r(0)])}{\mathrm{vol}(B_r(0))},$$

where vol denotes the Euclidean volume in $\mathcal{R}^d$. This gives a measure on each C, which does not depend on small r. The collection of all measures defined in this way is called the *transverse invariant measure* induced by μ. It is invariant in the sense that if V is a Borel subset of C and $x \in \mathbb{R}^d$ is such that $V - x$ is a Borel subset of another local transversal C', then $\nu(V - x) = \nu(V)$. Conversely, the measure μ of any box B written as $C[D]$ may be computed by the equation

$$\mu(C[D]) = \mathrm{vol}(D) \times \nu(C).$$

For a tower system $(\mathcal{B}_n)_{n\ge 0}$ where $\mathcal{B}_n = \{C_{n,i}[D_{n,i}]\}_{i=1}^{t_n}$ from (Z.4), Lemma 11 and the definition of transverse invariant measures, we get

$$\nu(C_{n-1,j}) = \sum_{i=1}^{t_n} \nu(C_{n,i}) m_{i,j}^{(n)}. \tag{4.1}$$

Fix $n \in \mathbb{N}$. From the relation $\mu(C_{n,i}[D_{n,i}]) = \mathrm{vol}(D_{n,i})\nu(C_{n,i})$ and the fact that $\mathcal{B}_n$ is a box decomposition, it follows that

$$\sum_{j=1}^{t_n} \mathrm{vol}(D_{n,j})\nu(C_{n,j}) = 1. \tag{4.2}$$

4.2. Unique ergodicity and speed of convergence

When the system $(\Omega, \mathbb{R}^d)$ has a unique translation invariant probability measure, the system is called *uniquely ergodic*. The unique ergodicity implies combinatorial properties for the Delone set. The dynamical system $(\Omega, \mathbb{R}^d)$ is uniquely ergodic, if and only if any Delone set $X \in \Omega$ has *uniform patch frequencies*, i.e., any patch $\mathtt{P}$ occurs with a positive frequency; more precisely: Let $X_{\mathtt{P}}$ be the set of occurrences of the patch $\mathtt{P}$ in X, and let $(D_N)_N$ be a nested sequence of d-cube D_N of side N, then the following limit exists.

$$\lim_{N\to\infty} \frac{\sharp X_{\mathtt{P}} \cap D_N}{\mathrm{vol}(D_N)} =: \mathrm{freq}(\mathtt{P}).$$

The number $\mathrm{freq}(\mathtt{P})$ is called the *frequency* of $\mathtt{P}$. Notice the difference with the standard Birkhoff ergodic Theorem that asserts a convergence only for almost all Delone set of the hull.

Theorem 15. *Let X be an aperiodic linearly repetitive Delone set of $\mathbb{R}^d$ and Ω its hull. Then the system $(\Omega, \mathbb{R}^d)$ is uniquely ergodic.*

The original proof is due to Lagarias and Pleasants in [LP2]. By using the identification between a transverse invariant measure and the inverse limit of top homologies of branched manifolds, the authors in [BBG] show that in the case described in Theorem 14, the system is uniquely ergodic. This proof is independent of the original one.

Actually for linearly repetitive system, we can be much more precise and give informations on the speed of convergence of the limit. For instance the following is a stronger result of Lagarias and Pleasants [LP2], that implies the unique ergodicity.

Theorem 16 ([LP2]). *Let X be a linearly repetitive Delone set of $\mathbb{R}^d$. There exists a $\delta > 0$ such that, for every patch $\mathtt{P}$ of X, there is a number* $\mathrm{freq}(\mathtt{P})$ *so that*

$$\left| \frac{X_{\mathtt{P}} \cap Dom_N}{\mathrm{vol}(Dom_N)} - \mathrm{freq}(\mathtt{P}) \right| = O(N^{-\delta}),$$

where Dom_N is either a d-cube with side N or a ball of radius N. The O-constant may depend on the patch $\mathtt{P}$.

In [AC], a proof of this theorem is given using the structure Theorem 14 and relating the constant δ with the matrices M_n by the following way

$$\delta = d - \log_K \left(1 - \sup_n ||M_n||_1^{-1} ||M_{n+1}||_1^{-1}\right),$$

where $\log_K$ denotes the logarithm in base K.

4.3. Non-mixing properties

A translation invariant probability measure μ on a the hull Ω of a Delone set is said to be *measurably strongly mixing* if for any Borel sets A, B in Ω,

$$\lim_{\|v\|\to\infty} \mu((A-v)\cap B) = \mu(A)\mu(B). \tag{4.3}$$

In this section, we show the following proposition which is analogous to theorem of Dekking and Keane [DK] for substitutive subshifts.

Proposition 17 ([C0]). *Let X be a linearly repetitive Delone set of $\mathbb{R}^d$ and Ω its hull. Then the system $(\Omega, \mathbb{R}^d)$ is not measurably strongly mixing.*

The proof's strategy is the same as for self-similar tiling in [So1] or for linear recurrent Cantor system in [CDHM]. But we need sharp estimates on the transverse measures of clopen sets, provided by Theorem 13.

Assume that the Delone set X is aperiodic and linearly repetitive with constant L. Let μ be the unique translation invariant probability measure on the hull Ω, and let ν be the associated transverse invariant measure. Let $(\mathcal{B}_n)_{n\geq 0}$ be the tower system given by Theorem 13 where for each integer n, $\mathcal{B}_n = \{C_{n,i}[D_{n,i}]\}_{i=1}^{t_n}$.

Lemma 18. *For the tower system of Ω given by Theorem 13, we have*

$$\inf_{\substack{n\geq 1\\ 1\leq i\leq n}} \mathrm{vol}(D_{n,i})\nu(C_{n,i}) > \left(\frac{K_1}{KK_2}\right)^d =: c > 0.$$

Proof. With equation 4.1, for any $1 \leq i \leq t_n$, we get

$$\nu(C_{n,i}) \geq \sum_{j=1}^{t_{n+1}} \nu(C_{n+1,j}). \tag{4.4}$$

By definition, for any $1 \leq i \leq t_n$, the domain $D_{n,i}$ contains a ball or radius $r_{\mathrm{int}}(\mathcal{B}_n)$ and for $1 \leq j \leq t_{n+1}$ the domain $D_{n+1,j}$ is included in a ball of radius $R_{\mathrm{ext}}(\mathcal{B}_{n+1})$. Thus, as in the proof of Theorem 14, we deduce from Theorem 13

$$\frac{\mathrm{vol}(D_{n+1,j})}{\mathrm{vol}(D_{n,i})} \leq \left(\frac{R_{\mathrm{ext}}(\mathcal{B}_{n-1})}{r_{\mathrm{int}}(\mathcal{B}_{n-1})}\right)^d \leq \left(K\frac{K_2}{K_1}\right)^d = c^{-1}. \tag{4.5}$$

Thus it follows from (4.2), that for any $n \geq 0$ and $1 \leq i \leq t_n$

$$\mathrm{vol}(D_{n,i})\nu(C_{n,i}) \geq \sum_{j=1}^{t_{n+1}} c\mathrm{vol}(D_{n+1,j})\nu(C_{n+1,j}) = c. \qquad \square$$

For the tower system $(\mathcal{B}_n)_n$, we define as in Definition 3.4, for integers $p \geq n > 0$

$$O_{i,j}^{(p,n)} := \{x \in D_{p,i} \mid C_{p,i} - x \subseteq C_{n-1,j}\}, \text{ for } 1 \leq i \leq t_p; 1 \leq j \leq t_{n-1} \tag{4.6}$$

and

$$m_{i,j}^{(p,n)} = \sharp O_{i,j}^{(p,n)}.$$

Then it is straightforward to check that the $t_p \times t_{n-1}$ matrix

$$(m_{i,j}^{(p,n)})_{i,j} = M_p \cdots M_n.$$

Lemma 19. *For the tower system of* Ω *given by Theorem* 13, *we have for* $n \geq 2$, *and* $1 \leq j \leq t_n$

$$\liminf_{p\to+\infty} \min_{1\leq i\leq t_p} \frac{m_{i,j}^{(p,n)}}{\mathrm{vol}(D_{i,p})} \geq \nu(C_{n-1,j}) \left(\frac{K_1}{K_2}\right)^d c.$$

Proof. Let $X \in \cap_{n\geq 0} C(n)$. By the unique ergodicity, we have

$$\lim_{R\to+\infty} \frac{1}{\mathrm{vol}(B_R(0))} \sharp\{B_R(0) \cap \mathcal{R}_{C_{n,j}}(X)\} = \nu(C_{n,j}). \tag{4.7}$$

Since for every $p > n$, the set $C_p \subset C_{p-1,1}$, we get for any $1 \leq i \leq t_p$,

$$m_{i,j}^{(p,n)} \geq m_{1,j}^{(p-1,n)} \geq \sharp\{D_{p-1,j} \cap \mathcal{R}_{C_{n-1,j}}(X)\}.$$

Hence we conclude by this inequality and inequality (4.5) that

$$\begin{aligned}\liminf_{p\to+\infty} \min_{1\leq i\leq t_p} \frac{m_{i,j}^{(p,n)}}{\mathrm{vol}(D_{p,i})} &\geq \liminf_{p\to+\infty} \frac{\sharp\{D_{p-1,j} \cap \mathcal{R}_{C_{n-1,j}}(X)\}}{\mathrm{vol}(D_{p,i})}\\ &\geq c \liminf_p \frac{\sharp\{D_{p-1,j} \cap \mathcal{R}_{C_{n-1,j}}(X)\}}{\mathrm{vol}(D_{p-1,j})}\\ &\geq c \lim_p \frac{\sharp\{B_{r_{\mathrm{int}}(\mathcal{B}_{p-1})}(0) \cap \mathcal{R}_{C_{n-1,j}}(X)\}}{\mathrm{vol}(B_{\frac{K_2}{K_1} r_{\mathrm{int}}(\mathcal{B}_{p-1})}(0))},\end{aligned}$$

since $D_{p-1,j}$ contains the ball $B_{r_{\mathrm{int}}(\mathcal{B}_{p-1})}(0)$ and is contained in the ball $\Longleftarrow$ disp

$$B_{R_{\mathrm{ext}}(\mathcal{B}_{p-1})}(0) \subset B_{\frac{K_2}{K_1} r_{\mathrm{int}}(\mathcal{B}_{p-1})}(0).$$

We obtain the conclusion by the equality (4.7). □

Now we are able to prove Proposition 17.

Proof of Proposition 17. Let n be an integer such that $\nu(C_n) < \left(\frac{K_1}{K_2}\right)^d c^2$. For $p \geq n$, Let $\mathcal{F}_{p,1} \subset \mathbb{R}^d$ be the set of vector v such that there exists a $1 \leq j \leq t_p$ satisfying $C_{p,1} - v \cap C_{p,j} \neq \emptyset$ and $D_{p,j} - v \cap D_{p,1} \neq \emptyset$. Let $\tilde{C}(n,v) = (C_{n,1} - v) \cap C_{n,1}$. We will show that

$$\liminf_{p\to\infty} \inf_{v\in\mathcal{F}_{p,1}} \nu(\tilde{C}(n,v)) > \nu(C_{n,1})^2$$

which implies that the system $(\Omega, \mathbb{R}^d)$ is not strongly mixing.
For $x \in O_{1,1}^{(p,n+1)} = \{x \in D_{p,1} \mid C_{p,1} - x \subseteq C_{n,1}\}$, and $v \in \mathcal{F}_{p,1}$, we have by (Z.1) and by i) in Theorem 13

$$C_{p+1,1} - (v + x) \subset C_{p+1} - x \subset C_{p,1} - x \subset C_{n,1}.$$

Thus for any $x \in O_{1,1}^{(p,n+1)}$ and $v \in \mathcal{F}_{p,1}$ we get $C_{p+1,1} - x \subset \tilde{C}(n,v)$. Then

$$\nu(\tilde{C}(n,v)) \geq \sharp O_{1,1}^{(p,n+1)} \nu(C_{p+1,1}) = m_{1,1}^{(p,n+1)} \nu(C_{p+1,1}).$$

By Lemma 19, we obtain

$$\begin{aligned}
&\liminf_{p\to\infty} \inf_{v\in\mathcal{F}_{p,1}} \nu(\tilde{C}(n,v)) \\
&\quad \geq \liminf_{p\to\infty} \frac{m_{1,1}^{(p,n+1)}}{\operatorname{vol}(D_{1,p})} \nu(C_{p+1,1}) \operatorname{vol}(D_{1,p}) \\
&\quad \geq \nu(C_{n,1}) c \left(\frac{K_1}{K_2}\right)^d \liminf_{p\to\infty} \nu(C_{p+1,1}) \operatorname{vol}(D_{1,p}) \\
&\quad \geq \nu(C_{n,1}) c \left(\frac{K_1}{K_2}\right)^d \liminf_{p\to\infty} \nu(C_{p+1,1}) \operatorname{vol}(D_{1,p+1}) c \text{ by inequality (4.5)} \\
&\quad \geq \nu(C_{n,1}) \left(\frac{K_1}{K_2}\right)^d c^2 > \nu(C_{n,1})^2. \qquad \square
\end{aligned}$$

4.4. Subadditive ergodic theorem

In Section 4.2 we recall that the linearly repetitive systems are uniquely ergodic. Actually such systems satisfy also a subadditive ergodic theorem. Let $\mathcal{B}(\mathbb{R}^d)$ denotes the family of bounded subsets in $\mathbb{R}^d$. A real-valued function $F\colon \mathcal{B}(\mathbb{R}^d) \to \mathcal{R}$ is called *subadditive* if

$$F(Q_1 \cup Q_2) \leq F(Q_1) + F(Q_2)$$

for any disjoint sets $Q_1, Q_2 \in \mathcal{B}(\mathbb{R}^d)$. For a Delone set X, the function F is called *X-invariant* if

$$F(Q) = F(Q+t) \quad \text{whenever } Q \in \mathcal{B}(\mathbb{R}^d) \text{ and } t + (Q \cap X) = (t+Q) \cap X.$$

For instance, given a patch $\mathtt{P}$ of the Delone set X, the function $B \in \mathcal{B}(\mathbb{R}^d) \mapsto -X_{\mathtt{P}} \cap B$ where $X_{\mathtt{P}}$ denotes the set of occurrences of the patches $\mathtt{P}$ in X, is a subadditive X-invariant function.

Theorem 20 ([DL, BBL]). *Let X be a linearly repetitive Delone set in $\mathbb{R}^d$. Then X satisfies the* uniform ergodic theorem*:* i.e.*, for any X-invariant subadditive function F and any nested sequence $(D_n)_n$ of d-cubes with side-lengths going to infinity as n goes to infinity, the following limit exists*

$$\lim_{n\to+\infty} \frac{F(D_n)}{\operatorname{vol}(D_n)},$$

and is independent of the sequence $(D_n)_n$.

It is then easy to deduce from this result that the associated dynamical system is uniquely ergodic. The converse is false, in [DL], the authors give an example of a Sturmian sequence that does not satisfy the subadditive ergodic theorem. They prove also a more stronger form of this theorem.

The *lower density* $\underline{\nu}(\mathtt{P})$ of a R-patch $\mathtt{P}$ is the quantity

$$\underline{\nu}(\mathtt{P}) := \lim\inf_{n\to\infty} \frac{\sharp X_{\mathtt{P}} \cap B_n(0)}{\mathrm{vol}(B_n(0))} \mathrm{vol}(B_R(0)).$$

The results in [BBL] have this direct corollary.

Proposition 21. *If X is a repetitive (r_X, R_X) Delone set verifying the uniform subadditive ergodic theorem, then X satisfies* positivity of weights, i.e.*,*

$$\inf_{\mathtt{P} \text{ is an } R\text{-patch},\ R\geq R_X} \underline{\nu}(\mathtt{P}) > 0.$$

Notice that in dimension 1, the positivity of weights property is sufficient to ensure the unique ergodiciy (see [Bo]). Actually, one can deduce from Lemma 18 that a linearly repetitive Delone set satisfies the positivity of weights.

4.5. A characterization of linear repetitivity

In [Le02], D. Lenz characterizes the subshifts that admit a uniform subadditive ergodic Theorem by uniform positivity of weights. This can be considered as an averaged version of linear repetitivity. For Delone systems, it is shown in [Bes, BBL] that the linear repetitivity is equivalent to positivity of weights plus some balancedness of the shape of patterns. For a Voronoï cell V of a Delone set, let us define:

$$r_{\mathrm{int}} := \sup\{r > 0; V \text{ contains a ball of radius } r\}.$$
$$R_{\mathrm{ext}} := \inf\{R > 0; V \text{ is contained in a ball of radius } R\}.$$

The *distorsion* of V is the constant $\lambda(V) := R_{\mathrm{ext}}(V)/r_{\mathrm{int}}(V)$.

Theorem 22 ([BBL]). *Let X be an aperiodic Delone set in $\mathbb{R}^d$ of finite type. Then X is linearly repetitive if and only if for any R-patch $\mathtt{P}$ of X, $R > 0$: the set $X_{\mathtt{P}}$ of occurrences of $\mathtt{P}$ is a $(r_{\mathtt{P}}, R_{\mathtt{P}})$-Delone set such that*

(i) $\sup_{\mathtt{P}, x\in X_{\mathtt{P}}} \lambda(V_x) < +\infty$ *where V_x denotes the Voronoï cell of x.*
(ii) *The Delone set X satisfies the positivity of weights* (*see Proposition* 21).

One can find in [BBL] another similar equivalent condition to linear repetitivity. Notice that in dimension $d = 1$, the distorsion of any compact Voronoï cell is equal to 1. Thus the condition (ii) is equivalent to the linear repetitivity.

For an aperiodic linearly repetitive Delone set, the properties (i)–(ii) arise from the properties recalled in Subsections 2.1 and 4.4.

Let us also mention in Chapter **On the non commutative geometry for tilings**, a characterization of Sturmian sequences that are linearly repetitive by using metrics arising from the Connes distance.

5. Factors of linearly repetitive system

A *factor map* between two Delone systems $(\Omega_1, \mathbb{R}^d)$ and $(\Omega_2, \mathbb{R}^d)$ is a continuous surjective map $\pi : \Omega_1 \to \Omega_2$ such that $\pi(X - v) = \pi(X) - v$, for every $X \in \Omega_1$ and $v \in \mathbb{R}^d$.

In symbolic dynamics it is well known that topological factor maps between subshifts are always given by sliding-block-codes. An equivalent notion for the Delone system is the *local derivability*: *i.e.*, there exists a constant $s_0 > 0$ such that for any radius $R > 0$, if two Delone sets $X, Y \in \Omega_1$ satisfy $X \cap B_{R+s_0}(0) = Y \cap B_{R+s_0}(0)$ then $\pi(X) \cap B_R(0) = \pi(Y) \cap B_R(0)$. However there are examples of factor maps on Delone systems that are not sliding-block codes ([Pe, RS]). Nevertheless, the following lemma shows that factor maps between Delone systems are not far from being sliding-block-codes. Similar results can be found in [CD, CDP, HRS].

Lemma 23. *Let X_1 and X_2 be two Delone sets. Suppose X_1 has finite local complexity and $\pi : \Omega_{X_1} \to \Omega_{X_2}$ is a factor map. Then, there exists a constant $s_0 > 0$ such that for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ satisfying the following: For any $R \geq R_\varepsilon$, if X and X' in Ω_{X_1} satisfy*

$$X \cap B_{R+s_0}(0) = X' \cap B_{R+s_0}(0),$$

then

$$(\pi(X) - v) \cap B_R(0) = \pi(X') \cap B_R(0)$$

for some $v \in B_\varepsilon(0)$.

Proof. The Delone set X_2 has also finite local complexity because Ω_{X_2} is compact. Let r_0 and R_0 be positive constants such that X_2 is a (r_0, R_0)-Delone set. Since all the elements of Ω_{X_2} are (r_0, R_0)-Delone sets, if two different points y_1, y_2 of $\mathbb{R}^d$ satisfy $(X - y_1) \cap B_R(a) = (X - y_2) \cap B_R(a)$ for some $X \in \Omega_{X_2}$, $a \in \mathbb{R}^d$ and $R > R_0$, then $\|y_1 - y_2\| \geq \frac{r_0}{2}$ (for the details see [So1]).

Let $0 < \delta_0 < \min\{\frac{r_0}{4}, \frac{1}{R_0}\}$. Since π is uniformly continuous, there exists $s_0 > 1$ such that if X and X' in Ω_{X_1} verify $X \cap B_{s_0}(0) = X' \cap B_{s_0}(0)$ then

$$(\pi(X) - v) \cap B_{\frac{1}{\delta_0}}(0) = \pi(X') \cap B_{\frac{1}{\delta_0}}(0),$$

for some $v \in B_{\delta_0}(0)$. Let $0 < \varepsilon < \delta_0$. By uniform continuity of π, there exists $0 < \delta < \frac{1}{s_0}$ such that if X and X' in Ω_{X_1} verify $X \cap B_{\frac{1}{\delta}}(0) = X' \cap B_{\frac{1}{\delta}}(0)$ then

$$(\pi(X) - v) \cap B_{\frac{1}{\varepsilon}}(0) = \pi(X') \cap B_{\frac{1}{\varepsilon}}(0), \tag{5.1}$$

for some $v \in B_\varepsilon(0)$. Now fix $R \geq R_\varepsilon = \frac{1}{\delta} - s_0$, and let X and X' be two Delone sets in Ω_{X_1} satisfying

$$X \cap B_{R+s_0}(0) = X' \cap B_{R+s_0}(0). \tag{5.2}$$

Observe that X and X' satisfy (5.1), and $(X - a) \cap B_{s_0}(0) = (X' - a) \cap B_{s_0}(0)$, for every a in $B_R(0)$. The choice of s_0 ensures that

$$(\pi(X) - a - t(a)) \cap B_{\frac{1}{\delta_0}}(0) = (\pi(X') - a) \cap B_{\frac{1}{\delta_0}}(0), \tag{5.3}$$

for some $t(a) \in B_{\delta_0}(0)$. Let us prove the map $a \mapsto t(a)$ is locally constant. For $a \in B_R(0)$, let $0 < s_a < \frac{1}{\delta_0} - R_0$ be such that $B_{s_a}(a) \subseteq B_R(0)$. Every $a' \in B_{s_a}(0)$ verifies $B_{\frac{1}{\delta_0}-\|a'\|}(-a') \subset B_{\frac{1}{\delta_0}}(0)$. Let $a' \in B_{s_a}(0)$. This inclusion and (5.3) imply

$$(\pi(X) - a - a' - t(a)) \cap B_{\frac{1}{\delta_0}-\|a'\|}(-a') = (\pi(X') - a - a') \cap B_{\frac{1}{\delta_0}-\|a'\|}(-a'). \quad (5.4)$$

On the other hand, from the definition of the map $a \mapsto t(a)$ we deduce

$$(\pi(X) - a - a' - t(a + a')) \cap B_{\frac{1}{\delta_0}}(0) = (\pi(X') - a - a') \cap B_{\frac{1}{\delta_0}}(0),$$

which implies

$$(\pi(X)-a-a'-t(a+a'))\cap B_{\frac{1}{\delta_0}-\|a'\|}(-a') = (\pi(X')-a-a')\cap B_{\frac{1}{\delta_0}-\|a'\|}(-a'). \quad (5.5)$$

Since $\|t(a) - t(a + a')\| \leq \frac{r_0}{2}$, from equations (5.4), (5.5) and the remark of the beginning of the proof we conclude $t(a) = t(a+a')$ for every $a' \in B_s(0)$. Therefore the map $a \mapsto t(a)$ is constant on $B_{s_a}(a)$.

Furthermore, due to $\delta_0 > \varepsilon$ and (5.2), Equation (5.1) implies there exists $v \in B_\varepsilon(0)$ such that

$$(\pi(X) - v) \cap B_{\frac{1}{\delta_0}}(0) = \pi(X') \cap B_{\frac{1}{\delta_0}}(0). \quad (5.6)$$

For $a = 0$, from (5.3) and (5.6) we have that $t(0) = v$ or $\|v - t(0)\| \geq \frac{r_0}{2}$. Since $\|t(0) - v\| \leq \delta_0 + \varepsilon < 2\delta_0 < \frac{r_0}{2}$, we conclude $t(0) = v$ and then $t(a) = v$ for every $a \in B_R(0)$. This property together with (5.3) and (5.6) imply that

$$(\pi(X) - v) \cap B_R(0) = \pi(X') \cap B_R(0).$$

This conclude the proof. □

Lemma 24 ([CD, Lem. 3]). *Let X_1 and X_2 be two Delone sets with finite local complexity. If $\pi : \Omega_{X_1} \to \Omega_{X_2}$ is a factor map and X_1 is linearly repetitive, then $(\Omega_{X_2}, \mathbb{R}^d)$ is linearly repetitive.*

Proof. Let $X \in \Omega_{X_1}$. Consider $0 < \varepsilon < 1$ and $s_0, R(\varepsilon) > 0$ the positive constants of Lemma 23 associated to ε. Since X is linearly repetitive with some constant L, for any $y \in \mathbb{R}^d$ there exists $v \in B_{L(R+s_0)}(y)$ such that $(X - v) \cap B_{R+s_0}(0) = X \cap B_{R+s_0}(0)$. From Lemma 23, there exists $t \in B_\varepsilon(0)$ such that $(\pi(X) - v - t) \cap B_R(0) = \pi(X) \cap B_R(0)$. This implies that any ball of radius $L(R + s_0) + 2\varepsilon$ in $\pi(X)$ contains a copy of $\pi(X) \cap B_R(0)$. Since $Ls_0 + 2\varepsilon$ is smaller than the constant $Ls_0 + 2$, it follows that $\pi(X)$ is linearly repetitive. □

Actually from the proofs of Lemmas 4 and 24 we can get a uniform bound on the linear repetitivity constant of the factor system.

Lemma 25. *Let X_1 and X_2 be two Delone sets with finite local complexity. If $\pi : \Omega_{X_1} \to \Omega_{X_2}$ is a factor map and X_1 is linearly repetitive with constant $L > 1$, then there exists $R_\pi > 0$ such that for every $R > R_\pi$ and every R-patch $\mathtt{P}$ of X_2, a copy of $\mathtt{P}$ appears in every ball of radius $3LR$ of X_2 and any two occurrences of $\mathtt{P}$ in X_2 are at distance at least $R/4L$.*

5.1. Finite number of aperiodic Delone systems as factors

The aim of this section is to prove the following theorem that is a generalization of a result in [Du1] in the context of subshifts.

Theorem 26 ([CDP, Thm. 12]). *Let $L > 1, d \geq 1$. There exists a constant $N(L, d)$ such that any linearly repetitive Delone set X of $\mathbb{R}^d$ with constant L, has at most $N(L, d)$ aperiodic Delone system factors of $(\Omega_X, \mathbb{R}^d)$ up to conjugacy.*

The bound $N(L, d)$ is essentially due to the constants arising in Lemmas 6 and 7. The proof relies on a generalization of these lemmas and on the specific structure of the factor maps for linearly repetitive Delone systems.

The next result says that factor maps between linearly repetitive Delone systems are finite-to-one. A proof of that result in the context of subshifts and Delone systems can be found in [Du1] and in [CDP, Prop. 5] respectively. Here we include the proof in the case where the factor map is a sliding-block-code.

Proposition 27. *Let X be a linearly repetitive Delone set with constant L. There exists a constant $C > 0$ (depending only on L) such that If X' is an aperiodic Delone set and $\pi : (\Omega_X, \mathbb{R}^d) \to (\Omega_{X'}, \mathbb{R}^d)$ is a factor map, then for every $Y \in \Omega_{X'}$, the fiber $\pi^{-1}(\{Y\})$ contains at most C elements.*

Proof. For simplicity we will assume that π is a sliding-block-code. That means there exists $s_0 > 0$ such that if X_1 and $X_2 \in \Omega_X$ verify $X_1 \cap B_{R+s_0}(0) = X_2 \cap B_{R+s_0}(0)$ for an $R > 0$, then $\pi(X_1) \cap B_R(0) = \pi(X_2) \cap B_R(0)$. From Lemma 24 the Delone set X' is linearly repetitive, and if R is sufficiently large, Lemma 25 implies that for any $x \in \mathbb{R}^d$ a copy of the patch $X' \cap B_R(x)$ appears in $X' \cap B_{3LR}(y)$, for every $y \in \mathbb{R}^d$. Let $Y \in \Omega_{X'}$ and $X_1, \ldots, X_n$ be different Delone sets in $\in \pi^{-1}(\{Y\})$. Because these Delone sets are different, for every sufficiently large R, the patches $X_i \cap B_R(0)$ are pairwise distinct. Linear repetitivity of X ensures the existence of points $v_1, \ldots, v_n \in B_{LR}(0)$ such that each $X - v_i \cap B_R(0)$ is a copy of $X_i \cap B_R(0)$, for every $1 \leq i \leq n$. This implies that $\pi(X) - v_i \cap B_{R-s_0}(0) = Y \cap B_{R-s_0}(0)$. From this and Lemma 25 we get that $\|v_i - v_j\| \geq \frac{R-s_0}{4L}$, from which we deduce that $n \leq C$, where C is a constant that depends only on L. $\square$

The following proposition is a straightforward generalization of Lemma 21 in [Du1]. A proof in our setting can be found in [CDP, Prop. 6]. Here we omit the proof.

Proposition 28. *Let $(\Omega, \mathbb{R}^d)$ be a minimal Delone system and $\phi_1 : (\Omega, \mathbb{R}^d) \to (\Omega_1, \mathbb{R}^d)$, $\phi_2 : (\Omega, \mathbb{R}^d) \to (\Omega_2, \mathbb{R}^d)$ be two factor maps. Suppose that $(\Omega_2, \mathbb{R}^d)$ is non-periodic and ϕ_1 is finite-to-one. If there exist $X, Y \in \Omega$ and $v \in \mathbb{R}^d$ such that $\phi_1(X) = \phi_1(Y)$ and $\phi_2(X) = \phi_2(Y - v)$, then $v = 0$.*

We have already defined the notion of return vector of a patch, now let us define the notion of return vector of a Voronoï cell of a patch. For a patch $\mathtt{P}$ of X and $v \in X_{\mathtt{P}}$, $V_{\mathtt{P},v}$ denotes the Voronoï cell of the point v of the Delone set $X_{\mathtt{P}}$. We say that $w \in \mathbb{R}^d$ is a *return vector of* $V_{\mathtt{P},v} \cap X$ if $(X - w) \cap V_{\mathtt{P},v} = X \cap V_{\mathtt{P},v}$. We set for $n \geq 1, v \in X_{\mathtt{P}}$,

$$\mathtt{P}_{n,w,v} \text{ the patch } (X - w - v) \cap B_{L^n R}(0).$$

Notice that $\mathtt{P}_{n,w,v} + v + w$ is a patch of X. When there is no confusion about n and v, we write $\mathtt{P}_w$ instead of $\mathtt{P}_{n,w,v}$.

The following lemma generalizes Lemma 6.

Lemma 29. *Let $n \in \mathbb{N}^*$ and X be an aperiodic linearly repetitive Delone set with constant L. There exists a constant $C(n, L) > 0$ such that for every sufficiently large $R > 0$ and every R-patch $\mathtt{P}$, the collection $\{P_{n,w,v} : w$ is a return vector of $V_{\mathtt{P},v} \cap X\}$ has at most $C(n, L)$ elements, for every $v \in X_{\mathtt{P}}$.*

Proof. Let $\mathtt{P} = X \cap B_R(x_{\mathtt{P}})$ and $v \in X_{\mathtt{P}}$. Lemma 4 implies that the Voronoï cell $V_{\mathtt{P},v}$ contains the ball $B_{\frac{R}{2(L+1)}}(v)$. Then for every pair of return vectors u and w of $V_{\mathtt{P},v}$, the patches $\mathtt{P}_u$ and $\mathtt{P}_w$ coincides at the ball $B_{\frac{R}{2(L+1)}}(0)$. The proof concludes using the fact that in $X \cap B_{2L(L^n R)}(0)$ there is at least one copy of each patch $\mathtt{P}_w$, $\mathtt{P}_u$ and applying Lemma 4 to the return vectors of the patch $\mathtt{P}_w \cap B_{\frac{R}{2(L+1)}}(0)$. □

Proof of Theorem 26. It is enough to suppose that X is an aperiodic linearly repetitive Delone set with constant $L > 1$. Let $n \in \mathbb{N}$ be such that

$$L^n - 1 - 12L - 64L^2 > 1. \tag{5.7}$$

We call $M(n, L)$ the number of coverings of a set with $c(L)c(n, L)$ elements, where $c(L)$ and $c(n, L)$ are the constants of Lemma 6 and Lemma 29 respectively. For every $1 \leq i \leq M(n, L) + 1$, let X_i be a non-periodic Delone set such that there exists a topological factor map $\pi_i : \Omega_X \to \Omega_{X_i}$, and let $X_0 = X$. We will show there exist $1 \leq i < j \leq M(n, L) + 1$ such that $(\Omega_{X_i}, \mathbb{R}^d)$ and $(\Omega_{X_j}, \mathbb{R}^d)$ are conjugate.

Since $M(n, L)$ is finite, we can take the same constant $s_0 > 0$ and R_π of Lemmas 23 and 25 respectively, associated to each π_i. Fix $0 < \varepsilon < 1$. Let $R > \sup\{s_0, R_\pi + \varepsilon, 17L\}$ be sufficiently large such that Lemma 6 and Lemma 29 are applicable to R-patches of X, and such that Lemma 23 is applicable to ε and each π_i.

Consider the patch $\mathtt{P} = B_R(0) \cap X$, and $v_1, \ldots, v_N \in X_{\mathtt{P}}$ such that for every $v \in X_{\mathtt{P}}$, there exist $1 \leq i \leq N$ and $u \in \mathbb{R}^d$ satisfying $V_{\mathtt{P},v} \cap X = (V_{\mathtt{P},v_i} \cap X) + u$. Roughly speaking, every set of the kind $V_{\mathtt{P},v} \cap X$ is a translated of some set $V_{\mathtt{P},v_i} \cap X$. Since $R > R_1$, Lemma 6 ensures $N \leq c(L)$.

For every $1 \leq j \leq N$, let $w_{j,1}, \ldots, w_{j,m_j}$ be return vectors of $V_{\mathtt{P},v_j} \cap X$, chosen in order that for every return vector w of $V_{\mathtt{P},v_j} \cap X$, there exists $1 \leq i \leq m_j$ such that $\mathtt{P}_{n,w,v_j}$ is equal to $\mathtt{P}_{n,w_{j,i},v_j} =: \mathtt{P}_{w_{j,i}}$. Since $R > R_1$, Lemma 29 implies that $m_j \leq c(n, L)$, for every $1 \leq j \leq N$. Therefore, the collection

$$\mathcal{F} = \{\mathtt{P}_{w_{j,l}} : 1 \leq l \leq m_j,\ 1 \leq j \leq N\}$$

contains at most $c(L)c(n, L)$ elements.

We define $R' = (L^n - 1)R - \varepsilon - 4LR$. The choice of n ensures that $R' > 0$.

For every $1 \le i \le M(n, L) + 1$, we define the following relation on $\mathcal{F}$:

$\mathtt{P}_{w_{j,l}} \leftrightarrow_i \mathtt{P}_{w_{k,m}}$ if and only if for every $X', X'' \in \Omega_X$ such that $X' \cap B_{L^n R}(0) = \mathtt{P}_{w_{j,l}}$ and $X'' \cap B_{L^n R}(0) = \mathtt{P}_{w_{k,m}}$, there exist $v \in B_{2\varepsilon}(0)$ and $w \in B_{4LR}(0)$ such that $\pi_i(X') \cap B_{R'}(0) = (\pi_i(X'') + v + w) \cap B_{R'}(0)$.

Since $L^n R - s_0 \ge (L^n - 1)R \ge R$, from Lemma 23 it follows this relation is reflexive, so non empty. Since the cardinal of $\mathcal{F}$ is bounded by $c(L)c(n, L)$, there are at most $M(n, L)$ different relations of this kind. So, there exist $1 \le i < j < M(n, L) + 1$ such that $\leftrightarrow_i = \leftrightarrow_j$.

In the sequel, we will prove that $(\Omega_{X_i}, \mathbb{R}^d)$ and $(\Omega_{X_j}, \mathbb{R}^d)$ are conjugate. For that, it is sufficient to show that if $Y, Z \in \Omega_X$ are such that $\pi_i(Y) = \pi_i(Z)$ then $\pi_j(Y) = \pi_j(Z)$.

Let Y and Z be two Delone sets in Ω_X such that $\pi_i(Y) = \pi_i(Z)$. Without loss of generality, we can suppose that 0 is an occurrence of $\mathtt{P}$ in Y and in $Z - u_0$, where u_0 is some point in $B_{4LR}(0)$. The patches of Y and Z are translated of the patches of X. This implies there exist $1 \le q_0, r_0 \le N$ such that

$$Y \cap B_{L^n R}(0) = \mathtt{P}_{w_{q_0,l_0}} \text{ and } (Z - u_0) \cap B_{L^n R}(0) = \mathtt{P}_{w_{r_0,k_0}},$$

for some $1 \le l_0 \le m_{q_0}$ and $1 \le k_0 \le m_{r_0}$.

It is possible to show that $\mathtt{P}_{w_{q_0,l_0}} \leftrightarrow_i \mathtt{P}_{w_{r_0,k_0}}$ and $\mathtt{P}_{w_{q_0,l_0}} \leftrightarrow_j \mathtt{P}_{w_{r_0,k_0}}$ for R sufficiently large (see Claim 1 in the proof of [CDP, Thm. 12]).

Let s be any other occurrence of $\mathtt{P}$ in Y. Repeating the same argument for $Y + s$ and $Z + s$, we deduce there exist $u_s \in B_{4LR}(0)$ and $1 \le q_s, r_s \le N$ such that

$$(Y + s) \cap B_{L^n R}(0) = \mathtt{P}_{w_{q_s,l_s}} \text{ and } (Z + s - u_s) \cap B_{L^n R}(0) = \mathtt{P}_{w_{r_s,k_s}},$$

for some $1 \le l_s \le m_{q_s}$ and $1 \le k_s \le m_{r_s}$. Then we get $\mathtt{P}_{w_{q_s,l_s}} \leftrightarrow_j \mathtt{P}_{w_{r_s,k_s}}$. This implies there exist $t_s \in B_{2\varepsilon}(0)$ and $w_s \in B_{4LR}(0)$ such that

$$\pi_j(Y + s) \cap B_{R'}(0) = (\pi_j(Z + s - u_s) + t_s + w_s) \cap B_{R'}(0).$$

Showing that $w_s - u_s + t_s$ does not depend on s (see Claim 2 in the proof of [CDP, Thm. 12]), we get there exists $y \in \mathbb{R}^d$ such that for every occurrence s of $\mathtt{P}$ in Y,

$$\pi_j(Y + s) \cap B_{R'}(0) = (\pi_j(Z + s) + y) \cap B_{R'}(0), \text{ and then}$$
$$\pi_j(Y) \cap B_{R'}(s) = (\pi_j(Z) + y) \cap B_{R'}(s).$$

The diameter of the Voronoï cells of $\mathtt{P}$ is less than $4LR$ (see 2.1), which is less than R'. Hence,

$$\pi_j(Y) = \pi_j(Z) + y.$$

We conclude with Propositions 27 and 28. □

5.2. Factors on groups and cocycles

Cocycles and cohomological equations play an important role in the study of factors dynamical systems, time change for flows orbit equivalence, ... We adapt this notion to the context of Delone system $(\Omega, \mathbb{R}^d)$. Let G denotes the group $\mathbb{R}^m$ or $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. A *continuous G-cocycle* is a continuous function $\alpha\colon \Omega \times \mathbb{R}^d \to G$ so that

$$\alpha(Y, v+w) = \alpha(Y, v) + \alpha(Y+v, w) \qquad \text{for all } Y \in \Omega, v, w \in \mathbb{R}^d.$$

An important question which appears in many problems, is to known if the *cohomological equation*

$$\alpha(Y, v) = \psi(Y+x) - \psi(Y)$$

has a measurable, continuous solution $\psi : \Omega \to G$. This solution is called a *transfer function* and if it exists, α is called a *coboundary.*

In Section 5.2.2 we will give a necessary and sufficient condition to find solutions to the cohomological equation for linearly repetitive Delone systems. We will focus on *transversally locally constant* cocycle α, *i.e.*: if there exists $r, R > 0$ such that for any $Y, Y' \in \Omega$ and $x \in B_R(0)$,

$$\text{if } Y \cap B_R(0) = Y' \cap B_R(0) \text{ then } \alpha(Y, x) = \alpha(Y', x).$$

More generally a cocycle α is *transversally Hölder* if there exist constants $K > 0$ and $\delta \in (0,1)$ such that for all $r > 0$, $Y, Y' \in \Omega$ and $x \in B_r(0)$,

$$\text{if } Y \cap B_R(0) = Y' \cap B_R(0) \text{ then } |\alpha(Y, x) - \alpha(Y', x)| \le K r^{-\delta}.$$

5.2.1. Examples of cohomological equations. Let us see first some dynamical problems where the cohomological equation appears.

Let us denote by $\langle .,. \rangle$ the usual inner product in $\mathbb{R}^d$ and μ be an ergodic $\mathbb{R}^d$ invariant probability measure on the hull Ω. A vector $\lambda \in \mathbb{R}^d$ is a *measurable eigenvalue* of the system $(\Omega, \mathbb{R}^d)$ if there exists a measurable function $\psi\colon \Omega \to \mathbb{S}^1$ such that

$$\psi(Y+v) = e^{2i\pi\langle \lambda, v\rangle}\psi(Y) \quad \text{for all } v \in \mathbb{R}^d \text{ and } \mu-a.e.\ Y \in \Omega.$$

If the function ψ is continuous, then λ is called a *continuous eigenvalue.* The map $(Y, v) \mapsto e^{2i\pi\langle \lambda, v\rangle}$ is a $\mathbb{S}^1$-cocycle over $(\Omega, \mathbb{R}^d)$. Then passing in additive notation $\mathbb{T}^1$, we have λ is a measurable (resp. continuous) eigenvalue of $(\Omega, \mathbb{R}^d)$ if and only if there is a measurable (resp. continuous) solution $\psi : \Omega \to \mathbb{T}^1$ to the cohomological equation

$$\langle \lambda, v\rangle = \psi(Y+v) - \psi(Y) \bmod \mathbb{Z}.$$

A continuous eigenvalue gives then a topological factor on the closure of an orbit in the one-dimensional torus $\mathbb{T}^1$. More generally, one can consider the closure $\mathbf{O}$ of an orbit of a n-rotations on the n-torus $\mathbb{T}^n$, $n \le d$, that are factors of the system $(\Omega, \mathbb{R}^d)$. More precisely, take $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ and let $\mathcal{A} : \mathbb{R}^d \times \mathbb{T}^n \to \mathbb{T}^n$ be the continuous action defined by

$$\mathcal{A}(v, x) = x + [v, \theta] \qquad \text{where } [v, \theta] = (v_1\theta_1, \ldots, v_n\theta_n).$$

The map $(Y, v) \mapsto [v, \theta]$ is a $\mathbb{T}^n$-cocycle over $(\Omega, \mathbb{R}^d)$. It is standard to show that the system $(\mathbf{O}, \mathcal{A})$ is a topological factor of $(\Omega, \mathbb{R}^d)$ if and only if there exists a continuous solution $\psi : \Omega \to \mathbb{T}^n$ to the cohomological equation

$$[v, \theta] = \psi(Y + v) - \psi(Y).$$

5.2.2. Characterization of continuous coboundary. A seminal work for the characterization of continuous eigenvalues of symbolic systems given by a primitive substitution, is in [H]. The authors of [CDHM, BDM1] generalize these results to the linearly recurrent symbolic systems and to finite rank systems in [BDM2]. An extension to $\mathbb{Z}^d$-action on a Cantor set is presented in [CGM]. We present here a part of the results in [C] that treat only continuous cocycles and generalizes the results of [CGM].

For a box decomposition $\mathcal{B} = \{C_i[D_i]\}_{i=1}^t$ (see Section 3.2.1), a *first retrun vector to* $C = \cup_i C_i$ is a vector $v \in \mathbb{R}^d$ with label $(i, j) \in \{1, \ldots, t\}^2$, such that

$$C_i - v \cap C_j \neq \emptyset \text{ and } C_i[D_i] \cap C_j[D_j] \neq \emptyset.$$

We denote by $\mathcal{F}$ the set of first return vectors to C associated with $\mathcal{B}$, and by $C(v) = C_i \cap (C_j + v)$ for a return vector v with label (i, j).

A tower system $(\mathcal{B}_n = \{C_{n,i}[D_{n,i}]\}_{i=1}^{t_n})_n$ is *well distributed* if it satisfies the properties i)–iii) in Theorem 13 and moreover for every $n \geq 0$, and every first return vector $v \in \mathcal{F}_n$ with label (i, j) there are x and x' in $D_{n+1,1}$ such that for $X \in \bigcap_n C_n$, $X - x \in C_{n,i}$ and $X - x' \in C_{n,j}$ and $v = x - x'$.

It is straightforward to check that this extra condition holds when each $D_{n+1,i}$ is big enough: more precisely when for any $n \geq 0$

$$r_{\text{int}}(\mathcal{B}_{n+1}) \geq (R_{\text{rec}}(\mathcal{B}_n) + R_{\text{ext}}(\mathcal{B}_n))\, L \geq M_X(R_{\text{rec}}(\mathcal{B}_n) + R_{\text{ext}}(\mathcal{B}_n)). \qquad (5.8)$$

For a linearly repetitive Delone set X, it is direct to check that for a constant K big enough, the tower system given by Theorem 13, satisfies inequality (5.8). Thus any linearly repetitive Delone system admits a well-distributed tower system. In the following $|\cdot|$ denotes the usual distance to the origin when $G = \mathbb{R}^m$ or $\mathbb{T}^m$.

Theorem 30 ([C]). *Let X be a linearly repetitive Delone set in $\mathbb{R}^d$, G be the group $\mathbb{R}^m$ or $\mathbb{T}^m$, α be a continuous G-cocycle over $(\Omega, \mathbb{R}^d)$, and $(\mathcal{B}_n)_{n \geq 0}$ be a well-distributed tower system. Then α is a tansversally Hölder coboundary with continuous transfer function if and only if the series*

$$\sum_{n \geq 0} \sup_{\substack{v \in \mathcal{F}_n \\ \omega \in C_n(v)}} |\alpha(\omega, v)|$$

converges, where each $\mathcal{F}_n$ denotes the set of first return vectors associated with $\mathcal{B}_n$.

In [C] appears also similar necessary conditions for a cocycle to be a coboundary on a general Delone system (without the assumption of linear repetitivity).

5.2.3. Characterization of the measurable eigenvalues. To be more complete on the problem of eigenvalues, let us mention that a characterization of measurable eigenvalues of linearly recurrent Cantor system is given in [BDM1] and measurable coboundary for linearly repetitive Delone systems in [C0].

Theorem 31 ([C0]). *Let $(\Omega, \mathbb{R}^d)$ be a linearly repetitive Delone system, μ be the unique invariant measure, G be the group $\mathbb{R}^m$ or $\mathbb{T}^m$, α be a transversally locally constant G-cocycle over $(\Omega, \mathbb{R}^d)$, and $(\mathcal{B}_n)_{n\geq 0}$ be a tower system well-distributed. Then the following are equivalent.*

1. *The series $\sum_{n\geq 0} \sup_{\substack{v\in\mathcal{F}_n \\ \omega\in C_n(v)}} |\alpha(\omega, v)|^2$ converges, where each $\mathcal{F}_n$ denotes the set of first return vectors associated with $\mathcal{B}_n$.*
2. *There exists a measurable function $\psi\colon \Omega \to G$ such that for μ-a-e $X \in \Omega$,*
$$\alpha(X, v) = \psi(X - v) - \psi(X), \qquad \text{for all } v \in \mathbb{R}^d.$$
Moreover $\psi \in L^2(\Omega, \mathbb{R}^m, \mu)$ when $G = \mathbb{R}^m$.

6. Bi-Lipschitz equivalence to a lattice

Let X_1 and X_2 be two Delone sets in $\mathbb{R}^d$. We say that they are *bi-Lipschitz equivalent* if there exists a homeomorphism $\phi : X_1 \to X_2$ and a constant $\Delta \geq 1$ such that $\forall x, x' \in X, x \neq x'$
$$\frac{1}{\Delta} \leq \frac{\|\phi(x) - \phi(x')\|}{\|x - x'\|} \leq \Delta.$$
The map ϕ is then called a *bi-Lipschitz homeomorphism* between X_1 and X_2.

The problem to know whether two Delone sets are bi-Lipschitz equivalent was raised by Gromov in [Gro93], and boiled down in Toledo's review [Tol] to the following question for the two-dimensional Euclidean space: *Is every Delone set in $\mathbb{R}^2$ bi-Lipschitz equivalent to $\mathbb{Z}^2$?* Counterexamples to this question were given independently by Burago and Kleiner [BK] and McMullen [McM]. Moreover, McMullen also showed that when relaxing the bi-Lipschitz condition to a Hölder one, all Delone set (with or without finite local complexity) in $\mathbb{R}^d$ are equivalent. Later, Burago and Kleiner [BK1] gave a sufficient condition for a Delone set to be bi-Lipschitz equivalent to $\mathbb{Z}^2$ and asked the following question: *If one forms a Delone set in the plane by placing a point in the center of each tile of a Penrose tiling, is the resulting set bi-Lipschitz equivalent to $\mathbb{Z}^2$?* They studied the more general question of knowing whether a Delone set arising from a cut-and-project tiling is bi-Lipschitz equivalent to $\mathbb{Z}^2$ (recall that the Penrose tiling is also a cut-and-project tiling [Bru]) and solved it in some cases that do not include the case of Penrose tilings, thus leaving the former question open. Recently, Solomon [Solo] gave a positive answer for Penrose tiling by using the fact that it can be constructed using substitutions. In fact, Solomon proved that each Delone set arising from a primitive self-similar tiling in $\mathbb{R}^2$ is bi-Lipschitz to $\mathbb{Z}^2$.
The following result was proved in [ACG1].

Theorem 32. *Every linearly repetitive Delone set in $\mathbb{R}^d$ is bi-Lipschitz equivalent to $\mathbb{Z}^d$.*

Notice that Theorem 32 is trivial when the dimension $d = 1$ since, in this case, every Delone set (with no extra assumptions) is bi-Lipschitz equivalent to $\mathbb{Z}$. As an application of the work of Laczkovich [L], Solomon in [Solo] showed also that for every self-similar tiling of $\mathbb{R}^d$ of Pisot type there is a *bounded displacement* between its associated Delone set X and $\beta\mathbb{Z}^d$ for a $\beta > 0$ (i.e., there is a bijection $\phi\colon X \to \beta\mathbb{Z}^d$ such that $\Phi - Id$ is bounded).

The strategy of the proof of Theorem 32 is the following. First consider the easy case where all the Voronoï cells V of a Delone set X have a unit volume. Thus any finite union of N Voronoï cells meet at least N unit squares, and conversely N unit squares meet at least N Voronoï cells. So by the transfinite form of Hall's marriage lemma, there exists a bijection between the collection of Vornoï cells and the units squares, so that any cell intersects its image. This defines a map $\phi\colon X \to \mathbb{Z}^d$ such that $\phi - Id$ is bounded.
For the general case, we need to consider the measurable function $f\colon \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(x) = \sum_{y : x \in V_y} \frac{1}{\operatorname{vol} V_y} \qquad x \in \mathbb{R}^d,$$

where V_y denotes the Voronoï cell of the point $y \in X$. If $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a bi-Lipschitz map so that its Jacobian determinant is f, standard calculus show us that the image $\phi(V)$ of any Voronoï cell V of X has volume 1. The proof of Theorem 32 consists then to generalize to all dimension d a sufficient condition given by Burago and Kleiner [BK1] in dimension 2 to solve the equation $\det \mathrm{D}\phi = f$ with ϕ an unknown bi-Lipschitz map. This condition involves analytical tools and the density deviation of the points of X with respect to its average. This last point is controlled by the Lagarias and Pleasants Theorem 16.

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ON THE SIMPLICITY OF HOMEOMORPHISM GROUPS OF A TILABLE LAMINATION

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Abstract. We show that the identity component of the group of homeomorphisms that preserve all leaves of a $\mathbb{R}^d$- tilable lamination is simple. Moreover, in the one dimensional case, we show that this group is uniformly perfect. We obtain a similar result for a dense subgroup of homeomorphisms.

1. Introduction

In this paper it is shown that the connected component of the identity of the group $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ of all leaf-preserving homeomorphisms of a minimal tilable lamination Ω in any dimension is a simple group. We also prove that this group is equal to the group of homeomorphisms that are isotopic to the identity and that is open in $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$.

Similar results were obtained in the 60's by G. Fisher [7] for the group of all homeomorphisms of a closed topological manifold of dimension smaller or equal than three. The algebraic simplicity for groups of homeomorphisms and diffeomorphisms of manifolds has been widely studied in the literature: In 1961, R. Anderson [2], generalizing the work of G. Fisher [7], showed the group of stable homeomorphisms of a manifold is simple. Later, D. Epstein [6] established sufficient conditions on a group of homeomorphisms, for the commutator subgroup to be simple. This means that a group satisfying Epstein's conditions is simple if and only if it is perfect (*i.e.* its commutators subgroup is the whole group).

It is also worth mentioning the works of M. Herman [9], W. Thurston [18] and J. Mather [12] who provided a nearly complete classification for the simplicity of diffeomorphism groups on manifolds.

Given a smooth foliation $\mathcal{F}$ over a manifold M, T. Rybicki [15] and T. Tsuboi [16] studied the simplicity and perfectness of the identity component of the group $G_{\mathcal{F}}(M)$ of all leaf preserving diffeomorphisms of $(M, \mathcal{F})$. Notice here that these groups do not satisfy Epstein's conditions.

On the other hand, tilable laminations have been recently introduced as a geometric model for the study of non-periodic tilings [3]. They also appear as suspensions of minimal Cantor $\mathbb{Z}^d$-actions, like minimal subshifts. In addition, they include some classical laminated spaces as the dyadic solenoid. These spaces are locally homeomorphic to the product of an open set in $\mathbb{R}^d$ and a Cantor set. In other words, these are laminated spaces with a Cantor transversal. They are also

1991 *Mathematics Subject Classification.* 57S05, 37C85 .

Key words and phrases. leaf preserving homeomorphisms, tilable lamination.

J. Aliste-Prieto acknowledges financial support from Fondecyt Iniciación 11121510 and Anillo DySyRf ACT-1103. S. Petite acknowledges financial support from the ANR SUBTILE 0879. This work is part of the program MathAmSud DYSTIL 12Math-02.

endowed with a natural $\mathbb{R}^d$-action, which we we call the translation flow. Like in the case of foliations of manifolds, groups of homeomorphisms on tilable laminations in general do not satisfy Epstein's conditions.

We denote by $D(\Omega)$ the group of *deformations*, that is, $D(\Omega)$ is the path connected component of the identity in $\mathrm{Homeo}(\Omega)$ endowed with the C^0-topology. We conjointly consider the following class of homeomorphisms, which arises naturally in the context of non-periodic tilings, namely the group of homeomorphisms preserving the vertical structure (see the precise definition in Section 2) denoted $\mathrm{Homeo}_{vsp}(\Omega)$. Roughly speaking, such homeomorphisms map any small Cantor transversal into a Cantor transversal. Notice that $\mathrm{Homeo}_{vsp}(\Omega)$ is a dense subgroup of $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$. Let $D_{vsp}(\Omega)$ denote the path connected component of the identity in $\mathrm{Homeo}_{vsp}(\Omega)$. Our aim is to show the simplicity of these groups.

Another motivation for studying these groups comes from topological orbit equivalence theory: two tilable laminations Ω_1, Ω_2 are orbit equivalent if there is a homeomorphism between the spaces mapping any orbit onto an orbit. Because of the totally disconnected transversal structure, Ω_1 and Ω_2 are orbit equivalent if and only if they are homeomorphic. A difficult result of Rubin [14] asserts that the group $\mathrm{Homeo}(\Omega)$ is a complete invariant of the orbit equivalence class of the lamination Ω: Any algebraic group isomorphism of these groups is induced by an homeomorphism on the topological laminations. It follows that the group $D(\Omega)$ is an invariant of flow equivalence.

For a topological group G, we denote by G^0 the connected component of the identity in G.

Theorem 1.1. *Let Ω be a minimal tilable lamination. Let G be either* $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ *or* $\mathrm{Homeo}_{vsp}(\Omega)$*. Then,*

(1) $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega) = D(\Omega)$ *and* $\mathrm{Homeo}^0_{vsp}(\Omega) = D_{vsp}(\Omega)$*;*
(2) G^0 *is open in G;*
(3) G^0 *is simple.*

Moreover, when the translation flow is expansive, e.g. for tiling spaces, the connected component of the identity in $\mathrm{Homeo}(\Omega)$ is the group of deformations.

Proposition 1.2. *Let Ω be a minimal tilable lamination. If the translation flow is expansive, then the identity component* $\mathrm{Homeo}^0(\Omega)$ *is equal to $D(\Omega)$ and is open in* $\mathrm{Homeo}(\Omega)$*.*

The proof of Theorem 1.1 follows the same strategy as in [7] for the triangulated manifolds (see [4] for a recent survey). In the next section we recall basic properties of tilable laminations and their homeomorphisms. By using a generalization of the Schoenflies Theorem due to R. Edwards and R. Kirby, we show, in Section 3, the groups under consideration satisfy the partition property (called also fragmentation property), and we prove Proposition 1.2 and the items (1), (2) of Theorem 1.1. We give in Section 4 a sufficient condition for a commutator subgroup of $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ to be simple. Next, we prove in Section 5 that the groups $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ and $\mathrm{Homeo}_{vsp}(\Omega)$ are perfect and we conclude the proof of Theorem 1.1 with the main result of Section 4. In the last section, we show, for the one-dimensional case, that these groups are *uniformly perfect*: more precisely, any element can be written as a product of two commutators in the group. This last result is similar to [8] for C^∞ leaf preserving diffeomorphisms of C^∞ foliations.

2. Preliminaries

2.1. **Background on tilable lamination.** We recall here some basic properties of tilable laminations and we refer to [3] for a more detailed exposition. Let Ω be a compact metric space. Assume that there exist a cover of Ω by open sets U_i and homeomorphisms called *charts* $h_i \colon U_i \to D_i \times C_i$ where C_i is a topological space and D_i is an open set of $\mathbb{R}^d$. These open sets and charts define an *atlas* of a *flat lamination* if the transition maps $h_j \circ h_i^{-1}$ read on their domains of definition

$$h_{i,j}(t,c) = (t + a_{i,j}, \gamma_{i,j}(c)), \tag{1}$$

where the $a_{i,j}$'s are elements of $\mathbb{R}^d$ and the maps $\gamma_{i,j}$ are continuous. Two atlases are *equivalent* if their union is also an atlas.

A flat lamination is the data of a compact metric space Ω together with an equivalence class of atlases $\mathcal{L}$. A *box* is the domain of a chart in the maximal atlas of $\mathcal{L}$. For any point x in a box B with coordinates (t_x, c_x) in the chart h, the set $h^{-1}(D \times \{c_x\})$ is called the *slice* and the set $h^{-1}(\{t_x\} \times C)$ is called the *vertical* of x in B. Since a transition map transforms slices into slices and verticals into verticals, these definitions make sense. As usual, a *leaf* of Ω is the smallest connected set that contains all the slices it intersects. From (1), it is clear that each leaf is a manifold with a flat Riemannian metric.

Definition 2.1. A *tilable lamination* $(\Omega, \mathcal{L})$ (or a $\mathbb{R}^d$-solenoid) is a flat lamination such that

- every leaf of $\mathcal{L}$ is isometric to $\mathbb{R}^d$.
- There exists a transversal Ξ (a compact subset of Ω such that for any leaf L of $\mathcal{L}$, $L \cap \Xi$ is non empty and a discrete subset with respect the manifold topology of the leaf L) which is a Cantor set.

For short, we will speak about Ω as a tilable lamination when there is no confusion. If every leaf is dense in Ω, we say that the lamination is *minimal*. By (1), the action by translations on $\mathbb{R}^d$ can be transported to a local action (also by translations) along the slices. In fact, these local translations induce a continuous and free $\mathbb{R}^d$-action T over Ω, see [3] for details. We refer to this action as the *translation flow* over Ω. To simplify the notations we write $\omega - t := T(t, \omega)$ for ω in Ω and t in $\mathbb{R}^d$. It is easy to see that the leaves of the lamination coincide with the orbits of the translation flow. Again by (1), the canonical orientation on $\mathbb{R}^d$ induce an orientation on each leaf of Ω. Given a box B that reads $h^{-1}(D \times C)$ in a chart h, by identifying a vertical in B with the Cantor set C, we can write $B = T(C, D) = C - D$, thus avoiding the explicit reference to the chart h.

Basic examples of minimal tilable laminations are given by the suspensions of minimal $\mathbb{Z}^d$ action on a Cantor set with locally constant ceiling functions. The tilable lamination structure also appears in the dynamical systems associated to non-periodic repetitive tilings and Delone sets of the Euclidean space, see [3]. In these examples, the translation flow is *expansive* in the following sense (see [13]).

Definition 2.2. Let $\eta > 0$. The translation flow of a tilable lamination Ω is said to be η-expansive if when one has points $x, y \in \Omega$ and a homeomorphism $h \colon \mathbb{R}^d \to \mathbb{R}^d$ satisfying $h(0) = 0$ and $d(x - t, y - h(t)) < \eta$ for all $t \in \mathbb{R}^d$, then there must exist $t_0 \in B_\eta(0)$ such that $x - t_0 = y$.

The translation flow is said *expansive*, if it is η-expansive for some constant η.

This last property will allow us to show, in the next section, that any homeomorphism that is close enough to the identity must by leaf-preserving.

A box in Ω is said to be *internal* if its closure is included in another box of Ω. In all the rest of the paper, any box will be supposed to internal. An internal box B is said to be *of ball type* if it can be written as $B = C - D$, where D is an open ball in $\mathbb{R}^d$. Instead, if D is a d-cube $(a_1, b_1) \times \cdots \times (a_d, b_d)$ in $\mathbb{R}^d$, then B is said to be a box of *cubic* type. In this case, and if f is a ℓ-face $(0 \leq \ell \leq d)$ of the cube D, then the set $C - f$ is said to be an *ℓ-vertical boundary* of B. A *box cover* of Ω is a cover $\{B_i\}_i$ of Ω, where each B_i is a box. Box covers of ball type and cubic type are defined in the same way.

Definition 2.3. A collection of boxes $\mathcal{B} = \{B_i\}_{i=0}^t$ in Ω is a *box decomposition*, if the following assertions hold:

(1) the B_i's are pairwise disjoint,
(2) the closures of the B_i's form a cover of Ω.

Also, if the boxes B_i are of cubic type, then $\mathcal{B}$ is a box decomposition of cubic type.

Box decompositions were introduced in [3] as a tool in the study of tilable laminations. The key lemma (see bellow) asserts that any box cover of cubic type can be turned into a box decomposition of cubic type. It follows that every tilable lamination admits a box decomposition of cubic type.

Lemma 2.4 ([3])**.** *Let Ω be a tilable lamination and $\{B_i\}_{i=0}^t$ be a box cover of cubic type of Ω. Then, there exists a box decomposition of cubic type $\mathcal{B}' = \{B'_i\}_{i=0}^n$, such that, for all i, whenever B'_i intersects B_j for some j, then it is included in B_j.*

The union of all the ℓ-vertical boundaries $(0 \leq \ell \leq d)$ of all the boxes of a box decomposition of cubic type $\mathcal{B}$ is called the *ℓ-skeleton* of $\mathcal{B}$.

2.2. **Homeomorphisms of tilable laminations.** Let Ω be a tilable lamination and denote by $\mathrm{Homeo}(\Omega)$ the set of homeomorphisms of Ω. We endow it with the C^0-topology, which is induced by the distance

$$\delta(f, g) = \sup_{x \in \Omega} d(f(x), g(x)) + \sup_{x \in \Omega} d(f^{-1}(x), g^{-1}(x)), \quad f, g \in \mathrm{Homeo}(\Omega).$$

The *support* of a homeomorphism f in $\mathrm{Homeo}(\Omega)$ is defined by

$$\operatorname{supp} f = \overline{\{x \in \Omega \mid f(x) \neq x\}}.$$

It is easy to see that $\operatorname{supp} f$ is f-invariant and $\operatorname{supp} \phi f \phi^{-1} = \phi(\operatorname{supp} f)$ for every $\phi \in \mathrm{Homeo}(\Omega)$.

Since the verticals of a tilable lamination Ω are totally disconnected, the path-connected components coincide with the leaves of the lamination. Thus, every element of $\mathrm{Homeo}(\Omega)$ maps each leaf onto a (possibly different) leaf. We define $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ be the group of all leaf-preserving homeomorphisms of Ω. Recall that a homeomorphism f of Ω is *homotopic to the identity*, if there exists a continuous map $F : [0, 1] \times \Omega \to \Omega$ such that $F(0, \cdot) = Id$ and $F(1, \cdot) = f$. If, in addition, $F(t, \cdot)$ is a homeomorphism of Ω for each $t \in \Omega$, then we say that f is *isotopic to the identity* or a *deformation*. The set $D(\Omega)$ denotes the group of all the deformations. Clearly, homeomorphisms that are homotopic to the identity belong to $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$. If Ω is minimal, then the converse is also true.

Theorem 2.5 ([1],[11]). *Let Ω be a minimal tilable lamination. Then every $f \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ is homotopic to the identity. In particular, for every $f \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$, there is a continuous map $\Phi_f \colon \Omega \to \mathbb{R}^d$, called the* displacement *of f, which is uniquely defined by the equation*

$$f(\omega) = \omega - \Phi_f(\omega) \quad \textit{for all } \omega \in \Omega.$$

We say the displacement of f is smaller than ε when $||\Phi||_\infty < \varepsilon$.

When the translation flow is expansive, we get the following refinement.

Proposition 2.6. *Let Ω be tilable lamination. Suppose its translation flow is η-expansive. Then,* $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ *is open in* $\mathrm{Homeo}(\Omega)$.

Proof. Define $B = \{f \in \mathrm{Homeo}(\Omega) \mid \delta(f, Id) < \eta\}$ and take any $f \in B$. Since the translation flow is free and homeomorphisms map leaves onto leaves, for every $\omega \in \Omega$ there is a continuous map $h : \mathbb{R}^d \to \mathbb{R}^d$ such that $f(\omega) - s = f(\omega - h(s))$ for all $s \in \mathbb{R}^d$.

Thus,

$$d(f(\omega) - s, \omega - h(s)) \leq \delta(f, Id) < \eta \quad \text{for all } s \in \mathbb{R}^d.$$

It follows from the expansivity of Ω that there exists a $t_0 \in \mathbb{R}^d$ such that $f(\omega) = \omega - t_0$. Since ω was arbitrary, this means that f preserves each leaf and thus $f \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$, which means that $B \subset \mathrm{Homeo}_{\mathcal{L}}(\Omega)$. The fact that $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ is open now follows from a standard argument. $\square$

Corollary 2.7. *If the translation flow on the tilable lamination Ω is expansive, then* $\mathrm{Homeo}^0(\Omega)$ *is open and*

$$\mathrm{Homeo}^0(\Omega) = \mathrm{Homeo}^0_{\mathcal{L}}(\Omega).$$

Proof. Since the connected component is the greatest connected set containing the identity, we have $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega) \subset \mathrm{Homeo}^0(\Omega)$. By Proposition 2.6 and the connexity property, we get $\mathrm{Homeo}^0(\Omega) \subset \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ and so $\mathrm{Homeo}^0(\Omega) \subset \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$, which concludes the proof. $\square$

In the context of laminations arising from the study of non-periodic tilings, an important class of homeomorphisms is given by homeomorphisms with the following property.

Definition 2.8. A homeomorphism $f \in \mathrm{Homeo}(\Omega)$ preserves the vertical structure if, given a point x in a vertical C of a box B and a vertical C' of a box B' containing $f(x)$, then there is a clopen subset $\tilde{C} \subset C$ containing x such that for every $y \in \tilde{C}$, $f(y) \in C'$.

Alternatively, provided that Ω is minimal, a map $f \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ preserves the vertical structure if and only if its deplacement Φ is transversally locally constant. In the context of non-periodic repetitive tilings, this notion corresponds to the notion of *strong pattern-equivariance* (see [10]) of the map $t \mapsto \Phi_f(\omega - t)$ for any fixed $\omega \in \Omega$. We denote by $\mathrm{Homeo}_{vsp}(\Omega)$ the collection of homeomorphisms preserving the vertical structure. It is plain to check that $\mathrm{Homeo}_{vsp}(\Omega)$ is dense in $\mathrm{Homeo}(\Omega)$. We will denote by $D_{vsp}(\Omega)$ the path-connected component of the identity in $\mathrm{Homeo}_{vsp}(\Omega)$.

3. Partition property

Definition 3.1. A group G of homeomorphisms of Ω satisfies the *partition property* if for every box cover $\{B_i\}_{i=0}^t$ of Ω, and for any $f \in G$, there exists a decomposition $f = g_1 \cdots g_\ell$ where $g_i \in G$ and $\operatorname{supp} g_i \subset B_{j(i)}$ for $i = 1, \ldots, \ell$.

In this section, following [7] and using the box decomposition structure of tilable laminations we show:

Proposition 3.2. *Let Ω be a minimal tilable lamination. The two groups $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ and $\mathrm{Homeo}^0_{vsp}(\Omega)$ satisfy the partition property.*

We will also show assertions (1) and (2) of Theorem 1.1. To prove this result, we will use several lemmas. We start by showing that every map having its support included in a box of ball type is a deformation.

Lemma 3.3. *Let Ω be a minimal tilable lamination and B be a box of ball type. Any map $g \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ (resp. in $\mathrm{Homeo}_{vsp}(\Omega)$), with support in the interior of B is a deformation of the identity (resp. $g \in D_{vsp}(\Omega)$).*

Proof. The proof is classical by using the Alexander's trick. We can assume that the closure $\overline{B}$ of the box reads $h^{-1}(D \times C)$ in a chart h, with D a closed ball in $\mathbb{R}^d$ of radius $r > 0$ centered at the origin. Since the support of the map g is in B, the map g preserves any slice of the box B. So, for any $c \in C$, let $g_c \colon D \to D$ be the map defined by $g(h^{-1}(t, c)) = h^{-1}(g_c(t), c)$ for $t \in D$. Now, for any $t \in [0, 1]$, let $F_t \colon D \times C \to D \times C$ be the map

$$F_t(x, c) = \begin{cases} ((1-t)g_c(\frac{x}{1-t}), c) & \text{if } |x| < r(1-t) \\ (x, c) & \text{if } |x| \geq r(1-t). \end{cases}$$

It is plain to check the map $h^{-1}F_t h$ gives an isotopy between the identity and the map g.

In the case where $g \in \mathrm{Homeo}_{vsp}(\Omega)$, up to subdivide the clopen set C, we can assume that the map g_c is independent of c. Hence the isotopy is also in $\mathrm{Homeo}_{vsp}(\Omega)$. □

Given two subsets $A \subset B$ of a topological space X, an *embedding* $f \colon A \to B$ is a continuous and injective map. This embedding is *proper* if $f^{-1}(\partial B) = A \cap \partial B$. The next theorem says that any proper embedding of a neighborhood of a compact set K into a ball, sufficiently close to the identity, can be isotoped to an embedding which is the identity on K. Moreover the isotopy depends continuously of the embedding. This theorem, true in any dimension, generalizes a version of the *Schoenflies Theorem.*

Theorem 3.4. [5] *Let D be a (closed or open) ball in $\mathbb{R}^d$, $K \subset D$ a compact subset and U a neighborhood of K in D. Then, for any proper embedding $f \colon U \to D$ close enough of the identity (for the C^0 topology), there exists a continuous map $H \colon U \times [0, 1] \to D$ such that:*

- *For any $t \in [0, 1]$, $H(\cdot, t) \colon U \to D$ is a proper embedding.*
- *$H(\cdot, 0) = f(\cdot)$ and $H(\cdot, 1)_{|K} = Id_{|K}$.*
- *There is a compact neighborhood K_2 of K in U, such that for any $t \in [0, 1]$, $H(\cdot, t)_{|U \setminus K_2} = f(\cdot)_{|U \setminus K_2}$.*
- *H depends continuously on f for the C^0 topology.*

Applied in our context, a first consequence is that any map close to the identity can be interpolated by a map with a support in a box.

Lemma 3.5. *Let Ω be a tilable lamination, let B be a box of ball type and B' be a box with closure included in B. Then there exists an $\varepsilon > 0$ such that for any homeomorphism $f \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ (resp. $\mathrm{Homeo}_{vsp}(\Omega)$) with a displacement smaller than ε, there exists a map $g \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ (resp. $\mathrm{Homeo}_{vsp}(\Omega)$) with $\operatorname{supp} g \subset B$ such that $g_{|B'} = f_{|B'}$. Moreover, the displacement of g depends continuously on f for the C^0-topology.*

Proof. Without loss of generality, we may assume that B reads $h^{-1}(D_3 \times C)$ in a chart h with D_3 a ball in $\mathbb{R}^d$. Assume that $D_1 \times C$ is a compact neighborhood of $h(B')$ with $D_1 \subset D_3$ a compact subset and let $D_2 \subset D_3$ be a neighborhood of D_1.

We consider a map $f \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ with a displacement smaller than ε (defined later). By continuity, for a small enough ε, the set $f(h^{-1}(D_2 \times C))$ is in B, and for any $c \in C$, $f(h^{-1}(D_2 \times \{c\})) \subset h^{-1}(D_3 \times \{c\})$. For any $c \in C$, let $f_c \colon D_2 \to D_3$ be the embedding defined by $f_c(\cdot) = (hfh^{-1})(\cdot, c)$ and let K be a compact neighborhood of ∂D_2 proper in $U = D_2 \setminus D_1$. For an ε small enough and for any $c \in C$, Theorem 3.4 applied to the maps $f_c \colon U \to D_3$, gives us embeddings $h_c = H_c(\cdot, 1) \colon U \to D_3$. We define then the maps $\bar{f}_c \colon D_3 \to D_3$ by $\bar{f}_{c|U} := h_c$ and $\bar{f}_{c|D_1} := f_{c|D_1}$ and $\bar{f}_{c|D_3 \setminus D_2} := Id_{|D_3 \setminus D_2}$ Let $\bar{f} \colon B \to B$ be the map defined by $\bar{f} \circ h^{-1}(t, c) = h \circ (\bar{f}_c(t), c)$ for any $(t, c) \in D_3 \times C$. By construction, $\bar{f}$ is a homeomorphism, $\bar{f}_{|h^{-1}(D_1 \times C)} = f_{|h^{-1}(D_1 \times C)}$ and $\bar{f}_{|\partial B} = Id_{|\partial B}$. So $\bar{f}$ can be extended by the identity to all the tilable lamination Ω to define a homeomorphism. This gives the map g.

Here again, when $f \in \mathrm{Homeo}_{vsp}(\Omega)$, up to subdividing the clopen set C, we can assume that the map f_c is independent of c. So the same is true for $\bar{f}$ and it belongs to $\mathrm{Homeo}_{vsp}(\Omega)$. $\square$

Proposition 3.6. *Let Ω be a minimal tilable lamination and $\mathcal{B} = \{B_i\}_{i=1}^k$ be a box cover of Ω. Then, there are $\varepsilon > 0$ and an integer $\ell > 0$ such that for every $f \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ (resp. in $\mathrm{Homeo}_{vsp}(\Omega)$) with displacement smaller than ε, there exists a decomposition $f = g_1 \cdots g_\ell$ with $g_i \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ (resp. $\mathrm{Homeo}_{vsp}(\Omega)$) and $\operatorname{supp} g_i \subset B_{j(i)}$.*

Proof. Let $\mathcal{B}' = \{B'_0, B'_1, \ldots, B'_m\}$ be the box decomposition of cubic type given by Lemma 2.4. Up to subdividing each box B'_i into smaller boxes, we can assume that the closure of every box B'_i is included in a box $B_{j(i)}$. We will construct, by induction on $0 \leq i \leq d$, a homeomoprhism f_i which equals the identity on a neighborhood of the i-skeleton of $\mathcal{B}'$ and equals f outside. At each step, we use Lemma 3.5 to approximate f_{i-1} by maps with support in a small box.

For any 0-vertical boundary V of a box B'_i, let $B_V^{(0)}$ be a box containing V in its interior and included in a box $B_{j(i)}$. Let $B_1^{(0)}, \ldots, B_n^{(0)}$ be the collection of these boxes containing all the 0-vertical boundaries. Up to refine the boxes $B_i^{(0)}$, we can assume that they are pairwise disjoint. The union of all theses boxes $B_i^{(0)}$ covers the 0-skeleton of $\mathcal{B}'$.

Step 0. Applying Lemma 3.5 to any box $B_i^{(0)}$ and any neighborhood of the 0-vertical $V \cap B_i^{(0)}$ (when not empty), we get, for an ε small enough, a $g_i \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ with $\operatorname{supp} g_i \subset B_i^{(0)}$ such that $g_i = f$ on a neighborhood of $V \cap B_i^{(0)}$. It follows that the maps $g_1, \dots, g_n$ commute; and $f_0 = g_1^{-1} \circ \cdots \circ g_n^{-1} \circ f$ is the identity in a neighborhood U_0 of the 0-skeleton of $\mathcal{B}'$. Moreover the displacement of f_0 continuously depends on the displacement of f.

Step i. $1 \leq i \leq d-1$. Let us assume that $f_i \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ equals the identity on a neighborhood U_{i-1} of the $i-1$-skeleton of $\mathcal{B}'$. We do the same as for the former step. Let $B_1^{(i)}, \dots, B_{n_i}^{(i)}$ be a collection of boxes such that any i-vertical boundary V of $\mathcal{B}'$ is in the interior of a box $B_j^{(i)} \subset B_{t(j)}$. Up to refine the boxes $B_j^{(i)}$, we may assume that the sets $B_j^{(i)} \setminus U_{i-1}$ are pairwise disjoint. Applying Lemma 3.5 to any box $B_j^{(i)}$ and to a neighborhood of the i-vertical $(V \setminus U_{i-1}) \cap B_j^{(i)}$ (when not empty), we have, for an ε small enough, a $g_j^{(i)} \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ with $\mathrm{supp}\, g_j^{(i)} \subset B_j^{(i)}$ such that $g_j^{(i)} = f$ on a neighborhood of $(V \setminus U_{i-1}) \cap B_j^{(i)}$. We get that the maps $g_1^{(i)}, \dots, g_{n_i}^{(i)}$ commute; and $f_i = (g_1^{(i)})^{-1} \circ \cdots \circ (g_{n_i}^{(i)})^{-1} \circ f_{i-1}$ is the identity in a neighborhood of the i-skeleton.

Hence the homeomorphism f_{d-1} preserves each box B_i', and f_{d-1} can be written as the composition of homeomorphisms with support in each box of the decomposition $\mathcal{B}'$. Moreover if $f \in \mathrm{Homeo}_{vsp}(\Omega)$, then $f_{d-1} \in \mathrm{Homeo}_{vsp}(\Omega)$ also. This proves the proposition. □

The following proposition shows the assertions (1) and (2) of Theorem 1.1.

Proposition 3.7. *Let Ω be a minimal tilable lamination. Then, $D(\Omega)$ is open in $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ and $D(\Omega) = \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$. Similarly, $D_{vsp}(\Omega)$ is open in $\mathrm{Homeo}_{vsp}(\Omega)$ and $D_{vsp}(\Omega) = \mathrm{Homeo}^0_{vsp}(\Omega)$.*

Proof. Let $\mathcal{B} = \{B_i\}_{i=1}^k$ be a box cover of ball type of Ω. Let ρ be the Lebesgue number of $\mathcal{B}$ and consider an atlas $\mathcal{B}' = \{B_j'\}_{j=1}^{\ell}$ of Ω such that the diameter of any box B_j' is smaller than ρ. Then by Proposition 3.6, any map $f \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ with displacement small enough, can be written as a product of maps $g_i \in \mathrm{Homeo}_{\mathcal{L}}(\Omega)$ with support in a $B_{j(i)}'$. Thus by Lemma 3.3 we get that any g_i is in $D(\Omega)$, and finally $f \in D(\Omega)$. This means that the identity lies in the interior of $D(\Omega)$. Standard arguments on topological groups show then that $D(\Omega)$ is open and closed in $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ and $D(\Omega) = \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$. The proof is similar for the group $\mathrm{Homeo}_{vsp}(\Omega)$. □

Finally, we can obtain the proof of Proposition 3.2.

of Proposition 3.2. Let H be either $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ or $\mathrm{Homeo}^0_{vsp}(\Omega)$ and let $\mathcal{B}$ be an atlas of Ω. Proposition 3.7 gives us a constant $\eta > 0$ such that any element of $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ (resp. in $\mathrm{Homeo}_{vsp}(\Omega)$) with a displacement smaller than η is in $D(\Omega)$ (resp. $D_{vsp}(\Omega)$). Up to refining the covering $\mathcal{B}$, we can assume that every element of $\mathcal{B}$ has a diameter smaller than η. Since the group H is connected, it is enough to show the partition property for any $f \in H$ with an arbitrary small displacement ε. By taking the ε given by Proposition 3.6, we can write f as a product of elements g_i in $\mathrm{Homeo}_{\mathcal{L}}(\Omega)$ with displacement smaller than η. By Proposition 3.7, we get $g_i \in H$ and this shows the partition property. □

4. Simplicity of the commutator subgroup

Epstein's result [6] asserts that if a group of homeomorphism is factorizable and acts transitively on open sets, then its commutator subgroup is simple. Let Ω be a minimal tilable lamination and let G be a subgoup of $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$. In general, we cannot expect G to act transively on open sets. We need to replace the transivity condition with another one which is more adapted to laminated spaces. Thus, we give here, a sufficient condition on a subgroup of $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ so that the derived subgroup is simple.

Theorem 4.1. *Let Ω be a minimal $\mathbb{R}^d$ tilable lamination. Let $G \subset \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ be a group such that:*

i) *G satisfies the partition property.*
ii) *For any boxes $B_1 = C - D_1, B_2 = C - D_2$ of ball type whose closures lie in a box $B = C - K$, there exists a $g \in G'$ such that $B_2 \subset g(B_1)$.*

Then the derived group $G' = [G, G]$ is simple.

Proof. Let N be a non-trivial normal subgroup of G'. We have to show that $N = G'$.

Lemma 4.2. *There exists an atlas $\mathcal{B}$ of the solenoid Ω such that for every box B of $\mathcal{B}$ there is a map $n_B \in N$ such that B and $n_B(B)$ are disjoint.*

Proof. Let $Id \neq n \in N$. There is a box $B_0 = C_0 - D_0$ of ball type such that B_0 and $n(B_0)$ are disjoint. Since the translation flow is free and minimal and translations have the vertical structure preserving property, it follows that for every $x \in \Omega$, there is a clopen subset $C_x \subset C_0$ and an open ball $D_x \subset \mathbb{R}^d$ with $D_0 \subset D_x$ such that $C_x - D_x$ is a box of Ω containing the point x.

Let $\tilde{B}_x$ be a box included in $C_x - D_x$ containing the point x. By hypothesis ii), there is a $g \in G'$ such that $\tilde{B}_x \subset g(C_x - D_0)$. It is then straightforward to check that the box $\tilde{B}_x$ is disjoint from its image by the map $g \circ n \circ g^{-1} \in N$. The collection of boxes $\{\tilde{B}_x\}_{x \in \Omega}$ satisfies the condition of the statement. □

Let $\mathcal{B}$ be the finite cover given by Lemma 4.2 of the tilable lamination Ω by boxes and let $\rho > 0$ be its Lebesgue number. Let us recall that for any ball of radius ρ in Ω there exists a box of $\mathcal{B}$ containing this ball. Let $\mathcal{B}_1$ be a box cover of cubic type of Ω, equivalent to $\mathcal{B}$, and such that any box has a diameter smaller than ρ. It follows that when two boxes B_1, B_2 of $\mathcal{B}_1$ are intersecting, there exists a box B of $\mathcal{B}$ containing $B_1 \cup B_2$.

The following is an algebraic lemma due to T. Tsuboi [17].

Lemma 4.3 ([17, Lemma 3.1])**.** *Let B be a box and n be an homeomorphism such that $n(B) \cap B = \emptyset$. Then for any homeomorphisms $a, b \in G$ with supports in B, the commutator $[a, b]$ can be written as a product of 4 conjugates of n and n^{-1}.*

Proof. Let $h = n^{-1}an$, since the supports are disjoint, we have $hb = bh$. So we get

$$\begin{aligned} aba^{-1}b^{-1} &= nhn^{-1}bnh^{-1}n^{-1}b^{-1} \\ &= nhn^{-1}h^{-1}hbnh^{-1}b^{-1}bn^{-1}b^{-1} \\ &= n(hn^{-1}h^{-1})(bhnh^{-1}b^{-1})(bn^{-1}b^{-1}). \end{aligned}$$

□

Now, for each $B \in \mathcal{B}_1$, let G_B be the subgroup of G of homeomorphisms with support in B, and let H be the subgroup of G generated by all the G_B with $B \in \mathcal{B}_1$. By the partition property (item i)), the groups H and G are the same. It is well-known that the commutator subgroup of H is generated by the conjugates of commutators of elements in a generating set of H. So to prove the theorem, we just have to show for any boxes B_1, B_2 in $\mathcal{B}_1$, and for $f_1 \in G_{B_1}$ and $f_2 \in G_{B_2}$, that the commutator $[f_1, f_2]$ belongs to N.

If the boxes B_1 and B_2 do not intersect, then every point of Ω is fixed by either f_1 or f_2, which means that $[f_1, f_2] = id$ and thus belong to N.

Suppose now that B_1 and B_2 intersect and let B be a box in $\mathcal{B}$ containing $B_1 \cup B_2$. By Lemma 4.2, there exists a $n \in N$ such that $B \cap n(B) = \emptyset$. Thus by Lemma 4.3, we have $[f_1, f_2] \in N$, and then $H' = N$. □

5. Perfectness

To show that the groups $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ and $\mathrm{Homeo}^0_s(\Omega)$ are simple, we will first prove in this section they are perfect. For this, we need the next lemma stating a transitivity of the action of the group $D_{vsp}(\Omega)$ on specific boxes of a same box. This is a reinforcement of condition ii) in Theorem 4.1. We will deduce then the perfectness. This will imply, together with the partition property, that these groups satisfy the conditions of Theorem 4.1, and henceforth they are simple.

Lemma 5.1. *Let $B_1 = C - D_1$ and $B_2 = C - D_2$ be two boxes of ball type, and let $B_0 = C - V$ be a box containing the closures of B_1 and B_2. Then there exists a $g \in D_{vsp}(\Omega)$ such that $B_2 = g(B_1)$.*

Proof. Let h be the chart associated to the box $C - V$. The boxes B_1 and B_2 read respectively $h^{-1}(D_1 \times C)$ and $h^{-1}(D_2 \times C)$, with D_1, D_2 two balls in V. Up to composing with a translation T_ρ, we may assume that $D_1 \subset D_2 \subset V$ or $D_2 \subset D_1 \subset V$. In both cases, it is straightforward to construct a homeomorphism $\psi \in D_{vsp}(\Omega)$ with support in $C - V$ such that $\psi(h^{-1}(D_1) \times C)) = h^{-1}(D_2 \times C)$. □

The proof of the next theorem follows directly the ideas of Frédéric Le Roux (see Theorem 1.1.3 in [4] for a proof on a surface). On a manifold, this shows directly that the group of homeomorphism is simple. Here, because of the lack of homogeneity, it enables just to show the groups $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ and $\mathrm{Homeo}^0_{vsp}(\Omega)$ are perfect.

Theorem 5.2. *For Ω a minimal $\mathbb{R}^d$-tilable lamination, the groups $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ and $\mathrm{Homeo}^0_{vsp}(\Omega)$ are perfect.*

Proof. Let H denotes either $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ or $\mathrm{Homeo}^0_{vsp}(\Omega)$. We have to show that any element of H can be written as a commutators product. It is simple to find two non commuting elements $a, b \in D_{vsp}(\Omega)$ with supports in a box $B \subset \Omega$ of ball type. So the element $g = [a, b] \in H$ is not the identity. We will show that $N(g)$, the normal subgroup generated by g, contains all the elements f of H with support in the box B. Recall that a conjugate of a commutator is still a commutator, it will follow that f can be written as a finite product of commutators. Since the box B is arbitrary, we get the conclusion by the partition property (Proposition 3.2).

We have $g \neq Id$, so we consider a box $B' \subset B$ such that $g(B')$ and B' are disjoint. In a chart h, we may assume that B' reads $h^{-1}(B_r(0) \times C)$ where $B_r(0)$ denotes the Euclidean ball in $\mathbb{R}^d$ of radius $r > 0$ centered at 0. For any integer $n \geq 0$, we define a nested sequence of boxes $B_n := h^{-1}(B_{r/2^{n+1}}(0) \times C)$. It is simple to construct an element ψ of $D_{vsp}(\Omega)$ with support in B', such that $\psi(B_n) = B_{n+1}$ of $n \geq 0$. We get then that the homeomorphism $k = [\psi, g] \in N(g)$ satisfies $k(B_n) = B_{n+1}$ for $n \geq 0$ and $\operatorname{supp} k \subset B' \cup g(B')$ (k is the product of ψ and $g\psi^{-1}g^{-1}$that have disjoint supports).

Let $A_n = B_n \setminus B_{n+1}$ and let us show that $N(g)$ contains all the element of H with support in A_1. For any $\phi \in H$ with a support in A_1, we claim that ϕk and k are conjugate: notice, we have $k = \phi^{-1}(\phi k)\phi$ on A_0, $\bigsqcup_{n\geq 0} A_n = \bigsqcup_{n\geq 0} g^n(A_0) = \bigsqcup_{n\geq 0} (\phi k)^n(A_0)$ and $k_{|B\setminus B_0} = \phi k_{|B\setminus B_0}$; It is then standard to check that the continuous homeomorphism $\tilde{\phi}$ defined by

$$\tilde{\phi}_{|A_n} := (\phi k)^n \phi k^{-n}_{|A_n} \text{ for any } n \geq 0 \qquad \text{and} \qquad \tilde{\phi}_{|\Omega\setminus B_0} = Id,$$

can be extended by continuity to $\overline{\bigcup_{n\geq 0} A_n} = B_0$, is in H and satisfies $k = \tilde{\phi}^{-1}(\phi k)\tilde{\phi}$ on Ω. We get then $\phi k \in N(k) \subset N(g)$, so $\phi \in N(g)$.

Finally, let $f \in H$ with a support in B. By Lemma 5.1, there exists a $\phi \in H$ such that the support of $\phi f \phi^{-1}$ is in A_1. So by the last result we have $f \in N(g)$. $\square$

We have then the groups $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ and $\mathrm{Homeo}^0_{vsp}(\Omega)$ equal their commutator groups. So by Lemma 5.1 and Theorem 4.1, we get the main result: Theorem 1.1.

6. Uniform perfectness in dimension one

Theorem 5.2 asserts that any homeomorphism of a tilable lamination Ω is a product of commutators. For the one dimension, we can be more precise.

Theorem 6.1. *For Ω a minimal $\mathbb{R}$-tilable lamination, any element of $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ (resp. $D_{vsp}(\Omega)$) can be written as a product of two commutators of $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ (resp. $D_{vsp}(\Omega)$).*

Before proving this theorem, we need some technical lemmas. The first one solves the problem of perfectness for homeomorphisms with support in a box. Recall that any map $f \in \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ preserves the orientation, so, if Φ denotes its displacement, for any $\omega \in \Omega$, the map $\mathbb{R} \ni t \mapsto t + \Phi(\omega - t) \in \mathbb{R}$ is increasing.

Lemma 6.2. *Let Ω be a minimal $\mathbb{R}$-tilable lamination and let $f \in \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ (resp. $\mathrm{Homeo}^0_{vsp}(\Omega)$) with support included in a box B. Then there exists a homeomorphism $g \in \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ (resp. $\mathrm{Homeo}^0_{vsp}(\Omega)$) with support in B such that $gfg^{-1} = f^2$. In particular $f = [g, f]$.*

Proof. Since f preserves the orientation, it preserves each slice of the box $B = C - I$, with C a clopen set and I an interval. For any $x \in C$, we denote by $f_x \colon I \to I$ the increasing map induced on the slice of x: i.e. defined by $f_x(t) = t + \Phi(x - t)$ so that $f(x - t) = x - f_x(t)$ for any $t \in I$.

For any $z_0 \in \{z \in B; f \neq Id\}$, let the vertical C_{z_0} be $C - t_0$ where z_0 writes $x_0 - t_0$ with $x_0 \in C$, $t_0 \in I$. We define the local strip

$$V_{z_0} := \{x - t;\ x \in C, t_0 \leq t < f_x(t_0)\}.$$

By the definition, the sets $\{f^n(V_{z_0}) = V_{f^n(z_0)}\}_{n\in\mathbb{Z}}$ are pairwise disjoints, and $\cup_{n\in\mathbb{Z}} f^n(V_{z_0})$ is a f-invariant open set. Hence there exist a collection at most countable of points $\{z_n\}_{n\geq 0} \subset \{f \neq Id\}$ and local strips $V_n = V_{z_n}$ such that

$$\mathrm{supp}\, f = \cup_{n\geq 0}\overline{\cup_{p\in\mathbb{Z}} f^p(V_n)} \text{ and the sets } \{\cup_{p\in\mathbb{Z}} f^p(V_n)\}_{n\geq 0} \text{ are pairwise disjoint.}$$

Notice that since f preserves the orientation, a point is fixed by f if and only it is a fixed point of f^2. So we have $\mathrm{supp}\, f = \mathrm{supp}\, f^2$.

For each $n \geq 0$, $z_n = x_n - t_n$ with $t_n \in I$, $x_n \in C$, let $h_n \colon [t_n, f_{z_n}(t_n)) \to [t_n, f^2_{z_n}(t_n))$ be the bijective affine map fixing t_n. It is then straightforward to check that the continuous map g_n defined on $\cup_{p\in\mathbb{Z}} f^p(V_n)$ by

$$g_{n|f^p(V_n)} := f^{2p} \circ h_n \circ f^{-p},$$

can be continuously extended by the identity to $\partial \cup_{p\in\mathbb{Z}} f^p(V_n)$ and satisfies $f^2 \circ g_n = g_n \circ f$ where it is defined. Hence we can define a homeomorphism g on Ω with support in B such that $g_{|\cup_{p\in\mathbb{Z}} f^p(V_n)} = g_n$ for every $n \geq 0$. Notice furthermore that g is in $\mathrm{Homeo}^0_{vsp}(\Omega)$ when f is. $\square$

The next lemma, is a version of lemma 3.5 without the condition to be close of the identity.

Lemma 6.3. *Let Ω be a minimal $\mathbb{R}$-tilable lamination and let $f \in \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ (resp. $\mathrm{Homeo}^0_{vsp}(\Omega)$). Suppose that $B' = C - J$ and $B = C - I$ are boxes of cubic type such that the closure of $f(B') \cup B'$ is contained in B. Then, there exists a g in $D(\Omega)$ (resp. $D_{vsp}(\Omega)$) with support contained in B such that $f|_{B'} = g|_{B'}$.*

Proof. Without loss of generality, we may assume that I, J are two open intervals such that $0 \in \overline{J} \subset I$. Consider $\eta : I \to [0,1]$ a continuous function that is equal to zero on the boundary of I, is equal to one on J and affine on each component of $I \setminus J$. Let ϕ be the displacement function of f, and define $\psi \colon \Omega \to \mathbb{R}$ by

$$\psi(x - t) = \eta(t)\phi(x - t) \quad \text{for any } x \in C, t \in I,$$

and by zero on the complement of B. It is clear that ψ is a continuous function. Thus, $g(x) := x - \psi(x)$ is continuous and coincides with f on B', and since $J \subset I$, it is also increasing by the choice of η. It is plain to check $g \in D(\Omega)$. $\square$

Lemma 6.4. *Let Ω ba a minimal $\mathbb{R}$-tilable lamination and let $f \in \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ (resp. $\mathrm{Homeo}^0_{vsp}(\Omega)$). Then there exist two boxes $B' \subset B$ and two homeomorphisms $f_1, f_2 \in \mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ (resp. $\mathrm{Homeo}^0_{vsp}(\Omega)$) such that*

- $\operatorname{supp} f_2 \subset B$;
- $f_1|_{B'} = Id|_{B'}$;
- $f = f_1 \circ f_2$.

Proof. Let x be a point of Ω. The points x and $f(x)$ are in the same leaf, so they belong to a same box $B = C - I$ of cubic type. By continuity, there exists a small box $x \in B' = C - J$ such that the closures of $B', f(B')$ are in B. Let f_2 be the map given by Lemma 6.3, and let $f_1 = f \circ f_2^{-1}$. It is straightforward to check they satisfy the conditions of the lemma. $\square$

Next we need a topological lemma on one dimensional tilable laminations. If $B = C - (a, b)$ is a box of cubic type, for an element $x \in C - b$, its *return time to* $C - a$ is

$$\tau_{C-a}(x) = \inf\{t > 0;\ x - t \in C - a\}.$$

By minimality, $\tau_{C-a}(x)$ is finite for any $x \in C - b$, and the map $\tau_{C-a} \colon C - b \to \mathbb{R}$ is locally constant, hence continuous.

Lemma 6.5. *Let Ω be a $\mathbb{R}$-tilable lamination, and let $B = C - (a, b)$ be a box of cubic type. Then the following map is an homeomorphsim.*

$$\begin{array}{ccc} \{(x,t);\ x \in C - b,\ 0 \leq t \leq \tau_{C-a}(x)\} & \longrightarrow & \Omega \setminus B \\ (x,t) & \mapsto & x - t. \end{array}$$

The proof is plain.

of Theorem 6.1. Let us denote by H the group $\mathrm{Homeo}^0_{\mathcal{L}}(\Omega)$ or $\mathrm{Homeo}^0_{vsp}(\Omega)$ and let $f \in H$. Let f_1 and f_2 be the homeomorphims in H and B, B' be the boxes given by Lemma 6.4. From Lemma 6.5 applied to the box $B' = C - (a, b)$, and since the map τ_{C-a} is locally constant, there exists a clopen partition $\{C_1, \ldots, C_\ell\}$ of C such that for any i, $\tau_{C-a}|C_i$ is constant, equals to τ_i and $\{C_i - [0, \tau_i]\}_{i=1}^{\ell}$ is a covering of $\Omega \setminus B$ by closed boxes with interior pairwise disjoint.

Hence, the map f_1 preserves any box $C_i - [0, \tau_i]$, so it can be written as a product of maps $g_1 \cdots g_\ell$, where any $g_i \in H$ and $\operatorname{supp} g_i \subset C_i - [0, \tau_i]$. By Lemma 6.2, f_2 is a commutator and any g_i is a commutator $[a_i, b_i]$ where the homeomorphisms $a_i, b_i \in H$ have their support in the box $C_i - [0, \tau_i]$. Since two homeomorphisms with disjoint interior of supports commute, we have

$$f_1 = \prod_{i=1}^{\ell} g_i = \prod_{i=1}^{\ell} [a_i, b_i] = [\prod_{i=1}^{\ell} a_i, \prod_{i=1}^{\ell} b_i].$$

It follows that f may be written as a product of two commutators. $\square$

Acknowledgments. It is a pleasure for S. Petite to acknowledge A. Rivière for all the discussions on the subtleties of the Schoenflies Theorem.

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LINEARLY REPETITIVE DELONE SYSTEMS HAVE A FINITE NUMBER OF NON PERIODIC DELONE SYSTEMS FACTORS.

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Abstract. In this paper we prove linearly repetitive Delone systems have finitely many Delone system factors up to conjugacy. This result is also applicable to linearly repetitive tiling systems.

1. Introduction

The concepts of tiling dynamical system and Delone dynamical system are extensions to $\mathbb{R}^d$-actions of the notion of subshift (see [Ro]). Classical examples are those generated by self-similar tilings, as the Penrose one, which have been extensively studied since the 90's. For details and references see for example [Ro, So1]. Systems arising from self-similar tilings are known to be linearly repetitive (see [So2, Lemma 2.3]), this means there exists a positive constant L, such that every pattern of diameter D appears in every ball of radius LD in any tiling of the system. This concept has been first defined in [LP]. Linearly repetitive tiling and Delone systems can be seen as a generalization to $\mathbb{R}^d$-actions of the notion of linearly recurrent subshift introduced in [DHS].

We study the factor maps between Delone systems. The main result is the following: linearly repetitive Delone systems have finitely many Delone system factors up to conjugacy. As noticed in [So3], tiling systems are topologically conjugate to Delone systems. This conjugacy also preserves linear repetitivity. Consequently, the results that we present can be easily extended to linearly repetitive tiling systems.

The main result of this paper was obtained in the context of subshifts in [Du1]. A key tool used in [Du1], is the existence of sliding-block-codes for factor maps between subshifts (Curtis-Hedlund-Lyndon Theorem). Unlike subshifts, factor maps between two tiling systems are not always sliding-block-codes (see [Pe] and [RS]). The lack of this property appears to be the main difficulty of this work. To surmount this obstacle, we carefully dissect continuity of factor maps, by means of Voronoï cells and return vectors.

This paper is organized as follows: In Section 2 we recall basic concepts and results about Delone systems. In Section 4 we show the factor maps

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1991 *Mathematics Subject Classification.* 37B50,

Key words and phrases. Delone sets, tiling systems, factor maps, linearly repetitive, Voronoï cell.

from linearly repetitive Delone systems to Delone systems are finite-to-one. Finally, Section 5 is devoted to the proof of the main theorem.

2. Definitions and background

In this section we give the basic definitions and properties concerning Delone sets. For more details we refer to [LP] and [Ro]. Let r and R be two positive real numbers. A (r,R)-*Delone set* X is a discrete subset of $\mathbb{R}^d$ satisfying the following two properties:

(1) *Uniform discreteness*: each open ball of radius $r>0$ in $\mathbb{R}^d$ contains at most one point of X.
(2) *Relative density*: each closed ball of radius $R>0$ in $\mathbb{R}^d$ contains at least one point of X.

A (r,R)-Delone set X, in short a *Delone set*, has *finite local complexity* if $X-X$ is *locally finite*, i.e. the intersection of $X-X$ with any bounded set is finite.
The translation by a vector $v\in\mathbb{R}^d$ of a Delone set X, is the Delone set $X-v$ obtained after translating every point of X by $-v$. Observe that $X-v$ has finite local complexity if and only if X has finite local complexity. A Delone set is said to be *non periodic* if $X-v=X$ implies $v=0$.
Let $R>0$ and X be a Delone set. We say that $P\subseteq X$ is the R-*patch* of X centered at the point $y\in\mathbb{R}^d$ if

$$P=X\cap B_R(y),$$

where $B_R(y)$ denotes the open ball of a radius R centered at y. If there is no confusion, we refer to a R-patch of X merely as a patch. A *sub-patch* of the patch P is a patch of X included in P. A patch Q is a *translated* of the patch P if there exists $v\in\mathbb{R}^d$ such that $P-v=Q$. The vector $v\in\mathbb{R}^d$ is a *return vector* of the patch P in X if $P-v$ is a patch of X. An *occurrence* of the patch P of X centered at $y\in\mathbb{R}^d$ is a point $w\in\mathbb{R}^d$ such that $y-w$ is a return vector of P. Observe the patch $P-(y-w)$ is the translated of P centered at w.
The R-*atlas* $\mathcal{A}_X(R)$ of X is the collection of all the R-patches centered at a point of X translated to the origin. More precisely:

$$\mathcal{A}_X(R)=\{X\cap B_R(x)-x;\ x\in X\}.$$

The atlas $\mathcal{A}_X$ of X is the union of all the R-atlases, for $R>0$. Notice that X has finite local complexity if and only if $\mathcal{A}_X(R)$ has finite local complexity for every $R>0$.
The Delone set X is *repetitive* if for each $R>0$ there is a finite number $M>0$, such that for every closed ball B of radius M the set $B\cap X$ contains a translated patch of every R-patch of X. Observe that any repetitive Delone set has necessarily finite local complexity.
The *Voronoï cell* of a point $x\in X$ is the compact subset

$$V_x=\{y\in\mathbb{R}^d;\ ||x-y||\leq||x'-y||\ \text{ for any } x'\in X\}.$$

Notice that if X is a Delone set has finite local complexity, then each Voronoï cell of X is a polyhedra, and there is a finite number of Voronoï cells of X up to translations.

2.1. **Delone systems.** We denote by $\mathcal{D}$ the collection of the Delone sets of $\mathbb{R}^d$. The group $\mathbb{R}^d$ acts on $\mathcal{D}$ by translations:

$$(v, X) \mapsto X - v \text{ for } v \in \mathbb{R}^d \text{ and } X \in \mathcal{D}.$$

Furthermore, this action is continuous with the topology induced by the following distance: take X, X' in $\mathcal{D}$, and define A the set of $\varepsilon \in (0, \frac{1}{\sqrt{2}})$ such that there exist v and v' in $B_\varepsilon(0)$ with

$$(X - v) \cap B_{1/\varepsilon}(0) = (X' - v') \cap B_{1/\varepsilon}(0),$$

we set

$$d(X, X') = \begin{cases} \inf A & \text{if } A \neq \emptyset \\ \frac{1}{\sqrt{2}} & \text{if } A = \emptyset. \end{cases}$$

Roughly speaking, two Delone sets are close if they have the same pattern in a large neighborhood of the origin, up to a small translation.
A *Delone system* is a pair $(\Omega, \mathbb{R}^d)$ such that Ω is a translation invariant closed subset of $\mathcal{D}$. The orbit closure of a Delone set X in $\mathcal{D}$ is the set $\Omega_X = \overline{\{X + v : v \in \mathbb{R}^d\}}$. This is invariant by the $\mathbb{R}^d$-action, and, it is compact if and only if X has finite local complexity (see [Ro] and [Ru]). Every $X' \in \Omega_X$ is a (r, R)-Delone set if X is a (r, R)-Delone set, and for any real $R > 0$, we have $\mathcal{A}_{X'}(R) \subset \mathcal{A}_X(R)$. If all the orbits are dense in Ω_X, the Delone system $(\Omega_X, \mathbb{R}^d)$ is said to be *minimal*. It is shown in [Ro] that the Delone set X is repetitive if and only if the system $(\Omega_X, \mathbb{R}^d)$ is minimal. In that case, for any $X' \in \Omega_X$ and any $R > 0$ the R-atlases $\mathcal{A}_{X'}(R), \mathcal{A}_X(R)$ are the same. If in addition, X is non periodic, then every Delone set in Ω_X is non periodic.
A *factor map* between two Delone systems $(\Omega_1, \mathbb{R}^d)$ and $(\Omega_2, \mathbb{R}^d)$ is a continuous surjective map $\pi : \Omega_1 \to \Omega_2$ such that $\pi(X - v) = \pi(X) - v$, for every $X \in \Omega_1$ and $v \in \mathbb{R}^d$.
In symbolic dynamics it is well-known that topological factor maps between subshifts are always given by sliding-block-codes. There are examples which show that this result can not be extended to Delone systems ([Pe], [RS]). The following lemma shows that factor maps between Delone systems are not far from being sliding-block-codes. A similar result can be found in [HRS].

Lemma 1. *Let X_1 and X_2 be two Delone sets. Suppose X_1 has finite local complexity and $\pi : \Omega_{X_1} \to \Omega_{X_2}$ is a factor map. Then, there exists a constant $s_0 > 0$ such that for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ satisfying the following: For any $R \geq R_\varepsilon$, if X and X' in Ω_{X_1} verify*

$$X \cap B_{R+s_0}(0) = X' \cap B_{R+s_0}(0),$$

then

$$(\pi(X) - v) \cap B_R(0) = \pi(X') \cap B_R(0)$$

for some $v \in B_\varepsilon(0)$.

Proof. The Delone set X_2 has also finite local complexity because Ω_{X_2} is compact. Let r_0 and R_0 be a positive constant such that X_2 is a (r_0, R_0)-Delone set. Since all the elements of Ω_{X_2} are (r_0, R_0)-Delone sets, if two different points y_1, y_2 of $\mathbb{R}^d$ satisfy $(X - y_1) \cap B_R(a) = (X - y_2) \cap B_R(a)$ for some $X \in \Omega_{X_2}$, $a \in \mathbb{R}^d$ and $R > R_0$, then $\|y_1 - y_2\| \geq \frac{r_0}{2}$ (for the details see [So1]).

Let $0 < \delta_0 < \min\{\frac{r_0}{4}, \frac{1}{R_0}\}$. Since π is uniformly continuous, there exists $s_0 > 1$ such that if X and X' in Ω_{X_1} verify $X \cap B_{s_0}(0) = X' \cap B_{s_0}(0)$ then

$$(\pi(X) - v) \cap B_{\frac{1}{\delta_0}}(0) = \pi(X') \cap B_{\frac{1}{\delta_0}}(0),$$

for some $v \in B_{\delta_0}(0)$. Let $0 < \varepsilon < \delta_0$. By uniform continuity of π, there exists $0 < \delta < \frac{1}{s_0}$ such that if X and X' in Ω_{X_1} verify $X \cap B_{\frac{1}{\delta}}(0) = X' \cap B_{\frac{1}{\delta}}(0)$ then

$$(\pi(X) - v) \cap B_{\frac{1}{\varepsilon}}(0) = \pi(X') \cap B_{\frac{1}{\varepsilon}}(0), \tag{2.1}$$

for some $v \in B_\varepsilon(0)$. Now fix $R \geq R_\varepsilon = \frac{1}{\delta} - s_0$, and let X and X' be two Delone sets in Ω_{X_1} satisfying

$$X \cap B_{R+s_0}(0) = X' \cap B_{R+s_0}(0). \tag{2.2}$$

Observe that X and X' satisfy (2.1), and $(X - a) \cap B_{s_0}(0) = (X' - a) \cap B_{s_0}(0)$, for every a in $B_R(0)$. The choice of s_0 ensures that

$$(\pi(X) - a - t(a)) \cap B_{\frac{1}{\delta_0}}(0) = (\pi(X') - a) \cap B_{\frac{1}{\delta_0}}(0), \tag{2.3}$$

for some $t(a) \in B_{\delta_0}(0)$. Let us prove the map $a \to t(a)$ is locally constant. For $a \in B_R(0)$, let $0 < s_a < \frac{1}{\delta_0} - R_0$ be such that $B_{s_a}(a) \subseteq B_R(0)$. Every $a' \in B_{s_a}(0)$ verifies $B_{\frac{1}{\delta_0} - \|a'\|}(-a') \subset B_{\frac{1}{\delta_0}}(0)$. Let $a' \in B_{s_a}(0)$. This inclusion and (2.3) imply

$$(\pi(X) - a - a' - t(a)) \cap B_{\frac{1}{\delta_0} - \|a'\|}(-a') = (\pi(X') - a - a') \cap B_{\frac{1}{\delta_0} - \|a'\|}(-a'). \tag{2.4}$$

On the other hand, from the definition of the map $a \to t(a)$ we deduce

$$(\pi(X) - a - a' - t(a + a')) \cap B_{\frac{1}{\delta_0}}(0) = (\pi(X') - a - a') \cap B_{\frac{1}{\delta_0}}(0),$$

which implies

$$(\pi(X) - a - a' - t(a + a')) \cap B_{\frac{1}{\delta_0} - \|a'\|}(-a') = (\pi(X') - a - a') \cap B_{\frac{1}{\delta_0} - \|a'\|}(-a'). \tag{2.5}$$

Since $\|t(a) - t(a + a')\| \leq \frac{r_0}{2}$, from equations (2.4), (2.5) and the remark of the beginning of the proof we conclude $t(a) = t(a + a')$ for every $a' \in B_s(0)$. Therefore the map $a \mapsto t(a)$ is constant on $B_{s_a}(a)$.

Furthermore, due to $\delta_0 > \varepsilon$ and (2.2), Equation (2.1) implies there exists $v \in B_\varepsilon(0)$ such that

$$(\pi(X) - v) \cap B_{\frac{1}{\delta_0}}(0) = \pi(X') \cap B_{\frac{1}{\delta_0}}(0). \tag{2.6}$$

For $a = 0$, from (2.3) and (2.6) we have that $t(0) = v$ or $\|v - t(0)\| \geq \frac{r_0}{2}$. Since $\|t(0) - v\| \leq \delta_0 + \varepsilon < 2\delta_0 < \frac{r_0}{2}$, we conclude $t(0) = v$ and then $t(a) = v$ for every $a \in B_R(0)$. This property together with (2.3) and (2.6) imply that

$$(\pi(X) - v) \cap B_R(0) = \pi(X') \cap B_R(0).$$

This conclude the proof. $\square$

3. Example of a prime Delone system

In this section, we will study a specific Delone system, and we will prove this system is *prime* in the following sense: there exist no aperiodic Delone system as a strict factor. Starting from a multidimensional discrete odometer, we built a symbolic extension of it which is 2 to 1 for only one orbit and 1-1 for any others. The suspension of the symbolic system is then conjugate to a Delone system and is one extension of a suspension of the odometer system with the same properties as the first extension. Thanks this we show this Delone system is prime.

Let G be a topological group compact and totally disconnected group, such that $\mathbb{Z}^d$ is a dense subgroup.

For instance, let $(Z_n)_{n\geq 0}$ be a sequence of decreasing sub lattices with finite index in $\mathbb{Z}^d$ such that $\bigcap_{n\geq 0} Z_n = \{0\}$. For any n, the inclusion defines a canonical homomorphism $p_n \colon \mathbb{Z}^d/Z_{n+1} \to \mathbb{Z}^d/Z_n$.

The group G given by the inverse limit of the $(\mathbb{Z}^d/Z_n, p_n)_n$, namely

$$G = \{(x_n)_n,\ x_n \in \mathbb{Z}^d/Z_n \ \text{ and } x_n = p_n(x_{n+1})\}.$$

By the topology induced by the product topolgy , G is compact and totally disconnected. Let i be the map $i \colon \mathbb{Z}^d \ni z \mapsto ([z]_n)_n \in G$, where $([z]_n)$ denotes the class of z in $\mathbb{Z}^d/Z_n$. It is staightfrward to check that i is a homorphism, is 1-1 because $\bigcap_{n\geq 0} Z_n$ is trivial and the image is dense in G. The group $\mathbb{Z}^d$ acts on G by translation of element of $i(\mathbb{Z}^d)$.

Let $F \colon G \to \{0, 1\}$ be a map continuous in all points exept in 0. Let Γ_F be theà graph of F, $\Gamma_F = \{(g, F(g)),\ g \in G\}$. The closure of Γ_F for the product topology, is then the unon of Γ_F with the point $(0, \bar{F(0)})$ where $\bar{F(0)} = F(0) + 1 \mathrm{mod} 1$. Moreover, we suppose the map F to be invariant under no rotations, *i.e.*: $F(g + i(z)) \neq F(g)$ for any $g \in G$ and $z \in \mathbb{Z}^d$. For explicit examples See [ARTICLE MIC ou these].

Now, we consider the $\mathbb{Z}^d$ subshift X given by taking the closure in the product space $\{0, 1\}^{\mathbb{Z}^d}$ of the sequence $(F(i(z))_{z\in\mathbb{Z}^d}$. By the very hypothesis, the system $(X, \mathbb{Z}^d)$ is minimal and there exists a factor map $\pi \colon X \to G$. For any $g \in G$ is $\pi^{-1}(\{g\})$ is one point or two points if $g \in i(\mathbb{Z}^d)$.

The group $\mathbb{Z}^d$ acts also by integer translation on $\mathbb{R}^d$. The *suspension* of the $\mathbb{Z}^d$-action is the quotient by the diagonal $\mathbb{Z}^d$ action on $X \times \mathbb{R}^d$. In [BBG], it is shown that this suspension is conjugated to a Delone system.

It is straightforward to check that the factor map $\pi \colon X \to G$ extends to a continuous factor map from the suspension of X onto the susepension of G, which is 1-1 for every $\mathbb{Z}^d$-orbit in G exept for $i(\mathbb{Z}^d)$. We conclude thanks the following lemma.

Lemma 2. *Let $(X, \mathbb{R}^d)$ be a dynamical system and $\pi : X \to G$ a factor map onto an minimal equicontinuous action $(G, \mathbb{R}^d)$ such that π is 1-1 on every $\mathbb{R}^d$ orbit exept one, and 2-1 oherwise.*
If $(Y, \mathbb{R}^d)$ is a non periodic factor of $(X, \mathbb{R}^d)$, then $(Y, \mathbb{R}^d)$ is a factor of $(G, \mathbb{R}^d)$.

4. Preimages of factor maps.

In the rest of this paper we suppose that all the Delone sets have finite local complexity.
A Delone set X is *linearly repetitive* if there exists a constant $L > 0$ such that for every patch P in X, any ball of radius $L\mathrm{diam}(P)$ intersected with X contains a translated patch of P. In this instance we say that X is *linearly repetitive with constant* L. Notice the constant L must be greater or equal than 1, and if X is linearly repetitive with constant L, then it is linearly repetitive with constant L', for every $L' > L$. Every Delone set in the orbit closure of a linearly repetitive Delone set is linearly repetitive with the same constant. When X is linearly repetitive, we call $(\Omega_X, \mathbb{R}^d)$ a *linearly repetitive* Delone system.
The following lemma shows the factors of linearly repetitive systems are also linearly repetitive with a uniform control on the constants. This was already proven for subshifts in [Du1].

Lemma 3. *Let X be a linearly repetitive Delone set with constant L. If X' is a Delone set such that $(\Omega_{X'}, \mathbb{R}^d)$ is a topological factor of $(\Omega_X, \mathbb{R}^d)$, then there exists a constant $\tau_{X'} > 0$ such that if P is a patch of X' with $\mathrm{diam}(P) \geq \tau_{X'}$, then for any $y \in \mathbb{R}^d$, the set $X' \cap B_{5L\mathrm{diam}(P)}(y)$ contains a translated patch of P.*

Proof. Let $\pi : \Omega_X \to \Omega_{X'}$ be a topological factor, where X is a (r_X, R_X)-linearly repetitive Delone set with constant L, and X' is a $(r_{X'}, R_{X'})$-Delone set. We can assume that $\pi(X) = X'$. Let $s_0 > 0$ be the constant of Lemma 1. Fix $0 < \varepsilon < Ls_0$ and consider $R_\varepsilon > 0$ as in Lemma 1. We set

$$\tau_{X'} = \max\{s_0, R_\varepsilon, R_X, R_{X'}\}.$$

Let P be a patch in X' with $\mathrm{diam}(P) = D \geq \tau_{X'}$, and let $v \in P \subset X'$. Let $Q = (X - v) \cap B_{D+s_0}(0)$. Since $\mathrm{diam}(Q) \leq 2(D + s_0)$, for every $y \in \mathbb{R}^d$ there exists $w \in B_{2L(D+s_0)}(y)$ such that $(X - w) \cap B_{D+s_0}(0) = Q$. Then, from Lemma 1 there exists $t \in B_\varepsilon(0)$ such that

$$(X' - v) \cap B_D(0) = (X' - w - t) \cap B_D(0).$$

Since $(X' - v) \cap B_D(0)$ contains a translated of P, this shows that every ball of radius $2L(D + s_0) + \varepsilon \leq 5LD$ in X' contains a translated of P as sub-patch. $\square$

The next Lemma follows the same lines of Lemma 2.4 in [So2]. We show the set of occurrences of a R-patch of a linearly repetitive Delone set and its factors is uniformly discrete with a constant depending linearly on R.

Lemma 4. *Let X be a non periodic linearly repetitive Delone set with constant L, and let X' be a non periodic Delone set such that $(\Omega_{X'}, \mathbb{R}^d)$ is a topological factor of $(\Omega_X, \mathbb{R}^d)$. There exists a constant $M_{X'} > 0$ such that for every $R \geq M_{X'}$ and for every R-patch P of X', if $x \in \mathbb{R}^d \setminus \{0\}$ is a return vector of P, then $\|x\| \geq R/(11L)$.*

Proof. Let $R' > 0$ be a real such that any patch of the kind $X' \cap B_{R'}(y)$, with $y \in \mathbb{R}^d$, has diameter greater than $\tau_{X'}$, where $\tau_{X'}$ is the constant given by Lemma 3. Let $M_{X'} = 110LR' + R'$ and P be the R-patch $X' \cap B_R(v)$ with $R > M_{X'}$ and $v \in \mathbb{R}^d$. Suppose there exists $x \in \mathbb{R}^d$, with $0 < \|x\| < R/(11L)$, such that $P + x$ is a patch of X'. For any $y \in \mathbb{R}^d$, consider the patches

$$Q_y = X' \cap B_{R'}(y) \text{ and } S_y = X' \cap B_{R'+\|x\|}(y).$$

Since

$$\tau_{X'} \leq \operatorname{diam}(S_y) \leq 2(R' + \|x\|),$$

from Lemma 3, every ball of radius $10L(R' + \|x\|)$ intersected with X' contains a translated of S_y. By the very hypothesis, we have

$$10L(R' + \|x\|) < 10LR' + \frac{10R}{11} \leq \frac{R}{11} + \frac{10R}{11} = R.$$

This implies there exists $w \in \mathbb{R}^d$ such that $S_y + w$ is a sub-patch of $X' \cap B_R(v) = P$. Because $P + x$ is also a patch of X', we have $Q_y + w + x$ is also a patch of X' and a sub-patch of $S_y + w$. Hence $Q_y + w + x = Q_{y+x} + w$ and

$$Q_y + x = Q_{y+x}.$$

Since y is arbitrary, we conclude that $X' + x = X'$, which contradicts the non periodicity of X' if $x \neq 0$. $\square$

We recall the following definition: A factor map $\pi : (\Omega, \mathbb{R}^d) \to (\Omega', \mathbb{R}^d)$ is said to be *finite-to-one* (with constant D) if for all $y \in Y$ we have $|\pi^{-1}(\{y\})| \leq D$. The next result is a technical lemma we use in Proposition 6 to show that factor maps between linearly repetitive Delone systems are finite-to-one.

Lemma 5. *Let $\pi : (\Omega_X, \mathbb{R}^d) \to (\Omega_{X'}, \mathbb{R}^d)$ be a factor map, where X is a linearly repetitive Delone set with constant L, and X' is a non periodic Delone set. We denote by s_0 the constant given by Lemma 1.*
For every $0 < \varepsilon < \frac{s_0}{2}$, there exists a constant R_π such that for any $R > R_\pi$ there are at most $n \leq (55L^2)^d$ patches $P_1, \ldots, P_n$ satisfying for every $1 \leq i \leq n$ the following conditions:

i) $P_i = (X - w_i) \cap B_{R+s_0}(0)$, *for some* $w_i \in \mathbb{R}^d$,

ii) *If* $X"$ *belongs to* Ω_X *and* $X" \cap B_{R+s_0}(0) = P_i$, *then there exists* $v \in B_\epsilon(0)$ *such that*
$$(\pi(X") - v) \cap B_R(0) = \pi(X) \cap B_R(0),$$

iii) *The patch* $(X - w_i) \cap B_{R+s_0-2\epsilon}(0)$ *is not a sub-patch of* P_j, *for every* $1 \le j \le n$, $j \ne i$.

Proof. Let $0 < \varepsilon < \frac{s_0}{2}$, $R_\pi = \max\{s_0, M_{X'}, R_\epsilon\}$ and $R > R_\pi$, where $M_{X'}$ is the constant given by Lemma 4 and R_ϵ by Lemma 1. Let $P_1, \dots, P_n$ be n patches of X satisfying the conditions $i)$, $ii)$, $iii)$.
Let $1 \le i \le n$. We have
$$\mathrm{diam}(P_i) \le 2(R + s_0) \le 4R.$$
Linear repetitivity implies there exists $v_i \in B_{4LR}(0)$ such that
$$(X - v_i) \bigcap B_{R+s_0}(0) = P_i.$$
Then by $ii)$, there is $u_i \in B_\epsilon(0)$ satisfying
$$Q = (\pi(X - v_i) + u_i) \cap B_R(0) = (\pi(X) - v_i + u_i) \cap B_R(0),$$
where $Q = \pi(X) \cap B_R(0)$ (observe that Q does not depend on i). This means the set $Q + v_i - u_i$ is a patch of $\pi(X)$. As $\{v_i - u_i,\ 1 \le i \le n\}$ is included in $B_{4LR+\epsilon}(0)$ and $R > M_{X'}$, Lemma 4 implies the number of elements in $\{v_i - u_i,\ 1 \le i \le n\}$ is bounded by
$$\frac{\mathrm{vol}(B_{4LR+\epsilon}(0))}{\mathrm{vol}\left(B_{\frac{R}{11L}}(0)\right)} \le (55L^2)^d.$$
If n is greater than $(55L^2)^d$, then there exist $i \ne j$ such that $v_i - u_i = v_j - u_j$, and $||v_i - v_j|| < 2\epsilon$. This implies the patch $(X - v_i) \cap B_{R+s_0-2\epsilon}(0)$ is included in the patch $(X - v_j) \cap B_{R+s_0}(0) = P_j$, which contradicts the condition $iii)$. □

The next result was proven in [Du1] for subshifts. We use it with Proposition 7 to conclude the proof of the main theorem.

Proposition 6. *Let* X *be a linearly repetitive Delone set with constant* L. *If* $\pi : (\Omega_X, \mathbb{R}^d) \to (\Omega_{X'}, \mathbb{R}^d)$ *is a factor map such that* X' *is a non periodic Delone set, then* π *is finite-to-one with constant* $(55L^2)^d$.

Proof. Let $X'_0 \in \Omega_{X'}$. Suppose there exist $n > (55L^2)^d$ elements $X_1, \dots, X_n$ of Ω_X, such that $\pi(X_i) = X'_0$, for each $1 \le i \le n$. Since they are all different, there exists $R_0 > 0$ such that for any $R \ge R_0$, the patches $X_i \cap B_R(0)$ are pairwise distinct.
Let $0 < \varepsilon < \frac{s_0}{2}$ and R_π be the constant given by Lemma 5. Lemma 1 ensures that for any $Y \in \Omega_X$ satisfying $Y \cap B_R(0) = X_i \cap B_R(0)$, with $1 \le i \le n$ and $R > \max\{R_0, R_\epsilon + s_0, R_\pi + s_0\}$, there exists $v \in B_\epsilon(0)$ such that $(\pi(Y) - v) \cap B_{R-s_0}(0) = X'_0 \cap B_{R-s_0}(0)$. This means the patches $X_1 \cap B_R(0), \cdots, X_n \cap B_R(0)$ satisfy conditions $i)$ and $ii)$ of Lemma 5. Then

we deduce there exist different $i(R)$ and $j(R)$ in $\{1,\ldots,n\}$ such that the patch $X_{i(R)}\cap B_{R-2\epsilon}(0)$ is a sub-patch of $X_{j(R)}\cap B_R(0)$. In other words, there exists $v_R\in B_{2\epsilon}(0)$ such that $X_{i(R)}\cap B_{R-2\epsilon}(0)=(X_{j(R)}+v_R)\cap B_{R-2\epsilon}(0)$. By the pigeonhole principle, there exist different i_0 and j_0 in $\{1,\ldots,n\}$, and an increasing sequence $(R_p)_{p\geq 0}$, tending to ∞ with p, such that $i(R_p)=i_0$ and $j(R_p)=j_0$, for every $p\geq 0$. By compactness, we can also assume that $(v_{R_p})_{p\geq 0}$ converges to a vector v. Thus, for every $p\geq 0$ we get

$$X_{i_0}\cap B_{R_p-2\epsilon}(0)=(X_{j_0}+v_{R_p})\cap B_{R_p-2\epsilon}(0),$$

which implies that $X_{i_0}=X_{j_0}+v$ and $X'_0=\pi(X_{i_0})=\pi(X_{j_0}+v)=X'_0+v$. Since $X_{i_0}\neq X_{j_0}$, the vector v is different from zero, but this contradicts the non periodicity of X'_0. □

The following proposition is a straightforward generalization of Lemma 21 in [Du1].

Proposition 7. *Let $(\Omega,\mathbb{R}^d)$ be a minimal Delone system and $\phi_1:(\Omega,\mathbb{R}^d)\to(\Omega_1,\mathbb{R}^d)$, $\phi_2:(\Omega,\mathbb{R}^d)\to(\Omega_2,\mathbb{R}^d)$ be two factor maps. Suppose that $(\Omega_2,\mathbb{R}^d)$ is non periodic and ϕ_1 is finite-to-one. If there exist $X,Y\in\Omega$ and $v\in\mathbb{R}^d$ such that $\phi_1(X)=\phi_1(Y)$ and $\phi_2(X)=\phi_2(Y-v)$, then $v=0$.*

Proof. There exists a sequence $(v_i)_{i\in\mathbb{N}}\subset\mathbb{R}^d$ such that $\lim_{i\to+\infty}X-v_i=Y$. By compactness, we can suppose that the sequence $(Y-v_i)_{i\in\mathbb{N}}$ converges to a point $Y_2\in\Omega$. By continuity, we have $\phi_1(Y)=\phi_1(Y_2)$, and $\phi_2(Y)=\phi_2(Y_2)-v$. By compactness, we can suppose that the sequence of points $(Y_2-v_i)_{i\in\mathbb{N}}\subset\Omega$ converges to a point Y_3. So we have $\phi_1(Y_2)=\phi_1(Y_3)$ and $\phi_2(Y_2)=\phi_2(Y_3)-v$. Hence we construct by induction a sequence $(Y_n)_{n\in\mathbb{N}}\subset\Omega$ such that $\phi_1(Y_n)=\phi_1(Y_{n+1})$ and $\phi_2(Y_n)=\phi_2(Y_{n+1})-v$ for all $n\geq 1$. Since the map ϕ_1 is finite-to-one, there exist $i<j$ such that $Y_i=Y_j$. Then, we have

$$\begin{aligned}\phi_2(Y_i)=\phi_2(Y_{i+1})-v=\phi_2(Y_{i+2})-2v=\ldots&=\phi_2(Y_j)-(j-i)v\\&=\phi_2(Y_i)-(j-i)v.\end{aligned}$$

Since $(\Omega_2,\mathbb{R}^d)$ is non periodic, we conclude $v=0$. □

Remark. Following the lines of the proof of Proposition 7, this result can be generalized to $\mathbb{Z}^d$ or $\mathbb{R}^d$ actions, more precisely: Let G be $\mathbb{R}^d$ or $\mathbb{Z}^d$. Let (X,G) be a minimal dynamical system and $\phi_1:(X,G)\to(X_1,G)$, $\phi_2:(X,G)\to(X_2,G)$ be two factor maps. Suppose that (X_2,G) is free and ϕ_1 is finite-to-one. If there exist $x,y\in X$ and $g\in G$ such that $\phi_1(x)=\phi_1(y)$ and $\phi_2(x)=\phi_2(g.y)$, then g is the identity in G.

5. Number of factors of linearly repetitive Delone systems.

Let X be a Delone set having finite local complexity, and $P=X\cap B_R(x)$ be a patch of X. We define

$$X_P=\{v\in\mathbb{R}^d:P+v\text{ is a patch of }X\}.$$

Observe that 0 always belongs to X_P. It is straightforward to check that X_P is a Delone set when X is repetitive. Furthermore, X_P is a Delone set having finite local complexity because of $X_P - X_P \subset X - X$. Then we define the *Voronoï cell of* P associated to $v \in X_P$ as the Voronoï cell of $v + x \in X_P + x$. That is,

$$V_{P,v} = \{y \in \mathbb{R}^d : \|y - (x+v)\| \leq \|y - (x+u)\|, \forall u \in X_P\}.$$

Notice the Voronoï cell of P associated to $v \in X_P$ is the Voronoï cell of $v \in X_P$ translated by the vector x.

Remark 8. It follows from the definition that a (r, R)-Delone set X satisfies the following: for any $x \in X$, the diameter of the Voronoï cell V_x is smaller or equal to $2R$ and $B_{\frac{r}{2}}(x)$ is contained in V_x. If X is linearly recurrent with constant L, Lemma 4 implies for every sufficiently large R and every patch $P = X \cap B_R(x)$ of X, the collection X_P is a $(\frac{R}{11L}, 2LR)$-Delone set. Therefore, in this instance we have $\mathrm{diam}(V_{P,v}) \leq 4LR$ and $B_{\frac{R}{11L}}(x+v) \subseteq V_{P,v}$, for every $v \in X_P$.

In the next lemma, we bound the number of ways we can extend a given patch P to a bigger one. More precisely, this gives an upper bound of the number (up to translation) of R'-patches $X \cap B_{R'}(x)$, such that $X \cap B_R(x)$ is a translated of P.

Lemma 9. *Let X be a linearly repetitive Delone set with constant L, and consider $0 < R_1 < R_2$, with R_1 sufficiently large. Then there are at most $n \leq (44L^2)^d \left(\frac{R_2}{R_1}\right)^d$ patches $P_1, \cdots, P_n$ of X, up to translation, satisfying for every $1 \leq i \leq n$ the following two conditions:*

i) there exists $v_i \in \mathbb{R}^d$ such that $P_i = X \cap B_{R_2}(v_i)$.

ii) $(X - v_i) \cap B_{R_1}(0) = (X - v_j) \cap B_{R_1}(0)$, for every $1 \leq j \leq n$.

Proof. Applying Lemma 4 to the identity factor map on $(\Omega_X, \mathbb{R}^d)$, we deduce there exists $M_X > 0$, such that for every $R \geq M_X$ and $x \in \mathbb{R}^d$, the distance between two different occurrences of $P = X \cap B_R(x)$ is greater or equal to $R/(11L)$.

Let $M_X \leq R_1 < R_2$ and $n \in \mathbb{N}$. Suppose $P_1, \cdots, P_n$ are patches of X satisfying conditions $i)$ and $ii)$, and such that for every $1 \leq i \leq n$,

iii) P_i is not a translated of P_j, for every $j \in \{1, \cdots, n\} \setminus \{i\}$.

Condition $i)$ and linear repetitivity of X imply for every $1 \leq i \leq n$, there exists $w_i \in \mathbb{R}^d$ such that $P_i + w_i$ is a sub-patch of $X \cap B_{2LR_2}(0)$. From condition $ii)$ it follows that for every $1 \leq i \leq n$, the point $v_i + w_i$ is an occurrence of the patch $X \cap B_{R_1}(v_1)$ in the ball $B_{2LR_2}(0)$. Finally, by the choice of R_1, conditions $ii)$, $iii)$ and Lemma 4, for every i and j in $\{1, \cdots, n\}$ such that $i \neq j$, we get $\|v_i + w_i - (v_j + w_j)\| \geq \frac{R_1}{11L}$, which implies

$$n \leq \frac{\mathrm{vol}(B_{2LR_2}(0))}{\mathrm{vol}(B_{\frac{R_1}{22L}}(0))} = (44L^2)^d \left(\frac{R_2}{R_1}\right)^d,$$

and achieves the proof. □

The following lemma is certainly well-known, but we did not find any reference. This shows that a Voronoi cell of a point x in a (r, R)-Delone set X is completely determined by the points in $X \cap B_{4R}(x)$.

Lemma 10. *Let X be a (r, R)-Delone set. Then for every $x \in X$ one has*

$$V_x = \{y \in \mathbb{R}^d : \|x - y\| \leq \|x' - y\|, \text{ for every } x' \in X \cap B_{4R}(x)\}.$$

Proof. Let $C_x = \{y \in \mathbb{R}^d : \|x - y\| \leq \|x' - y\|, \text{ for every } x' \in X \cap B_{4R}(x)\}$. By definition of Voronoï cell, the inclusion $V_x \subseteq C_x$ is direct.
Observe the set C_x is convex because is obtained as intersection of convex sets. Now, suppose there exists $y \in C_x \setminus V_x$. Then there exist $x' \in X$, satisfying $V_x \cap V_{x'} \neq \emptyset$, and $z \in ([x, y] \cap V_{x'}) \setminus V_x$, where $[x, y]$ is the segment with extreme points x and y. Since $\|x - x'\| \leq 4R$ and $\|z - x'\| < \|z - x\|$, definition of C_x implies $z \notin C_x$, which contradicts the convexity of C_x. □

Lemma 11. *Let X be a non periodic linearly repetitive Delone set with constant L. There exists a positive constant $c(L)$ such that for every sufficiently large R and every patch $P = X \cap B_R(x)$, the collection $\{X \cap V_{P,v} : v \in X_P\}$ contains at most $c(L)$ elements up to translation.*

Proof. Let R be a big enough positive number, in order to apply Lemma 9 to $R_1 = R$ and $R_2 = 8LR$.
Let $x \in \mathbb{R}^d$, $P = X \cap B_R(x)$ and $v \in X_P$. Since $X_P + x$ is a Delone set with constant of uniform density equal to $2LR$ (see Remark 8), Lemma 10 implies $V_{P,v}$ is completely determined by the patch $X \cap B_{8RL}(v + x)$. Furthermore, the Voronoï cell $V_{P,v}$ is contained in the ball $B_{4RL}(v + x)$ (see Remark 8). Then it follows there are at most as many Voronoï cells of P and patches of the kind $X \cap V_{P,v}$, up to translation, as patches Q satisfying the following two conditions: $i)$ there exists $w \in \mathbb{R}^d$ such that $Q = X \cap B_{8RL}(w)$ and $ii)$ w is an occurrence of a translated of P. These two conditions and Lemma 9 imply there are at most

$$c(L) \leq (44L^2)^d \left(\frac{8LR}{R}\right)^d = (352L^3)^d$$

patches of the kind $X \cap V_{P,v}$ up to translation. □

We have already defined the notion of return vector of a patch, now let us define the notion of return vector of a Voronoï cell of a patch. For a patch $P = X \cap B_R(x)$ of X and $v \in X_P$, we say that $w \in \mathbb{R}^d$ is a *return vector of* $V_{P,v} \cap X$ if $(X - w) \cap V_{P,v} = X \cap V_{P,v}$. We set

$$P_{n,w,v} \text{ the patch } (X - w - x - v) \cap B_{L^n R}(0).$$

Notice that $P_{n,w,v} + v + w + x$ is a patch of X. When there is no confusion about n and v, we write P_w instead of $P_{n,w,v}$.

Lemma 12. *Let $n \in \mathbb{N}$ and X be a non periodic linearly repetitive Delone set with constant L. For every sufficiently large $R > 0$ and every R-patch P, the collection $\{P_w : w$ is a return vector of $V_{P,v} \cap X\}$ has at most $c(n,L)$ elements, for every $v \in X_P$.*

Proof. Let $R_1 = R$ and $R_2 = L^n R$ be sufficiently large positive numbers in order to apply Lemma 9. Let $P = X \cap B_R(x)$ be a patch of X and $v \in X_P$. Since $X_P + x$ is a Delone set with constant of uniform discreteness equal to $\frac{R}{11L}$, the Voronoï cell $V_{P,v}$ contains the ball $B_{\frac{R}{22}}(v+x)$. This implies for every pair of return vectors u and w of $V_{P,v}$ it holds that $P_w \cap B_{\frac{R}{22}}(0) = P_u \cap B_{\frac{R}{22}}(0)$. Thus, from Lemma 9 it follows there are at most

$$c(n,L) \leq (44L^2)^d \left(\frac{L^n R}{\frac{R}{22L}} \right)^d = (968L^{n+3})^d$$

patches of the kind P_w. $\square$

Let $n \in \mathbb{N}$. We call $M(n,L)$ the number of coverings of a set with $c(L)c(n,L)$ elements, where $c(L)$ and $c(n,L)$ are the constants of Lemma 11 and Lemma 12 respectively.

Theorem 13. *Let X be a linearly repetitive Delone set. There are finitely many Delone system factors of $(\Omega_X, \mathbb{R}^d)$ up to conjugacy. Moreover, the number of factors only depends on the linearly recurrence constant of X.*

Proof. Let X be a non periodic linearly repetitive Delone set with constant $L > 1$. Let $n \in \mathbb{N}$ be such that

$$L^n - 1 - 12L - 176L^2 > 1, \tag{5.1}$$

and let $R_1 > 1$ be a constant such that for every $R \geq R_1$, Lemma 11 and Lemma 12 are applicable to R-patches of X.
For every $1 \leq i \leq M(n,L)+1$, let X_i be a non periodic Delone set such that there exists a topological factor map $\pi_i : \Omega_X \to \Omega_{X_i}$, and let $X_0 = X$. Let M_{X_i} be the constant of Lemma 4 associated to X_i.
Fix $0 < \varepsilon < 1$. For every $1 \leq i \leq M(n,L)+1$, consider $R_\varepsilon^{(i)}$ and $s_0^{(i)}$ the constants of Lemma 1 associated to π_i. We define

$$R_\varepsilon = \max_i \{R_\varepsilon^{(i)}\},\ s_0 = \max_i \{s_0^{(i)}\} \text{ and } M = \max_i \{M_{X_i}\}.$$

Observe in an open ball of radius $r/22L$, there is at most one return vector of a r-patch of X_i, with $r \geq M$, for every $1 \leq i \leq M(n,L)+1$.
We take

$$R > \max\{R_\varepsilon, s_0, M+\varepsilon, R_1, 45L\},$$

Consider the patch $P = B_R(0) \cap X$, and $v_1, \cdots, v_N \in X_P$ such that for every $v \in X_P$, there exist $1 \leq i \leq N$ and $u \in \mathbb{R}^d$ satisfying $V_{P,v} \cap X = (V_{P,v_i} \cap X) + u$. Roughly speaking, every set of the kind $V_{P,v} \cap X$ is a translated of some set $V_{P,v_i} \cap X$. Since $R > R_1$, Lemma 11 ensures $N \leq c(L)$.

For every $1 \le j \le N$, let $w_{j,1}, \cdots, w_{j,m_j}$ be return vectors of $V_{P,v_j} \cap X$, chosen in order that for every return vector w of $V_{P,v_j} \cap X$, there exists $1 \le i \le m_j$ such that P_w is equal to $P_{w_{j,i}}$. Since $R > R_1$, Lemma 12 implies that $m_j \le c(n, L)$, for every $1 \le j \le N$. Therefore, the collection

$$\mathcal{F} = \{P_{w_{j,l}} : 1 \le l \le m_j,\ 1 \le j \le N\}$$

contains at most $c(L)c(n, L)$ elements.
Let R' be the constant given by

$$R' = (L^n - 1)R - \varepsilon - 4LR.$$

The choice of n ensures that $R' > 0$.
For every $1 \le i \le M(n, L) + 1$, we define the following relation on $\mathcal{F}$:

$P_{w_{j,l}} \mathcal{R}_i P_{w_{k,m}}$ if and only if for every $X', X'' \in \Omega_X$ such that $X' \cap B_{L^n R}(0) = P_{w_{j,l}}$ and $X'' \cap B_{L^n R}(0) = P_{w_{k,m}}$, there exist $v \in B_{2\varepsilon}(0)$ and $w \in B_{4LR}(0)$ such that $\pi_i(X') \cap B_{R'}(0) = (\pi_i(X'') + v + w) \cap B_{R'}(0)$.

Since $L^n R - s_0 \ge (L^n - 1)R \ge R > R_\varepsilon$, from Lemma 1 it follows this relation is reflexive, so non empty. Since the cardinal of $\mathcal{F}$ is bounded by $c(L)c(n, L)$, there are at most $M(n, L)$ different relations of this kind. So, there exist $1 \le i < j < M(n, L) + 1$ such that $\mathcal{R}_i = \mathcal{R}_j$.
In the sequel, we will prove that $(\Omega_{X_i}, \mathbb{R}^d)$ and $(\Omega_{X_j}, \mathbb{R}^d)$ are conjugate. For that, it is sufficient to show that if $Y, Z \in \Omega_X$ are such that $\pi_i(Y) = \pi_i(Z)$ then $\pi_j(Y) = \pi_j(Z)$.
Let Y and Z be two Delone sets in Ω_X such that $\pi_i(Y) = \pi_i(Z)$. Without loss of generality, we can suppose that 0 is an occurrence of P in Y and in $Z - u_0$, where u_0 is some point in $B_{4LR}(0)$. The patches of Y and Z are translated of the patches of X. This implies there exist $1 \le q_0, r_0 \le N$ such that

$$Y \cap B_{L^n R}(0) = P_{w_{q_0,l_0}} \text{ and } (Z - u_0) \cap B_{L^n R}(0) = P_{w_{r_0,k_0}},$$

for some $1 \le l_0 \le m_{q_0}$ and $1 \le k_0 \le m_{r_0}$

Claim 1: $P_{w_{q_0,l_0}} \mathcal{R}_i P_{w_{r_0,k_0}}$.

Proof of Claim 1: Let X' and X'' be two Delone sets in Ω_X such that $X' \cap B_{L^n R}(0) = P_{w_{q_0,l_0}}$ and $X'' \cap B_{L^n R}(0) = P_{w_{r_0,l_0}}$. Since $R \ge s_0$, $R \ge R_\varepsilon$ and

$$X' \cap B_{L^n R}(0) = Y \cap B_{L^n R}(0),\ X'' \cap B_{L^n R}(0) = (Z - u_0) \cap B_{L^n R}(0),$$

By the choice of n and R, Lemma 1 implies there exits z_1 and z_2 in $B_\varepsilon(0)$ such that

$$(\pi_i(X') + z_1) \cap B_{(L^n-1)R}(0) = \pi_i(Y) \cap B_{(L^n-1)R}(0), \text{ and}$$
$$(\pi_i(X'') + z_2) \cap B_{(L^n-1)R}(0) = \pi_i(Z - u_0) \cap B_{(L^n-1)R}(0).$$

Then we get

$$\begin{aligned}
&(\pi_i(X'') + z_2 + u_0) \cap B_{(L^n-1)R-4LR}(0)\\
=&\pi_i(Z) \cap B_{(L^n-1)R-4LR}(0)\\
=&\pi_i(Y) \cap B_{(L^n-1)R-4LR}(0)\\
=&(\pi_i(X') + z_1) \cap B_{(L^n-1)R-4LR}(0).
\end{aligned}$$

Therefore

$$(\pi_i(X'')+z_2+u_0-z_1)\cap B_{(L^n-1)R-4LR-\varepsilon}(0) = \pi_i(X')\cap B_{(L^n-1)R-4LR-\varepsilon}(0),$$

which implies that $P_{w_{q_0,l_0}}\mathcal{R}_i P_{w_{r_0,k_0}}$.

Since $\mathcal{R}_i = \mathcal{R}_j$, from Claim 1 we get $P_{w_{q_0,l_0}}\mathcal{R}_j P_{w_{r_0,k_0}}$.
Let s be any other occurrence of P in Y. Repeating the same argument for $Y+s$ and $Z+s$, we deduce there exist $u_s \in B_{4LR}(0)$ and $1 \le q_s, r_s \le N$ such that

$$(Y+s)\cap B_{L^nR}(0) = P_{w_{q_s,l_s}} \text{ and } (Z-u_s)\cap B_{L^nR}(0) = P_{w_{r_s,k_s}},$$

for some $1 \le l_s \le m_{q_s}$ and $1 \le k_s \le m_{r_s}$. Then from Claim 1 we get $P_{w_{q_s,l_s}}\mathcal{R}_j P_{w_{r_s,k_s}}$. This implies there exist $t_s \in B_{2\varepsilon}(0)$ and $w_s \in B_{4LR}(0)$ such that

$$\pi_j(Y+s)\cap B_{R'}(0) = (\pi_j(Z+s-u_s)+t_s+w_s)\cap B_{R'}(0).$$

Claim 2: The vector $w_s - u_s + t_s$ does not depend on s, i.e, there exists $y \in \mathbb{R}^d$ such that $w_s - u_s + t_s = y$ for every occurrence s of P in Y.

Proof of Claim 2: Let s_1 and s_2 be two occurrences of P in Y such that the Voronoï cells of s_1 and s_2, with respect to set of occurrences of P in Y, have common points in their borders. Since the diameter of these Voronoï cells is smaller or equal to $4RL$ (see remark 8), we get $\|s_1 - s_2\| \le 8LR$. Then Then

$$\begin{aligned}
&(\pi_j(Z) + s_1 + (s_2 - s_1) - u_{s_1} + t_{s_1} + w_{s_1}) \cap B_{R'-8LR}(0)\\
=\;&(\pi_j(Y) + s_1 + (s_2 - s_1)) \cap B_{R'-8LR}(0)\\
=\;&(\pi_j(Z) + s_2 - u_{s_2} + t_{s_2} + w_{s_2}) \cap B_{R'-8LR}(0).
\end{aligned}$$

This implies $(-u_{s_1} + t_{s_1} + w_{s_1}) - (-u_{s_2} + t_{s_2} + w_{s_2})$ is a return vector of a $(R'-8LR)$-patch of $\pi_j(Z)+s_2$. Since

$$R' - 8LR = R(L^n - 1 - 12L) - \varepsilon \ge R - \varepsilon > M,$$

Lemma 4 implies the non zero vectors of the $(R'-8LR)$-patches of $\pi_j(Z)+s_2$ have norm greater or equal to $(R'-8LR)/11L$. Thus, due to

$$\|-u_{s_1} + t_{s_1} + w_{s_1} - (-u_{s_2} + t_{s_2} + w_{s_2})\| \le 16LR + 4\varepsilon,$$

and

$$\begin{aligned} 11(16LR + 4\varepsilon) &= 176L^2R + 44L\varepsilon \\ &< (L^n - 1 - 12L - 1)R + 44L\varepsilon \\ &= R' - 8LR + \varepsilon - R + 44L\varepsilon \\ &< R' - 8LR + L - R + 44L < R' - 8LR, \end{aligned}$$

we deduce $-u_{s_1} + t_{s_1} + w_{s_1} = -u_{s_2} + t_{s_2} + w_{s_2}$, which shows Claim 2.

¿From Claim 2 we get there exists $y \in \mathbb{R}^d$ such that for every occurrence s of P in Y,

$$\pi_j(Y + s) \cap B_{R'}(0) = (\pi_j(Z + s) + y) \cap B_{R'}(0), \text{ and then}$$
$$\pi_j(Y) \cap B_{R'}(s) = (\pi_j(Z) + y) \cap B_{R'}(s).$$

¿From Remark 8, the diameter of the Voronoï cells of P is less than $4LR$, which is less than R'. Hence,

$$\pi_j(Y) = \pi_j(Z) + y.$$

We conclude with Lemma 6 and Proposition 7. □

Acknowledgments. MIC is grateful for the grant FONDECYT de Iniciación 11060002. She also thanks the Laboratoire Amiénois de Mathématiques Fondamentales et Appliquées, CNRS-UMR 6140, Université de Picardie Jules Verne, where part of this work was done. FD would like to thank the hospitality of Departamento de Matemática y Ciencia de la Computación de la Universidad de Santiago de Chile. MIC and FD were partially supported by Nucleus Millenius P04-069-F. The three authors thank the financial support of program CNRS/CONICYT 2008 N^o 21202.

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EIGENVALUES AND STRONG ORBIT EQUIVALENCE.

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Abstract. We give conditions on the subgroups of the circle to be realized as the subgroups of eigenvalues of minimal Cantor systems belonging to a determined strong orbit equivalence class. Actually, the additive group of continuous eigenvalues $E(X,T)$ of the minimal Cantor system (X,T) is a subgroup of the intersection $I(X,T)$ of all the images of the dimension group by its traces. We show, whenever the infinitesimal subgroup of the dimension group associated to (X,T) is trivial, the quotient group $I(X,T)/E(X,T)$ is torsion free. We give examples with non trivial infinitesimal subgroups where this property fails. We also provide some realization results.

1. Introduction

Two dynamical systems are orbit equivalent if there is a bijection between their phase spaces that preserves their structures (measure preserving, topological, etc.) and induces a one-to-one correspondence between their orbits. The notion of orbit equivalence arises first in the context of probability measure preserving group actions (measurable orbit equivalence), as a consequence of the study of von Neumann algebras [20, 29]. One of the most remarkable results in this theory establishes that there is only one orbit equivalence class among the free ergodic probability measure preserving actions of amenable groups [8, 23].
Motivated by the measurable orbit equivalence results, in particular, the characterization of the orbit equivalence classes in terms of von Neumann algebras [17, 18], Giordano, Putnam and Skau obtain in [12] one of the most important results in the context of the orbit equivalence from a topological point of view: the orbit equivalence classes of the minimal $\mathbb{Z}$-actions on the Cantor set are characterized in terms of the K_0 group of the associated C^*-algebra (see [26, 32] for an interplay between C^*-algebras and dynamics). As a consequence, they obtain that there are as many orbit equivalent classes as reduced simple dimension groups with distinguished order unit. Thus, unlike the measurable setting, in the topological context it is natural to ask for the dynamical properties which are preserved under orbit (or strong orbit) equivalence. For instance, in [15] it is shown that the set of invariant probability measures of a given minimal Cantor system is affinely isomorphic to the set of traces of the associated dimension group. Thus the set of invariant probability measures is preserved, up to affine homeomorphism, under strong orbit equivalence. On the contrary, within a strong orbit equivalence class it is possible to find a minimal Cantor systems having any possible entropy (see [2], [22] and [30] for the general case).
In this paper, we study the relation between (strong) orbit equivalence and the spectral properties of a system. We know from [22] that strong orbit equivalent minimal Cantor systems share the same subgroup of rational continuous eigenvalues. Thus, if the subgroup of rational continuous eigenvalues of a minimal Cantor system is not cyclic, then within its strong orbit equivalence class there is no mixing minimal Cantor systems. It is no longer true for the orbit equivalence as shown again in [22]. Indeed, Ormes proved (Theorem 8.2 in [22])

M. I. Cortez was partially funded by Anillo Research Project 1103 DySyRF and Fondecyt Research Project 1140213. She thanks the hospitality of the LAMFA UMR 7352 CNRS-UPJV and the "poste rouge" CNRS program.

that in a prescribed orbit equivalence class it is possible to realize any countable subgroup of the circle as a group of measurable eigenvalues.
In this work we investigate the case of non-rational eigenvalues and whether the dimension groups induce restrictions (other than those due to the rational eigenvalues) on the groups of eigenvalues that can be realized within this given strong orbit equivalence class.
It happens that a first restriction has been shown in [16] : the additive group of eigenvalues, $E(X,T)$, of a minimal Cantor system (X,T), is a subgroup of the intersection of all the images of the dimension group by its traces. Dynamically speaking, it is a subgroup of $I(X,T) = \cap_{\mu\in\mathcal{M}(X,T)} \left\{\int f d\mu | f \in C(X,\mathbb{Z})\right\}$, where $\mathcal{M}(X,T)$ is the set of T-invariant probability measures of (X,T) and $C(X,\mathbb{Z})$ is the set of continuous functions from X to $\mathbb{Z}$. An other proof of this observation can be found in [5] but it was not pointed out.
In this paper we prove the following strong restriction.

Theorem 1. *Suppose that (X,T) is a minimal Cantor system such that the infinitesimal subgroup of the dimension group $K^0(X,T)$ is trivial. Then the quotient group $I(X,T)/E(X,T)$ is torsion free.*

To illustrate this result, take $K^0(X,T) = \mathbb{Z} + \alpha\mathbb{Z} = I(X,T)$, with α irrational. This is the case for a Sturmian subshift. Then within the strong orbit equivalence class of (X,T) the only groups of continuous eigenvalues that can be realized are $\mathbb{Z}$, which will provide topologically weakly mixing minimal Cantor systems, and $\mathbb{Z} + \alpha\mathbb{Z}$. Moreover, both can be realized, in the first case using results in [22] and in the second case it is realized by a Sturmian subshift.
Relations between additive eigenvalues and topological invariants can be found in [28, 27, 24, 11], but they do not apply to Cantor systems.

In Section 2 we recall the concept of Kakutani-Rohlin partitions that will be necessary through this paper. The next section is concerns the notions and definitions we will need. In particular, we recall the algebraic notions and dynamical interpretations of dimension group, trace, infinitesimal and rational subgroup. Section 3 is devoted to the proof of our main result: Theorem 1. To this aim we use a precise description of entrance times with respect to some well-chosen Kakutani-Rohlin partitions. We follow the approach proposed in [3, 4] to tackle eigenvalue problems. Apart from Theorem 1, there are three results that could be of independent interest. We obtain a new necessary condition to be an eigenvalue (Proposition 9). We give an elementary proof of the fact that $E(X,T)$ is included in $I(X,T)$ (Proposition 11). For every α in $I(X,T)$, we show there exists a continuous function $f : X \to \mathbb{Z}$ such that $\alpha = \int f d\mu$ for all T-invariant measure μ (Lemma 12).
In the last section we provide realization examples around Theorem 1. In particular we give an example where the automorphism group (*i.e.* the group of self homeomorphism of the space commuting with the map T) is not invariant under orbit equivalence.

2. Definitions and background

2.1. **Dynamical systems.** We introduce here the notations and recall some classical facts. We refer to [25] for a more detailed expository. By a *topological dynamical system*, we mean a couple (X,T) where X is a compact metric space and $T : X \to X$ is a homeomorphism. We say that it is a *Cantor system* if X is a Cantor space; that is, X has a countable basis of its topology which consists of closed and open sets (clopen sets) and does not have isolated points. It is *minimal* if it does not contain any non empty proper closed T-invariant subset. A dynamical system (Y,S) is called a *factor* of (X,T) if there is a continuous and onto map $\phi : X \to Y$, called a *factor map*, such that $\phi \circ T = S \circ \phi$. If ϕ is one-to-one we say that ϕ is a *conjugacy* and that (X,T) and (Y,S) are *conjugate*. If (X,T) is minimal and $\phi : X \to Y$

is a factor map for which there exists $x \in X$ such that $\sharp\phi^{-1}(\phi(x)) = 1$, we say that ϕ is an *almost 1-1 factor map* and (X,T) is an *almost 1-1 extension* of (Y,S).
We denote by $\mathcal{M}(X,T)$ the set of all T-invariant probability measure μ, defined on the Borel σ-algebra $\mathcal{B}_X$ of X. For such a measure μ, the quadruple $(X,\mathcal{B},\mu_X,T)$ is called a *measurable dynamical system*. This system is called *ergodic* if any T-invariant measurable set has measure 0 or 1. Two measurable dynamical systems $(X,\mathcal{B},\mu,T)$ and $(Y,\mathcal{B}',\nu,S)$ are *measure theoretically conjugate* if we can find invariant subsets $X_0 \subset X$, $Y_0 \subset Y$ with $\mu(X_0) = \nu(Y_0) = 1$ and a bimeasurable bijective map $\psi\colon X_0 \to Y_0$ such that $S \circ \psi = \psi \circ T$ and $\mu(\psi^{-1}B) = \nu(B)$ for any $B \in \mathcal{B}'$.
A complex number λ is a *continuous eigenvalue* (resp. a *measurable eigenvalue*) of (X,T) if there exists a continuous (resp. integrable with respect to an invariant measure) function $f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called a *continuous eigenfunction*, associated to λ . Of course any continuous eigenvalue is a measurable one for any fixed measure. Hence, every eigenvalue is of modulus 1, i.e., belongs to the circle $\mathbb{S}^1 = \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$, and every eigenfunction has a constant modulus. Notice that any continuous eigenfunction provides a factor map from (X,T) to a rotation.
In this work we are mainly concerned with continuous eigenvalues $\lambda = \exp(2i\pi\alpha)$ of minimal Cantor systems. Such α is call an *additive continuous eigenvalue* of (X,T), and the set of additive continuous eigenvalue, denoted $E(X,T)$, is an additive subgroup of $\mathbb{R}$ called the *group of additive continous eigenvalues*. It is well-known that $E(X,T)$ is countable and contains $\mathbb{Z}$.
We say two dynamical systems (X,T) and (Y,S) are *orbit equivalent* (OE) whenever there exists a homeomorphism $\phi : X \to Y$ sending orbits to orbits: for all $x \in X$,

$$\phi\left(\{T^n x \mid n \in \mathbb{Z}\}\right) = \{S^n\phi(x) \mid n \in \mathbb{Z}\}.$$

This induces the existence of maps $\alpha : X \to \mathbb{Z}$ and $\beta : X \to \mathbb{Z}$ satisfying: for all $x \in X$,

$$\phi \circ T(x) = S^{\alpha(x)} \circ \phi(x) \text{ and } \phi \circ T^{\beta(x)}(x) = S \circ \phi(x).$$

When α and β have both at most one point of discontinuity, we say (X,T) and (Y,S) are *strongly orbit equivalent* (SOE). We recall below the seminal result in [12] that characterized these equivalence in terms of dimension groups.

Theorem 2. [12] *Let (X,T) and (Y,S) be two minimal Cantor dynamical systems. The following are equivalent:*

(1) *(X,T) and (Y,S) are strong orbit equivalent.*
(2) *$K^0(X,T)$ and $K^0(Y,S)$ are isomorphic as dimension groups with order units.*

The following are also equivalent:

(1) *(X,T) and (Y,S) are orbit equivalent.*
(2) *$K^0(X,T)/\mathrm{Inf}(K^0(X,T))$ and $K^(Y,S)/\mathrm{Inf}(K^0(Y,S))$ are isomorphic as dimension groups with order units.*

2.2. **Partitions and towers.** Sequences of partitions associated to minimal Cantor systems were used in [15] to build representations of such systems as adic transformations on ordered Bratteli diagrams. Here we do not introduce the whole formalism of Bratteli diagrams since we will only use the language describing the tower structure, even if both languages are very close. We recall some definitions and fix some notations.
For a minimal Cantor system (X,T), a *clopen Kakutani-Rohlin partition* (CKR partition) is a partition $\mathcal{P}$ of X given by

$$\mathcal{P} = \{T^j B(k);\ 1 \le k \le C,\ 0 \le j < h(k)\}, \tag{2.1}$$

where $C, h(1), \dots, h(k)$ are positive integers, and $B(1), \dots, B(C)$ are clopen subsets of X such that

$$\bigcup_{k=1}^{C} T^{h(k)} B(k) = \bigcup_{k=1}^{C} B(k).$$

The set $B = \bigcup_{1 \le k \le C} B(k)$ is called the *base* of $\mathcal{P}$. Let

$$\left\{ \mathcal{P}_n = \{ T^j B_n(k); 1 \le k \le C_n,\ 0 \le j < h_n(k) \} \right\}_{n \in \mathbb{N}} \tag{2.2}$$

be a sequence of CKR partitions. For every $n \in \mathbb{N}$, we denote B_n the base of $\mathcal{P}_n$. To be coherent with the notations of [3], we assume that $\mathcal{P}_0$ is the trivial partition, that is, $B_0 = X$, $C_0 = 1$ and $h_0(1) = 1$, and for the partition $\mathcal{P}_1$, $h_1(k) = 1$ for any integer $1 \le k \le C_1$.
We say that the sequence $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ is *nested* if it satisfies: for any integer $n \in \mathbb{N}$

(KR1) $B_{n+1} \subseteq B_n$;

(KR2) $\mathcal{P}_{n+1} \succeq \mathcal{P}_n$; i.e., for any $A \in \mathcal{P}_{n+1}$ there exists an atom $A' \in \mathcal{P}_n$ such that $A \subseteq A'$;

(KR3) $\bigcap_{n \in \mathbb{N}} B_n$ consists of a unique point;

(KR4) the sequence of partitions spans the topology of X.

In [15] it is proven that given a minimal Cantor system (X, T), there exists a nested sequence of CKR partitions fulfilling **(KR1)**–**(KR4)** with the following additional technical conditions: for any integer $n \ge 0$,

(KR5) for any $1 \le k \le C_n$, $1 \le l \le C_{n+1}$, there exists an integer $0 \le j < h_{n+1}(l)$ such that $T^j B_{n+1}(l) \subseteq B_n(k)$;

(KR6) $B_{n+1} \subseteq B_n(1)$.

We associate to the sequence $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$, the sequence of matrices $\{M_n\}_{n \ge 1}$, where $M_n = (m_n(l,k))_{1 \le l \le C_n, 1 \le k \le C_{n-1}}$ is given by

$$m_n(l,k) = \#\{0 \le j < h_n(l);\ T^j B_n(l) \subseteq B_{n-1}(k)\}.$$

Notice that **(KR5)** is equivalent to: for any $n \ge 1$, the matrix M_n has positive entries.
For $n \ge 0$, we set $H_n = (h_n(l); 1 \le l \le C_n)^T$. Since the sequence of partitions is nested, we have $H_n = M_n H_{n-1}$ for any $n \ge 1$. Notice also that, by the convention,

$$M_1 = H_1 = (1, \dots, 1)^T. \tag{2.3}$$

For $n > m \ge 0$, we define

$$P_{n,m} = M_n M_{n-1} \dots M_{m+1}, P_1 = M_1, \text{ and } P_{n+1} = P_{n+1,1}. \tag{2.4}$$

Clearly, we have the relations

$$P_{n,m}(l,k) = \#\big\{0 \le j < h_n(l);\ T^j B_n(l) \subseteq B_m(k)\big\}, \tag{2.5}$$

for $1 \le l \le C_n,\ 1 \le k \le C_m$, and

$$P_{n+1,m} H_m = H_{n+1} = P_{n+1} H_1. \tag{2.6}$$

Along the paper, we will strongly use a technique that we call *telescoping*: That is, starting from a sequence of CKR partitions $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ fulfilling **(KR1)**–**(KR6)**, we will consider an infinite subsequence of partitions satisfying an additional property. Actually, it is plain to check that any such subsequence of CKR partition satisfies also **(KR1)**–**(KR6)**. Moreover, the sequences of the associated matrices of the type $\{M_n\}_{n \in \mathbb{N}}$ and $\{P_{n,m}\}_{n > m \ge 0}$ are subsequences of the previous ones.

2.3. Dimension groups, traces, infinitesimals and rational subgroups.

2.3.1. *Dimension groups.* We recall here some basic definitions of the algebraic notion of dimension groups arising from the C^*-alegbras. Relations with dynamical systems will be explain in the next subsection. Most of the notions arises from [10].
By an *ordered group* we shall mean a countable abelian group G together with a subset G^+, called the *positive cone*, such that $G^+ - G^+ = G$, $G^+ \cap (-G^+) = \{0\}$ and $G^+ + G^+ \subset G^+$. We shall write $a \leq b$ if $b - a \in G^+$. We say that an ordered group is *unperforated* if $a \in G$ and $na \in G^+$ for some $a \in G$ and $n \in \mathbb{N}$ implies that $a \in G^+$. Observe that an unperforated group is torsion free. We say (G, G^+) is *acyclic* whenever G is not isomophic to $\mathbb{Z}$. By an *order unit* for (G, G^+) we mean an element $u \in G^+$ such that for every $a \in G$, $a \leq nu$ for some $n \in \mathbb{N}$.

Definition 1. A *dimension group* (G, G^+, u) with distinguished order unit u is an unperforated ordered group (G, G^+) satisfying the *Riesz interpolation property*, i.e., given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ $(i, j = 1, 2)$, there exists $c \in G$ with $a_i \leq c \leq b_j$, $(i, j = 1, 2)$.

We say that two dimension groups (G_1, G_1^+, u_1) and (G_2, G_2^+, u_2) are isomorphic whenever there exists an order isomorphism $\phi : G_1 \to G_2$, i.e., ϕ is a group isomorphism such that $\phi(G_1^+) = G_2^+$, and $\phi(u_1) = u_2$. An *order ideal* is a subgroup J such that $J = J^+ - J^+$ (where $J^+ = J \cap G^+$) and $0 \leq a \leq b \in J$ implies $a \in J$. A dimension group (G, G^+, u) is *simple* if it contains no non-trivial order ideals. It is easily seen that (G, G^+) is a simple dimension group if and only if every $a \in G^+ \backslash \{0\}$ is an order unit. Moreover, an unperforated simple ordered group is acyclic if and only if it satisfies the Riesz interpolation property (see [9]). Thus, the dimension groups are all acyclic.

2.3.2. *Traces.* Let (G, G^+, u) be a simple dimension group with distinguished order unit u. We say that a homomorphism $p : G \to \mathbb{R}$ is a *trace* (also called a *state*) if p is non negative (i.e., $p(G^+) \geq 0$) and $p(u) = 1$. We denote the collection of all traces on (G, G^+, u) by $S(G, G^+, u)$. Now $S(G, G^+, u)$ is a convex compact subset of the locally convex space $\mathbb{R}^G$ endowed with the product topology. In fact, one can show that $S(G, G^+, u)$ is a Choquet simplex. It is a fact (see [15]) that $S(G, G^+, u)$ determines the order on G. Actually,

$$G^+ = \{a \in G;\ p(a) > 0, \forall p \in S(G, G^+, u)\} \cup \{0\}.$$

As we will see later, the following group is fundamental in the study of continuous eigenvalues of minimal Cantor systems.

Definition 2. Let (G, G^+, u) be an ordered group with unit. We call *image subgroup* of (G, G^+, u) the subgroup of $\mathbb{R}$ given by

$$I(G, G^+, u) = \bigcap_{\tau \in S(G, G^+, u)} \tau(G).$$

2.3.3. *Infinitesimals.* Let (G, G^+) be a simple dimension group and let $u \in G^+ \backslash \{0\}$. We say that an element $a \in G$ is *infinitesimal* if $-\epsilon u \leq a \leq \epsilon u$ for all $0 < \epsilon \in \mathbb{Q}^+$ (for $\epsilon = \frac{p}{q}$, $p, q \in \mathbb{N}$, then $a \leq \epsilon u$ means that $qa \leq pu$).
It is easy to see that the definition does not depend upon the particular order unit u. An equivalent definition is: $a \in G$ is infinitesimal if $p(a) = 0$ for all $p \in S(G, G^+, u)$. The collection of infinitesimal elements of G forms a subgroup, *the infinitesimal subgroup of* G, which we denote by $\mathrm{Inf}(G)$.
Observe that the quotient group $G/\mathrm{Inf}(G)$ is also a simple dimension group for the induced order, and the infinitesimal subgroup of $G/\mathrm{Inf}(G)$ is trivial (see [15]). Furthermore, an order unit for G maps to an order unit for $G/\mathrm{Inf}(G)$. Moreover the traces space of G and $G/\mathrm{Inf}(G)$ are isomorphic.

When $S(G,G^+,u)$ consists of a unique trace, notice that $G/\mathrm{Inf}(G)$ is isomorphic to $(I(G,G^+,u), I(G,G^+,u)\cap \mathbb{R}^+, 1)$, as ordered groups with unit.

2.3.4. *Rational subgroups.* By a *rational group* H we shall mean a subgroup of $\mathbb{Q}$ that contains $\mathbb{Z}$. We say that H is a *cyclic* rational group if H is isomorphic to $\mathbb{Z}$. Clearly $(H, H\cap\mathbb{Q}^+, 1)$ is a simple dimension group with distinguished order unit 1. For a simple dimension group with order unit (G,G^+,u), we define the *rational subgroup* of G, denoted $\mathbb{Q}(G,G^+,u)$ (or $\mathbb{Q}(G,u)$ for short), by

$$\mathbb{Q}(G,u) = \{m/n;\ n\in\mathbb{N}^*, m\in\mathbb{Z}, \exists g\in G,\ ng = mu\}.$$

The notion of rational subgroup of a dimension group with distinguished order unit depends heavily upon the choice of the order unit.
Notice that for $n,m\in\mathbb{Z}$ and $g\in G$ such that $ng=mu$, one gets, for any trace τ, $\tau(g) = m/n$. Consequently, $\mathbb{Q}(G,u)$ is a subgroup of $I(G,G^+,u)$.

2.4. **Dynamical interpretation of dimension groups, traces, infinitesimals and rational subgroups.** We consider here (X,T) a minimal Cantor dynamical system.

2.4.1. *"Dynamical" dimension groups.* We denote by $C(X,\mathbb{Z})$ the set of continuous maps from X to $\mathbb{Z}$. Consider the map $\beta : C(X,\mathbb{Z}) \to C(X,\mathbb{Z})$ defined by $\beta f = f\circ T - f$ for all $f\in C(X,\mathbb{Z})$. The images of β are called *coboundaries.* Let $H(X,T)$ be the quotient group $C(X,\mathbb{Z})/\beta C(X,\mathbb{Z})$. The class of a function $f\in C(X,,\mathbb{Z})$ in this quotient is denoted by $[f]$. We call *order unit* the class $[1]$ of the constant function equal to 1.
The positive cone, $H^+(X,T,\mathbb{Z})$, is the set of classes of non-negative functions $C(X,\mathbb{N})$. Finally, the triple

$$K^0(X,T) = (H(X,T,\mathbb{Z}), H^+(X,T,\mathbb{Z}), [1]).$$

is an ordered group with order unit. It is moreover a dimension group and, which is less immediate, a converse also holds.

Theorem 3. [15] *If (X,T) is a minimal Cantor system, then $K^0(X,T)$ is a simple dimension group. Furthermore, if (G,G^+,u) is a simple dimension group then there is a minimal Cantor system (X,T) such that $K^0(X,T)$ and (G,G^+,u) are isomorphic.*

2.4.2. *Traces are invariant measures.* Given any invariant probability measure μ of the system (X,T), we associate a trace τ_μ on $K^0(X,T)$ defined by $\tau_\mu([f]) := \int f d\mu$ for any $f\in C(X,\mathbb{Z})$. It is shown in [15], that the map $\mu\mapsto\tau_\mu$ is an affine isomorphism from the space of T-invariant probability measures $\mathcal{M}(X,T)$ to the traces space $S(K^0(X,T))$.
We denote by $I(X,T)$ the image subgroup $I(K^0(X,T))$. Rephrasing the definition of the image subgroup in dynamical terms, it is clear that

$$I(X,T) = \bigcap_{\mu\in\mathcal{M}(X,T)} \left\{\int f d\mu; f\in C(X,\mathbb{Z})\right\}.$$

2.4.3. *Infinitesimals are functions with zero integral for all invariant measures.* We have seen that $\mathrm{Inf}(K^0(X,T)) = \{g\in K^0(X,T); \tau(g) = 0 \text{ for all traces } \tau\}$. Thus, due to the identification described before, we also have

$$\mathrm{Inf}(K^0(X,T)) = \left\{[f]\in K^0(X,T); \int f d\mu = 0 \text{ for all } \mu\in\mathcal{M}(X,T)\right\}.$$

Observe that if (X,T) is uniquely ergodic, then $K^0(X,T)/\mathrm{Inf}(K^0(X,T))$ is isomorphic to $(I(X,T), I(X,T)\cap\mathbb{R}^+, 1)$, as ordered groups with unit.

2.4.4. *The rational subgroup is the group of rational continuous eigenvalues.* For any $m/n \in \mathbb{Q}(K^0(X,T))$, there exists a class $[f]$ such that $n[f] = m[1_X]$ and thus $\int f d\mu = m/n$ for any T-invariant measure μ. Thus, we have the inclusion

$$\mathbb{Q}(K^0(X,T)) \subset I(X,T) \cap \mathbb{Q}.$$

The following theorem gives a clear dynamical interpretation of such elements.

Theorem 4. [12, 22] *Let (X,T) be a minimal Cantor system and let μ be any T-invariant measure. Then, a rational $\frac{p}{q}$ is an additive continuous eigenvalue of (X,T), i.e., belongs to $E(X,T)$, if and only if $\frac{p}{q} = \int f d\mu$ for some $f \in C(X,\mathbb{Z})$. Or, equivalently,*

$$E(X,T) \cap \mathbb{Q} = \mathbb{Q}(K^0(X,T)).$$

As observe in [12] (see also [22]), this implies the following result.

Corollary 5. *Let (X,T) and (Y,S) be two strong orbit equivalent minimal Cantor systems (i.e., $K^0(X,T)$ is isomorphic to $K^0(Y,S)$). Then, (X,T) and (Y,S) share the same rational continuous eigenvalues, that is*

$$E(X,T) \cap \mathbb{Q} = E(Y,S) \cap \mathbb{Q}.$$

3. Group of eigenvalues and image of traces

In this section (X,T) stands for a given minimal Cantor system. We fix a sequence $\{\mathcal{P}_n\}_n$ of CKR partitions of (X,T) satisfying (**KR1**)-(**KR6**) and (2.3). We recall such a partition always exists. Once it is fixed, we freely use the notations of Section 2.2.

3.1. **Some necessary conditions to be an eigenvalue.** The following results are fundamental in our study of eigenvalues of minimal Cantor systems. We set also some notations.

Lemma 6. [4, Theorem 3, Theorem 5] *Let (X,T) be a minimal Cantor system and let $\alpha \in E(X,T)$. Then, there exist an integer $m > 1$, a real vector v_m and an integer vector w_m such that*

(1) $\alpha P_m H_1 = v_m + w_m$, *and*
(2) $\sum_{n>m} \|P_{n,m} v_m\|_\infty < \infty$,

where $\|\cdot\|_\infty$ denotes the supremum norm.

For any given T-invariant probability measure μ of (X,T) we set

$$\mu_n = (\mu(B_n(k)))^T_{1\le k\le C_n}$$

and we call it the *measure vector* of (X,T). It is easy to check it fullfills, for all $1 \le m < n$, the following identities:

$$\mu_1^T H_1 = 1 \text{ and } \mu_m^T = \mu_n^T P_{n,m}. \tag{3.7}$$

Lemma 7. [4] *With the conditions and the notations of Lemma 6, for any T-invariant probability measure μ of (X,T), for any integer $n \ge m$,*

$$\alpha = \mu_m^T w_m \quad \text{and} \quad 0 = \mu_n^T P_{n,m} v_m.$$

Proof. For any integer $n > m > 1$, we set $v_n = P_{n,m} v_m$ and $w_n = P_{n,m} w_m$. Observe that Relation (3.7) and Lemma 6 imply for any integer $n > m$,

$$\alpha = \alpha\mu_1^T H_1 = \mu_m^T P_m H_1 \alpha = \mu_m^T w_m + \mu_m^T v_m = \mu_m^T w_m + \mu_n^T P_{n,m} v_m.$$

Since $\alpha - \mu_m^T w_m$ does not depend on n and $\lim_{n\to\infty} \|\mu_n^T P_{n,m} v_m\|_\infty = 0$, we deduce that $\mu_m^T v_m = \mu_n^T P_{n,m} v_m = 0$, for every $n > m$. $\square$

The following proposition and lemma will provide key arguments in the proof of our main result Theorem 1. For its proof we need to introduce some crucial quantities as it can be seen in the series of papers [5, 3, 4] and [7].
For any integer $n \in \mathbb{N}$, we define the *entrance time* $r_n(x)$ of a point $x \in X$ to the base B_n by $r_n(x) = \min\{j \geq 0; T^j x \in B_n\}$. The *suffix map of order* n is the map $s_n : X \to \mathbb{N}^{C_n}$ given by

$$(s_n(x))_k = \sharp\{j \in \mathbb{N};\ 0 \leq j < r_{n+1}(x), T^j x \in B_n(k)\}$$

for every $k \in \{1, \ldots, C_n\}$. A classical computation gives[1] (see for example [3])

$$r_n(x) = \sum_{k=1}^{n-1} \langle s_k(x), P_k H_1 \rangle, \tag{3.8}$$

where $\langle v, v' \rangle = v^T v'$ stands for the usual scalar product.

Proposition 8. [3] *Let (X, T) be a minimal Cantor system and let $\alpha \in \mathbb{R}$. The following conditions are equivalent,*

1. *α belongs to $E(X, T)$;*
2. *the sequence of functions $(\exp(2i\pi\alpha r_n(\cdot)))_n$ converges uniformly.*

Proposition 9. *With the conditions and the notations of Lemma 6, we have*

$$\max_{x \in X} |\langle s_n(x), P_{n,m} v_m \rangle| \to_{n \to +\infty} 0.$$

Proof. Let us denote by $|||\cdot|||$ the distance to the closest integer. The sequence $(|||\alpha r_n|||)_n$ is a uniform Cauchy sequence (Proposition 8) and $\alpha(r_{n+1} - r_n) = \langle s_n, \alpha P_n H_1 \rangle$. Therefore $|||\langle s_n, \alpha P_n H_1 \rangle|||$ converges to zero. We have $\alpha P_n H_1 = P_{n,m} v_m + P_{n,m} w_m$. Since $P_{n,m} w_m$ is an integer vector, we obtain $|||\langle s_n, \alpha P_n H_1 \rangle||| = |||\langle s_n, P_{n,m} v_m \rangle|||$. Consequently Lemma 6 (2) ensures for any $\epsilon \in (0, 1/8)$, there exists n_0 such that for all $n \geq n_0$ and all x

$$||P_{n,m} v_m|| < \epsilon < 1/8 \text{ and } |||\langle s_n(x), P_{n,m} v_m \rangle||| < \epsilon.$$

We may write $\langle s_n(x), P_{n,m} v_m \rangle = \epsilon_n(x) + E_n(x)$ with $|\epsilon_n(x)| < \epsilon$ and $E_n(x)$ an integer. Notice that $(\epsilon_n)_n$ uniformly converges to 0.
Consider the set $A = \{x \in X; E_n(x) = 0\}$. Observe that $\bigcap_n B_n$ is contained in A so it is non empty. It is not difficult to check that A is closed. Let us check it is T-invariant. We fix some $x \in A$. It is straightforward to verify that there are only three distincts possible cases for $s_n(x)$: $s_n(Tx) = s_n(x)$; $s_n(x) = 0$ and $s_n(Tx) \neq 0$; or $s_n(x) \neq 0$ with $s_n(Tx) = s_n(x) - e$ for some vector e from the canonical base. The first case is easy to handle. For the second one, notice it implies $x \in B_{n+1}$. So by the very definitions, there exist two vectors e_1, e_2 from the canonical base such that $s_n(Tx) = M_{n+1}^t e_1 - e_2$. It follows that

$$\begin{aligned} |\langle s_n(Tx), P_{n,m} v_m \rangle| =& |\langle M_{n+1}^t e_1, P_{n,m} v_m \rangle - \langle e_2, P_{n,m} v_m \rangle| = |\langle e_1, P_{n+1,m} v_m \rangle - \langle e_2, P_{n,m} v_m \rangle| \\ \leq& ||P_{n+1,m} v_m|| + ||P_{n,m} v_m|| < 1/4, \end{aligned}$$

so that $E_n(Tx) = 0$ and $Tx \in A$.
For the last case, consider

$$\begin{aligned} |E_n(Tx) - E_n(x)| =& |\epsilon_n(x) - \epsilon_n(Tx) + \langle s_n(Tx), P_{n,m} v_m \rangle - \langle s_n(x), P_{n,m} v_m \rangle| \\ \leq& \frac{1}{4} + |\langle e, P_{n,m} v_m \rangle| \leq \frac{1}{4} + ||P_{n,m} v_m|| \leq \frac{1}{2}. \end{aligned}$$

[1] Observe that by the conventions on $\mathcal{P}_0$ and $\mathcal{P}_1$, we have $s_0(x) = 0$ for any $x \in X$.

Therefore we have $E_n(Tx) = E_n(x) = 0$ and $Tx \in A$. Finally, by minimality, we obtain that $A = X$ which implies that

$$|\langle s_n(x), P_{n,m}v_m\rangle| = |\epsilon_n(x)| = |||\langle s_n(x), P_{n,m}v_m\rangle||| < \epsilon.$$

This achieves the proof. □

Lemma 10. *Let (X, T) be a minimal Cantor system. Then, for any $k \in \mathbb{Z}^*$ and $\alpha \in \mathbb{R}$ such that $k\alpha$ belongs to $E(X, T)$ and α does not, there exist an integer $m > 1$, a real vector v_m and an integer vector w_m such that*

1. $k\alpha P_m H_1 = w_m + v_m$,
2. $\sum_{n>m} \|P_{n,m}v_m\|_\infty < \infty$,
3. *for every measure $\mu \in \mathcal{M}(X, T)$ and integer $n \geq m$, $\langle \mu_n, P_{n,m}v_m\rangle = 0$.*
4. *the vector $\frac{1}{k}P_{n,m}w_m$ is not an integer vector for infinitely many integers $n > m$.*

Proof. From Lemma 6 and Lemma 7, there exist a positive integer m, a real vector v_m and an integer vector w_m satisfying the items (1), (2) and (3). From Proposition 8, the sequence $(k\alpha r_n)_n$ converges uniformly (mod $\mathbb{Z}$). Moreover, the relation (3.8) gives us for any integer $n > m + 1$

$$\begin{aligned}\alpha r_n(x) &= \sum_{i=1}^{n-1} \alpha\langle s_i(x), P_i H_1\rangle \\ &= \sum_{i=1}^{m}\langle s_i(x), \alpha P_i H_1\rangle + \sum_{i=m+1}^{n-1}\langle s_i(x), P_{i,m}\left(\frac{1}{k}v_m + \frac{1}{k}w_m\right)\rangle.\end{aligned}$$

Suppose that $\frac{1}{k}P_{i,m}w_m$ is an integer vector for any large enough integer i. To obtain a contradiction, by Proposition 8, it suffices to show that $(\alpha r_n)_n$ is a Cauchy sequence. As $(k\alpha r_n)_n$ converges uniformly (mod $\mathbb{Z}$), we deduce that $(\sum_{i=m+1}^{n}\langle s_i(x), P_{i,m}v_m\rangle)_n$ converges uniformly (mod $\mathbb{Z}$). Hence, given $\epsilon \in (0, \frac{1}{2})$, there exists an integer n_0 such that for any integer $n \geq n_0$, any integer $p \geq 0$ and $x \in X$, there exists an integer $E_p(x)$ such that

$$|\sum_{i=n}^{n+p}\langle s_i(x), P_{i,m}v_m\rangle - E_p(x)| < \frac{\epsilon}{4}.$$

By Proposition 9, we can assume that the integer n_0 is sufficiently large to have

$$\max_{x\in X}|\langle s_n(x), P_{n,m}v_m\rangle| < \frac{\epsilon}{4} \quad \forall n \geq n_0. \tag{3.9}$$

Now fix $n \geq n_0$. Notice that

$$\begin{aligned}E_{p+1}(x) - E_p(x) = \quad & E_{p+1}(x) - \textstyle\sum_{i=n}^{n+p+1}\langle s_i(x), P_{i,m}v_m\rangle - \left(E_p(x) - \sum_{i=n}^{n+p}\langle s_i(x), P_{i,m}v_m\rangle\right) \\ & + \langle s_{n+p+1}(x), P_{n+p+1,m}v_m\rangle.\end{aligned}$$

We deduce that $|E_{p+1}(x) - E_p(x)| < \epsilon < \frac{1}{2}$ and so $E_{p+1}(x) = E_p(x)$ for any $x \in X$, and $p \geq 0$.
The inequality (3.9) ensures that $E_0(x) = 0$, and thus $E_p(x) = 0$ for any $p \geq 0$. It follows that $(\sum_{i=m+1}^{n}\langle s_i(x), P_{i,m}v_m\rangle)_n$ is a uniform Cauchy type sequence in x, so the sequence $(\alpha r_n)_n$ converges uniformly (mod $\mathbb{Z}$). This gives a contradiction. □

3.2. **Group of eigenvalues versus image group of dimension group.** A fundamental fact for this work is the following proposition (Proposition 11). This has been previously shown in [16], but has been also obtained in [5] (Proposition 11) without to be claimed.

Proposition 11. *Let (X,T) be a minimal Cantor system. Then the set of additive continuous eigenvalues $E(X,T)$ is a subgroup of the image subgroup $I(X,T)$.*

Proof. It suffices to show that $E(X,T)$ is a subset of $I(X,T)$. From Lemma 7, there exist a positive integer m and a vector $w_m \in \mathbb{Z}^{C_m}$ such that for every invariant measure μ one gets

$$\alpha = \mu_m^T w_m = \int f d\mu,$$

where $f = \sum_{k=1}^{C_m} w_m(k) 1_{B_m(k)}$. This shows that α is in $I(X,T)$. □

For any $\alpha \in I(X,T)$, by definition, for every invariant measure μ there exists $f_\mu \in C(X,\mathbb{Z})$ (thus depending on μ) such that $\alpha = \int f_\mu d\mu$. In the next lemma, we show this function can be chosen independently of the invariant measures.

Lemma 12. *Let (X,T) be a minimal Cantor system. If α belongs to the image subgroup $I(X,T)$, then there exists a function $g \in C(X,\mathbb{Z})$ such that $\int g d\mu = \alpha$, for any measure $\mu \in \mathcal{M}(X,T)$.*

Proof. If $\mathcal{M}(X,T)$ is a singleton then the result is obvious. From now, we will assume that $\mathcal{M}(X,T)$ contains at least two elements. For any $g \in C(X,\mathbb{Z})$, we define

$$\mathcal{M}_g = \left\{ \mu \in \mathcal{M}(X,T);\ \int g d\mu = \alpha \right\}.$$

Observe that $\mathcal{M}_g$ is convex and closed with respect to the weak* topology in $\mathcal{M}(X,T)$. From the definition of $I(X,T)$, it is clear we have

$$\mathcal{M}(X,T) = \bigcup_{g \in C(X,\mathbb{Z})} \mathcal{M}_g.$$

Since $C(X,\mathbb{Z})$ is countable, Baire's theorem implies there exists a map $g_0 \in C(X,\mathbb{Z})$ such that $\mathcal{M}_{g_0}$ has a non empty interior. It follows that $I\colon \mu \mapsto \int g_0 d\mu$ is an affine map which is constant on an open set of $\mathcal{M}(X,T)$.
We get the conclusion by showing this map is constant. To prove this, let μ_0 be in the interior of $\mathcal{M}_{g_0}$, and let μ_1 be another measure in $\mathcal{M}(X,T)$. The map $t \in [0,1] \mapsto I(t\mu_0 + (1-t)\mu_1) \in \mathbb{R}$ is an affine map taking at least two times the same value α. So, it is a constant map and $I(\mu_0) = I(\mu_1) = \alpha$. Since the measure μ_1 is arbitrary, this concludes the proof. □

Remark 13. Observe that from [14, Lemma 2.4], Lemma 12 implies that for any $\alpha \in I(X,T) \cap (0,1)$ there exists a clopen set U such that $\alpha = \mu(U)$ for any T-invariant probability measure μ. In particular this is true when α is in $E(X,T)$.

By Theorem 3, this lemma can of course be rephrased in terms of dimension group (G,G^+,u). Let $\tilde{G}$ denote the group $\{g \in G;\ \tau(g) = \tau'(g) \text{ for every traces } \tau,\tau' \in S(G,G^+,u)\}$. Notice that the unit u and any infinitesimal in $\mathrm{Inf}(G)$ belong to $\tilde{G}$.

Corollary 14. *Let (G,G^+,u) be a simple dimension group. Then for any trace $\tau \in S(G,G^+,u)$, the morphism*

$$\tau\colon \tilde{G} \to I(G,G^+,u)$$

is a surjective order preserving morphism. In particular the dimensions groups $(\tilde{G}/\mathrm{Inf}(G), \tilde{G} \cap G^+/\mathrm{Inf}(G), [u])$, with $[u]$ the class of the unit, and $(I(G,G^+,u), I(G,G^+,u) \cap \mathbb{R}^+, 1)$ are isomorphic.

We are now able to prove our main theorem (Theorem 1).

Proof of Theorem 1. Suppose $I(X,T)/E(X,T)$ is not torsion free. Then, there exist $\alpha \in I(X,T)\setminus E(X,T)$ and an integer $k>1$ such that $k\alpha \in E(X,T)$. From Lemma 10, this implies there exist a positive integer m, vectors $w_m \in \mathbb{Z}^{C_m}$ and $v_m \in \mathbb{R}^{C_m}$ such that $k\alpha P_m H_1 = v_m + w_m$. We recall, we set $v_n = P_{n,m} v_m$ and $w_n = P_{n,m} w_m$ for every $n \geq m$, and the vector v_n is orthogonal to the vector of measures μ_n.
Since α is not a continuous eigenvalue, Lemma 10 implies there must be infinitely many n's such that the following set is not empty:

$$I_n = \{i \in \{1, \cdots, C_n\} : w_n(i) \text{ is not divisible by } k\}.$$

Telescoping the sequence of CKR partitions if needed, we can assume that $I_n \neq \emptyset$ for every sufficiently large n. For $1 \leq i \leq C_n$, we write

$$w_n(i) = k a_n(i) + b_n(i),$$

where $a_n(i) \in \mathbb{Z}$ and $b_n(i)$ is some integer in $\{0, \ldots, k-1\}$. Thus, the index i is in I_n if and only if $b_n(i) \neq 0$. Observe that for any $n \geq m$

$$\alpha = \alpha \mu_1^T H(1) = \mu_n^T P(n) H(1) \alpha = \frac{1}{k} \mu_n^T (v_n + w_n) = \frac{1}{k} \mu_n^T w_n = \frac{1}{k} \mu_n^T b_n + \mu_n^T a_n, \tag{3.10}$$

Since α is in $I(X,T)$, Lemma 12 implies there exists a function $f_1 \in C(X,\mathbb{Z})$ such that for every invariant probability measure μ,

$$\alpha = \int f_1 d\mu.$$

On the other hand,

$$\mu_n^T a_n = \int f_2 d\mu,$$

where $f_2 = \sum_{i=1}^{C_n} a_n(i) 1_{B_n(i)}$. Thus Equation (3.10) implies that for every invariant probability measure μ,

$$\frac{1}{k} \mu_n^T b_n = \int (f_1 - f_2) d\mu = \int f d\mu,$$

where $f = f_1 - f_2$. Thus we obtain,

$$\int k f d\mu = \mu_n^T b_n = \int \sum_{i=1}^{C_n} b_n(i) 1_{B_n(i)} d\mu.$$

Hence the map

$$h = kf - \sum_{i=1}^{C_n} b_n(i) 1_{B_n(i)}$$

belongs to $\mathrm{Inf}(X,T)$ that is assumed to be trivial. Consequently, there exists a map $g \in C(X,\mathbb{Z})$ such that $h = g - g \circ T$.
Choose $p \geq n$ such that f is constant on any atom of the partition $\mathcal{P}_p$ and such that the function g is constant on the base B_p. This is always possible as the sequence $\{\mathcal{P}_j\}_j$ satisfies (**KR1-KR6**). Let x be an element in $B_p \subset X$ and let $1 \leq i \leq C_p$ be such that $x \in B_p(i)$. We have then

$$
\begin{aligned}
0 = & g(x) - g(T^{h_p(i)}x) = \sum_{j=0}^{h_p(i)-1} h(T^j x) = \sum_{j=0}^{h_p(i)-1} kf(T^j x) - \sum_{j=0}^{h_p(i)-1} \sum_{l=1}^{C_n} b_n(l) 1_{B_n(l)}(T^j x) \\
= & \left(k \sum_{j=0}^{h_p(i)-1} f(T^j x) \right) - \sum_{l=1}^{C_n} b_n(l) \sum_{j=0}^{h_p(i)-1} 1_{B_n(l)}(T^j x) \\
= & \left(k \sum_{j=0}^{h_p(i)-1} f(T^j x) \right) - \sum_{l=1}^{C_n} b_n(l) P_{p,n}(i,l) \\
= & \left(k \sum_{j=0}^{h_p(i)-1} f(T^j x) \right) - (P_{p,n} b_n)(i).
\end{aligned}
$$

It follows that all the coordinates of $P_{p,n}b_n$ are divisible by k. On the other hand, for every $i \in I_p$ we have

$$
w_p(i) = (P_{p,n}w_n)(i) = P_{p,n}(ka_n + b_n)(i) = k(P_{p,n}a_n)(i) + (P_{p,n}b_n)(i),
$$

which contradicts that I_p is non empty. □

4. Examples

In the sequel we will construct various examples of minimal Cantor systems starting with an "abstract" simple dimension group (G, G^+, u) having some fixed properties, and then we will make use of Theorem 3 to have the existence of a minimal Cantor system (X, T) having this prescribed simple dimension group. Thus, we will most of the times avoid to mention this Theorem and we will identify $K^0(X, T)$ to (G, G^+, u).
In some examples we will need some classical definition we will not recall and that can be found in any book on *Ergodic Theory*, we refer the reader to [25] and [13].

4.1. **(Measurable) eigenvalues are not related with (strong) orbit equivalence.** At the difference of continuous additive eigenvalues, irrational additive measurable eigenvalues can not be interpreted in terms of dimension group. Counter examples mainly come from the powerful result obtained by N. Ormes in [22] (Theorem 6.1), generalizing Jewett-Krieger Theorem to strong orbit equivalence classes.

Theorem 15 ([22])**.** *Let (X, T) be a minimal Cantor systems and μ be an ergodic S-invariant Borel probability measure. Let (Y, S, ν) be an ergodic measurable dynamical system of a non-atomic Lebesgue probability space (Y, ν) such that $\exp(2i\pi/p)$ is an eigenvalue of (Y, S, ν) for any element $\frac{1}{p} \in \mathbb{Q}(K^0(X, T))$. Then, there exists a minimal Cantor system (X, T') strongly orbit equivalent to (X, T) such that (X, S', μ) is measurably conjugate to (Y, S, ν).*

In the same paper Ormes obtained the following remarkable generalization of Dye's theorem [8].

Theorem 16 ([22])**.** *Let (Y_1, S_1, ν_1) and (Y_2, S_2, ν_2) be ergodic dynamical systems of non-atomic Lebesgue probability spaces. There are minimal Cantor systems (X, T_1) and (X, T_2) and a Borel probability measure μ on X which is T_1 and T_2 invariant such that:*

1. *(X, T_i, μ) is measurably conjugate to (Y_i, S_i, ν_i), for $i = 1, 2$,*
2. *(X, T_1) is strongly orbit equivalent to (X, T_2) by the identity map.*

For example, one can take (Y_1, S_1, ν_1) and (Y_2, S_2, ν_2) with any countable groups (eventually trivial) G_1 and G_2 of measurable eigenvalues. The theorem asserts, there are strongly orbit equivalent minimal Cantor systems (X, T_1) and (X, T_2) that are measurably conjugate to the two previous dynamical systems, respectively. Of course the groups of measurable eigenvalues of (X, T_1) and (X, T_2) are respectively G_1 and G_2.

4.2. **Rational eigenvalues are not preserved under orbit equivalence.** Let $G = \mathbb{Z}\times\mathbb{Q}$, $u = (1, 1)$ and $a \in (0, 1) \cap \mathbb{Q}$. We set $G^+ = \{v \in G : \tau_a(v) > 0\} \cup \{0\}$, where $\tau_a(v) = av(1) + (1 - a)v(2)$, for every $v = (v(1), v(2)) \in G$. It is straightforward to check that (G, G^+, u) is a simple dimension group verifying the following:

- $\mathbb{Q}(G, G^+, u) = \mathbb{Z}$,
- $I(G, G^+, u) = \mathbb{Q}$,
- $S(G, G^+, u) = \{\tau_a\}$ and
- $G/\mathrm{Inf}(G)$, $I(G, G^+, u)$ and $\mathbb{Q}$ are isomorphic.

From Theorem [15] there exists minimal Cantor systems having (G, G^+, u) as a dimension group (up to isomorphism).
Proposition 11 implies that $E(X, T) \subset I(X, T) = I(G, G^+, u) = \mathbb{Q}$. Using Theorem 4 one obtains $E(X, T) = \mathbb{Q}(G, G^+, u) = \mathbb{Z}$. Thus, for every minimal Cantor system (X, T) such that $K^0(X, T)$ is isomorphic to (G, G^+, u), the group of eigenvalues $E(X, T)$ is equal to $\mathbb{Z}$.
Nevertheless $I(X, T) = \mathbb{Q}$, so $I(X, T)/E(X, T) = \mathbb{Q}/\mathbb{Z}$ is a torsion group. It is not difficult to see that $\mathrm{Inf}(K^0(X, T))$ is not trivial.

Remark 17. Thus, Theorem 1 is not true when $\mathrm{Inf}(K^0(X, T))$ is not trivial.

As (X, T) is uniquely ergodic, then, as observe in Section 2.4.3, $K^0(X, T)/\mathrm{Inf}(K^0(X, T))$ is isomorphic to $(I(X, T), I(X, T) \cap \mathbb{R}^+, 1) = (\mathbb{Q}, \mathbb{Q}^+, 1)$.
On the other hand, since every minimal Cantor system (Y, S) whose simple dimension group is isomorphic to $(\mathbb{Q}, \mathbb{Q}^+, 1)$ verifies $E(Y, S) = \mathbb{Q}$ (due to Proposition 11 and Theorem 4) and is orbit equivalent to (X, T) (by [12, Theorem 2.2]), we deduce that the continuous rational eigenvalues are not invariant under orbit equivalence, unlike strong orbit equivalence.

Remark 18. This example also shows that the groups of eigenvalues which are realizable among the class associated to G (i.e., a class of strong orbit equivalence) are not necessarily realizable by systems in the class determined by $G/\mathrm{Inf}(G)$ (i.e., the corresponding class of orbit equivalence) and viceversa.

4.3. $I(X, T)/E(X, T)$ **can be a torsion group, even if the rational subgroup of** $I(X, T)$ **is cyclic.** Let $\alpha \in (0, 1)$ be an irrational number such that $\alpha^2 \in \mathbb{Z} + \alpha\mathbb{Z}$, e.g. the inverse of the golden mean. Let (X, T) be the Sturmian subshift with angle 2α (we refer to [19] for the definition). Let us recall that its dimension group is isomorphic to $(\mathbb{Z}+2\alpha\mathbb{Z}, \mathbb{Z}+2\alpha\mathbb{Z}\cap\mathbb{R}^+, 1)$.
Let (Y, S) be any system such that its simple dimension group is isomorphic to $(\mathbb{Z} + \alpha\mathbb{Z}, \mathbb{Z} + \alpha\mathbb{Z} \cap \mathbb{R}^+, 1)$. These groups having a unique trace, (Y, S) and (X, T) are uniquely ergodic.
The rational subgroup of $\mathbb{Z} + \alpha\mathbb{Z}$ being $\mathbb{Z}$, Theorem 16 ensures that we can choose (Y, S) having no non trivial measurable eigenfunctions; i.e., (Y, S) is *weakly mixing*. We consider the product system $(X \times Y, T \times S)$ with the product action, i.e. $T \times S(x, y) = (Tx, Sy)$ for any $x \in X, y \in Y$.
We will show that

(1) for the image subgroup, we have $I(X \times Y, T \times S) = \mathbb{Z} + \alpha\mathbb{Z}$,
(2) for the set of additive eigenvalues $E(X \times Y, T \times S) = \mathbb{Z} + 2\alpha\mathbb{Z}$.

Thus the quotient group $I(X \times Y, T \times S)/E(X \times Y, T \times S)$ will be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We call π_X and π_Y the projections of $X \times Y$ on X and Y respectively, and μ and ν the unique invariant probability measures of (X,T) and (Y,S) respectively. Let λ be the Lebesgue measure in $\mathbb{S}^1$. We denote $(\mathbb{S}^1, R_{2\alpha})$ the rotation by angle $2\pi 2\alpha$ on the circle $\mathbb{S}^1$.
Since (Y,S) is weakly mixing, it is disjoint from $(\mathbb{S}^1, R_{2\alpha})$ (see Theorem 6.27 in [13]) and thus, the product measure $\lambda \times \nu$ is the unique invariant probability measure of $(\mathbb{S}^1 \times Y, R_{2\alpha} \times S)$. Since (X,T) and $(\mathbb{S}^1, R_{2\alpha})$ are measure theoretically conjugate, clearly $\mu \times \nu$ is the unique invariant probability measure of $(X \times Y, T \times S)$. Thus $(X \times Y, T \times S)$ is uniquely ergodic. Notice moreover that every open set has a positive measure. The Ergodic Theorem ensures then that the system $(X \times Y, T \times S)$ is minimal.
One can checks, for example using Lemma 2.6 in [14], that

$$\begin{aligned} \{\mu(A);\ A \text{ clopen subset of } X\} &= (\mathbb{Z} + 2\alpha\mathbb{Z}) \cap [0,1], \text{ and} \\ \{\nu(B);\ B \text{ clopen subset of } Y\} &= (\mathbb{Z} + \alpha\mathbb{Z}) \cap [0,1]. \end{aligned}$$

Also notice that any clopen set $C \subseteq X \times Y$ is a finite union of clopen sets of the kind $A \times B$, where A and B are clopen subsets of X and Y respectively. Hence, by the very definition of α, we get

$$I(X \times Y, T \times S) = \langle \{\mu(A)\nu(B) : A \subseteq X, B \subseteq Y \text{ clopen subsets}\} \rangle = \mathbb{Z} + \alpha\mathbb{Z}.$$

Remark 19. By Theorem 2, the system $(X \times Y, T \times S)$ is orbit equivalent to the initial Sturmian subshift (X,T). Observe their automorphism groups are nevertheless distincts: At the difference of the Sturmian subshift [21], the product system $(X \times Y, T \times S)$ admits automorphisms, namely $\mathrm{Id} \times S$ and $T \times \mathrm{Id}$, that are not power of the action $T \times S$.

Fundamental properties of Sturmian subshifts ensure there exists $\phi : X \to \mathbb{S}^1$ an almost 1-1 factor map from (X,T) to $(\mathbb{S}^1, R_{2\alpha})$. The function $\phi \circ \pi_X$ is then a factor map of the product system. This shows that $2\alpha \in E(X \times Y, T \times S)$.
We will show that α is not an additive eigenvalue of the system $(X \times Y, T \times S)$.
Suppose there exists a continuous eigenfunction $f \colon X \times Y \to \mathbb{S}^1$, such that $f \circ T \times S = e^{2i\pi\alpha} f$. Since the map $Id \times S$, product of the identity with the map S, commutes with the product action, the map $f \circ Id \times S$ is also a continuous eigenfunction associated with the same eigenvalue. So there is a constant $\lambda \in \mathbb{S}^1$ such that $f \circ Id \times S = \lambda f$. It follows for any $x \in X$, the map $y \mapsto f(x,y)$ is a continuous eigenfunction of the system (Y,S) associated to the eigenvalue λ. The system (Y,S) being weakly mixing, we get $\lambda = 1$ and $f(x,y)$ does not depend on y, we denote this last value $f(x)$. So the map $x \mapsto f(x)$ is a continuous eigenfunction of the system (X,T) associated with the eigenvalue $e^{2i\pi\alpha}$. This is impossible because this Sturmian subshift is an almost one-to-one extension of $(\mathbb{S}^1, R_{2\alpha})$.

We conclude that $E(X \times Y, T \times S) = \mathbb{Z} + 2\alpha\mathbb{Z}$.

According to Theorem 1, the infinitesimal subgroup of $K^0(X \times Y, T \times S)$ must be non-trivial. Let us give an example of a non-trivial infinitesimal element in the system $(X \times Y, T \times S)$.

From Lemma 12, there exists a function $g \in C(Y, \mathbb{Z})$ such that $\int g d\nu = 2\alpha$.
Claim. For any function $f \in C(X, \mathbb{Z})$ such that $\int f d\mu = \int g d\nu$, the function $F : (x,y) \in X \times Y \mapsto f(x) - g(y)$ is a non trivial infinitesimal.

A standard computation show us that $\int_{X \times Y} f(x) - g(y) d\mu \times \nu = 0$. Therefore it remains to prove that it is not a coboundary of $C(X \times Y, \mathbb{Z})$.
Let us assume that the function $f - g$ is such a coboundary. Then, there exists a function $H \in C(X \times Y)$ such that

$$f(x) - g(y) = H(x,y) - H(Tx, Sy) \qquad \forall x \in X, y \in Y.$$

By taking the integral of the former equality for the measure μ, we obtain

$$\begin{aligned} g(y) = & \int f d\mu - \int H(x,y)d\mu(x) + \int H(Tx,Sy)d\mu(x) \\ = & \int g d\nu - \int H(x,y)d\mu(x) + \int H(x,Sy)d\mu(x). \end{aligned}$$

By the Lebesgue's dominated convergence Theorem, the function $h\colon y \mapsto \int H(x,y)d\mu(x)$ is continuous. So $g - \int g d\nu = h \circ S - h$ is a real coboundary. Then, g taking integer values, the function $y \mapsto \exp\,(2i\pi h(y))$ defines a continuous eigenfunction associated to the additive eigenvalue $-\int g d\nu = -2\alpha$ for the system (Y,S). This is impossible because this system is weakly mixing. This proves our claim.

5. Some results about realization.

Definition 3. Let (G, G^+, u) be a simple dimension group with distinguished order unit. We define $\mathcal{E}(G, G^+, u)$ as the collection of all the subgroups Γ of $\mathbb{R}$ for which there exists a minimal Cantor system (X,T) such that $K^0(X,T)$ and $E(X,T)$ are isomorphic to (G, G^+, u) and Γ respectively.

In this section we are interested in a characterization of the family $\mathcal{E}(G, G^+, u)$, for a given simple dimension group. Most of our results are based in [31].

Remark 20. Proposition 11 implies that the elements in $\mathcal{E}(G, G^+, u)$ are subgroups of $I(G, G^+, u)$. If in addition $\inf(G) = \{0\}$, from Theorem 1 we get the following:

$$\mathcal{E}(G, G^+, u) \subseteq \{\Gamma : \text{ subgroup of } I(G, G^+, u) \text{ such that } I(G, G^+, u)/\Gamma \text{ is torsion free } \}.$$

5.1. **Basic example.** Let α be an irrational number. Consider $G = \mathbb{Z} + \alpha\mathbb{Z}$, $G^+ = G \cap \mathbb{R}^+$ and $u = 1$. Since the infinitesimal subgroup of (G, G^+, u) is trivial, the collection $\mathcal{E}(G, G^+, u)$ is a subfamily of $\{\mathbb{Z}, \mathbb{Z}+\alpha\mathbb{Z}\}$. It is known that the dimension group associated to the Sturmian subshift with angle α is isomorphic to (G, G^+, u) (see [19] [6]). Moreover, it is an almost 1-1 extension of the rotation with angle α. Hence its subgroup of eigenvalues equals G. On the other hand, Theorem 15 implies there exists a minimal Cantor system having no non trivial eigenvalues whose dimension group is isomorphic to (G, G^+, u). Thus we get

$$\mathcal{E}(G, G^+, u) = \{\mathbb{Z}, \mathbb{Z} + \alpha\mathbb{Z}\}.$$

5.2. **Eigenvalues and dimension subgroups.** Let (G, G^+, u) and (H, H^+, w) be two simple dimension group with distinguished order unit. An *order embedding* is a monomorphism $i : H \to G$ such that $i(h) \in G^+$ if and only if $h \in H^+$ and $i(w) = u$. An order embedding always induces an affine homomorphism $i^* : S(G) \to S(H)$ by

$$i^*(\tau)(h) = \tau(i(h)), \text{ for every } \tau \in S(G) \text{ and } h \in H.$$

We use the next two lemmas to show that a minimal Cantor system and any of its almost 1-1 extensions share their maximal equicontinuous factor.
The second lemma is a converse of the first one. A proof of the next result, in a more general context, can be found in [1, Chapter 9].

Lemma 21. *Let $\pi : (X,T) \to (Y,S)$ be a proximal extension of minimal Cantor systems, then (X,T) and (Y,S) have the same maximal equicontinuous factor.*

Lemma 22. *Let $\pi : (X,T) \to (Y,S)$ be an almost 1-1 extension of compact systems, such that (Y,S) is minimal. Then π is a proximal extension.*

Proof. Let $y \in Y$ be an element having only one pre-image by π. If π is injective, then the result is trivial. We can assume then there exist $x' \neq x''$ in X such that $\pi(x') = \pi(x'') = y' \in Y$. Suppose that x' and x'' are not proximal. This means there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$,

$$d(T^n(x'), T^n(x'')) > \varepsilon. \tag{5.11}$$

Since (Y, S) is minimal, there exists a subsequence $(S^{n_i}(y'))_{i\geq 0}$ of the orbit of y' that converges to y. By compactness, taking subsequences if needed, we can suppose $(T^{n_i}(x'))_{i\geq 0}$ and $(T^{n_i}(x''))_{i\geq 0}$ converging to some z' and z'' respectively. Inequality (5.11) ensures that $z' \neq z''$, and since π is continuous, we have $\lim_{i\to\infty} \pi(T^{n_i}(x')) = \pi(z')$ and $\lim_{i\to\infty} \pi(T^{n_i}(x'')) = \pi(z'')$. On the other hand, the choice of $(n_i)_{i\geq 0}$ implies that

$$\pi(z') = \lim_{i\to\infty} \pi(T^{n_i}(x')) = \lim_{i\to\infty} S^{n_i}(y') = y,$$
$$\pi(z'') = \lim_{i\to\infty} \pi(T^{n_i}(x'')) = \lim_{i\to\infty} S^{n_i}(y'') = y,$$

which contradicts the fact that y has only one pre-image. □

Let us recall a consequence of [31, Theorem 1.1] (see Corollary 1.2 in [31]).

Theorem 23. *Suppose that (Y, S) is a uniquely ergodic minimal Cantor system and (G, G^+, u) is a simple dimension group with distinguished order unit satisfying the following assumptions:*

(i) *there is an order embedding $i : K^0(Y, S) \to (G, G^+, u)$,*
(ii) *$G/i(K^0(Y, S))$ is torsion free.*

Then there exists a minimal Cantor system (X, T) such that $K^0(X, T)$ is isomorphic to (G, G^+, u) and such that there is an almost one-to-one factor map $\pi : (X, T) \to (Y, S)$.

The next result is a direct consequence of Lemma 21, Lemma 22 and Theorem 23, but stated in terms of dimension groups.

Proposition 24. *Let (G, G^+, u) and (H, H^+, w) be two simple dimension group with distinguished order unit, such that (H, H^+, w) has a unique trace. Suppose there exists an order embedding $i : H \to G$ with $G/i(H)$ torsion free. Then $\mathcal{E}(H, H^+, w)$ is a subfamily of $\mathcal{E}(G, G^+, u)$.*

Recall that for any countable dense subgroup Γ of $\mathbb{R}$ containing $\mathbb{Z}$, the dimension group $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ has only one trace and no non trivial infinitesimal (see [10]).

Proposition 25. *Let (G, G^+, u) be a simple dimension group with distinguished order unit and with a trivial infinitesimal subgroup. Then, for any subgroup Γ of $I(G, G^+, u)$ with $\mathbb{Z} \subseteq \Gamma$ and $I(G, G^+, u)/\Gamma$ torsion free, the family $\mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ is contained in $\mathcal{E}(G, G^+, u)$.*

Proof. Let $\tau : \tilde{G} \to I(G, G^+.u)$ be the morphism given by Corollary 14. Since the infinitesimal subgroup is trivial, it is an isomorphism. Let (H, H^+, u) be the dimension group image of $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ by the inverse τ^{-1}. It is easy to check the group $G/\tilde{G}$ is torsion free, so by hypothesis, the quotient group G/H is also torsion free. Moreover, $S(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ has only one element because $S(H, H^+, u) \simeq S(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$. The desired result follows from Proposition 24. □

As a consequence of these results, we obtain the following characterization.

Corollary 26. *The following are equivalent:*

(1) *For any countable dense subgroup Γ of $\mathbb{R}$ containing $\mathbb{Z}$, there is a Cantor minimal system (X, T) with $E(X, T) = \Gamma$, and $K^0(X, T) \simeq (\Gamma, \Gamma \cap \mathbb{R}^+, 1)$, i.e.*

$$\Gamma \in \mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1).$$

(2) *For any simple dimension group with distinguished order unit (G, G^+, u) and with no non trivial infinitesimal,*

$$\mathcal{E}(G, G^+, u) = \{\Gamma; \Gamma \text{ is a subgroup of } I(G, G^+, u) \text{ such that } I(G, G^+, u)/\Gamma \text{ is torsion free}\}.$$

Proof. It is obvious that (1) is a consequence of (2). Proposition 25 and (1) imply (2). □

Proposition 27. *Let (G, G^+, u) be a simple dimension group with distinguished order unit and let Γ be a subgroup of $I(G, G^+, u)$ verifying the following:*

- *Γ is generated by a family of rationally independent numbers containing 1.*
- *$I(G, G^+, u)/\Gamma$ is torsion free.*

Then $\mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ is contained in $\mathcal{E}(G, G^+, u)$.

Remark that Proposition 27 includes the case where Γ is finitely generated. Example 4.2 shows that Proposition 27 becomes false whenever the group Γ is not generated by a rationally independent family.

Proof. Let $\{\alpha_i\}_{i\geq 0}$ be a rationally independent family generating Γ. Without loss of generality, we can assume that $\alpha_0 = 1$. From Lemma 12, for every $i \in \mathbb{N}$, there exists a $g_i \in G$ such that $\tau(g_i) = \alpha_i$, for every $\tau \in S(G, G^+, u)$. We choose $g_0 = u$.
For any $\alpha \in \Gamma$, there exists a unique sequence of integers $(m_i)_{i\geq 0}$ with $m_i = 0$ except for a finite number of i's such that $\alpha = \sum_{i\geq 0} m_i\alpha_i$. This implies that the function $\phi : \Gamma \to G$ given by $\phi(\alpha) = \sum_{i\geq 0} m_i g_i$ is a well defined one-to-one homomorphism. It is not difficult to see that this is an order embedding such that $G/\phi(\Gamma)$ is torsion free. Since $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ has only one trace, from Proposition 24 we obtain the desired property. □

5.3. **Necessary conditions for $\Gamma \in \mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$.** Let (G, G^+, u) be a simple dimension group with distinguished order unit having no non trivial infinitesimal and with a unique trace. So it is isomorphic to the dimension group $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$, where $\Gamma = I(G, G^+, u)$. We suppose here, that $\Gamma \in \mathcal{E}(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$. That is, there exists a uniquely ergodic minimal Cantor system (X, T) whose dimension group is isomorphic to $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ and such that $E(X, T) = \Gamma$.
We use the notations of the sections 2.2 and 3.1. Recall that for every $n \geq 1$, $\mu_n = (\mu_n(k))_{k=1}^{C_n}$ denotes the vector of measures of the bases of the partition $\mathcal{P}_n$, corresponding to the unique invariant probability measure μ of (X, T) (see Section 2.2 for definitions). Since (X, T) is uniquely ergodic, the group Γ is generated by $\{\mu_n(k) : 1 \leq k \leq C_n, n \geq 1\}$, which implies that for every $m \geq 1$ and every $1 \leq k \leq C_m$ the number $\mu_{m,k}$ is in $E(X, T)$. From Lemma 6 we get that for every n sufficiently large, there exist an integer vector w_n and a real vector v_n such that

$$P_n H_1 \mu_m(k) = v_n + w_n \text{ and } \sum_{l\geq n} \|P_{l,n} v_n\|_\infty < \infty. \tag{5.12}$$

Multiplying by μ_n^T the first equation we get but the normalization conditions (see Equation (3.7) and Lemma 7)

$$\mu_m(k) = \mu_n^T w_n.$$

On the other hand, we have

$$\mu_m(k) = \mu_n^T P_{n,m}(\cdot, k).$$

Since the infinitesimal subgroup is trivial, the previous two equations implies that for n sufficiently large

$$w_n = P_{n,m}(\cdot, k). \tag{5.13}$$

Equations (5.12) and (5.13) imply

$$\sum_{n\geq 1} \max_i \|h_{n,i}\mu_m - P^T_{n,m}(\cdot,i)\|_\infty < \infty. \tag{5.14}$$

On the other hand, the unique ergodicity of the system (X,T) implies that the rows of the matrix $P_{n,m}$ converges with n (after normalization) to μ_m. That is

$$\lim_{n\to\infty} \max_i \left\| \mu_m - \frac{1}{h_{n,i}} P^T_{n,m}(\cdot,i) \right\|_\infty = 0.$$

Thus from (5.14), we deduce that if $E(X,T) = \Gamma$, then the rate of convergence of the rows of $P_{n,m}$ to the direction generated by μ_m has to be extremely fast.

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G-ODOMETERS AND THEIR ALMOST 1-1 EXTENSIONS.

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ABSTRACT. In this paper we recall the concepts of G-odometer and G-subodometer for G-actions, where G is a discrete finitely generated group, which generalize the notion of odometer in the case $G = \mathbb{Z}$. We characterize the G-regularly recurrent systems as the minimal almost 1-1 extensions of subodometers, from which we deduce that the family of the G-Toeplitz subshifts coincides with the family of the minimal symbolic almost 1-1 extensions of subodometers. We determine the continuous eigenvalues of these systems. When G is amenable and residually finite, a characterization of the G-invariant measures of these systems is given.

1. INTRODUCTION

It is known that an almost 1-1 extension of a minimal equicontinuous system always has this system as the maximal equicontinuous factor. This justifies the study of these almost 1-1 extensions. The aim of this paper is to study the extensions of a particular type of equicontinuous systems: the G-odometers, where G is a discrete finitely generated group, like for example a non abelian free group. The notion of G-odometer generalizes the notion of odometer, or adding-machine, in the case $G = \mathbb{Z}$.

An example of extensions of $\mathbb{Z}$-odometers are the Toeplitz flows, which were introduced by Jacobs and Keane in [JK]. Toeplitz flows have been extensively studied in different contexts and they have been used to provide series of examples with interesting dynamical properties (see for example [Do], [GJ], [Wi]). Markley and Paul characterize them in [MP] as the minimal almost 1-1 extensions of odometers and a proof of this theorem is given in [DL] by Downarowicz and Lacroix (see also [Au]). Let us mention also an example of F. Krieger in [Kr] where he constructs, for a residually finite and amenable group G, a G-Toeplitz sequence with an arbitrary entropy.

Following the work developed in [Co] for $G = \mathbb{Z}^d$, we prove that for a discrete finitely generated group G, the G-Toeplitz systems are the symbolic minimal almost 1-1 extensions of G-odometers. The main difficulties lie in the fact that we consider non-abelian groups and therefore the used techniques are not straight generalizations of the $\mathbb{Z}$-case. Unlike in the abelian case, there appear some degenerated systems that we call subodometers.

This paper is organized as follows: in Section 2, we give some basic definitions relevant for the study of topological dynamical systems. We recall also the generalized notions of odometer and subodometer and we identify the set of eigenvalues of these systems. In Section 3, we introduce the notions of regularly recurrent systems and strongly regularly recurrent systems. We characterize them as the minimal almost 1-1 extensions of subodometers and odometers respectively. In the particular case where G is amenable and residually finite, we show in Section 4 that the set of invariant probability measures of a G-regularly recurrent Cantor system can be represented as an inverse limit. In Section 5, in the case when G is a residually finite group, we introduce a notion of semicocycles and we show that an almost 1-1 extension of a G-subodometer is conjugated to the action of G on some semicocycle. Finally in Section

1991 *Mathematics Subject Classification.* Primary: 54H20; Secondary: 37B50 .

Key words and phrases. Almost 1-1 extensions, odometer, Toeplitz system, discrete group actions.

6, we consider a particular family for a discrete group G: the G-Toeplitz arrays, which is a particular family of semicocycles when G is residually finite. We prove, by giving an explicit construction, that this family coincides with the family of symbolic almost 1-1 extensions of the G-subodometers.

Acknowledgment: It is a great pleasure for S.P. to thank the Instituto de Matemáticas U.N.A.M. of Cuernavaca (México), where part of this work has been done, for their warm hospitality. M.I. Cortez acknowledges financial support from proyecto Fondecyt de Iniciación 11060002 and Nucleus Millennium P04-069-F. The authors thank the anonymous referee for its constructive remarks.

2. Basic definitions and background

In this article, by a *topological dynamical system* we mean a pair (X, G), where G is a topological group which acts, by homeomorphism, on a compact metric space (X, d). Given $g \in G$ and $x \in X$ we will identify g with the associated homeomorphism and we denote by $g.x$ the action of g on x. The dynamical system (X, G) is *free* if $g.x = x$ for some $x \in X$ implies $g = e$, where e is the neutral element in G. For a syndetic subgroup Γ of G, the Γ-*orbit* of $x \in X$ is $O_\Gamma(x) = \{\gamma.x : \gamma \in \Gamma\}$ and the Γ-*system associated* to x is $(\Omega_\Gamma(x), \Gamma)$, where $\Omega_\Gamma(x)$ is the closure of $O_\Gamma(x)$ and the action of Γ on $\Omega_\Gamma(x)$ is the restriction to Γ and $\Omega_\Gamma(x)$ of the action of G on X. The set of *return times* of $x \in X$ to $A \subseteq X$ is $T_A(x) = \{g \in G : \, g.x \in A\}$. The topological dynamical system (X, G) is *minimal* if the orbit of any $x \in X$ is dense in X, and it is said to be *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ satisfy $d(x, y) < \delta$ then $d(g.x, g.y) < \varepsilon$ for all $g \in G$. We say that (X, G) is an *extension* of (Y, G), or that (Y, G) is a *factor* of (X, G), if there exists a continuous surjection $\pi : X \to Y$ such that π preserves the action. We call π a *factor map*. When the factor map is bijective, we say that (X, G) and (Y, G) are *conjugate*. The factor map π is an *almost 1-1 factor map* and (X, G) is an *almost 1-1 extension* of (Y, G) by π if the set of points having one pre-image is residual (contains a dense G_δ set) in Y. In the minimal case it is equivalent to the existence of a point with one pre-image.
The set $\mathcal{M}_G(X)$ of *invariant probability measures* of X is the set of probability measures μ defined on $\mathcal{B}(X)$, the Borel σ-algebra of X, such that $\mu(g.B) = \mu(B)$ for all $g \in G$ and $B \in \mathcal{B}(X)$.

2.1. G-odometers and G-subodometers.

In all the following, we will denote by G a discrete group generated by a finite family and by e its neutral element.

Definition 1. *A discrete finitely generated group G is called residually finite if and only if there exists a sequence $\Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_n \supset \ldots$ of subgroups Γ_n with finite index in G such that:*
$$\bigcap_n \Gamma_n = \{e\}.$$

A trivial example of a residually finite subgroup is the group of integers $\mathbb{Z}$, for example by taking the groups $\Gamma_n = n!\mathbb{Z}$. Less trivial examples are given by the fundamental groups of connected oriented compact graph. When $\pi : S_2 \to S_1$ is a finite covering of an oriented compact connected graph S_2 onto a compact graph S_1, the application π induces an homomorphism π_* from the fundamental group of S_2 to the fundamental group of S_1. The image of the morphism π_* is a subgroup of the fundamental group of S_1. The index of this subgroup is then the number of preimages of one point for the map π. Let us denote by $\widetilde{S_1}$ the universal cover of S_1. Consider a sequence $(S_n, \pi_n)_n$ of finite coverings $\pi_n : S_{n+1} \to S_n$ of compact connected and oriented graphs S_n such that for each n the injectivity radius of $\widetilde{S_1}$ onto S_n

goes to infinity when n goes to infinity. The sequence of fundamental groups of graphs S_n satisfies then the condition of Definition 1. More generally, we have the following result of Mal'cev [Ma]:

Theorem 1. [Ma] *For any integer n and any field $\mathbb{K}$ with characteristic null, every finitely generated subgroup of the group of invertible matrices $GL(n, \mathbb{K})$ is a residually finite group.*

In particular, the free groups $\mathbb{F}_n$ with n generators, the groups of surfaces and the braids group B_n generated by n elements are residually finite groups.
Let us denote, for a subgroup H of G, by G/H the set of right classes of H in G. It is important to note that G acts on G/H by left multiplication. Now we will prove the useful following lemma:

Lemma 1. *Let G be a group. If H is a subgroup of G with index in G equal to n (i.e. the cardinal of the quotient space G/H is n) then there exists a normal subgroup K of G contained in H such that the cardinality of G/K divides $n!$.*

Proof. The group G acts on G/H by left multiplication. This action defines an homomorphism ρ from G to the permutation group of n elements. The kernel of this application is a normal subgroup of G contained in H and its index in G divides the cardinal of permutations of n elements. □

As a corollary, G is a residually finite group if and only if there exists a sequence $H_1 \supset \ldots \supset H_n \supset \ldots$ of normal subgroups of G with finite index in G such that $\bigcap_n H_n = \{e\}$.

Let us consider a discrete group G generated by a finite family, and a decreasing sequence (for the inclusion) $(\Gamma_i)_{i\geq 0} \subseteq G$ of subgroups with finite index in G (we do not ask $\bigcap_{i\geq 0} \Gamma_i = \{e\}$) and let $\pi_i : G/\Gamma_{i+1} \to G/\Gamma_i$ be the function induced by the inclusion $\Gamma_{i+1} \subset \Gamma_i$, $i \geq 0$. Consider the inverse limit

$$\overleftarrow{G} = \lim_{\leftarrow i}(G/\Gamma_i, \pi_i).$$

More precisely, $\overleftarrow{G}$ is defined as the subset of the product $\Pi_{i\geq 0} G/\Gamma_i$ consisting of the elements $\mathbf{g} = (g_i)_{i\geq 0}$ such that $\pi_i(g_{i+1}) = g_i$ for all $i \geq 0$.
Every G/Γ_i is endowed with the discrete topology and $\Pi_{i\geq 0} G/\Gamma_i$ with the product topology. Thus $\overleftarrow{G}$ is a compact metrizable space whose topology is spanned by the cylinder sets

$$[i; a] = \{\mathbf{g} \in \overleftarrow{G} : g_i = a\}, \text{ with } a \in G/\Gamma_i \text{ and } i \geq 0.$$

The space $\overleftarrow{G}$ is a totally disconnected, it is a Cantor set when $G/\cap_{i\geq 0} \Gamma_i$ is infinite and a finite set when $G/\cap_{i\geq 0} \Gamma_i$ is finite.
The group G acts continuously on $\overleftarrow{G}$ by left multiplication, namely for $\mathbf{g} = (g_i)_i \in \overleftarrow{G}$ and $h \in G$,

$$h.\mathbf{g} = (h._i g_i)_i,$$

where $h._i$ denotes the action on G/Γ_i given by $h._i g\Gamma_i = hg\Gamma_i$, for every $h \in G$ and $g \in G$.
Since for all $h \in G$ and for all cylinders $[i; a]$ we have

$$h.([i; a]) \subseteq [i; h._i a_i],$$

the topological dynamical system $(\overleftarrow{G}, G)$ is equicontinuous. Moreover, every orbit for this action is dense, then $(\overleftarrow{G}, G)$ is a minimal equicontinuous system.

Definition 2. *We call $(\overleftarrow{G}, G)$ a G-subodometer system* or simply a subodometer. If in addition, every Γ_i is normal, we say that $(\overleftarrow{G}, G)$ is a G-odometer system or simply an odometer.*

It is straightforward to show that for a point $\mathbf{g} = (g_i)_i$ of a subodometer $\overleftarrow{G}$, its stabilizer for the G-action is the group $\bigcap_i \tilde{g}_i \Gamma_i \tilde{g}_i^{-1}$, where $\tilde{g}_i$ is a representing element of the class $g_i \in \mathbf{G}/\Gamma_i$ in G, for $i \geq 0$. Hence, when G is a residually finite group and $\bigcap_{i\geq 0} \Gamma_i = \{e\}$, for $\mathbf{e} = (e_i)_i \in \overleftarrow{G}$, where e_i is the projection of the neutral element of G on G/Γ_i, its stabilizer is trivial. This does not mean necessarily that the action of G on $\overleftarrow{G}$ is free. If furthermore, all the groups Γ_i are normal subgroups of G, then the stabilizer of every point of a G-odometer is trivial and the action of G is free. For this reason, when G is residually finite and $\bigcap_{i\geq 0} \Gamma_i = \{e\}$, the G-odometer $\lim_{\leftarrow i}(G/\Gamma_i, \pi_i)$ will be called a *free G-odometer*.

If $(\overleftarrow{G}, G)$ is an odometer then the set $\overleftarrow{G}$ is a group equipped with the multiplication defined by

$$\mathbf{g}.\mathbf{h} = (g_i \cdot_i h_i)_{i\geq 0},$$

where $\cdot_i$ denotes the multiplication operation induced on G/Γ_i by the multiplication on G. notice that for a free odometer $(\overleftarrow{G}, G)$, the group G is then a dense subgroup of $\overleftarrow{G}$.
Notice that for all $\mathbf{g}$ in a cylinder set $[i; a]$ of an odometer $\overleftarrow{G} = \lim_{\leftarrow i}(G/H_i, \pi_i)$, the set of return times of $\mathbf{g}$ to $[i; a]$ is H_i. Throughout this paper we will use this property and we will identify $\overleftarrow{G}$ with $(\overleftarrow{G}, G)$.

Lemma 2. *Let $\overleftarrow{G}_j = \lim_{\leftarrow i}(G/H_i^j, \pi_i)$ be two subodometers ($j = 1, 2$). Let $\mathbf{e_j}$ ($j = 1, 2$) be the element $(e_i^j)_i \in \overleftarrow{G}_j$ where e_i^j denotes the class of the neutral element $e \in G$ in G/H_i^j. There exists a factor map $\pi : (\overleftarrow{G}_1, G) \to (\overleftarrow{G}_2, G)$ such that $\pi(\mathbf{e_1}) = \mathbf{e_2}$ if and only if for every H_i^2 there exists some H_k^1 such that $H_k^1 \subseteq H_i^2$.*

Proof. If $\pi : \overleftarrow{G}_1 \to \overleftarrow{G}_2$ is a factor map then by continuity, given $i \geq 0$ and e_i^2 in G/H_i^2, there exists $k \geq 0$ such that $[k; e_k^1] \subseteq \pi^{-1}[i; e_i^2]$. Let $v \in H_k^1$. We have that $v.\mathbf{g} \in [k; e_k^1]$ for all $\mathbf{g} \in [k; e_k^1]$, which implies that

$$\pi(v.\mathbf{g}) = v.\pi(\mathbf{g}) \in [i; e_i^2].$$

Since $\pi(\mathbf{g}) \in [i; e_i^2]$ and $T_{[i;e_i^2]}(\pi(\mathbf{g})) = H_i^2$, we get $v \in H_i^2$.
Suppose that for every $i \geq 0$ there exists $H_{n_i}^1 \subseteq H_i^2$. Since the sequences $(H_i^j)_{i\geq 0}$, $j = 1, 2$, are decreasing, we can take $n_i \leq n_{i+1}$ for all $i \geq 0$. The function $\pi : \overleftarrow{G}_1 \to \overleftarrow{G}_2$ defined by $\pi((g_i)_{i\geq 0}) = (j_{n_i}(g_{n_i}))_{i\geq 0}$ where $j_{n_i} : G/H_{n_i}^1 \to G/H_i^2$ is the function induced by the inclusion $H_{n_i}^1 \subseteq H_i^2$, is a factor map. □

By a straightforward application of the former lemma and Lemma 1, we get

Proposition 1. *If $(\lim_{\leftarrow i}(G/\Gamma_i, \pi_i), G)$ is a G-subodometer, then there exists a G-odometer which is an extension of this subodometer.*

Proposition 2. *Let $\overleftarrow{G}$ be a G-odometer and (X, G) a dynamical system. If there exists a factor map from $\overleftarrow{G}$ onto X, then there exists a closed subgroup H of $\overleftarrow{G}$ such that the dynamical system $(\overleftarrow{G}/H, G)$ is conjugated to (X, G).*

*Note that this definition is not a profinite completion of the group G because here, we consider only a sequence of decreasing subgroups.

In particular this proposition says that a subodometer is conjugate to the quotient of an odometer by a closed subgroup.

Proof. Let us denote by p the factor map $\overleftarrow{G} \to X$, and by $\mathbf{e}$ the neutral element of $\overleftarrow{G}$. Let H be the closed subset $p^{-1}(p(e))$ of $\overleftarrow{G}$. For $\mathbf{g} = (g_i)_i$ and $\mathbf{h} = (h_i)_i$ in H, we have:

$$p(\mathbf{hg}) = \lim_i p(h_i g_i) = \lim_i h_i.p(g_i) = \lim_i h_i.p(e) = \lim_i p(h_i) = p(e).$$

With the same technique we get:

$$p((\mathbf{g})^{-1}) = \lim p(g_i^{-1}e) = \lim_i g_i^{-1}.p(e) = \lim_i g_i^{-1}.p(g_i) = p(e).$$

So $\mathbf{gh}$ and $\mathbf{g}^{-1}$ belong to H, and H is a group.
Now let us see that $p^{-1}(p(\mathbf{g})) = \mathbf{g}H$ for any $\mathbf{g} \in \overleftarrow{G}$. Let $\mathbf{h}$ be in H, we have:

$$p(\mathbf{gh}) = \lim_i p(g_i h_i) = \lim_i g_i.p(h_i) = \lim_i g_i.p(e) = p(\mathbf{g}).$$

Then $\mathbf{g}H \subset p^{-1}(p(\mathbf{g}))$.
Let $\mathbf{h} \in \overleftarrow{G}$ be such that $p(\mathbf{h}) = p(\mathbf{g})$. Then $\lim_i p(g_i) = \lim_i p(h_i)$ and $p(e) = \lim_i g_i^{-1}.h_i.p(e) = p(\mathbf{g}^{-1}\mathbf{h})$. So $\mathbf{g}^{-1}\mathbf{h}$ belongs to H and $p^{-1}(p(\mathbf{g})) = \mathbf{g}H$. Therefore, the map p factorizes onto a homeomorphism from $\overleftarrow{G}/H$ to X. □

2.2. **Eigenvalues of odometers and subodometers.** Let (X, μ, G) be a measure-theoretic dynamical system with a left action of G. A character χ is a homomorphism from G to the group $\mathbb{S}^1$, the set of complex numbers with modulus 1. Since the group G is equipped with the discrete topology, every character is a continuous map.
A character is an *eigenvalue* of X if there exists $f \in L^2_\mu(X) \setminus \{0\}$ such that $f(g.x) = \chi(g)f(x)$ for all $x \in X$ and $g \in G$. We call f an *eigenfunction* associated to χ. We say that an eigenvalue is a *continuous eigenvalue* if it has an associated continuous eigenfunction.

Since a G-odometer $\overleftarrow{G}$ is a compact group, the normalized left invariant Haar measure λ of $\overleftarrow{G}$ is the only probability measure of $\overleftarrow{G}$ invariant under the action of G. Thus the system $(\overleftarrow{G}, G)$ is uniquely ergodic. Any G-subodometer, as a factor of some G-odometer, is alo uniquely ergodic. Thus when we speak about a subodometer $\overleftarrow{G}$ as a measure-theoretic dynamical system, we mean $\overleftarrow{G}$ equipped with the unique invariant probability measure λ for the action of G.

Proposition 3. *Let $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ be a subodometer. The set of eigenvalues of $\overleftarrow{G}$ is given by $E_G = \bigcup_{n\geq 0}\{$character $\chi : G \to \mathbb{S}^1,\ \chi(\gamma) = 1$ for all $\gamma \in \Gamma_n\}$. Moreover, every eigenvalue of $\overleftarrow{G}$ is a continuous eigenvalue.*

Proof. For $n \geq 0$ we call $C_n = [n; e]$. Since $v, w \in G$ satisfy $v.C_n = w.C_n$ if and only if w and v belong to the same class in G/Γ_n, it makes sense to write $v.C_n$ for $v \in G/\Gamma_n$. Notice that the collection $\mathcal{P}_n = \{v.C_n : v \in G/\Gamma_n\}$ is a clopen partition of G.
Let $\chi \in E_G$ and let $n \geq 0$ be such that $\chi(\gamma) = 1$ for all $\gamma \in \Gamma_n$. This means that χ is constant on each class of G/Γ_n, which implies that $f = \sum_{v\in G/\Gamma_n} \chi(v)1_{v.C_n}$ is a well defined continuous function that verifies $f(h.\mathbf{g}) = \chi(h)f(\mathbf{g})$ for all $\mathbf{g} \in \overleftarrow{G}$ and $h \in G$.
Let χ be an eigenvalue of $\overleftarrow{G}$ and let $f \in L^2_\lambda(\overleftarrow{G}) \setminus \{0\}$ be an associated eigenfunction. For $g \in G$ we have that

$$\chi(g)\left(\int_{C_n} f d\lambda\right) = \int_{g.C_n} f d\lambda.$$

Since $C_n = \gamma.C_n$ for all $\gamma \in \Gamma_n$, it holds that

$$\chi(g)\left(\int_{C_n} f d\lambda\right) = \int_{C_n} f d\lambda \quad \text{for all } g \in \Gamma_n. \tag{1}$$

Observe that

$$\mathbb{E}(f|\mathcal{P}_n) = \sum_{g\in K_n} \frac{\chi(g)}{\lambda(C_n)} \left(\int_{C_n} f d\lambda\right) 1_{g.C_n},$$

for a finite set $K_n \subset G$ containing at least one element of each class of G/Γ_n. Since $\mathcal{B}(\mathcal{P}_n) \uparrow \mathcal{B}(\overleftarrow{G})$, by the increasing Martingale Theorem, we have that $\mathbb{E}(f|\mathcal{P}_n)$ converges to f in $L^2_\lambda(\overleftarrow{G})$. Because $f \neq 0$, this implies there exists $m \geq 0$ such that $\int_{C_m} f d\lambda \neq 0$ and, by (1), we conclude that $\chi(\gamma) = 1$ for all $\gamma \in \Gamma_m$, which means that $\chi \in E_G$. □

3. Characterization of minimal almost 1-1 extensions of odometers

Let (X, G) and (Y, G) be two topological dynamical systems. (Y, G) is said to be the *maximal equicontinuos factor* of (X, G) if it is an equicontinuos factor of (X, G) such that for any other equicontinuous factor (Y', G) of (X, G) there exists a factor map $\pi : Y \to Y'$ that satisfies $\pi \circ f = f'$, with $f : X \to Y$ and $f' : X \to Y'$ factor maps.
It is well known that every topological dynamical system has a maximal equicontinuous factor and if (X, G) is a minimal almost 1-1 extension of a minimal equicontinuous system (Y, G), then (Y, G) is the maximal equicontinuous factor of (X, G) (for more details see [Au]).

3.1. **Regularly recurrent systems.** A subset S of G is said to be *syndetic* if there exists a compact subset K of G such that $G = K.S = \{k.s : s \in S,\ k \in K\}$. Because we consider a discrete group G, a subset S of G is syndetic if and only there exists a finite subset K of G such that $G = K.S$. It is important to note that a subgroup Γ of G is syndetic if and only if G/Γ is finite.
Let (X, G) be a topological dynamical system and let $x \in X$. The point x is *uniformly recurrent* if for every open neighborhood V of x the set $T_V(x)$ is syndetic. It is well known that $(\Omega_G(x), G)$ is minimal if and only if x is uniformly recurrent.
A point $x \in X$ is *regularly recurrent* if for every open neighborhood V of x there is a syndetic subgroup Γ of G such that $\Gamma \subseteq T_V(x)$. We say that a system is *regularly recurrent* if it is the orbit closure of a regularly recurrent point.
Similarly, we say that a point $x \in X$ is *strongly regularly recurrent* if for every open neighborhood V of x there is a closed neighborhood $W \subset V$ of x such that $T_W(x)$ is a syndetic normal subgroup of G. We say that a system is *strongly regularly recurrent* if it is the orbit closure of a strongly regularly recurrent point. Obviously, a strongly regularly recurrent point is a regularly recurrent point. Regularly recurrent systems are minimal.
The subodometers are examples of regularly recurrent systems, whose any point is regularly recurrent. In the same way, the odometers are strongly regularly recurrent systems whose any point is strongly regularly recurrent.
In this section, we will show that regularly recurrent systems are exactly the minimal almost 1-1 extensions of the subodometers and strongly regularly recurrent systems are the minimal almost 1-1 extensions of the odometers. From that we will conclude that a group G admits an action that is both strongly regularly recurrent and free if and only if G is residually finite.

Lemma 3. *Let (X, G) be a minimal topological dynamical system and let $x \in X$. If $\Gamma \subseteq G$ is a syndetic subgroup of G then $(\Omega_\Gamma(x), \Gamma)$ is minimal.*

Proof. Let H be a syndetic normal subgroup of G contained in Γ (Lemma 1). The group G acts by the natural product action on the compact space $X \times G/H$. Pick a minimal set M

in $X \times G/H$ for this action. The canonical projection on the first coordinate is a factor map that maps M onto a minimal subset of X hence onto X. Thus for every $x \in X$ there exists a point $(x, [a]) \in M$, where $[a]$ denotes the H-class of an element $a \in G$. By minimality of M, this point is uniformly recurrent. The right multiplication by $[a^{-1}]$ on the second axis is a conjugacy that sends the minimal set M onto a minimal set M' that contains $(x, [e])$. The canonical projection $G/H \to G/\Gamma$ induces a factor map from $X \times G/H$ onto $X \times G/\Gamma$ for the product action of G. So it maps M' onto a minimal set of $X \times G/\Gamma$ that contains the point $(x, [e]_\Gamma)$ where $[e]_\Gamma$ denotes the Γ-class of the neutral element e. This implies that for any neighborhood $V \subseteq X$ of x, the set $T_{V\times\{[e]_\Gamma\}}(x, [e]_\Gamma) = \{g \in G : g.x \in V, g \in \Gamma\}$ is syndetic. □

Lemma 4. *Let (X, G) be a topological dynamical system and let $x \in X$ be a regularly recurrent point. For every closed neighborhood V of x there exists a syndetic subgroup Γ of G such that $\Gamma \subseteq T_V(x)$ and $\{w(\Omega_\Gamma(x))\}_{w\in G/\Gamma}$ is a clopen partition of X.*
Moreover, if x is strongly regularly recurrent, the former group Γ is normal.

Proof. Let $V \subseteq X$ be a closed neighborhood of a regularly recurrent point x and let $\Gamma' \subseteq G$ be a subgroup with finite index such that $\Gamma' \subseteq T_V(x)$. Let us consider the normal subgroup $H \subset \Gamma'$ given by Lemma 1. By Lemma 3, the set $\Omega_H(x)$ is closed and minimal for the action of H. Since H is normal, for any $g \in G$, the set $g.\Omega_H(x)$, which equals $\Omega_H(g.x)$, is also closed, invariant and minimal for the H-action. Therefore if $w.\Omega_H(x) \cap u.\Omega_H(x) \neq \emptyset$ for $u, w \in G$, we have $w.\Omega_H(x) = u.\Omega_H(x)$.
Furthermore, if u and $w \in G$ are in the same H-class, then we have also $w.\Omega_H(x) = u.\Omega_H(x)$. Since H is syndetic and the G-orbit of x is dense, we have $X = \bigsqcup_{u\in K} u.\Omega_H(x)$, for some finite set $K \subset G$.
Let Γ be the group

$$\Gamma = \{g \in G : \; g.\Omega_H(x) = \Omega_H(x)\}.$$

We have $H \subset \Gamma$, so Γ is syndetic. Since $\Omega_\Gamma(x) = \Omega_H(x)$, we have $\Gamma \subset T_V(x)$ and for any $g \in G$ $g.\Omega_\Gamma(x)$ and $\Omega_\Gamma(x)$ are disjoint or equal because they are minimal closed H-invariant sets. Thus we get :

(1) $g.\Omega_\Gamma(x) = g'.\Omega_\Gamma(x)$ if and only if $g \in g'\Gamma$.
(2) $T_{g.\Omega_\Gamma(x)}(y) = g\Gamma g^{-1}$ for every $y \in g.\Omega_\Gamma(x)$.

It holds that for $w \in G/\Gamma$, $w.\Omega_\Gamma(x)$ is well defined and $\{w.\Omega_\Gamma(x)\}_{w\in G/\Gamma}$ is a clopen partition of X.

When x is a strongly regularly recurrent point of X, we follow the same proof with H being the normal subgroup $T_W(x)$ given by a clopen neighborhood $W \subset V$ of x. Due to this strong property, we have that the group Γ equals H and thus Γ is a normal subgroup of G. □

Corollary 1. *Let (X, G) be a topological dynamical system and let $x \in X$. The point x is regularly recurrent if and only if there exists a fundamental system $(C_i)_{i\geq 0}$ of clopen neighborhoods of x ($\cap_i C_i = \{x\}$), such that for all $y \in C_i$ the set of return times of y to C_i is a syndetic subgroup Γ_i of G, for every $i \geq 0$.*
Moreover, x is strongly regularly recurrent if and only if the groups Γ_i are normal.

Proof. If $x \in X$ has a fundamental system of neighborhoods as written above, it is a (resp. strongly) regularly recurrent point.

The sequences $(C_i)_i$ and $(\Gamma_i)_i$ are defined by induction. If x is a (resp. strongly) regularly recurrent point, let C_1 be the space X and Γ_1 be the group G.
So, given C_i and Γ_i, we take an open neighborhood V_{i+1} of x, whose the closure is strictly contained in C_i. By Lemma 4, we obtain a syndetic (resp. normal) group Γ_{i+1} with $\Gamma_{i+1} \subseteq$

$T_{\overline{V}_{i+1}}(x)$ and $\{w(\Omega_{\Gamma_{i+1}}(x))\}_{w\in G/\Gamma_{i+1}}$ is a clopen partition of X. Clearly, we have $\Gamma_{i+1}\subset\Gamma_i$. We set $C_{i+1}=\Omega_{\Gamma_{i+1}}(x)$ which is a clopen set with $T_{C_{i+1}}(y)=\Gamma_{i+1}$ for all $y\in\Gamma_{i+1}$.
Since $\lim_{i\to\infty}\operatorname{diam}(V_i)=0$, we obtain that $(C_i)_{i\geq 0}$ is a fundamental system of clopen neighborhoods of x. □

Theorem 2. *A minimal topological dynamical system (X,G) is an almost 1-1 extension of a subodometer $\overleftarrow{G}$ by π if and only if (X,G) is a regularly recurrent system.*
A minimal topological dynamical system (X,G) is an almost 1-1 extension of an odometer $\overleftarrow{G}$ by π if and only if (X,G) is a strongly regularly recurrent system.
Moreover, the set of regularly recurrent or strongly regularly recurrent points of X is exactly the pre-image of the set of points in $\overleftarrow{G}$ which have only one pre-image by π.

Proof. Let (X,G) be a minimal almost 1-1 extension of a subodometer $\overleftarrow{G}=\lim_{\leftarrow i}(G/\Gamma_i,\pi_i)$. Let $\pi: X\to\overleftarrow{G}$ be the almost 1-1 factor map and let $x\in X$ be such that $\{x\}=\pi^{-1}(\{\pi(x)\})$. Since π is continuous, if $\pi(x)=(a_i)_{i\geq 0}\in\overleftarrow{G}$ then $(\pi^{-1}([i;a_i]))_i$ is a decreasing sequence of clopen neighborhoods of x that satisfies

$$\bigcap_{i\geq 0}\pi^{-1}([i;a_i])=\{x\}.$$

We know that for every $\mathbf{g}\in[i;a_i]$, the set $T_{[i;a_i]}(\mathbf{g})$ is a group conjugated to Γ_i, therefore for all y in $\pi^{-1}([i;a_i])$, we have $T_{\pi^{-1}([i;a_i])}(y)$ is a group conjugated to Γ_i. So, by Corollary 1 we conclude that x is a regularly recurrent point of X. When $\overleftarrow{G}$ is an odometer, the groups Γ_i are normal and the point x is then a strongly regularly recurrent point of X.
Let (X,G) be a regularly recurrent system and let $x\in X$ be a regularly recurrent point. By Corollary 1 there exists a decreasing sequence $(C_i)_{i\geq 0}$ of clopen neighborhoods of x such that $\bigcap_{i\geq 0}C_i=\{x\}$, and there is a syndetic (resp. normal) subgroup Γ_i such that $T_{C_i}(y)=\Gamma_i$ for all $y\in C_i$, $i\geq 0$. Since $C_{i+1}\subseteq C_i$, we have that $\Gamma_{i+1}\subseteq\Gamma_i$, $i\geq 0$. So, we can define the subodometer $\overleftarrow{G}=\lim_{\leftarrow i}(G/\Gamma_i,\pi_i)$. We define $\pi: X\to\overleftarrow{G}$ by $\pi=(f_i)_{i\geq 0}$ where f_i is the continuous map $f_i: X\to G/\Gamma_i$ given by $f_i(y)=[z]$, where $[z]$ denotes the Γ_i-class of $z\in G$, if and only if $y\in z.C_i$ for $y\in X$, $z\in G$ and $i\geq 0$. The function π is a factor map, and, since $\bigcap_{i\geq 0}C_i=\{x\}$, we have that $\pi^{-1}\{\mathbf{e}\}=\{x\}$. So, π is an almost 1-1 extension. When x is strongly regularly recurrent, the groups Γ_i are normal and $\overleftarrow{G}$ is an odometer.
If $\pi': X\to\overleftarrow{G'}$ is another almost 1-1 factor map and $\overleftarrow{G'}$ an subodometer or an odometer, $\overleftarrow{G}$ and $\overleftarrow{G'}$ are maximal equicontinuous factors of (X,G), therefore, they are conjugate. Thus there exists a factor map $\pi'':\overleftarrow{G'}\to\overleftarrow{G}$ such that $\pi''\circ\pi'=\pi$, which implies that $\pi'^{-1}\{x\}=\pi^{-1}\{\pi''(x)\}$ for any x of $\overleftarrow{G'}$. We conclude that the set of regularly recurrent or strongly regularly recurrent points is exactly the pre-image of the points in $\overleftarrow{G}$ which have only one pre-image. □

By a straightforward application of Theorem 2 we get the following corollaries.

Corollary 2. *Every point of a system (X,G) is regularly recurrent if and only if (X,G) is conjugate to a subodometer.*
Similarly, every point of (X,G) is strongly regularly recurrent if and only if (X,G) is conjugate to an odometer.

Corollary 3. *A discrete finitely generated group G admits a strongly regularly recurrent free action on a compact metric space if and only if G is residually finite.*

Corollary 4. *Let (X,G) be a regularly recurrent system and let $\overleftarrow{G}$ be its maximal equicontinuous factor. The set of continuous eigenvalues of X is E_G.*

Proof. It is clear that E_G is contained in the set of continuous eigenvalues of X. Conversely, if χ is a continuous eigenvalue of X we can take $f : X \to \mathbb{S}^1$ an associated continuous eigenfunction which is a factor map between (X,G) and the dynamical system $(f(X),G)$, where the action of $g \in G$ on $\exp(2i\pi x) \in f(X)$ is given by $g.\exp(2i\pi x) = \chi(g)\exp(2i\pi x)$, which is an isometry. Thus the system $(f(X),G)$ is equicontinuous and therefore there exists a factor map $\pi : \overleftarrow{G} \to f(X)$. Since π is an eigenfunction associated to χ we conclude that $\chi \in E_G$. □

4. Regularly recurrent Cantor systems with G amenable.

We say that a topological dynamical system (X,G) is a *regularly recurrent Cantor system* if it is regularly recurrent and X is a Cantor set. In this section we suppose that (X,G) is a regularly recurrent Cantor system.

Proposition 4. *Let (X,G) be a regularly recurrent Cantor system. There exists a sequence*

$$(\mathcal{P}_n = \{w.C_{n,k} : w \in D_n,\ 1 \le k \le k_n\})_{n\ge 0},$$

of finite clopen partitions of X, where $D_n \subseteq G$ and $C_{n,k} \subseteq X$ is a clopen set, satisfying, for every $n \ge 0$, the following:

1. $C_{n+1} \subseteq C_n = \bigcup_{k=1}^{k_n} C_{n,k} \subset X$.
2. *There exists a syndetic subgroup Γ_n of G such that D_n is a subset of G containing exactly one representing element of each class in G/Γ_n and such that $T_{C_n}(x) = \Gamma_n$, for all $x \in C_n$.*
3. *$\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$.*
4. *The family of sets $\{\mathcal{P}_n,\ n \ge 0\}$ spans the topology of X.*

Proof. The idea of the proof (the same as used in [HPS] and [Pu]) is to show that any minimal Cantor $\mathbb{Z}$-system has a nested sequence of clopen Kakutani-Rohlin partitions.
We recall the algorithm introduced in [Pu] to generate a Kakutani-Rohlin partition finer than a given one. Let $\mathcal{R}$ be a finite clopen partition of X. Suppose that

$$\mathcal{Q} = \{w.C_j : w \in D,\ 1 \le j \le k\},$$

is another finite clopen partition of X, and that there exists a syndetic subgroup Γ of G such that $D = \{w_1, \cdots, w_l\}$ is a subset of G containing exactly one representing element of each class in G/Γ, and that the set of return times of any point in $C = \bigcup_{j=1}^{k} C_j$ to C is equal to Γ. The following algorithm produces a partition $\mathcal{R} \wedge \mathcal{Q} = \{w.B_j : w \in D,\ 1 \le j \le d\}$ verifying

- $\mathcal{R} \wedge \mathcal{Q}$ is finer than $\mathcal{R}$ and $\mathcal{Q}$.
- $C = \bigcup_{j=1}^{d} B_j$

Step 1: let $1 \le j \le k$. Consider $A_{1,j,i_1}, \cdots, A_{1,j,i_{l_{1,j}}}$, the sets in $\mathcal{R}$ such that

$$w_1^{-1}.A_{1,j,i_s} \cap C_j \ne \emptyset, \text{ for every } 1 \le s \le l_{1,j}.$$

We denote by $B_{1,1}, \cdots, B_{1,k_1}$, with $k_1 = \sum_{j=1}^{k} l_{1,j}$, the elements of the collection

$$\{w_1^{-1}.A_{1,j,i_s} \cap C_j : 1 \le s \le l_{1,j}, 1 \le j \le k\}.$$

We have that $\mathcal{Q}_1 = \{w.B_{1,j} : w \in D,\ 1 \le j \le k_1\}$ is a clopen finite partition of X. In addition, for every $1 \le i \le k_1$ there exist $1 \le j \le k$ and $1 \le s \le l_{1,j}$ such that $w_1.B_{1,i} \subseteq A_{1,j,i_s}$, $w_1.B_{1,i} \subseteq w_1.C_j$ and $\bigcup_{s=1}^{l_j} B_{1,i} = C_j$. In other words, we have obtained a clopen partition $\mathcal{Q}_1 = \{w.B_{1,j} : w \in D,\ 1 \le j \le k_1\}$, satisfying

- For every $1 \leq j \leq k_1$, there exist A in $\mathcal{R}$ and B in $\mathcal{Q}$ such that $w_1.B_{1,j}$ is contained in $A \cap B$.
- $\bigcup_{j=1}^{k_1} B_{1,j} = C$.

Now, for $2 \leq n \leq l$, we suppose that the step $n-1$ has produced a finite clopen partition $\mathcal{Q}_{n-1} = \{w.B_{n-1,j} : w \in D,\, 1 \leq j \leq k_{n-1}\}$ such that

- For every $1 \leq j \leq k_{n-1}$ and every $1 \leq i \leq n-1$, there exists A in $\mathcal{R}$ and $B \in \mathcal{Q}$ such that $w_i.B_{n-1,j}$ is contained in $A \cap B$.
- $\bigcup_{j=1}^{k_{n-1}} B_{n-1,j} = C$.

<u>*Step n:*</u> let $1 \leq j \leq k_{n-1}$. Consider $A_{n,j,i_1}, \cdots, A_{n,j,i_{l_{n,j}}}$, the sets in $\mathcal{R}$ such that

$$w_n^{-1}.A_{n,j,i_s} \cap B_{n-1,j} \neq \emptyset, \text{ for every } 1 \leq s \leq l_{n,j}.$$

We denote by $B_{n,1}, \cdots, B_{n,k_n}$, with $k_n = \sum_{j=1}^{k} l_{n,j}$, the elements in the collection

$$\{w_n^{-1}.A_{j,i_s} \cap B_{n-1,j} : 1 \leq s \leq l_{n,j}, 1 \leq j \leq k_{n-1}\}.$$

We have $\mathcal{Q}_n = \{w.B_{n,j} : w \in D,\, 1 \leq j \leq k_n\}$ is a clopen finite partition of X. In addition, for every $1 \leq l \leq k_n$ there exist $1 \leq j \leq k_{n-1}$ and $1 \leq s \leq l_{n,j}$ such that $B_{n,l} \subseteq B_{n-1,j}$ and $B_{n,l} \subseteq w_n^{-1}.A_{n,j,s}$. This implies that for every $1 \leq i \leq n-1$, $w_i.B_{n,l} \subseteq w_i.B_{n-1,j}$ and by hypothesis, $w_i.B_{n,l}$ is contained in a subset A_i in $\mathcal{R}$. Since $\bigcup_{l=1}^{k_n} B_{n,l} = \bigcup_{i=1}^{k_{n-1}} B_{n-1,i}$, the partition $\mathcal{Q}_n$ satisfies

- For every $1 \leq j \leq k_n$ and $1 \leq i \leq n$, there exist A in $\mathcal{R}$ and B in $\mathcal{Q}$ such that $w_i.B_{n,j}$ is contained in $A \cap B$.
- $\bigcup_{j=1}^{k_n} B_{n,j} = C$.

This implies that after the step l, we obtain a partition

$$\mathcal{R} \wedge \mathcal{Q} = \mathcal{Q}_l = \{w.B_{l,j} : w \in D,\, 1 \leq j \leq k_l\},$$

which is finer than $\mathcal{R}$ and $\mathcal{Q}$, and which satisfies $\bigcup_{j=1}^{k_l} B_{n,j} = C$.

Now we use this algorithm to prove the Proposition 4 . From Corollary 1, there exists a decreasing sequence $(C_n)_{n\geq 0}$ of clopen subsets of X and a decreasing sequence $(\Gamma_n)_{n\geq 0}$ of syndetic subgroups of G such that $|\bigcap_{n\geq 0} C_n| = 1$ and $T_{C_n}(x) = \Gamma_n$ for all $x \in C_n$.
For every $n \geq 0$, we take a subset D_n of G containing exactly one representing element in each class of G/Γ_n, and we define

$$\mathcal{Q}_n = \{w.C_n : w \in D_n\}.$$

The collection $\mathcal{Q}_n$ is a finite clopen partition of X.
Since X is a Cantor set, it is always possible to take a sequence $(\mathcal{R}_n)_{n\geq 0}$ of finite clopen partitions of X which spans its topology.
We construct the desired sequence $(\mathcal{P}_n)_{n\geq 0}$ as follows:

- We set $\mathcal{P}_0 = \mathcal{R}_0 \wedge \mathcal{Q}_0$.
- For $n > 0$. First, we set $\mathcal{P}'_n = \mathcal{R}_n \wedge \mathcal{Q}_n$, and then $\mathcal{P}_n = \mathcal{P}_{n-1} \wedge \mathcal{P}'_n$.

From this construction we get

$$(\mathcal{P}_n = \{w.C_{n,j} : w \in D_n,\, 1 \leq j \leq k_n\})_{n\geq 0},$$

a sequence of finite clopen partition of X satisfying, for every $n \geq 0$:

(i) $\mathcal{P}_n$ is finer than $\mathcal{P}_{n-1}$ and $\mathcal{R}_n$.
(ii) $\bigcup_{j=1}^{k_n} C_{n,j} = C_n$.

The condition (i) implies $(\mathcal{P}_n)_{n\geq 0}$ is a nested sequence and that it spans the topology of X. The condition (ii) implies that this sequence verifies conditions 1. and 2. of Proposition 4. □

Let us recall that a group G is amenable if and only if any continuous G-action on a compact metric space admits an invariant probability measure. We have the following characterization: The group G is amenable if and only if it has a Følner sequence, that is, a sequence $(F_n)_{n\geq 0}$ of finite subsets of G such that for every $g \in G$

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$

Let (X, G) be a regularly recurrent Cantor system with G amenable, so this action admits an invariant probability measure. Consider the sequence of finite clopen partitions of X as in Proposition 4:

$$(\mathcal{P}_n = \{w.C_{n,k} : w \in D_n,\ 1 \leq k \leq k_n\})_{n\geq 0}.$$

Let $n \geq 0$. The incidence matrix between $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ is $A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{Z}^+)$ defined by

$$A_n(i,j) = |\{w \in D_{n+1} : w.C_{n+1,j} \subseteq C_{n,i}\}|.$$

Notice that $\sum_{i=1}^{k_n} A_n(i,j) = q_{n,j}$ is the number of $w \in D_{n+1}$ such that $w.C_{n+1,j} \subseteq C_n$. Since the set of return times of the points in C_n to C_n is equal to Γ_n, the number $q_{n,j}$ does not depend on j and it is equal to the number of $w \in D_{n+1}$ which are in Γ_n. So, $q_{n,j} = \frac{|D_{n+1}|}{|D_n|}$ for every $1 \leq j \leq k_{n+1}$. Consider the set

$$\triangle_n = \{(x_1, \cdots, x_{k_n}) \in (\mathbb{R}^+)^{k_n} : \sum_{i=1}^{k_n} x_i = \frac{1}{|D_n|}\}.$$

Since, for every $1 \leq j \leq k_{n+1}$, $\sum_{i=1}^{k_n} A_n(i,j) = \frac{|D_{n+1}|}{|D_n|}$, the map $A_n : \triangle_{n+1} \to \triangle_n$ is well defined by ordinary matrix multiplication.

Because $(\mathcal{P}_n)_{n\geq 0}$ is a countable collection of clopen sets that spans the topology of X, any invariant measure defined on this family of sets extends to a unique invariant measure on the Borel σ-algebra of X. So, any invariant measure μ on $(\mathcal{P}_n)_{n\geq 0}$ must verify

$$\mu(C_{n,i}) = \sum_{j=1}^{k_{n+1}} A_n(i,j)\mu(C_{n+1,j}), \text{ for every } 1 \leq i \leq k_n \text{ and } n \geq 0 \ ,$$

and it is completely determined by this relation. In other words, we can identify an invariant measure with an element in the inverse limit $\lim_{\leftarrow n}(\triangle_n, A_n)$. In the next Proposition we provide a sufficient condition for the reversed identification.

Remark 1. From [We], and more explicitely in [Kr], we have the following theorem.

Theorem 3 (Weiss)**.** *Let G be a numerable and amenable group and $(\Gamma_n)_{n\in\mathbb{N}}$ a nested sequence of normal subgroups s.t. $\bigcap_n \Gamma_n$ is trivial. Then there exist a Følner sequence $(D_n)_{n\in\mathbb{N}}$ of G and a subsequence $(\Gamma_{\varphi(n)})_n$ of $(\Gamma_n)_n$ s.t.:*

- *Each D_n contains exactly one representing element in each class of $G/\Gamma_{\varphi(n)}$*
- *$D_n \subset D_{n+1}$.*
- *$\bigcup_n D_n = G$.*

We deduce that in the case of an amenable group G with $\bigcap_n \Gamma_n$ trivial, up to take a subsequence, it is possible to take the sequence $(D_n)_{n\geq 0}$, defined as in Proposition 4, as a Følner sequence.

Theorem 4. *If G is amenable and the sequence $(D_n)_{n\geq 0}$ is Følner then $\mathcal{M}_G(X)$ is affinely-homeomorphic to* $\lim_{\leftarrow n}(\triangle_n, A_n)$.

Proof. Let $((x_{n,1},\cdots,x_{n,k_n}))_{n\geq 0}$ be an element in $\lim_{\leftarrow n}(\triangle_n, A_n)$. It defines a probability measure on X by setting

$$\mu(u.C_{n,i}) = x_{n,i}, \text{ for every } 1\leq i\leq k_n,\ u\in D_n \text{ and } n\geq 0.$$

To show this measure is invariant it is sufficient to show that for every $n\geq 0$, $1\leq k\leq k_n$ and $v\in G$, $\mu(v.C_{n,k}) = \mu(C_{n,k}) = x_{n,k}$.

Fix $v\in G$ and $m > n\geq 0$. Consider the sets

$$J(m,n,k,l) = \{w\in D_m : w.C_{m,l}\subseteq C_{n,k}\},$$

$J_1(m,n,k,l) = \{w\in J(m,n,k,l) : vw\in D_m\}$ and $J_2(m,n,k,l) = J(m,n,k,l)\setminus J_1(m,n,k,l)$. We have

$$v.C_{n,k} = \bigcup_{l=1}^{k_m}\bigcup_{w\in J(m,n,k,l)} vw.C_{m,l},$$

and then

$$\mu(v.C_{n,k}) = \sum_{l=1}^{k_m}\sum_{w\in J(m,n,k,l)}\mu(vw.C_{m,l}) = \sum_{l=1}^{k_m}\sum_{w\in J_1(m,n,k,l)}\mu(vw.C_{m,l}) + \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(vw.C_{m,l}).$$

Since $\mu(u.C_{m,l}) = \mu(C_{m,l})$ for $u\in D_m$, we get

$$\begin{aligned}\mu(v.C_{n,k}) &= \sum_{l=1}^{k_m}|J_1(m,n,k,l)|\mu(C_{m,l}) + \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(vw.C_{m,l})\\ &= \mu(C_{n,k}) - \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(C_{m,l}) + \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(vw.C_{m,l}).\end{aligned}$$

Thus we have

$$|\mu(v.C_{n,k}) - \mu(C_{n,k})| \leq \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(C_{m,l}) + \sum_{l=1}^{k_m}\sum_{w\in J_2(m,n,k,l)}\mu(vw.C_{m,l}).$$

Because $J_2(m,n,k,l)\subset\{w\in D_m : vw\notin D_m\}$, we have

$$\begin{aligned}|\mu(v.C_{n,k}) - \mu(C_{n,k})| &\leq \sum_{\{w\in D_m :\, vw\notin D_m\}}\sum_{l=1}^{k_m}\mu(C_{m,l}) + \sum_{\{w\in D_m :\, vw\notin D_m\}}\sum_{l=1}^{k_m}\mu(vw.C_{m,l})\\ &= \sum_{\{w\in D_m :\, vw\notin D_m\}}\mu\left(\bigcup_{l=1}^{k_m}C_{m,l}\right) + \sum_{\{w\in D_m :\, vw\notin D_m\}}\mu\left(\bigcup_{l=1}^{k_m}vw.C_{m,l}\right).\end{aligned}$$

Since $|\{w\in D_m : vw\notin D_m\}|\leq |v.D_m\triangle D_m|$ and $\mu\left(\bigcup_{l=1}^{k_m}C_{m,l}\right) = \mu\left(\bigcup_{l=1}^{k_m}vw.C_{m,l}\right) = \frac{1}{|D_m|}$, we have

$$|\mu(v.C_{n,k}) - \mu(C_{n,k})| \leq \frac{2|v.D_m\triangle D_m|}{|D_m|}.$$

So, because $(D_n)_{n\geq 0}$ is Følner, we get $\mu(v.C_{n,k}) = \mu(C_{n,k})$.

∎

5. Semicocycles

The notion of a semicocycle has been extensively used in the theory of one-dimensional Toeplitz flows (see [Do]). In this section it is not used but we develop it for actions of a residually finite discrete group G for further utility.

Recall that for a residually finite group G and a decreasing sequence $(\Gamma_i)_{i\geq 0}$ of syndetic subgroups of G with $\bigcap_{i\geq 0}\Gamma_i = \{e\}$, the stabilizer of $\mathbf{e} = (e_i)_{i\geq 0}$ in the free G-subodometer $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ is trivial. This defines an immersion τ of G into $\overleftarrow{G}$.

Definition 3. *Let $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ be a G-subodometer with $\bigcap_{i\geq 0}\Gamma_i = \{e\}$ and let K be a compact metric space. A function $f : G \to K$ is a* semicocycle on $\overleftarrow{G}$ *if it is continuous with respect $\Theta_{\overleftarrow{G}}$, where $\Theta_{\overleftarrow{G}}$ is the topology on G inherited from $\overleftarrow{G}$ (we identify $\tau(G)$ with G).*

The functions $f : G \to K$ may be seen as elements of the topological dynamical system (K^G, G), where K^G is endowed with the metrizable product topology, and the left-action of $\gamma \in G$ on $f = (f(g))_{g\in G} \in K^G$ is the shift action, defined by $\gamma.f \in K^G$, where $\gamma.f(g) = f(g\gamma)$ for every $g \in G$.

The proofs of Theorems 5 and 6 below follow the same ideas as used in [Do] for $G = \mathbb{Z}$.

Theorem 5. *If $f \in K^G$ is a semicocycle on some subodometer $\overleftarrow{G}$ then f is a regularly recurrent point of (K^G, G).*

Proof. Fix $\epsilon > 0$ and a finite set C in G. The pair (ϵ, C) determines a basic open set V in the Tychonov topology. Since f is continuous on G for the topology induced by the odometer $\overleftarrow{G}$, there exists $\delta > 0$ such that for every $g \in C$ and $g' \in G$, $\mathrm{dist}(g, g') < \delta$ (for the metric inherited from $\overleftarrow{G}$) implies $\mathrm{d}(f(g), f(g')) < \epsilon$ in K. By definition of a subodometer, there exist a finite index subgroup Γ of G and a factor map $\pi : \overleftarrow{G} \to G/\Gamma$ such that for any element w of G/Γ, $\pi^{-1}(w)$ is a clopen subset of $\overleftarrow{G}$ with diameter smaller than δ. Furthermore, for any $y \in \pi^{-1}(w)$, $T_{\pi^{-1}(w)}(y)$ is a group conjugated to Γ. Let us consider now the finite index normal subgroup $H = \cap_{g\in G} g\Gamma g^{-1}$. Since Γ is of finite index in G, there is just a finite number of groups conjugated to Γ and the former intersection is a finite intersection. The group H is a subgroup of any group of the kind $T_{\pi^{-1}(w)}(y)$ with $w \in G/\Gamma, y \in \pi^{-1}(w)$. Thus, $\mathrm{dist}(n'.g, n') < \delta$ for any $g \in H$, $n' \in G$, by the normality of H. Hence $d(f(n'.g), f(n')) < \epsilon$ for any $g \in H$ and $n' \in G$. We have proved that the H-orbit of f is contained in V and then f is a regularly recurrent point of K^G. □

Proposition 2 and Theorem 5 imply that $(\Omega_G(f), G)$ is a minimal almost 1-1 extension of some free subodometer, where $\Omega_G(f)$ represents the orbit closure of a semicocycle f in K^G with a trivial stabilizer under the action of G. Notice that $\overleftarrow{G}$ needs not to be the maximal equicontinuous factor of $(\Omega_G(f), G)$, as we will see later.

Let $f \in K^G$ be a semicocycle on a G-subodometer $\overleftarrow{G}$. Since we have identified the group G with G embedded in $\overleftarrow{G}$, it makes sense to define F to be the closure of the graph of f in $\overleftarrow{G} \times K$ endowed with the product topology, $F = \overline{\{(g, f(g)) : g \in G\}} \subseteq \overleftarrow{G} \times K$. Let $F(\mathbf{g})$ be the set $\{k \in K : (\mathbf{g}, k) \in F\}$ for $\mathbf{g} \in \overleftarrow{G}$.

We call C_f the set of $\mathbf{g} \in \overleftarrow{G}$ such that $|F(\mathbf{g})| = 1$ and $D_f = \overleftarrow{G} \setminus C_f$. Since f is continuous we have that $F(g) = \{g\}$ for all $g \in G$. Thus C_f is the subset where f can be continuously extended by $f(\mathbf{g}) = F(\mathbf{g})$.

The semicocycle f is said to be *invariant under no rotation* if for every $\mathbf{h}_1 \neq \mathbf{h}_2 \in \overleftarrow{G}$ there exists a $g \in G$ such that $F(g.\mathbf{h}_1) \neq F(g.\mathbf{h}_2)$.

Theorem 6. *Let (X, G) be a minimal topological dynamical system and $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ be a G-subodometer with $\bigcap_{i\geq 0} \Gamma_i = \{e\}$. There exists an almost 1-1 factor π of (X, G) onto $(\overleftarrow{G}, G)$ with $|\pi^{-1}(\mathbf{e})| = 1$ if and only if (X, G) is conjugated to $(\Omega_G(f), G)$, where f is a semicocycle on $\overleftarrow{G}$, invariant under no rotation.*

Proof. Consider the system $(\Omega_G(f), G)$. By definition, for every $x \in \Omega_G(f)$, there exists a sequence $(g_i)_i \subset G$ such that for each $h \in G$, $\lim_i f(hg_i) = x(h)$. Let $\mathbf{j} \in \overleftarrow{G}$ be an accumulation point of the sequence $(g_i)_i$. We have $x(h) \in F(h.\mathbf{j})$. By a straightforward calculation, we check that for each such $\mathbf{j}$, the set $\{(h.\mathbf{j}, x(h))| \ h \in G\}$ is a dense subset of F. Since f is invariant under no rotation, $\mathbf{j}$ is determined for any x in an unique way. So we have proved that if $g_i.f \to x$ then $g_i \to \mathbf{j}$. The map $\pi : x \in \Omega_G(f) \mapsto \mathbf{j} \in \overleftarrow{G}$ is a continuous extension onto $\Omega_G(f)$ of the application $g.f \mapsto g$. It is straightforward to check that π is a factor map that sends f to $\mathbf{e}$. If $\pi(x) = \mathbf{e}$ then $x(h) \in F(h) = \{f(h)\}$ and $x(h) = f(h)$ for any $h \in G$. Since the system $(\Omega_G(f), G)$ is minimal, π is an almost 1 to 1 factor map.
Conversely, consider a minimal almost 1-1 extension (X, G) of a G-subodometer and $\pi : X \to \overleftarrow{G}$ the associated factor map. Consider $x \in X$ such that $\pi(x)$ has a singleton fiber by π. The same holds for all the elements of its G-orbit. The map $f : g \in G \mapsto \pi^{-1}(g.\pi(x)) = g.\pi^{-1}(x) \in X$ is continuous for the induced topology on G, it is then a semicocycle. This is straightforward to check that $F(\mathbf{j}) = \pi^{-1}(\mathbf{k})$ where $\mathbf{k} \in \overleftarrow{G}$ is the limit point of the sequence $(g_i.\pi(x))_i$ with (g_i) a sequence of G that converges to $\mathbf{j}$. The set $\pi^{-1}(\mathbf{k})$ does not depend of the choice of the sequence (g_i). It is then straightforward to show that f is invariant under no rotation. The conjugating map from $(\Omega_G(f), G)$ onto (X, G) is the projection onto the neutral element coordinate: $\phi \mapsto \phi(\mathbf{e})$. By a standard way, we check this application is a homeomorphism which commutes with the G-action. □

Corollary 5. *A topological dynamical system (X, G) is a minimal almost 1-1 extension of a free odometer $(\overleftarrow{G}, G)$ if and only if it is conjugated to $(\Omega_G(f), G)$, where f is a semicocycle on G, invariant under no rotation.*

Proof. For a factor map $p : X \to \overleftarrow{G}$ and any point $x \in X$, by a right multiplication by $p(x)^{-1}$, we obtain again a factor map that sends the point x to $\mathbf{e}$. The result follows from Theorem 6. □

6. G-Toeplitz Arrays

In this section we assume that G is a discrete finitely generated group. For a finite alphabet Σ equipped with the discrete topology, we consider the left action of G on Σ^G, continuous with respect to the product topology, defined for $x = (x(g))_{g\in G} \in \Sigma^G$ and $\gamma \in G$ by :

$$\gamma.x(g) = x(g\gamma) \text{ for any } g \in G.$$

Recall that we denote by $\Omega_G(x)$ the closure of the G-orbit of x for this action. In this section, we consider regularly recurrent systems of the kind $(\omega_G(x), G)$.

For a syndetic group $\Gamma \subseteq G$ and $x = (x(g))_{g\in G} \in \Sigma^G$ we define:

$$Per(x, \Gamma, \sigma) = \{g \in G : x(g\gamma) = \sigma \ \text{ for all } \gamma \in \Gamma\}, \quad \sigma \in \Sigma,$$

$$Per(x, \Gamma) = \bigcup_{\sigma\in\Sigma} Per(x, \Gamma, \sigma).$$

Clearly for two subgroups Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2$, we have $Per(x, \Gamma_2, \sigma) \subset Per(x, \Gamma_1, \sigma)$. When $Per(x, \Gamma) \neq \emptyset$ we say that Γ is a *group of periods of* x. Furthermore, $Per(x, \Gamma)$ is invariant under the right multiplication by an element of Γ. We say that x is a *G-Toeplitz array* (or simply a Toeplitz array) if for all $g \in G$ there exists a syndetic subgroup $\Gamma \subseteq G$ such that $g \in Per(x, \Gamma)$.

Proposition 5. *The following statements concerning $x \in \Sigma^G$ are equivalent:*

(1) *x is a Toeplitz array.*
(2) *There exists a sequence of syndetic subgroups $(\Gamma_n)_{n\geq 0}$, such that $\Gamma_{n+1} \subset \Gamma_n$ for all $n \geq 0$ and $G = \cup_n Per(x, \Gamma_n)$.*
(3) *x is regularly recurrent.*

Proof. Let D_n be the ball of radius n in G centered at the neutral element.
Suppose that x is a Toeplitz array. Since for any two groups Z_1, Z_2 of periods of x we have $Per(x, Z_1) \subset Per(x, Z_1 \cap Z_2)$, for any $n \geq 0$, there exists a syndetic subgroup Z_n such that $D_n \subset Per(x, Z_n)$. Let $\Gamma_0 = Z_0$ and $\Gamma_{n+1} = \Gamma_n \cap Z_n$. The sequence $(\Gamma_n)_n$ satisfies the statement (2).
Let $(\Gamma_n)_n$ be a sequence as in statement (2). Let C_n be the set $\{y \in \Sigma^G : y(D_n) = x(D_n)\}$ for all $n \geq 0$, $(C_n)_{n\geq 0}$ is a fundamental system of clopen neighborhoods of x. Since D_n is contained in $Per(x, \Gamma_n)$, the set of return times of x to C_n contains Γ_n which implies that x is regularly recurrent.
Suppose that x is regularly recurrent. For $n \geq 0$ we take Γ_n a syndetic subgroup of G such that $\Gamma_n \subseteq T_{C_n}(x)$. It holds that G is equal to $\bigcup_{n\geq 0} Per(x, \Gamma_n)$, which means that x is a Toeplitz array. □

A subshift (X, G) is a *G-Toeplitz system* (or simply a Toeplitz system) if there exists a Toeplitz array x such that $X = \Omega_G(x)$. From Theorem 2 and Proposition 5 we conclude that the family of minimal subshifts which are almost 1-1 extensions of subodometers coincides with the family of Toeplitz systems.
In order to know the maximal equicontinuous factor of a given Toeplitz system, we will introduce the concepts of essential group of periods and period structure.

Definition 4. *Let $x \in \Sigma^G$. A syndetic group $\Gamma \subset G$ is called an* essential group of periods of x *if $Per(x, \Gamma, \sigma) \subseteq Per(g.x, \Gamma, \sigma)$ for every $\sigma \in \Sigma$ implies that $g \in \Gamma$.*

Lemma 5. *If Γ is an essential group of periods of x then every group of periods Γ' satisfying $Per(x, \Gamma) \subseteq Per(x, \Gamma')$ is contained in Γ.*

Proof. Let Γ be an essential group of periods of x. Suppose that Γ' is a group of periods such that $Per(x, \Gamma) \subseteq Per(x, \Gamma')$. For $w \in Per(x, \Gamma, \sigma)$ and $g \in \Gamma'$ we have $w\gamma g \in Per(x, \Gamma', \sigma)$ for every $\gamma \in \Gamma$. This implies that $x(w\gamma g) = g.x(w\gamma) = \sigma$ for every $\gamma \in \Gamma$, which means that $w \in Per(g.x, \Gamma', \sigma)$. Because Γ is essential, we conclude that $g \in \Gamma$ and then $\Gamma' \subseteq \Gamma$. □

Remark 2. From Lemma 5 we deduce that the family of the essential groups of periods is contained in the family of the groups generated by essential periods introduced in [Co] for the case $G = \mathbb{Z}^d$.
Notice that for any $x \in \Sigma^G$, $g \in G$ and any group $\Gamma \subset G$, we have the relation $Per(g.x, \Gamma, \sigma) = Per(x, g^{-1}\Gamma g, \sigma)g^{-1}$ for any $\sigma \in \Sigma$. This relation will be useful in the following to characterize the essential groups of periods.

In the following Lemma we show the existence of essential groups of periods.

Lemma 6. *Let $x \in \Sigma^G$. If $\Gamma \subseteq G$ is a group of periods of x then there exists an essential group $K \subseteq G$ of periods of x such that $Per(x, \Gamma) \subseteq Per(x, K)$.*

Proof. Let $\Gamma \subseteq G$ be a group of periods of x and Γ' be a syndetic normal subgroup of Γ. We denote by $\hat{\Gamma}'$ the collection of shifted groups:

$$\bigcup_{g\in G}\{Hg : H \text{ syndetic subgroup of } G \text{ such that } Per(x,\Gamma',\sigma) \subseteq Per(x, g^{-1}Hg,\sigma)g^{-1}, \forall \sigma \in \Sigma\}.$$

Let K be the group generated by the elements of the union of all sets in $\hat{\Gamma}'$. Let $w \in Per(x,\Gamma',\sigma)$. For any $\gamma \in \Gamma'$ and any $Hg \in \hat{\Gamma}'$, $w\gamma$ belongs to $Per(x,\Gamma',\sigma) \subseteq Per(x, g^{-1}Hg,\sigma)g^{-1}$. This implies that for every $hg \in Hg \in \hat{\Gamma}'$ we have $w\gamma hg \in Per(x, g^{-1}Hg, \sigma)$. Since Γ' is a normal subgroup, we get for any $\gamma \in \Gamma'$ and any $hg \in Hg \in \hat{\Gamma}'$, $x(whg\gamma) = \sigma$, which means that $whg \in Per(x,\Gamma',\sigma)$. Thus we obtain that for any $h_1g_1,\dots,h_ng_n$ with h_ig_i belonging to a set in $\hat{\Gamma}'$ and $w \in Per(x,\Gamma',\sigma)$, we have $x(wh_1g_1\dots h_ng_n) = \sigma$. In other words, $Per(x,\Gamma',\sigma)$ is contained in $Per(x,K,\sigma)$. So, we have $Per(x,\Gamma,\sigma) \subseteq Per(x,\Gamma',\sigma) \subseteq Per(x,K,\sigma)$. If $g \in G$ is such that $Per(x,K,\sigma) \subseteq Per(g.x,K,\sigma) = Per(x,g^{-1}Kg,\sigma)g^{-1}, \forall \sigma \in \Sigma$, then Kg belongs to $\hat{\Gamma}'$, which implies that g is in K. □

Corollary 6. *Let $x \in \Sigma^G$ be a Toeplitz array. There exists a sequence $(\Gamma_n)_{n\geq 0}$ of essential group of periods of x such that $\Gamma_{n+1} \subseteq \Gamma_n$ and $\bigcup_{n\geq 0} Per(x,\Gamma_n) = G$.*

Proof. From Proposition 5 (2) we conclude there exists a decreasing sequence $(\Gamma'_n)_{n\geq 0}$ of syndetic groups of periods of x such that $\bigcup_{n\geq 0} Per(x,\Gamma'_n) = G$. We set Γ_0 an essential group of periods of x such that $Per(x,\Gamma'_0) \subseteq Per(x,\Gamma_0)$. For $n > 0$ we set $\Gamma''_n = \Gamma'_n \cap \Gamma_{n-1}$ which is a syndetic subgroup of G, and since $Per(x,\Gamma_{n-1})$ and $Per(x,\Gamma'_n)$ are contained in $Per(x,\Gamma''_n)$, Γ''_n is a group of periods of x. Thus, by Lemma 6, there exists an essential group of periods Γ_n, such that $Per(x,\Gamma_{n-1}) \subseteq Per(x,\Gamma''_n) \subseteq Per(x,\Gamma_n)$. Since Γ_{n-1} is an essential group of periods, from Lemma 5 we get $\Gamma_n \subseteq \Gamma_{n-1}$. Because $\bigcup_{n\geq 0} Per(x,\Gamma'_n) = G$, we deduce $\bigcup_{n\geq 0} Per(x,\Gamma_n) = G$. □

Definition 5. *A sequence of groups as in Corollary 6 is called a* period structure *of x.*

In the sequel, we will show that from a period structure $(\Gamma_n)_{n\geq 0}$ of a G-Toeplitz array x it is possible to construct a sequence of nested finite clopen partitions of $\Omega_G(x)$. From this sequence of partitions it will be easy to define an almost 1-1 factor map between the Toeplitz system $(\Omega_G(x), G)$ and the odometer $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$.
Let $x \in \Sigma^G$ be a Toeplitz array, let $y \in \Omega_G(x)$ and let $\Gamma \subseteq G$ be group of periods of y. We define the set:

$$C_\Gamma(y) = \{x' \in \Omega_G(x) : Per(x',\Gamma,\sigma) = Per(y,\Gamma,\sigma),\ \forall\ \sigma \in \Sigma\}.$$

Lemma 7. *$C_\Gamma(y) = \gamma.C_\Gamma(y)$ for every $\gamma \in \Gamma$.*
For every $x' \in C_\Gamma(y)$, we have $\Omega_\Gamma(x') \subseteq C_\Gamma(y)$.

Proof. Let $x' \in \gamma.C_\Gamma(y)$. There exists $x'' \in C_\Gamma(y)$ such that $x' = \gamma.x''$. If $g \in Per(x',\Gamma,\sigma)$ then $x'(g\gamma') = \sigma$ for every $\gamma' \in \Gamma$. In particular, we have

$$\sigma = x'(g\gamma'\gamma^{-1}) = \gamma^{-1}.x'(g\gamma') = x''(g\gamma'),\ \forall \gamma' \in \Gamma,$$

which implies $Per(x',\Gamma,\sigma) \subseteq Per(x'',\Gamma,\sigma) = Per(y,\Gamma,\sigma)$. On the other hand, if $g \in Per(x'',\Gamma,\sigma)$ then

$$\sigma = x''(g\gamma') = x''(g\gamma'\gamma) = \gamma.x''(g\gamma') = x'(g\gamma'),\ \forall \gamma' \in \Gamma,$$

which implies that $Per(y,\Gamma,\sigma) \subseteq Per(x',\Gamma,\sigma)$. Thus we obtain that $\gamma.C_\Gamma(y) \subseteq C_\Gamma(y)$. Since this is true also for γ^{-1}, we conclude that $\gamma.C_\Gamma(y) = C_\Gamma(y)$.

To show the second point, let us first consider $x' \in \Omega_\Gamma(y)$. It is straightforward to show that for any σ, $Per(y,\Gamma,\sigma) \subseteq Per(x,\Gamma,\sigma)$. Since $\Omega_\Gamma(y)$ is a minimal Γ-invariant closed set (Lemma 3), y belongs to $\Omega_\Gamma(x')$ and therefore $Per(x',\Gamma,\sigma) = Per(y,\Gamma,\sigma)$. So we have $\Omega_\Gamma(y) \subseteq C_\Gamma(y)$. To conclude, notice that $C_\Gamma(y) = C_\Gamma(x')$ for any $x' \in C_\Gamma(y)$, so we get $\Omega_\Gamma(x') \subseteq C_\Gamma(x')$ for any $x' \in C_\Gamma(y)$. □

We will use the following convention: for a Γ-periodic subset C of $\Omega_G(x)$, i.e. such that $w.C = w'.C$ whenever $w^{-1}w' \in \Gamma$, we will write $v.C$ instead of $w.C$, where v is the projection of w to G/Γ.

Proposition 6. *Let $x \in \Sigma^G$ be a Toeplitz array and let $y \in \Omega_G(x)$. If $\Gamma \subseteq G$ is an essential group of periods of y then $\Omega_\Gamma(y) = C_\Gamma(y)$ and $\{w.C_\Gamma(y)\}_{w\in G/\Gamma}$ is a clopen partition of $\Omega_G(x)$.*

Proof. By Lemma 7, we have $\Gamma \subseteq T_{C_\Gamma}(x')$ for every $x' \in C_\Gamma(y)$. In the sequel, we will show that for an essential group of periods Γ, we have $T_{C_\Gamma(y)}(x') = \Gamma$ for every $x' \in C_\Gamma(y)$.

Suppose that $g \in G$ satisfies $g.y \in C_\Gamma(y)$. This implies $Per(g.y,\Gamma,\sigma) = Per(y,\Gamma,\sigma)$ for every $\sigma \in \Sigma$. Since Γ is an essential group of periods of y, we obtain $g \in \Gamma$ and $T_{C_\Gamma(y)}(y) = \Gamma$. By Lemma 3, for any $x' \in \Omega_G(x)$, the set $\Omega_\Gamma(x')$ is a minimal Γ-invariant set, hence by syndicity of Γ and by minimality of the G-action, $\Omega_G(x)$ is a finite and disjoint union of minimal Γ-invariant sets. So the sets $\Omega_\Gamma(x')$ are clopen sets and by Lemma 7, $C_\Gamma(y)$ is a finite union of clopen sets. By minimality of the G-action, it is straightforward to check that $T_{C_\Gamma(y)}(x') = \Gamma$ for every $x' \in C_\Gamma(y)$. Thus we get that $\{w.C_\Gamma(y)\}_{w\in G/\Gamma}$ is a collection of disjoint sets. Moreover, this collection is a partition of $\Omega_G(x)$ because $w.\Omega_\Gamma(y) \subseteq w.C_\Gamma(y)$ for every $w \in G/\Gamma$ and $\{w.\Omega_\Gamma(y)\}_{w\in G/\Gamma}$ is a covering of $\Omega_G(x)$. This also implies that $\Omega_\Gamma(x) = C_\Gamma(x)$. □

Proposition 7. *Let $x \in \Sigma^G$ be a Toeplitz array. If $(\Gamma_n)_{n\geq 0}$ is a period structure of x then the subodometer $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ is the maximal equicontinuous factor of $(\Omega_G(x), G)$.*

Proof. By Proposition 6, if $(\Gamma_n)_{\geq 0}$ is period structure of the Toeplitz array x, then $(\{C_{g.\Gamma_n}(x) : g \in G/\Gamma_n\})_{n\geq 0}$ is a sequence of nested clopen partitions of $\Omega_G(x)$. This implies that the function $f_n : \Omega_G(x) \to G/\Gamma_n$ given by $f_n(y) = g$ if and only if $y \in g.C_{\Gamma_n}(x)$ is a well defined continuous function, $y \in \Omega_G(x)$, $n \geq 0$. The function $\pi : \Omega_G(x) \to \overleftarrow{G}$ given by $\pi = (f_n)_{n\geq 0}$ is a factor map. Since, by definition $\bigcap_{n\geq 0} C_{\Gamma_n}(x) = \{x\}$, we have that $\pi^{-1}\{\mathbf{e}\} = \{x\}$ and then π is an almost 1-1 factor map. □

Theorem 7. *For every subodometer $\overleftarrow{G}$ there exists a Toeplitz array $x \in \{0,1\}^G$ such that $\overleftarrow{G}$ is the maximal equicontinuous factor of $(\Omega_G(x), G)$.*

Proof. Let $\overleftarrow{G} = \lim_{\leftarrow n}(G/\Gamma_n, \pi_n)$ be a subodometer with $\Gamma_0 = G$. We distinguish two cases:
Case 1. There exists $m \geq 0$ such that $\Gamma_n = \Gamma_m$ for all $n \geq m$. In this case $\overleftarrow{G}$ is the finite group G/Γ_m and then every minimal almost 1-1 extension will be conjugate to $\overleftarrow{G}$. For example, $x \in \{0,1\}^G$ defined by $x(v) = 0$ for all $v \in \Gamma_m$ and $x(v) = 1$ if not, provides a Toeplitz sequence x such that $\overleftarrow{G}$ is the maximal equicontinuous factor of the system associated to x.

Case 2. For every $m \geq 0$ there exists $n > m$ such that $\Gamma_n \neq \Gamma_m$. In this case we can take a subsequence $(\Gamma_n)_{n\geq 0}$ such that $\Gamma_{n+1} \neq \Gamma_n$ and $[\Gamma_n : \Gamma_{n+1}] \geq 2$ for all $n \geq 0$. By Proposition 2, $\overleftarrow{G}$ is conjugate to the subodometer obtained from this sequence. In order to construct the Toeplitz array x we will consider a sequence $(D_n)_{n\geq 0}$ of compact subsets of G such that:

- for each n, D_n is a fundamental domain of Γ_n (i.e. D_n contains an unique element of each class of G/Γ_n). The set D_0 is the singelton set $\{e\}$.
- For each n, $D_n \subset D_{n+1}$ and $D_{n+1} = \bigsqcup_{k\in K_n} D_n.k$ for some finite set $K_n \subset G$ containing the neutral element e of G. By assumption, the cardinal of K_n is bigger than 2.

- $\bigcup_{g\in\cap_n\Gamma_n}\bigcup_{n\geq 0} D_n.g = G.$

We define now a sequence of subsets of G $(S_n)_{n\geq 0}$ by induction. Let S_0 be the singleton $\{e\}$. Let v_1 be an element of D_1 distinct from e and let $S_1=\{v_1\}$. For $n>1$, let S_n be the set $v_{n-1}.\Gamma_{n-1}\cap(D_n\setminus D_{n-1})$ and let v_n be a point in S_n. We define then $x\in\{0,1\}^G$ by :

$$x(w)=\left\{\begin{array}{ll} 0 & \text{if } w \text{ belongs to } \cup_{n\geq 0}S_{2n}.\Gamma_{2n+1}, \\ 1 & \text{else.}\end{array}\right. \tag{2}$$

Notice that $x(w)=1$ for the element w of $\cup_{n\geq 0}S_{2n+1}.\Gamma_{2n+2}$. Since $\displaystyle\bigcup_{j\in\{j,\ 0\leq 2j+1\leq n\}} S_{2j}\Gamma_{2j+1}$ $\subseteq Per(x,\Gamma_n,0)$ and $(D_{n-1}\setminus \displaystyle\bigcup_{j\in\{j,\ 0\leq 2j+1\leq n\}} S_{2j}\Gamma_{2j+1})\subseteq Per(x,\Gamma_n,1)$ for any $n\geq 1$, it holds that $D_{n-1}\subseteq Per(x,\Gamma_n)$, and for any $g\in\bigcap_n\Gamma_n$, we have also $D_{n-1}.g\subseteq Per(x,\Gamma_n)$. Thus, we get $G=\bigcup_{n\geq 0}Per(x,\Gamma_n)$ and x is a Toeplitz array. To conclude that $\overleftarrow{G}$ is the maximal equicontinuous factor of the system associated to x, by Proposition 7, it is enough to show that $(\Gamma_n)_{n\geq 0}$ is a period structure of x.

Let us prove by induction on n that Γ_n is an essential group of periods of x. For $n=0$, $\Gamma_0=G$ and this is obviously true. Suppose now that $n>0$ and that Γ_{n-1} is an essential group of periods. Let $g\in G$ be such that $Per(x,\Gamma_n,\sigma)\subset Per(g.x,\Gamma_n,\sigma)$, for all σ of $\{0,1\}$. Since $\Gamma_n\subset\Gamma_{n-1}$, we have $Per(x,\Gamma_{n-1},\sigma)\subset Per(x,\Gamma_n,\sigma)$. Let γ_{n-1} in Γ_{n-1}, there exist $\gamma\in D_n$ and $\gamma_n\in\Gamma_n$ such that $\gamma_{n-1}=\gamma\gamma_n$. For w in $Per(x,\Gamma_{n-1},\sigma)$, we have $w\gamma_{n-1}\gamma_n^{-1}=w\gamma$ belongs to $Per(x,\Gamma_{n-1},\sigma)\subset Per(g.x,\Gamma_n,\sigma)$. So we have $\sigma=g.x(w\gamma)=g.x(w\gamma.\gamma_n)=g.x(w.\gamma_{n-1})$ and therefore $w\in Per(g.x,\Gamma_{n-1},\sigma)$ for all $w\in Per(x,\Gamma_{n-1},\sigma)$. By the hypothesis of induction we get that g belongs to Γ_{n-1}.
By the definition of x, the element v_{n-1} belongs to $Per(x,\Gamma_n,\sigma)$ with $\sigma=x(v_{n-1})$, so $x(v_{n-1}.g)=\sigma$. Since $g\in\Gamma_{n-1}$ and by the construction of x, g belongs to Γ_n and so Γ_n is an essential group of periods of x. □

Remark 3. It is interesting to note that when $\overleftarrow{G}$ is a free odometer, the action of G on $\overleftarrow{G}$ is free and minimal. The G-Toeplitz array x, constructed as above, is such that $(\Omega_G(x),G)$ is an almost 1-1 extension of the system $(\overleftarrow{G},G)$, so the action of G on $\Omega_G(x)$ is also free and minimal. All the elements of $\Omega_G(x)$ are not stable for the G-action.

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INVARIANT MEASURES AND ORBIT EQUIVALENCE FOR GENERALIZED TOEPLITZ SUBSHIFTS.

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Abstract. We show that for every metrizable Choquet simplex K and for every group G, which is infinite, countable, amenable and residually finite, there exists a Toeplitz G-subshift whose set of shift-invariant probability measures is affine homeomorphic to K. Furthermore, we get that for every integer $d > 1$ and every Toeplitz flow (X, T), there exists a Toeplitz $\mathbb{Z}^d$-subshift which is topologically orbit equivalent to (X, T).

1. Introduction

The *Toeplitz subshifts* are a rich class of symbolic systems introduced by Jacobs and Keane in [21], in the context of $\mathbb{Z}$-actions. Since then, they have been extensively studied and used to provide series of examples with interesting dynamical properties (see for example [7, 8, 17, 27]). Generalizations of Toeplitz subshifts and some of their properties to more general group actions can be found in [3, 5, 9, 22, 23]. For instance, in [5] Toeplitz subshifts are characterized as the minimal symbolic almost 1-1 extensions of odometers (see [13] for this result in the context of $\mathbb{Z}$-actions). In this paper, we give an explicit construction that generalizes the result of Downarowicz in [7], to Toeplitz subshifts given by actions of groups which are amenable, countable and residually finite. The following is our main result.

Theorem A. *Let G be an infinite, countable, amenable and residually finite group. For every metrizable Choquet simplex K and any G-odometer O, there exists a Toeplitz G-subshift whose set of invariant probability measures is affine homeomorphic to K and such that it is an almost 1-1 extension of O.*

Typical examples of the groups G involved in this theorem are the finitely generated subgroups of upper triangular matrices in $GL(n, \mathbb{C})$.
The strategy of Downarowicz in [7], is to construct an affine homeomorphism between an arbitrary metrizable Choquet simplex K and a subset of the space of invariant probability measures of the full shift $\{0, 1\}^{\mathbb{Z}}$. Then he shows it coincides with the space of invariant probability measures of a Toeplitz subshift $Y \subseteq \{0, 1\}^{\mathbb{Z}}$. To do this, he uses the structure of metric space of the space of measures. In this paper we consider the representation of K as an inverse limit of finite dimensional simplices with linear transition maps $(M_n)_n$. Then we use this transition maps to construct Toeplitz G-subshifts having sequences of Kakutani-Rokhlin partitions with $(M_n)_n$ as the associated sequence of incidence matrices. Our approach is closer to the strategy used in [17] by Gjerde and Johansen, and deals with the combinatorics of Følner sequences.

2000 *Mathematics Subject Classification.* Primary: 37B10; Secondary: 37B05.
Key words and phrases. Toeplitz subshift, discrete group actions, invariant measures, orbit equivalence.
M. I. Cortez acknowledges financial support from proyecto Fondecyt 1100318. This work is part of the project PICS "dynamics and combinatorics".

We obtain, furthermore some consequences in the orbit equivalence problem. Two minimal Cantor systems are (topologically) orbit equivalent, if there exists an orbit-preserving homeomorphism between their phase spaces. Giordano, Matui, Putnam and Skau show in [15] that every minimal $\mathbb{Z}^d$-action on the Cantor set is orbit equivalent to a minimal $\mathbb{Z}$-action. It is still unknown if every minimal action of a countable amenable group on the Cantor set is orbit equivalent to a $\mathbb{Z}$-action. Nevertheless it is clear that the result in [15] can not be extended to any countable group. For instance, by using the notion of cost, Gaboriau [14] proves that if two free actions of free groups $\mathbb{F}_n$ and $\mathbb{F}_p$ are (even measurably) orbit equivalent then their rank are the same i.e. $n = p$. Another problem is to know which are the $\mathbb{Z}$-orbit equivalence classes that the $\mathbb{Z}^d$-actions (or more general group actions) realize. We give a partial answer for this question. As a consequence of the proof of Theorem A we obtain the following result.

Theorem B. *Let $(X, \sigma|_X, \mathbb{Z})$ be a Toeplitz $\mathbb{Z}$-subshift. Then for every $d \geq 1$ there exists a Toeplitz $\mathbb{Z}^d$-subshift which is orbit equivalent to $(X, \sigma|_X, \mathbb{Z})$.*

This paper is organized as follows. Section 2 is devoted to introduce the basic definitions. For an amenable discrete group G and a decreasing sequence of finite index subgroups of G with trivial intersection, we construct in Section 3 an associated sequence $(F_n)_{n\geq 0}$ of fundamental domains, so that it is Følner and each F_{n+1} is tilable by translated copies of F_n. In Section 4 we construct Kakutani-Rokhlin partitions for generalized Toeplitz subshifts, and in Section 5 we use the fundamental domains introduced in Section 3 to construct Toeplitz subshifts having sequences of Kakutani-Rokhlin partitions with a prescribed sequence of incidence matrices. This construction improves and generalizes that one given in [4] for $\mathbb{Z}^d$-actions, and moreover, allows to characterize the associated ordered group with unit. In Section 6 we give a characterization of any Choquet simplex as an inverse limit defined by sequences of matrices that we use in Section 5 (they are called "managed" sequences). Finally, in Section 7 we use the previous results to prove Theorems A and B.

2. Basic definitions and background

In this article, by a *topological dynamical system* we mean a triple (X, T, G), where T is a continuous left action of a countable group G on the compact metric space (X, d). For every $g \in G$, we denote T^g the homeomorphism that induces the action of g on X. The unit element of G will be called e. The system (X, T, G) or the action T is *minimal* if for every $x \in X$ the orbit $o_T(x) = \{T^g(x) : g \in G\}$ is dense in X. We say that (X, T, G) is a *minimal Cantor system* or a minimal Cantor G-system if (X, T, G) is a minimal topological dynamical system with X a Cantor set.
An *invariant probability measure* of the topological dynamical system (X, T, G) is a probability Borel measure μ such that $\mu(T^g(A)) = \mu(A)$, for every Borel set A. We denote by $\mathcal{M}(X, T, G)$ the space of invariant probability measures of (X, T, G).

2.1. **Subshifts.** For every $g \in G$, denote $L_g : G \to G$ the left multiplication by $g \in G$. That is, $L_g(h) = gh$ for every $h \in G$. Let Σ be a finite alphabet. Σ^G denotes the set of all the functions $x : G \to \Sigma$. The (left) *shift action* σ of G on Σ^G is given by $\sigma^g(x) = x \circ L_{g^{-1}}$, for every $g \in G$. Thus $\sigma^g(x)(h) = x(g^{-1}h)$. We consider Σ endowed with the discrete topology and Σ^G with the product topology. Thus every σ^g is a homeomorphism of the Cantor set Σ^G. The topological dynamical system (Σ^G, σ, G) is called the full G-shift on Σ. For every finite subset D of G and $x \in \Sigma^G$, we denote $x|_D \in \Sigma^D$ the restriction of x to D. For $F \in \Sigma^D$ (F is a function from D to Σ) we denote by $[F]$ the set of all $x \in \Sigma^D$ such that $x|_D = F$. The set $[F]$ is called the *cylinder* defined by F, and it is a clopen set (both open and closed). The collection of all the sets $[F]$ is a base of the topology of Σ^G.

Definition 1. *A subshift or G-subshift of Σ^G is a closed subset X of Σ^G which is invariant by the shift action.*

The topological dynamical system $(X, \sigma|_X, G)$ is also called subshift or G-subshift. See [2] for details.

2.1.1. *Toeplitz G-subshifts.* An element $x \in \Sigma^G$ is a *Toeplitz sequence*, if for every $g \in G$ there exists a finite index subgroup Γ of G such that $\sigma^\gamma(x)(g) = x(\gamma^{-1}g) = x(g)$, for every $\gamma \in \Gamma$.
A subshift $X \subseteq \Sigma^G$ is a *Toeplitz subshift* or Toeplitz G-subshift if there exists a Toeplitz sequence $x \in \Sigma^G$ such that $X = \overline{o_\sigma(x)}$. It is shown in [5], [22] and [23] that a Toeplitz sequence x is *regularly recurrent*, *i.e.* for every neighborhood V of x there exists a finite index subgroup Γ of G such that $\sigma^\gamma(x) \in V$, for every $\gamma \in \Gamma$. This condition is stronger than almost periodicity, which implies minimality of the closure of the orbit of x (see [1] for details about almost periodicity).

2.2. **Inverse and direct limit.** Given a sequence of continuous maps $f_n\colon X_{n+1} \to X_n, n \geq 0$ on topological spaces X_n, we denote the associated *inverse limit* by

$$\begin{aligned}\varprojlim_n (X_n, f_n) &= X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots \\ &:= \{(x_n)_n; x_n \in X_n,\ x_n = f_n(x_{n+1})\ \forall n \geq 0\}.\end{aligned}$$

Let us recall that this space is compact when all the spaces X_n are compact and the inverse limit spaces associated to any increasing subsequences $(n_i)_i$ of indices are homeomorphic.
In a similar way, we denote for a sequence of maps $g_n\colon X_n \to X_{n+1}, n \geq 0$ the associated *direct limit* by

$$\begin{aligned}\varinjlim_n (X_n, g_n) &= X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} \cdots \\ &:= \{(x, n), x \in X_n,\ n \geq 0\}/\sim,\end{aligned}$$

where two elements are equivalent $(x, n) \sim (y, m)$ if and only if there exists $k \geq m, n$ such that $g_k \circ \ldots \circ g_n(x) = g_k \circ \ldots \circ g_m(x)$. We denote by $[x, n]$ the equivalence class of (x, n). When the maps g_n are homomorphisms on groups X_n, then the direct limit inherits a group structure.

2.3. **Odometers.** A group G is said to be *residually finite* if there exists a nested sequence $(\Gamma_n)_{n\geq 0}$ of finite index normal subgroups such that $\bigcap_{n\geq 0}\Gamma_n$ is trivial. For every $n \geq 0$, there exists then a canonical projection $\pi_n\colon G/\Gamma_{n+1} \to G/\Gamma_n$. The *$G$-odometer* or *adding machine* O associated to the sequence $(\Gamma_n)_n$ is the inverse limit

$$O := \varprojlim_n (G/\Gamma_n, \pi_n) \quad = \quad G/\Gamma_0 \xleftarrow{\pi_0} G/\Gamma_1 \xleftarrow{\pi_1} G/\Gamma_2 \xleftarrow{\pi_2} \cdots.$$

We refer to [5] for the basic properties of such a space. Let us recall that it inherits a group structure through the quotient groups G/Γ_n and it contains G as a subgroup thanks the injection $G \ni g \mapsto ([g]_n) \in O$, where $[g]_n$ denotes the class of g in G/Γ_n. Thus the group G acts by left multiplication on O. When there is no confusion, we call this action also odometer. It is equicontiuous, minimal and the left Haar measure is the unique invariant probability measure. Notice that this action is free: the stabilizer of any point is trivial. The Toeplitz G-subshifts are characterized as the subshifts that are minimal almost 1-1 extensions of G-odometers [5].

2.4. **Ordered groups.** For more details about ordered groups and dimension groups we refer to [12] and [18].
An *ordered group* is a pair (H, H^+), such that H is a countable abelian group and H^+ is a subset of H verifying $(H^+) + (H^+) \subseteq H^+$, $(H^+) + (-H^+) = H$ and $(H^+) \cap (-H^+) = \{0\}$ (we use 0 as the unit of H when H is abelian). An ordered group (H, H^+) is a *dimension group* if for every $n \in \mathbb{Z}^+$ there exist $k_n \geq 1$ and a positive homomorphism $A_n : \mathbb{Z}^{k_n} \to \mathbb{Z}^{k_{n+1}}$, such that (H, H^+) is isomorphic to (J, J^+), where J is the direct limit

$$\varinjlim_n (\mathbb{Z}^{k_n}, A_n) = \mathbb{Z}^{k_0} \xrightarrow{A_0} \mathbb{Z}^{k_1} \xrightarrow{A_1} \mathbb{Z}^{k_2} \xrightarrow{A_2} \cdots,$$

and $J^+ = \{[v, n] : a \in (\mathbb{Z}^+)^{k_n}, n \in \mathbb{Z}^+\}$. The dimension group is *simple* if the matrices A_n can be chosen strictly positive.
An *order unit* in the ordered group (H, H^+) is an element $u \in H^+$ such that for every $g \in H$ there exists $n \in \mathbb{Z}^+$ such that $nu - g \in H^+$. If (H, H^+) is a simple dimension group then each element in $H^+ \setminus \{0\}$ is an order unit. A *unital ordered group* is a triple (H, H^+, u) such that (H, H^+) is an ordered group and u is an order unit. An isomorphism between two unital ordered groups (H, H^+, u) and (J, J^+, v) is an isomorphism $\phi : H \to J$ such that $\phi(H^+) = J^+$ and $\phi(u) = v$. A *state* of the unital ordered group (H, H^+, u) is a homomorphism $\phi : H \to \mathbb{R}$ so that $\phi(u) = 1$ and $\phi(H^+) \subseteq \mathbb{R}^+$. The *infinitesimal subgroup* of a simple dimension group with unit (H, H^+, u) is

$$\inf(H) = \{a \in H : \phi(a) = 0 \text{ for all state } \phi\}.$$

It is not difficult to show that $\inf(H)$ does not depend on the order unit.
The quotient group $H/\inf(H)$ of a simple dimension group (H, H^+) is also a simple dimension group with positive cone

$$(H/\inf(H))^+ = \{[a] : a \in H^+\}.$$

The next result is well-known. The proof is left to the reader.

Lemma 1. *Let (H, H^+) be a simple dimension group equals to the direct limit*

$$\lim_{\to n}(\mathbb{Z}^{k_n}, M_n) = \mathbb{Z}^{k_0} \xrightarrow{M_0} \mathbb{Z}^{k_1} \xrightarrow{M_1} \mathbb{Z}^{k_2} \xrightarrow{M_2} \cdots.$$

Then for every $z = (z_n)_{n\geq 0}$ in the inverse limit

$$\lim_{\leftarrow n}((\mathbb{R}^+)^{k_n}, M_n^T) = (\mathbb{R}^+)^{k_0} \xleftarrow{M_0^T} (\mathbb{R}^+)^{k_1} \xleftarrow{M_1^T} (\mathbb{R}^+)^{k_2} \xleftarrow{M_2^T} \cdots,$$

the function $\phi_z : H \to \mathbb{R}$ given by $\phi([n, v]) =< v, z_n >$, for every $[n, v] \in H$, is well defined and is a homomorphism of groups such that $\phi_z(H^+) \subseteq \mathbb{R}^+$. Conversely, for every group homomorphism $\phi : H \to \mathbb{R}$ such that $\phi(H^+) \subseteq \mathbb{R}^+$, there exists a unique $z \in \lim_{\leftarrow n}((\mathbb{R}^+)^{k_n}, M_n^T)$ such that $\phi = \phi_z$.

The following lemma is a preparatory lemma to prove Theorem A and B.

Lemma 2. *Let (H, H^+, u) be a simple dimension group with unit given by the following direct limit*

$$\lim_{\to n}(\mathbb{Z}^{k_n}, A_n) = \mathbb{Z} \xrightarrow{A_0} \mathbb{Z}^{k_1} \xrightarrow{A_1} \mathbb{Z}^{k_2} \xrightarrow{A_2} \cdots,$$

with unit $u = [1, 0]$. Suppose that $A_n > 0$ for every $n \geq 0$. Then (H, H^+, u) is isomorphic to

$$\mathbb{Z} \xrightarrow{\tilde{A}_0} \mathbb{Z}^{k_1+1} \xrightarrow{\tilde{A}_1} \mathbb{Z}^{k_2+1} \xrightarrow{\tilde{A}_2} \cdots,$$

where $\tilde{A}_0$ is the $(k_1+1)\times 1$-dimensional matrix given by

$$\tilde{A}_0 = \begin{pmatrix} A_0(1,\cdot) \\ A_0(1,\cdot) \\ A_0(2,\cdot) \\ \vdots \\ A_0(k_1,\cdot) \end{pmatrix},$$

and $\tilde{A}_n$ is the $(k_{n+1}+1)\times(k_n+1)$ dimensional matix given by

$$\tilde{A}_n = \begin{pmatrix} 1 & A_n(1,1)-1 & A_n(1,2) & \cdots & A_n(1,k_n) \\ 1 & A_n(1,1)-1 & A_n(1,2) & \cdots & A_n(1,k_n) \\ 1 & A_n(2,1)-1 & A_n(2,2) & \cdots & A_n(2,k_n) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & A_n(k_{n+1},1)-1 & A_n(k_{n+1},2) & \cdots & A_n(k_{n+1},k_n) \end{pmatrix}, \text{ for every } n\geq 0.$$

Proof. For $n\geq 1$, consider M_n the $(k_n+1)\times k_n$-dimensional matrix given by

$$M_n(\cdot,k) = \begin{cases} \vec{e}_{n,1}+\vec{e}_{n,2} & \text{if} \quad k=1 \\ \vec{e}_{k+1} & \text{if} \quad 3\leq k\leq k_n \end{cases},$$

where $\vec{e}_{n,1},\cdots,\vec{e}_{n,k_n+1}$ are the canonical vectors in $\mathbb{R}^{k_n+1}$. Let B_n be the $k_{n+1}\times(k_n+1)$-dimensional matrix defined by

$$B_n(i,j) = \begin{cases} 1 & \text{if} \quad j=1 \\ A_n(i,1)-1 & \text{if} \quad j=2 \\ A_n(i,j-1) & \text{if} \quad 3\leq j\leq k_n+1 \end{cases}$$

We have $A_n = B_nM_n$ and $\tilde{A}_n = M_{n+1}B_n$ for every $n\geq 1$, and $\tilde{A}_0 = M_1A_0$.
Thus the Bratteli diagrams defined by the sequences of matrices $(A_n)_{n\geq 0}$ and $(\tilde{A}_n)_{n\geq 0}$ are contractions of the same diagram. This shows that the respective dimension groups with unit are isomorphic (see [16] or [10]). □

2.5. **Associated ordered group and orbit equivalence.** Let (X,T,G) be a topological dynamical system such that X is a Cantor set and T is minimal. The ordered group associated to (X,T,G) is the unital ordered group

$$\mathcal{G}(X,T,G) = (D_m(X,T,G), D_m(X,T,G)^+, [1]),$$

where

$$D_m(X,T,G) = C(X,\mathbb{Z})/\{f\in C(X,\mathbb{Z}) : \int f d\mu = 0, \forall\mu\in\mathcal{M}(X,T,G)\},$$

$$D_m(X,T,G)^+ = \{[f] : f\geq 0\},$$

and $[1]\in D_m(X,T,G)$ is the class of the constant function 1.

Two topological dynamical systems (X_1,T_1,G_1) and (X_2,T_2,G_2) are (topologically) *orbit equivalent* if there exists a homeomorphism $F: X_1\to X_2$ such that $F(o_{T_1}(x)) = o_{T_2}(F(x))$ for every $x\in X_1$.
In [15] the authors show the following algebraic caracterization of orbit equivalence.

Theorem 1 ([15], Theorem 2.5). *Let $(X,T,\mathbb{Z}^d)$ and $(X',T',\mathbb{Z}^m)$ be two minimal actions on the Cantor set. Then they are orbit equivalent if and only if*

$$\mathcal{G}(X,T,\mathbb{Z}^d)\simeq\mathcal{G}(X',T',\mathbb{Z}^m)$$

as isomorphism of unital ordered group.

3. Suitable Følner sequences.

Let G be a residually finite group, and let $(\Gamma_n)_{n\geq 0}$ be a nested sequence of finite index normal subgroup of G such that $\bigcap_{n\geq 0}\Gamma_n=\{e\}$.
For technical reasons it is important to notice that since the groups Γ_n are normal, we have $g\Gamma_n=\Gamma_n g$, for every $g\in G$.
To construct a Toeplitz G-subshift that is an almost 1-1 extension of the odometer defined by the sequence $(\Gamma_n)_n$, we need a "suitable" sequence $(F_n)_n$ of fundamental domains of G/Γ_n. More precisely, each F_{n+1} has to be tileable by translated copies of F_n. To control the simplex of invariant measures of the subshift, we need in addition the sequence $(F_n)_n$ to be Følner. We did not find in the specialized litterature a result ensuring these conditions.

3.1. **Suitable sequence of fundamental domains.** Let Γ be a normal subgroup of G. By a *fundamental domain* of G/Γ, we mean a subset $D\subseteq G$ containing exactly one representative element of each equivalence class in G/Γ.

Lemma 3. *Let $(D_n)_{n\geq 0}$ be an increasing sequence of finite subsets of G such that for every $n\geq 0$, $e\in D_n$ and D_n is a fundamental domain of G/Γ_n. Let $(n_i)_{i\geq 0}\subseteq\mathbb{Z}^+$ be an increasing sequence. Consider $(F_i)_{i\geq 0}$ defined by $F_0=D_{n_0}$ and*

$$F_i=\bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1} \text{ for every } i\geq 1.$$

Then for every $i\geq 0$ we have the following:

(1) $F_i\subseteq F_{i+1}$ and F_i is a fundamental domain of G/Γ_{n_i}.
(2) $F_{i+1}=\bigcup_{v\in F_{i+1}\cap\Gamma_{n_i}} vF_i$.

Proof. Since $e\in D_{n_i}$, the sequence $(F_i)_{i\geq 0}$ is increasing.
$F_0=D_{n_0}$ is a fundamental domain of G/Γ_{n_0}. We will prove by induction on i that F_i is a fundamental domain of G/Γ_{n_i}. Let $i>0$ and suppose that F_{i-1} is a fundamental domain of $G/\Gamma_{n_{i-1}}$.
Let $v\in D_{n_i}$. There exist then $u\in F_{i-1}$ and $w\in\Gamma_{n_{i-1}}$ such that $v=wu$. Let $z\in D_{n_i}$ and $\gamma\in\Gamma_{n_i}$ be such that $w=\gamma z$. Since $z\in\Gamma_{n_{i-1}}\cap D_{n_i}$ and $v=\gamma zu$, we conclude that F_i contains one representing element of each class in G/Γ_{n_i}.
Let $w_1,w_2\in F_i$ be such that there exists $\gamma\in\Gamma_{n_i}$ verifying $w_1=\gamma w_2$. By definition, $w_1=v_1u_1$ and $w_2=v_2u_2$, for some $u_1,u_2\in F_{i-1}$ and $v_1,v_2\in D_{n_i}\cap\Gamma_{n_{i-1}}$. This implies that u_1 and u_2 are in the same class of $G/\Gamma_{n_{i-1}}$. Since F_{i-1} is a fundamental domain, we have $u_1=u_2$. From this we get $v_1=\gamma v_2$, which implies that $v_1=v_2$. Thus we deduce that F_i contains at most one representing element of each class in G/Γ_{n_i}. This shows that F_i is a fundamental domain of G/Γ_{n_i}.
To show that $D_{n_i}\cap\Gamma_{n_{i-1}}\subseteq F_i\cap\Gamma_{n_{i-1}}$, observe that the definition of F_i implies that for every $v\in D_{n_i}\cap\Gamma_{n_{i-1}}$ and $u\in F_{i-1}$, $vu\in F_i$. Then for $u=e\in F_{i-1}$ we get $v=ve\in F_i$. Now suppose that $v\in F_i\cap\Gamma_{n_{i-1}}\subseteq F_i$. The definition of F_i implies there exist $u\in F_{i-1}$ and $\gamma\in D_{n_i}\cap\Gamma_{n_{i-1}}$ such that $v=\gamma u$. Since v and γ are in $\Gamma_{n_{i-1}}$, we get that $u\in\Gamma_{n_{i-1}}\cap F_{i-1}$. This implies that $u=e$ because $\Gamma_{n_{i-1}}\cap F_{i-1}=\{e\}$. □

In this paper, by Følner sequences we mean right Følner sequences. That is, a sequence $(F_n)_{n\geq 0}$ of nonempty finite sets of G is a Følner sequence if for every $g\in G$

$$\lim_{n\to\infty}\frac{|F_ng\triangle F_n|}{|F_n|}=0.$$

Observe that $(F_n)_{n\geq 0}$ is a right Følner sequence if and only if $(F_n^{-1})_{n\geq 0}$ is a left Følner sequence.

Lemma 4. *Suppose that G is amenable. There exists an increasing sequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(F_i)_{i\in\mathbb{Z}^+}$, such that*

i) $F_i \subseteq F_{i+1}$ *and* F_i *is a fundamental domain of* G/Γ_{n_i}*, for every* $i \geq 0$.
ii) $G = \bigcup_{i\geq 0} F_i$.
iii) $F_{i+1} = \bigcup_{v\in F_{i+1}\cap\Gamma_{n_i}} vF_i$*, for every* $i \geq 0$.

Proof. From [26, Theorem 1] (see [22, Proposition 4.1] for a proof in our context), there exists an increasing sequence $(m_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(D_i)_{i\in\mathbb{Z}^+}$ such that for every $i \geq 0$, $D_i \subseteq D_{i+1}$, D_i is a fundamental domain of G/Γ_{m_i}, and $G = \bigcup_{i\geq 0} D_i$. Up to take subsequences, we can assume that D_i is a fundamental domain of G/Γ_i, for every $i \geq 0$, and that $e \in D_0$.
We will construct the sequences $(n_i)_{i\geq 0}$ and $(F_n)_{n\geq 0}$ as follows:

Step 0*:* We set $n_0 = 0$ and $F_0 = D_0$.

Step i: Let $i > 0$. We assume that we have chosen n_j and F_j for every $0 \leq j < i$. We take $n_i > n_{i-1}$ in order that the following two conditions are verified:

$$\frac{|D_{n_i} g \bigtriangleup D_{n_i}|}{|D_{n_i}|} < \frac{1}{i|F_{i-1}|}, \text{ for every } g \in F_{i-1}. \tag{1}$$

$$D_{n_{i-1}} \subseteq \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1}. \tag{2}$$

Such integer n_i exists because $(D_n)_{n\geq 0}$ is a Følner sequence and F_{i-1} is a fundamental domain of $G/\Gamma_{n_{i-1}}$ (then $G = \bigcup_{v\in\Gamma_{n_{i-1}}} vF_{i-1}$).
We define

$$F_i = \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1}.$$

Lemma 3 ensures that $(F_i)_{i\geq 0}$ verifies *i)* and *iii)* of the lemma. The equation (2) implies that $(F_i)_{i\geq 0}$ verifies *ii)* of the lemma.
It remains to show that $(F_i)_{i\geq 0}$ is a Følner sequence.
By definition of F_i we have

$$(F_i \setminus D_{n_i}) \subseteq \bigcup_{g\in F_{i-1}} (D_{n_i} g \setminus D_{n_i}).$$

Then by equation (1) we get

$$\begin{aligned} \frac{|F_i \setminus D_{n_i}|}{|D_{n_i}|} &\leq \sum_{g\in F_{i-1}} \left(\frac{|D_{n_i} g \setminus D_{n_i}|}{|D_{n_i}|}\right) \\ &\leq \left(|F_{i-1}| \frac{1}{i|F_{i-1}|}\right) = \frac{1}{i}. \end{aligned}$$

Since

$$(|F_i \cap D_{n_i}| + |D_{n_i} \setminus F_i|) = |D_{n_i}| = |F_i| = |F_i \cap D_{n_i}| + |F_i \setminus D_{n_i}|,$$

we obtain

$$\frac{|D_{n_i} \setminus F_i|}{|D_{n_i}|} \leq \frac{1}{i}.$$

Let $g \in G$. Since

$$\begin{aligned} F_i g \setminus F_i &= [(F_i \cap D_{n_i})g \setminus F_i] \bigcup [(F_i \setminus D_{n_i})g \setminus F_i] \\ &\subseteq [(F_i \cap D_{n_i})g \setminus F_i] \bigcup (F_i \setminus D_{n_i})g \\ &\subseteq [D_{n_i}g \setminus (F_i \cap D_{n_i})] \bigcup (F_i \setminus D_{n_i})g, \end{aligned}$$

we have

$$\frac{|F_i g \setminus F_i|}{|F_i|} \leq \frac{|D_{n_i}g \setminus (F_i \cap D_{n_i})|}{|D_{n_i}|} + \frac{|(F_i \setminus D_{n_i})g|}{|D_{n_i}|} \leq \frac{|D_{n_i}g \setminus (F_i \cap D_{n_i})|}{|D_{n_i}|} + \frac{1}{i}. \tag{3}$$

On the other hand, the relation

$$D_{n_i}g \setminus D_{n_i} = D_{n_i}g \setminus [(D_{n_i} \cap F_i) \cup (D_{n_i} \setminus F_i)] = [D_{n_i}g \setminus (D_{n_i} \cap F_i)] \setminus (D_{n_i} \setminus F_i),$$

implies that

$$\begin{aligned} D_{n_i}g \setminus (F_i \cap D_{n_i}) &= [(D_{n_i}g \setminus (F_i \cap D_{n_i})) \cap (D_{n_i} \setminus F_i)] \bigcup [(D_{n_i}g \setminus (F_i \cap D_{n_i})) \setminus (D_{n_i} \setminus F_i)] \\ &= [(D_{n_i}g \setminus (F_i \cap D_{n_i})) \cap (D_{n_i} \setminus F_i)] \bigcup [D_{n_i}g \setminus D_{n_i}] \\ &\subseteq (D_{n_i} \setminus F_i) \bigcup (D_{n_i}g \setminus D_{n_i}), \end{aligned}$$

which ensures that

$$\frac{|D_{n_i}g \setminus (F_i \cap D_{n_i})|}{|D_{n_i}|} \leq \frac{|D_{n_i} \setminus F_i|}{|D_{n_i}|} + \frac{|D_{n_i}g \setminus D_{n_i}|}{|D_{n_i}|}. \tag{4}$$

From equations (3) and (4), we obtain

$$\frac{|F_i g \setminus F_i|}{|F_i|} \leq \frac{2}{i} + \frac{|D_{n_i}g \setminus D_{n_i}|}{|D_{n_i}|},$$

which implies

$$\lim_{i \to \infty} \frac{|F_i g \setminus F_i|}{|F_i|} = 0. \tag{5}$$

In a similar way we deduce that

$$F_i \setminus F_i g \subseteq [D_{n_i} \setminus (F_i \cap D_{n_i})g] \bigcup (F_i \setminus D_{n_i}),$$

$$D_{n_i} \setminus D_{n_i}g = [D_{n_i} \setminus (D_{n_i} \cap F_i)g] \setminus (D_{n_i} \setminus F_i),$$

and

$$D_{n_i} \setminus (F_i \cap D_{n_i})g \subseteq (D_{n_i} \setminus F_i) \bigcup (D_{n_i} \setminus D_{n_i}g).$$

Combining the last three equations we get

$$\frac{|F_i \setminus F_i g|}{|F_i|} \leq \frac{2}{i} + \frac{|D_{n_i} \setminus D_{n_i}g|}{|D_{n_i}|},$$

which implies

$$\lim_{i \to \infty} \frac{|F_i \setminus F_i g|}{|F_i|} = 0. \tag{6}$$

Equations (5) and (6) imply that $(F_i)_{i \geq 0}$ is Følner. □

The following result is a direct consequence of Lemma 4.

Lemma 5. *Let G be an amenable residually finite group and let $(\Gamma_n)_{n \geq 0}$ be a decreasing sequence of finite index normal subgroups of G such that $\bigcap_{n \geq 0} \Gamma_n = \{e\}$. There exists an increasing sequence $(n_i)_{i \geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(F_i)_{i \geq 0}$ of G such that*

(1) $\{e\} \subseteq F_i \subseteq F_{i+1}$ *and* F_i *is a fundamental domain of* G/Γ_{n_i}*, for every* $i \geq 0$.
(2) $G = \bigcup_{i\geq 0} F_i$.
(3) $F_j = \bigcup_{v\in F_j\cap\Gamma_{n_i}} vF_i$*, for every* $j > i \geq 0$.

Proof. The existence of the sequence of subgroups of G and the Følner sequence verifying (1), (2) and (3) for $j = i+1$ is direct from Lemma 4. Using induction, it is straightforward to show (3) for every $j > i \geq 0$. □

4. Kakutani-Rokhlin partitions for generalized Toeplitz subshifts

In this section G is an amenable, countable, and residually finite group.
Let Σ be a finite alphabet and let (Σ^G, σ, G) be the respective full G-shift.
For a finite index subgroup Γ of G, $x \in \Sigma^G$ and $a \in \Sigma$, we define

$$Per(x, \Gamma, a) = \{g \in G : \sigma^{\gamma}(x)(g) = x(\gamma^{-1}g) = a, \forall \gamma \in \Gamma\},$$

and $Per(x, \Gamma) = \bigcup_{a\in\Sigma} Per(x, \Gamma, a)$.
It is straightforward to show that $x \in \Sigma^G$ is a Toeplitz sequence if and only if there exists an increasing sequence $(\Gamma_n)_{n\geq 0}$ of finite index subgroups of G such that $G = \bigcup_{n\geq 0} Per(x, \Gamma_n)$ (see [5, Proposition 5]).
A *period structure* of $x \in \Sigma^G$ is an increasing sequence of finite index subgroups $(\Gamma_n)_{n\geq 0}$ of G such that $G = \bigcup_{n\geq 0} Per(x, \Gamma_n)$ and such that for every $n \geq 0$, Γ_n is an *essential group of periods*: This means that if $g \in G$ is such that $Per(x, \Gamma_n, a) \subseteq Per(\sigma^g(x), \Gamma_n, a)$ for every $a \in \Sigma$, then $g \in \Gamma_n$.
It is known that every Toeplitz sequence has a period structure (see for example [5, Corollary 6]). We construct in this section, thanks the period structure, a Kakutani-Rokhlin partition and we deduce a characterization of its ordered group.

4.1. Existence of Kakutani-Rokhlin partitions.

In this subsection we suppose that $x_0 \in \Sigma^G$ is a non-periodic Toeplitz sequence ($\sigma^g(x_0) = x_0$ implies $g = e$) having a period structure $(\Gamma_n)_{n\geq 0}$ such that for every $n \geq 0$,

(i) Γ_{n+1} is a proper subset of Γ_n,
(ii) Γ_n is a normal subgroup of G.

Every non-periodic Toeplitz sequence has a period structure verifying (i) [5, Corollary 6]. Condition (ii) is satisfied for every Toeplitz sequence whose Toeplitz subshift is an almost 1-1 extension of an odometer (in the general case these systems are almost 1-1 extensions of subodometers. See [5] for the details).
By Lemma 5 we can assume there exists a Følner sequence $(F_n)_{n\geq 0}$ of G such that

(F1) $\{e\} \subseteq F_n \subseteq F_{n+1}$ and F_n is a fundamental domain of G/Γ_n, for every $n \geq 0$.
(F2) $G = \bigcup_{n\geq 0} F_n$.
(F3) $F_n = \bigcup_{v\in F_n\cap\Gamma_i} vF_i$, for every $n > i \geq 0$.

We denote by X the closure of the orbit of x_0. Thus $(X, \sigma|_X, G)$ is a Toeplitz subshift.

Definition 2. *We say that a finite clopen partition* $\mathcal{P}$ *of* X *is a regular Kakutani-Rokhlin partition (r-K-R partition), if there exists a finite index subgroup* Γ *of* G *with a fundamental domain* F *containing* e *and a clopen* C_k*, such that*

$$\mathcal{P} = \{\sigma^{u^{-1}}(C_k) : u \in F, 1 \leq k \leq N\}$$

and

$$\sigma^{\gamma}(\bigcup_{k=1}^{N} C_k) = \bigcup_{k=1}^{N} C_k \textit{ for every } \gamma \in \Gamma.$$

To construct a regular Kakutani-Rokhlin partition of X, we need the following technical lemma.

Lemma 6. *Let $\mathcal{P}' = \{\sigma^{u^{-1}}(D_k) : u \in F, 1 \leq k \leq N\}$ be a r-K-R partition of X and $\mathcal{Q}$ any other finite clopen partition of X. Then there exists a r-K-R partition $\mathcal{P} = \{\sigma^{u^{-1}}(C_k) : u \in F, 1 \leq k \leq M\}$ of X such that*

(1) $\mathcal{P}$ is finer than $\mathcal{P}'$ and $\mathcal{Q}$,
(2) $\bigcup_{k=1}^{M} C_k = \bigcup_{k=1}^{N} D_k$.

Proof. Let $F = \{u_0, u_1, \cdots, u_{|F|-1}\}$, with $u_0 = e$.
We refine every set D_k with respect to the partition $\mathcal{Q}$. Thus we get a collection of disjoint sets

$$D_{1,1}, \cdots, D_{1,l_1}; \cdots; D_{N,1}, \cdots, D_{N,l_N},$$

such that each of these sets is in an atom of $\mathcal{Q}$ and $D_k = \bigcup_{j=1}^{l_k} D_{k,j}$ for every $1 \leq k \leq N$. Thus $\mathcal{P}_0 = \{\sigma^{u^{-1}}(D_{k,j}) : u \in F, 1 \leq j \leq l_k, 1 \leq k \leq N\}$ is a r-K-R partition of X. For simplicity we write

$$\mathcal{P}_0 = \{\sigma^{u^{-1}}(D_k^{(0)}) : u \in F, 1 \leq k \leq N_0\}.$$

We have that $\mathcal{P}_0$ verifies (2) and every $D_k^{(0)}$ is contained in atoms of $\mathcal{P}'$ and $\mathcal{Q}$.
Let $0 \leq n < |F| - 1$. Suppose that we have defined a r-K-R partition of X

$$\mathcal{P}_n = \{\sigma^{u^{-1}}(D_k^{(n)}) : u \in F, 1 \leq k \leq N_n\},$$

such that $\mathcal{P}_n$ verifies (2) and such that for every $0 \leq j \leq n$ and $1 \leq k \leq N_n$ there exist $A \in \mathcal{P}'$ and $B \in \mathcal{Q}$ such that

$$\sigma^{u_j^{-1}}(D_k^{(n)}) \subseteq A, B.$$

Now we refine every set $\sigma^{u_{n+1}^{-1}}(D_k^{(n)})$ with respect to $\mathcal{Q}$. Thus we get a collection of disjoint sets

$$D_{1,1}, \cdots, D_{1,s_1}; \cdots; D_{N_n,1}, \cdots, D_{N_n,s_{N_n}}$$

such that each of these sets is in an atom of $\mathcal{Q}$ and $\sigma^{u_{n+1}^{-1}}(D_k^{(n)}) = \bigcup_{j=1}^{s_k} D_{k,j}$, for every $1 \leq k \leq N_n$.
For every $1 \leq k \leq N_n$ and $1 \leq j \leq s_k$, let $C_{k,j} = \sigma^{u_{n+1}}(D_{k,j}) \subseteq D_k^{(n)}$. We have that

$$\mathcal{P}_{n+1} = \{\sigma^{u^{-1}}(C_{k,j}) : u \in F, 1 \leq j \leq s_k, 1 \leq k \leq N_n\}$$

is a r-K-R partition of X verifying (2) and such that for every $0 \leq i \leq n+1$, $1 \leq j \leq s_k$ and $1 \leq k \leq N_n$ there exist $A \in \mathcal{P}'$ and $B \in \mathcal{Q}$ such that

$$\sigma^{u_j^{-1}}(C_{k,j}) \subseteq A, B.$$

At the step $n = |F| - 1$ we get $\mathcal{P} = \mathcal{P}_{|F|-1}$ verifying (1) and (2). $\square$

Proposition 1. *There exists a sequence $(\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k}) : u \in F_n, 1 \leq k \leq k_n\})_{n \geq 0}$ of r-K-R partitions of X such that for every $n \geq 0$,*

(1) $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$,
(2) $C_{n+1} \subseteq C_n = \bigcup_{k=1}^{k_n} C_{n,k}$,
(3) $\bigcap_{n \geq 1} C_n = \{x_0\}$,
(4) The sequence $(\mathcal{P}_n)_{n \geq 0}$ spans the topology of X.

Proof. For every $n \geq 0$, let define

$$C_n = \{x \in X : Per(x, \Gamma_n, a) = Per(x_0, \Gamma_n, a) \forall a \in \Sigma\}.$$

From [5, Proposition 6] we get

$$C_n = \overline{\{\sigma^{\gamma}(x_0) : \gamma \in \Gamma_n\}},$$

and that $\mathcal{P}'_n = \{\sigma^{u^{-1}}(C_n) : u \in F_n\}$ is a clopen partition of X such that $\sigma^{\gamma}(C_n) = C_n$ for every $\gamma \in \Gamma_n$. Thus $\mathcal{P}'_n$ is a r-K-R partition of X. Furthermore, the sequence $(\mathcal{P}'_n)_{n\geq 0}$ verifies (1), (2) and (3).

For every $n \geq 0$, let $\mathcal{Q}_n = \{[B] \cap X : B \in \Sigma^{F_n}, [B] \cap X \neq \emptyset\}$. This is a finite clopen partition of X and $(\mathcal{Q}_n)_{n\geq 0}$ spans the topology of X.

We define $\mathcal{P}_0 = \{\sigma^{u^{-1}}(C_{0,k}) : u \in F_0, 1 \leq k \leq k_0\})$ the r-K-R partition finer than $\mathcal{P}'_0$ and $\mathcal{Q}_0$ given by Lemma 6. Now we take $\mathcal{P}''_n$ the r-K-R partition finer that $\mathcal{P}_{n-1}$ and $\mathcal{Q}_n$ given by Lemma 6, and we define

$$\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k}) : u \in F_n, 1 \leq k \leq k_n\},$$

the r-K-R partition finer than $\mathcal{P}' = \mathcal{P}'_n$ and $\mathcal{Q} = \mathcal{P}''_n$ given by Lemma 6. Thus $\mathcal{P}_n$ is finer than $\mathcal{P}_{n-1}$ and $\mathcal{Q}_n$. This implies that the sequence $(\mathcal{P}_n)_{n\geq 0}$ verifies (1) and (4). Since $\bigcup_{k=1}^{k_n} C_{n,k} = C_n$, we deduce that $(\mathcal{P}_n)_{n\geq 0}$ verifies (2) and (3). □

Remark 1. The sequence of partitions of Proposition 1 is a generalization to Toeplitz G-subshifts of the sequences of Kakutani-Rokhlin partitions for Toeplitz $\mathbb{Z}$-subshifts introduced in [17]. See [19] for more details about Kakutani-Rokhlin partitions for minimal $\mathbb{Z}$-actions on the Cantor set

Definition 3. *We say that a sequence $(\mathcal{P}_n)_{n\geq 0}$ of r-K-R partitions as in Proposition 1 is a nested sequence of r-K-R partitions of X.*

Let $(\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k}) : u \in F_n, 1 \leq k \leq k_n\})_{n\geq 0}$ be a sequence of nested r-K-R partitions of X.

For every $n \geq 0$ we define the matrix $M_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{Z}^+)$ as

$$M_n(i,k) = |\{\gamma \in F_{n+1} \cap \Gamma_n : \sigma^{\gamma^{-1}}(C_{n+1,k}) \subseteq C_{n,i}\}|,$$

We call M_n the *incidence matrix* of the partitions $\mathcal{P}_{n+1}$ and $\mathcal{P}_n$.

Let p be a positive integer. For every $n \geq 1$ we denote by $\triangle(n,p)$ the closed convex hull generated by the vectors $\frac{1}{p}e_1^{(n)}, \cdots, \frac{1}{p}e_n^{(n)}$, where $e_1^{(n)}, \cdots, e_n^{(n)}$ is the canonical base in $\mathbb{R}^n$. Thus $\triangle(n,1)$ is the unitary simplex in $\mathbb{R}^n$.

Observe that for every $n \geq 0$ and $1 \leq k \leq k_{n+1}$,

$$\sum_{i=1}^{k_n} M_n(i,k) = \frac{|F_{n+1}|}{|F_n|}.$$

This implies that $M_n(\triangle(k_{n+1}, |F_{n+1}|)) \subseteq \triangle(k_n, |F_n|)$.

The next result characterizes the maximal equicontinuous factor, the space of invariant probability measures and the associated ordered group of $(X, \sigma|_X, G)$ in terms of the sequence of incidence matrices of a nested sequence of r-K-R partitions.

Proposition 2. *Let $(\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k}) : u \in F_n, 1 \leq k \leq k_n\})_{n\geq 0}$ be a nested sequence of r-K-R partitions of X with an associated sequence of incidence matrices $(M_n)_{n\geq 0}$. Then*

(1) $(X, \sigma|_X, G)$ is an almost 1-1 extension of the odometer $O = \varprojlim_n (G/\Gamma_n, \pi_n)$,

(2) *there is an affine homeomorphism between the set of invariant probability measures of* $(X, \sigma|_X, G)$ *and the inverse limit* $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$,

(3) *the ordered group* $\mathcal{G}(X, \sigma|_X, G)$ *is isomorphic to* $(H/inf(H), (H/inf(H))^+, u + inf(H))$, *where* (H, H^+) *is given by*

$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$

where $M = |F_0|(1, \cdots, 1)$ *and* $u = [M^T, 0]$.

Proof. **1.** For every $x \in X$ and $n \geq 0$, let $v_n(x) \in F_n$ be such that $x \in \sigma^{v_n(x)^{-1}}(C_n)$. The map $\pi : X \to O$ given by $\pi(x) = (v_n(x)^{-1}\Gamma_n)_{n\geq 1}$ is well defined, is a factor map and verifies $\pi^{-1}(\pi(x_0)) = \{x_0\}$. This shows that $(X, \sigma|_X, G)$ is an almost 1-1 extension of O.

2. It is clear that for any invariant probability measure μ of $(X, \sigma|_X, G)$, the sequence $(\mu_n)_{n\geq 0}$, with $\mu_n = (\mu(C_{n,k}) : 1 \leq k \leq k_n)$, is an element of the inverse limit $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$. Conversely, any element $(\mu_{n,k} : 1 \leq k \leq k_n)_{m\geq 0}$ of such inverse limit, defines a probability measure μ on the σ-algebra generated by $(\mathcal{P}_n)_{n\geq 0}$, which is equal to the Borel σ-algebra of X because $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X and is countable. Since the sequence (F_n) is Følner, it is standard to check that the measure μ is invariant by the G-action.
The function $\mu \mapsto (\mu_n)_{n\geq 0}$ is thus an affine bijection between $\mathcal{M}(X, \sigma|_X, G)$ and the inverse limit $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$. Observe that this function is a homeomorphism with respect to the weak topology in $\mathcal{M}(X, \sigma|_X, G)$ and the product topology in the inverse limit.

3. We denote by $[k, -1]$ the class of the element $(k, -1) \in \mathbb{Z} \times \{-1\}$ in H.
Let $\phi : H \to D_m(X, \sigma|_X, G)$ be the function given by $\phi([v, n]) = \sum_{k=1}^{k_n} v_i[1_{C_{n,k}}]$, for every $v = (v_1, \cdots, v_{k_n}) \in \mathbb{Z}^{k_n}$ and $n \geq 0$, and $\phi([k, -1]) = k1_X$ for every $k \in \mathbb{Z}$. It is easy to check that ϕ is a well defined homomorphism of groups that verifies $\phi(H^+) \subseteq D_m(X, \sigma|_X, G)^+$. Since $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X, every function $f \in C(X, \mathbb{Z})$ is constant on every atom of $\mathcal{P}_n$, for some $n \geq 0$. This implies that ϕ is surjective. Lemma 1 and (2) of Proposition 2, imply that $Ker(\phi) = inf(H)$. Finally, ϕ induces a isomorphism $\widehat{\phi} : H/inf(H) \to D_m(X, \sigma|_X, G)$ such that $\widehat{\phi}((H/inf(H))^+) = D_m(X, \sigma|_X, G)^+$. Since $[1, -1] = [M^T, 0]$, we get $\phi([M^T, 0]) = [1_X]$. □

5. Kakutani-Rokhlin partitions with prescribed incidence matrices.

We say that a sequence of positive integer matrices $(M_n)_{n\geq 0}$ is *managed* by the increasing sequence of positive integers $(p_n)_{n\geq 0}$, if for every $n \geq 0$ the integer p_n divides p_{n+1}, and if the matrix M_n verifies the following properties:

(1) M_n has $k_n \geq 2$ rows and $k_{n+1} \geq 2$ columns;
(2) $\sum_{i=1}^{k_n} M_n(i, k) = \frac{p_{n+1}}{p_n}$, for every $1 \leq k \leq k_{n+1}$.

If $(M_n)_{n\geq 0}$ is a sequence of matrices managed by $(p_n)_{n\geq 0}$, then for each $n \geq 0$, $M_n(\triangle(k_{n+1}, p_{n+1})) \subseteq \triangle(k_n, p_n)$.
Observe that the sequences of incidence matrices associated to the nested sequences of r-K-R partitions defined in Section 4 are managed by $(|F_n|)_{n\geq 0}$.
In this Section we construct Toeplitz subshifts with nested sequences of r-K-R partitions whose sequences of incidence matrices are managed.

5.1. **Construction of the partitions.** In the rest of this Section G is an amenable and residually finite group. Let $(\Gamma_n)_{n\geq 0}$ be a decreasing sequence of finite index normal subgroup of G such that $\bigcap_{n\geq 0}\Gamma_n=\{e\}$, and let $(F_n)_{n\geq 0}$ be a Følner sequence of G such that

- (F1) $\{e\}\subseteq F_n\subseteq F_{n+1}$ and F_n is a fundamental domain of G/Γ_n, for every $n\geq 0$.
- (F2) $G=\bigcup_{n\geq 0}F_n$.
- (F3) $F_n=\bigcup_{v\in F_n\cap\Gamma_i}vF_i$, for every $n>i\geq 0$.

Lemma 5 ensures the existence of a Følner sequence verifying conditions (F1), (F2) and (F3).
For every $n\geq 0$, we call R_n the set $F_n\cdot F_n^{-1}\cup F_n^{-1}\cdot F_n$. This will enable us to define a "border" of each domain F_{n+1}.
Let Σ be a finite alphabet. For every $n\geq 0$, let $k_n\geq 3$ be an integer. We say that the sequence of sets $(\{B_{n,1},\cdots,B_{n,k_n}\})_{n\geq 0}$ where for any $n\geq 0$, $\{B_{n,1},\cdots,B_{n,k_n}\}\subseteq\Sigma^{F_n}$ is a collection of different functions, *verifies conditions* (C1)-(C4) if it verifies the following four conditions for any $n\geq 0$:

- (C1) $\sigma^{\gamma^{-1}}(B_{n+1,k})|_{F_n}\in\{B_{n,i}:1\leq i\leq k_n\}$, for every $\gamma\in F_{n+1}\cap\Gamma_n$, $1\leq k\leq k_{n+1}$.
- (C2) $B_{n+1,k}|_{F_n}=B_{n,1}$, for every $1\leq k\leq k_{n+1}$.
- (C3) For any $g\in F_n$ such that for some $1\leq k,k'\leq k_n$, $B_{n,k}(gv)=B_{n,k'}(v)$ for all $v\in F_n\cap g^{-1}F_n$, then $g=e$.
- (C4) $\sigma^{\gamma^{-1}}(B_{n+1,k})|_{F_n}=B_{n,k_n}$ for every $\gamma\in(F_{n+1}\cap\Gamma_n)\cap\left[F_{n+1}\setminus F_{n+1}g^{-1}\right]$, for some $g\in R_n$.

Example 1. To illustrate these conditions, let us consider the case $G=\mathbb{Z}$, $\Sigma=\{1,2,3,4\}$ and $\Gamma_n=3^{2(n+1)}\mathbb{Z}$ for every $n\geq 0$. The set

$$F_n=\left\{-\left(\frac{3^{2(n+1)}-1}{2}\right),-\left(\frac{3^{2(n+1)}-1}{2}\right)+1,\cdots,\left(\frac{3^{2(n+1)}-1}{2}\right)\right\}$$

is a fundamental domain of $\mathbb{Z}/\Gamma_n$. Furthermore we have

$$F_n=\bigcup_{v\in\{k3^{2n}:-4\leq k\leq 4\}}(F_{n-1}+v),$$

for every $n\geq 1$. This shows that sequence $(F_n)_{n\geq 0}$ satisfies (F1), (F2) and (F3).
Now let us consider the case where $k_n=4$ for every $n\geq 0$. We define $B_{0,k}(j)=k$ for every $j\in F_0$ and $1\leq k\leq 4$, and for $n\geq 1$,

$$B_{n,k}|_{F_{n-1}}=B_{n-1,1},\quad B_{n,k}|_{F_{n-1}+v}=B_{n-1,4}\text{ for }v\in\{-l\cdot 3^{2n},l\cdot 3^{2n}:l=3,4\}.$$

Thus they verify the conditions (C1) and (C4). We fill the rest of the $B_{n,k}|_{F_{n-1}+v}$ with $B_{n-1,3}$ and $B_{n-1,2}$ in order that $B_{n,1},\cdots,B_{n,4}$ are different. They satisfy conditions (C2) and (C4). The limit in $\Sigma^{\mathbb{Z}}$ of the functions $B_{n,1}$ is a $\mathbb{Z}$-Toeplitz sequence x. If X denotes the closure of the orbit of x, then we prove in the next lemma (in a more general setting) that

$$(\mathcal{P}_n=\{\sigma^j([B_{n,k}]\cap X):j\in F_n,1\leq k\leq 4\})_{n\geq 0}$$

is a sequence of nested Kakutani-Rokhlin partitions of the subshift X.

In the next lemma, we show that conditions (C1) and (C2) are sufficient to construct a Toeplitz sequence. The technical conditions (C3) (aperiodicity) and (C4) (also known as "forcing the border") will allow to construct a nested sequence of r-K-R partitions of X.

Lemma 7. *Let $(\{B_{n,1},\cdots,B_{n,k_n}\})_{n\geq 0}$ be a sequence that verifies conditions (C1)-(C4). Then:*

(1) *The set $\bigcap_{n\geq 0}[B_{n,1}]$ contains only one element x_0 which is a Toeplitz sequence.*

(2) *Let X be the orbit closure of x_0 with respect to the shift action. For every $n \geq 0$, let*
$$\mathcal{P}_n = \{\sigma^{u^{-1}}([B_{n,k}] \cap X) : 1 \leq k \leq k_n, u \in F_n\}.$$
Then $(\mathcal{P}_n)_{n\geq 0}$ is a sequence of nested r-K-R partitions of X.

Let $(M_n)_{n\geq 0}$ be the sequence of incidence matrices of $(\mathcal{P}_n)_{n\geq 0}$. Thus we have

(3) *The Toeplitz subshift $(X, \sigma|_X, G)$ is an almost 1-1 extension of the odometer $O = \varprojlim_n (G/\Gamma_n, \pi_n)$.*

(4) *There is an affine homeomorphism between the set of invariant probability measures of $(X, \sigma|_X, G)$ and the inverse limit $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$.*

(5) *The ordered group $\mathcal{G}(X, \sigma|_X, G)$ is isomorphic to $(H/inf(H), (H/inf(H))^+, u + inf(H))$, where (H, H^+) is given by*
$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$
with $M = |F_0|(1, \cdots, 1)$ and $u = [M^T, 0]$.

Proof. Condition (C2) implies that $\bigcap_{n\geq 0}[B_{n,1}]$ is non empty, and since $G = \bigcup_{n\geq 0} F_n$, there is only one element x_0 in this intersection. Let X be the orbit closure of x_0. For every $n \geq 0$ and $1 \leq k \leq k_n$, we denote $C_{n,k} = [B_{n,k}] \cap X$.

Claim: For every $m > n \geq 0$, $1 \leq k \leq k_m$ and $\gamma \in F_m \cap \Gamma_n$,
$$\sigma^{\gamma^{-1}}(B_{m,k})|_{F_n} \in \{B_{n,i} : 1 \leq i \leq k_n\}. \tag{7}$$
Condition (C1) implies that (7) holds when $n = m - 1$. We will show the claim by induction on n.
Suppose that for every $1 \leq k \leq k_m$ and $\gamma \in F_m \cap \Gamma_{n+1}$,
$$\sigma^{\gamma^{-1}}(B_{m,k})|_{F_{n+1}} \in \{B_{n+1,i} : 1 \leq i \leq k_{n+1}\}.$$
Let $g \in \Gamma_n \cap F_m$. Condition (F3) implies there exist $v \in \Gamma_{n+1} \cap F_m$ and $u \in F_{n+1}$ such that $g = vu$. Thus we get
$$\sigma^{g^{-1}}(B_{m,k})|_{F_n} = \sigma^{u^{-1}v^{-1}}(B_{m,k}) = \sigma^{v^{-1}}(B_{m,k})|_{uF_n}.$$
Since $u \in \Gamma_n \cap F_{n+1}$, condition (F3) implies that $uF_n \subseteq F_{n+1}$. Then by hypothesis, there exists $1 \leq l \leq k_{n+1}$ such that
$$\sigma^{v^{-1}}(B_{m,k})|_{uF_n} = B_{n+1,l}|_{uF_n},$$
which is equal to some $B_{n,s}$, by (C1). This shows the claim.

From (7) we deduce that $\sigma^{\gamma^{-1}}(x_0)|_{F_n} \in \{B_{n,i} : 1 \leq i \leq k_n\}$, for every $\gamma \in \Gamma_n$. Thus if g is any element in G, and $u \in F_n$ and $\gamma \in \Gamma_n$ are such that $g = \gamma u$, then
$\sigma^{g^{-1}}(x_0) = \sigma^{u^{-1}}(\sigma^{\gamma^{-1}}(x_0)) \in \sigma^{u^{-1}}(C_{n,k})$, for some $1 \leq k \leq k_n$. It follows that
$$\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k}) : 1 \leq k \leq k_n, u \in F_n\}$$
is a clopen covering of X.
From condition (C2) and (7) we get that $\sigma^{\gamma^{-1}}(x_0)|_{F_{n-1}} = B_{n-1,1}$ for any $\gamma \in \Gamma_n$, which implies that $F_{n-1} \subseteq Per(x_0, \Gamma_n)$. This shows that x_0 is Toeplitz.
Now we will show that $\mathcal{P}_n$ is a partition. Suppose that $1 \leq k, l \leq k_n$ and $u \in F_n$ are such that $\sigma^{u^{-1}}(C_{n,k}) \cap C_{n,l} \neq \emptyset$. Then there exist $x \in C_{n,k}$ and $y \in C_{n,l}$ such that $\sigma^{u^{-1}}(x) = y$. From this we have $x(uv) = y(v)$ for every $v \in G$. In particular, $x(uv) = y(v)$ for every $v \in F_n \cap u^{-1}F_n$, which implies $B_{n,k}(uv) = B_{n,l}(v)$ for every $v \in F_n \cap u^{-1}F_n$. From condition (C3) we get $u = e$ and $k = l$. This ensures that the

set of return times of x_0 to $\bigcup_{k=1}^{k_n} C_{n,k}$, *i.e.* the set $\{g \in G : \sigma^{g^{-1}}(x_0) \in \bigcup_{k=1}^{k_n} C_{n,k}\}$, is Γ_n. From this it follows that $\mathcal{P}_n$ is a r-K-R partition. From (C1) we have that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and that $C_{n+1} \subseteq \bigcup_{k=1}^{k_n} C_{n,k} = C_n$. By the definition of x_0 we have that $\{x_0\} = \bigcap_{n\geq 0} C_n$.

Now we will show that $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X. Since every $\mathcal{P}_n$ is a partition, for every $n \geq 0$ and every $x \in X$ there are unique $v_n(x) \in F_n$ and $1 \leq k_n(x) \leq k_n$ such that

$$x \in \sigma^{v_n(x)^{-1}}(C_{n,k_n(x)}).$$

The collection $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X if and only if $(v_n(x))_{n\geq 0} = (v_n(y))_{n\geq 0}$ and $(k_n(x))_{n\geq 0} = (k_n(y))_{n\geq 0}$ imply $x = y$.

Let $x, y \in X$ be two sequences such that $v_n(x) = v_n(y) = v_n$ and $k_n(x) = k_n(y)$ for every $n \geq 0$. Let $g \in G$ be such that $x(g) \neq y(g)$.

We have then for any $n \geq 0$

$$\sigma^{v_n}(x)|_{F_n} = \sigma^{v_n}(y)|_{F_n} \in \{B_{n,i} : 1 \leq i \leq k_n\},$$

and then

$$x|_{v_n^{-1}F_n} = y|_{v_n^{-1}F_n}.$$

Thus by definition, we get $g \notin v_n^{-1}F_n$ for any n. We can take n sufficiently large in order that $g \in F_{n-1}$.

Let $\gamma \in \Gamma_n$ and $u \in F_n$ such that $v_n(x)g = \gamma u$. Observe that $ug^{-1} \notin F_n$. Indeed, if $ug^{-1} \in F_n$, then the relation $v_n(x) = \gamma u g^{-1}$ implies $\gamma = e$, but in that case we get $v_n(x)g = u \in F_n$ which is not possible by hypothesis. By the condition (C1), there exists an index $1 \leq i \leq k_n$ such that $\sigma^{\gamma^{-1}}(\sigma^{v_n}(x))|_{F_n} = B_{n,i}$ and then

$$x(g) = \sigma^{\gamma^{-1}}\sigma^{v_n}(x)(\gamma^{-1}v_n g) = B_{n,i}(u).$$

Let $\gamma' \in \Gamma_{n-1} \cap F_n$ and $u' \in F_{n-1}$ such that $u = \gamma' u'$. Since $\gamma' u' g^{-1} = ug^{-1} \notin F_n$, we get $\gamma' \in F_n \setminus F_n g u'^{-1}$. This implies that $\gamma' \in F_n \setminus F_n w$, for $w = gu'^{-1} \in R_{n-1}$ and $B_{n,i}(u) = B_{n-1,k_{n-1}}(u')$ by the condition (C4). Thus $x(g) = B_{n-1,k_{n-1}}(u')$. The same argument implies that $y(g) = B_{n-1,k_{n-1}}(u') = x(g)$ and we obtain a contradiction.

This shows that $(\mathcal{P}_n)_{n\geq 0}$ is a sequence of nested r-K-R partitions of X.

The point (3), (4) and (5) follows from Propositions 2. □

The next result shows that, up to telescope a managed sequence of matrices, it is possible to obtain a managed sequence of matrices with sufficiently large coefficient to satisfy the conditions of Lemma 7.

Lemma 8. *Let $(M_n)_{n\geq 0}$ be a sequence of matrices managed by $(|F_n|)_{n\geq 0}$. Let k_n be the number of rows of M_n, for every $n \geq 0$.*
Then there exists an increasing sequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ such that for every $i \geq 0$ and every $1 \leq k \leq k_{n_{i+1}}$,

(i) $R_{n_i} \subseteq F_{n_{i+1}}$,
(ii) *For every $1 \leq l \leq k_{n_i}$,*

$$M_{n_i} M_{n_i+1} \cdots M_{n_{i+1}-1}(l,k) > 1 + |\bigcup_{g\in R_{n_i}} F_{n_{i+1}} \setminus F_{n_{i+1}} g^{-1}|$$

If in addition there exists a constant $K > 0$ such that $k_{n+1} \leq K\frac{|F_{n+1}|}{|F_n|}$ for every $n \geq 0$, then the sequence $(n_i)_{i\geq 0}$ can be chosen in order that

(iii) $k_{n_{i+1}} < M_{n_i} \cdots M_{n_{i+1}-1}(i,k)$, *for every $1 \leq i \leq k_{n_i}$.*

Proof. We define $n_0 = 0$. Let $i \geq 0$ and suppose that we have defined n_j for every $0 \leq j \leq i$. Let $m_0 > n_i$ be such that for every $m \geq m_0$,

$$R_{n_i} \subseteq F_m.$$

Let $0 < \varepsilon < 1$ be such that $\varepsilon|R_{n_i}| < 1$. Since $(F_n)_{n\geq 0}$ is a Følner sequence, there exists $m_1 > m_0$ such that for every $m \geq m_1$,

$$\frac{|F_m \setminus F_m g^{-1}|}{|F_m|} < \frac{\varepsilon}{|F_{n_i+1}|}, \text{ for every } g \in R_{n_i}. \tag{8}$$

Since $\varepsilon|R_{n_i}| < 1$, there exists $m_2 > m_1$ such that for every $m \geq m_2$,

$$1 - \frac{|F_{n_i+1}|}{|F_m|} > \varepsilon|R_{n_i}|.$$

Then

$$\frac{|F_m|}{|F_{n_i+1}|} - 1 > \varepsilon|R_{n_i}|\frac{|F_m|}{|F_{n_i+1}|}.$$

Since the matrices M_n are positive, using induction on m and condition (2) for managed sequences, we get

$$M_{n_i}\cdots M_{m-1}(l,j) \geq \frac{|F_m|}{|F_{n_i+1}|}, \text{ for every } 1 \leq l \leq k_{n_i}, 1 \leq j \leq k_m.$$

Combining the last two equations we get

$$M_{n_i}\cdots M_{m-1}(l,j) - 1 > \varepsilon|R_{n_i}|\frac{|F_m|}{|F_{n_i+1}|},$$

and from equation (8), we obtain

$$M_{n_i}\cdots M_{m-1}(l,j) - 1 > |F_m \setminus F_m g^{-1}||R_{n_i}|, \text{ for every } g \in R_{n_i},$$

which finally implies that

$$M_{n_i}\cdots M_{m-1}(l,j) > |\bigcup_{g\in R_{n_i}} F_m \setminus F_m g^{-1}| + 1, \text{ for every } 1 \leq l \leq k_{n_i}, 1 \leq j \leq k_m.$$

Now, suppose there exists $K > 0$ such that $k_{m+1} \leq K\frac{|F_{m+1}|}{|F_m|}$ for every $m \geq 0$. The property (2) for managed sequences of matrices implies

$$M_{n_i}\cdots M_m(l,j) \geq \frac{|F_{m+1}|}{|F_{n_i+1}|} \text{ for every } m > n_i.$$

Let $m_3 > m_2$ be such that $K < \frac{|F_m|}{|F_{n_i+1}|}$ for every $m \geq m_3$. Then for every $m \geq m_3$ we have

$$k_{m+1} \leq K\frac{|F_{m+1}|}{|F_{n_i}|} \leq M_{n_i}\cdots M_m(l,j) \text{ for every } 1 \leq l \leq k_{n_i} \text{ and } 1 \leq j \leq k_{m+1}.$$

By taking $n_{i+1} \geq m_3$ we get the desired subsequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$. □

The following proposition shows that given a managed sequence, there exists a sequence of decorations verifying conditions (C1)-(C4). The aperiodicity condition (C3) is obtained by decorating the center of F_n in a unique way with respect to other places in F_n. A restriction on the number of columns of the matrices gives enough choices of coloring to ensure conditions (C3) and (C4).

Proposition 3. *Let $(M_n)_{n\geq 0}$ be a sequence of matrices which is managed by $(|F_n|)_{n\geq 0}$. For every $n \geq 0$, we denote by k_n the number of rows of M_n. Suppose in addition there exists $K > 0$ such that $k_{n+1} \leq K\frac{|F_{n+1}|}{|F_n|}$, for every $n \geq 0$. Then there exists a Toeplitz subshift $(X, \sigma|_X, G)$ verifying the following three conditions:*

(1) *The set of invariant probability measures of* $(X, \sigma|_X, G)$ *is affine homeomorphic to* $\varprojlim_n(\triangle(k_n, |F_n|), M_n)$.
(2) *The ordered group* $\mathcal{G}(X, \sigma|_X, G)$ *is isomorphic to* $(H/inf(H), (H/inf(H))^+, u + inf(H))$, *where* (H, H^+) *is given by*

$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$

with $M = |F_0|(1, \cdots, 1)$ *and* $u = [M^T, 0]$.
(3) $(X, \sigma|_X, G)$ *is an almost 1-1 extension of the odometer* $O = \varprojlim_n(G/\Gamma_n, \pi_n)$.

Proof. Let $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ be a sequence as in Lemma 8. Since $(M_n)_{n\geq 0}$ and the sequence $(M_{n_i} \cdots M_{n_{i+1}-1})_{i\geq 0}$ define the same inverse and direct limits, without loss of generality we can assume that for every $n \geq 0$ we have:

$$R_n \subseteq F_{n+1},$$

$$M_n(i,k) > 1 + |\bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1}| \text{ for every } 1 \leq i \leq k_n, 1 \leq k \leq k_{n+1},$$

and

$$k_{n+1} < \min\{M_n(i,j) : 1 \leq i \leq k_n, 1 \leq j \leq k_{n+1}\}.$$

Let $\tilde{M}$ be the $1 \times (k_0 + 1)$-dimensional matrix given by

$$\tilde{M}(\cdot, 1) = \tilde{M}(\cdot, 2) = M(\cdot, 1),$$

and $\tilde{M}(\cdot, k+1) = M(\cdot, k)$ for every $2 \leq k \leq k_0$. For every $n \geq 0$, consider the $(k_n + 1) \times (k_{n+1} + 1)$-dimensional matrix given by

$$\tilde{M}_n(\cdot, 1) = \tilde{M}_n(\cdot, 2) = \begin{pmatrix} 1 \\ M_n(1,1) - 1 \\ M_n(2,1) \\ \vdots \\ M_n(k_n, 1) \end{pmatrix}$$

and

$$\tilde{M}_n(\cdot, k+1) = \begin{pmatrix} 1 \\ M_n(1,k) - 1 \\ M_n(2,k) \\ \vdots \\ M_n(k_n, k) \end{pmatrix} \text{ for every } 2 \leq k \leq k_{n+1}.$$

Lemma 2 implies that the dimension groups with unit given by

$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$

and

$$\mathbb{Z} \xrightarrow{\tilde{M}^T} \mathbb{Z}^{k_0+1} \xrightarrow{\tilde{M}_0^T} \mathbb{Z}^{k_1+1} \xrightarrow{\tilde{M}_1^T} \mathbb{Z}^{k_2+1} \xrightarrow{\tilde{M}_2^T} \cdots,$$

are isomorphic.
Thus from Lemma 1 we get that $\varprojlim_n(\triangle(k_n, |F_n|), M_n)$ and $\varprojlim_n(\triangle(k_n + 1, |F_n|), \tilde{M}_n)$ are affine homeomorphic. Observe that $(\tilde{M}_n)_{\geq 0}$ is managed by $(|F_n|)_{n\geq 0}$ and verifies for every $n \geq 0$:

$$\tilde{M}_n(i,k) \geq 1 + |\bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1}| \text{ for every } 2 \leq i \leq k_n + 1, 1 \leq k \leq k_{n+1} + 1,$$

and

$$3 \leq k_{n+1} + 1 \leq \min\{M_n(i,j) : 2 \leq i \leq k_n + 1, 1 \leq j \leq k_{n+1} + 1\}.$$

Thus, by Lemma 7, to prove the proposition it is enough to find a Toeplitz subshift having a sequence of r-K-R-partitions whose sequence of incidence matrices is $(\tilde{M}_n)_{n\geq 0}$.

For every $n \geq 0$, we call l_n and l_{n+1} the number of rows and columns of $\tilde{M}_n$ respectively.

For every $n \geq 0$, we will construct a collection of functions $B_{n,1}, \cdots, B_{n,l_n} \in \Sigma^{F_n}$ as in Lemma 7, where $\Sigma = \{1, \cdots, l_0\}$.
For every $1 \leq k \leq l_0$ we define $B_{0,k} \in \Sigma^{F_0}$ by $B_{0,k}(g) = k$, for every $g \in F_0$. Observe that the collection $\{B_{0,1}, \cdots, B_{0,l_0}\}$ verifies condition (C3).

Let $n \geq 0$. Suppose that we have defined $B_{n,1}, \cdots, B_{n,l_n} \in \Sigma^{F_n}$ verifying condition (C3). For $1 \leq k \leq l_{n+1}$, we define

$$B_{n+1,k}|_{F_n} = B_{n,1},$$

and

$$\sigma^{s^{-1}}(B_{n+1,k})|_{F_n} = B_{n,l_n} \text{ for every } s \in \bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1} \cap \Gamma_n.$$

We fill the rest of the coordinates $v \in F_{n+1} \cap \Gamma_n$ in order that $\sigma^{v^{-1}}(B_{n+1,k})|_{F_n} \in \{B_{n,1}, \cdots, B_{n,l_n}\}$ and such that

$$|\{v \in F_{n+1} \cap \Gamma_n : \sigma^{v^{-1}}(B_{n+1,k})|_{F_n} = B_{n,i}\}| = \tilde{M}_n(i,k),$$

for every $2 \leq i \leq l_n$.
Since $\tilde{M}_n(1,k) = 1$, if $\sigma^{v^{-1}}(B_{n+1,k})|_{F_n} = B_{n,1}$ then $v = e$.
Notice that the number of such v is at least $\tilde{M}_n(2,k) + 1$, because there are at least $\tilde{M}_n(2,k)$ coordinates to be filled with $B_{n,2}$ and at least 1 coordinate to be filled with B_{n,l_n}. Thus we have at least $\tilde{M}_n(2,k)+1 \geq l_{n+1}$ different ways to fill the coordinates such that the functions $B_{n+1,1}, \cdots, B_{n+1,l_{n+1}}$ are pairwise different (the number of columns of $\tilde{M}_n$ which are equal to the k-column is at most the number of different functions that "respect the rules" of the k-column).
By construction, every function $B_{n+1,k}$ verifies (C1), (C2) and (C4). Let us assume there are $g \in F_{n+1}$ and $1 \leq k, k' \leq k_{n+1}$ such that $B_{n+1,k}(gv) = B_{n+1,k'}(v)$ for any v where it is defined, then by the induction hypothesis, $g \in \Gamma_n$. This implies $\sigma^{g^{-1}}(B_{n+1,k})|_{F_n} = B_{n+1,k'}|_{F_n} = B_{n,1}$ and then $g = e$. This shows that the collection $B_{n+1,1}, \cdots, B_{n+1,l_{n+1}}$ verifies (C3). We conclude applying Lemma 7. □

For positive integers $n_1, \cdots, n_k$, we denote by $(n_1, \cdots, n_k)!$ the corresponding multinomial coefficient. That is,

$$(n_1, \cdots, n_k)! = \frac{(n_1 + \cdots + n_k)!}{n_1! \cdots n_k!}.$$

Remark 2. In Proposition 3, to construct the collection of functions $(B_{n,1} \cdots, B_{n,l_n})_{n\geq 0}$ we just need that the number of columns of $\tilde{M}_n$ which are equal to $\tilde{M}_n(\cdot, k)$ does not exceed the number of possible ways to construct different functions $B \in \Sigma^{F_n}$ verifying $B|_{F_{n-1}} = B_{n-1,1}$ and $B|_{vF_{n-1}} = B_{n-1,l_{n-1}}$ for every $v \in \bigcup_{g\in R_{n-1}} F_n \setminus F_n g^{-1} \cap \Gamma_{n-1}$. In other words, it is possible to make this construction with $\tilde{M}_n$ verifying the following property: for every $1 \leq k \leq l_{n+1}$ the number of $1 \leq l \leq l_{n+1}$ such that $\tilde{M}_n(\cdot, l) = \tilde{M}_n(\cdot, k)$ is not grater than

$$(\tilde{M}_n(2,k), \cdots, \tilde{M}_n(l_n - 1, k), \tilde{M}_n(l_n, k) - |\bigcup_{g\in R_{n-1}} F_n \setminus F_n g^{-1} \cap \Gamma_{n-1}|)!$$

Among the hypothesis of Proposition 3, we ask a stronger condition on the number of columns of M_n which is stable under multiplication of matrices, unlike the condition that we mention in this remark.

6. Characterization of Choquet simplices

A compact, convex, and metrizable subset K of a locally convex real vector space is said to be a (metrizable) Choquet simplex, if for each $v \in K$ there is a unique probability measure μ supported on the set of extreme points of K such that $\int x d\mu(x) = v$.

In this section we show that any metrizable Choquet simplex is affine homeomorphic to the inverse limit defined by a managed sequence of matrices satisfying the additional restriction on the number of columns.

6.1. **Finite dimensional Choquet simplices.** For technical reasons, we have to separate the finite and the infinite dimensional cases.

Lemma 9. *Let K be a finite dimensional metrizable Choquet simplex with exactly $d \geq 1$ extreme points. Let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n \geq 0$ the integer p_n divides p_{n+1}, and let $k \geq \max\{2, d\}$. Then there exist an increasing subsequence $(n_i)_{i\geq 0}$ of indices and a sequence $(M_i)_{i\geq 0}$ of square k-dimensional matrices which is managed by $(p_{n_i})_{i\geq 0}$ such that K is affine homeomorphic to $\varprojlim_n (\triangle(k, p_{n_i}), M_i)$.*

Proof. Let $k \geq \max\{3, d\}$, we will define the subsequence $(n_i)_{i\geq 0}$ by induction on i through a condition explained later. For every $i \geq 0$, we define M_i the k-dimensional matrix by

$$M_i(l, j) = \begin{cases} \frac{p_{n_{i+1}}}{p_{n_i}} - k(k-1) & \text{if} \quad 1 \leq l = j \leq d \\ k & \text{if} \quad l \neq j, 1 \leq l \leq k \text{ and } 1 \leq j \leq d \\ M_i(l, d) & \text{if} \quad d < j \leq k. \end{cases}$$

We always suppose that n_{i+1} is sufficiently large in order to have $\frac{p_{n_{i+1}}}{p_{n_i}} - k(k-1) > 0$.
By the very definition, M_i is a positive matrix having $k \geq 3$ rows and columns; $\sum_{l=1}^{k} M_i(l, j) = \frac{p_{n_{i+1}}}{p_{n_i}}$ for every $1 \leq j \leq k$ and the range of M_i is at most d. Thus the convex set $\varprojlim_n (\triangle(k, p_{n_i}), M_i)$ has at most d extreme points.
If it has exactly d extreme points, it is affine homeomorphic to K. We will choose the sequence $(p_{n_i})_{i\geq 0}$ in order that $P = \bigcap_{i\geq 0} M_0 \cdots M_i(\triangle(k, p_{n_{i+1}}))$ has d extreme points, which implies that $\varprojlim_n (\triangle(k, p_{n_i}), M_i)$ has exactly d extreme points.
For every $i \geq 0$, the set $P_i = M_0 \cdots M_i(\triangle(k, p_{n_{i+1}}))$ is the closed convex set generated by the vectors $v_{i,1}, \cdots, v_{i,d}$, where

$$v_{i,l} = \frac{1}{p_{n_{i+1}}} M_0 \cdots M_i(\cdot, l), \text{ for every } 1 \leq l \leq d.$$

Since every $v_{i,l}$ is in $\triangle(k, p_{n_0})$, there exists a sequence $(i_j)_{j\geq 0}$ such that for every $1 \leq l \leq d$, the sequence $(v_{i_j,l})_{j\geq 0}$ converges to an element v_l in $\triangle(K, p_{n_0})$. Observe that P is the closed convex set generated by $v_1, \cdots, v_d$. Thus if $v_1, \cdots, v_d$ are linearly independent then P has d extreme points.
Since for every $1 \leq l \leq d$ we have $\sum_{j=1}^{k} \frac{1}{p_{n_{i+1}}} M_0 \cdots M_i(j, l) = \frac{1}{p_{n_0}}$, there exists a positive vector $\delta_l^{(i)} = (\delta_{1,l}^{(i)}, \cdots, \delta_{k,l}^{(i)})^T$ such that $\sum_{j=1}^{k} \delta_{j,l}^{(i)} = 1$ and such that for each $1 \leq j \leq k$

$$\frac{1}{p_{n_{i+1}}} M_0 \cdots M_i(j, l) = \delta_{j,l}^{(i)} \frac{1}{p_{n_0}}.$$

Thus if B_i is the matrix given by

$$B_i(\cdot,l) = \begin{cases} v_{i,l} & \text{if } 1 \leq l \leq d \\ \frac{1}{p_{n_0}} e_l^{(k)} & \text{if } d+1 \leq l \leq k. \end{cases},$$

then $B_i = DA_i$, where D is the k-dimensional diagonal matrix given by

$$D_i(l,l) = \frac{1}{p_{n_0}}, \text{ for every } 1 \leq l \leq k,$$

and A_i is the k-dimensional matrix defined by

$$A_i(\cdot,l) = \begin{cases} \delta_l^{(i)} & \text{if} \quad 1 \leq l \leq d \\ e_l^{(k)} & \text{if} \quad d+1 \leq l \leq k. \end{cases}.$$

If $\lim_{j\to\infty} A_j = A$ is invertible (A is the k-dimensional matrix whose columns are the vectors $\lim_{j\to\infty} \delta_l^{(i_j)}$ and the canonical vectors $e_{d+1}^{(k)}, \cdots, e_k^{(k)}$), then $v_1, \cdots, v_l$ are linearly independent. For this it is enough to show that A is strictly diagonally dominant (see the Levy-Desplanques Theorem in [20]).
Now we will define $(n_i)_{i\geq 0}$ in order that A is strictly diagonally dominant.
Let $\varepsilon \in (0, \frac{1}{4})$. Let $n_0 = 0$ and $n_1 > n_0$ such that for every $1 \leq l \leq d$,

$$\delta_{l,l}^{(0)} = 1 - \frac{p_{n_0}}{p_{n_1}} \sum_{j=1, j\neq l}^{k} M_0(j,l) = 1 - \frac{p_{n_0}}{p_{n_1}} k(k-1) \geq \frac{3}{4} + \varepsilon.$$

For $i \geq 1$ we choose $n_{i+1} > n_i$ in order that

$$\frac{1}{p_{n_{i+1}}} M_0 \cdots M_{i-1}(l,l) < \varepsilon \frac{1}{p_{n_0} k(k-1)2^i}, \text{ for every } 1 \leq l \leq d.$$

After a standart computation, for every $i \geq 1$ and $1 \leq l \leq d$ we get

$$\delta_{l,l}^{(i)} \geq \delta_{l,l}^{(i-1)} - \frac{p_{n_0}}{p_{n_{i+1}}} k(k-1) M_0 \cdots M_{i-1}(l,l),$$

which implies that

$$\delta_{l,l}^{(i)} \geq \delta_{l,l}^{(0)} - \varepsilon \sum_{j\geq 1} \frac{1}{2^j} \geq \frac{3}{4}.$$

It follows that $A(l,l) \geq \frac{3}{4}$ for every $1 \leq l \leq k$, and since the sum of the elements in a column of A is equal to 1, we deduce that A is strictly diagonally dominant. □

6.2. **Infinite dimensional Choquet simplices.** We use the following characterization of infinite dimensional metrizable Choquet simplex.

Lemma 10 ([24], Corollary p.186). *For every infinite dimensional metrizable Choquet simplex K, there exists a sequence of matrices $(A_n)_{n\geq 1}$ such that for every $n \geq 1$*

(1) $A_n(\triangle(n+1,1)) = \triangle(n,1)$,
(2) K is affine homeomorphic to $\varprojlim_n (\triangle(n,1), A_n)$.

Our strategy is to approximate the sequence of matrices $(A_n)_n$ by a managed sequence. Then we show that the associated inverse limits are affine homeomorphic. For this, we need the following classical density result, whose proof follows from the fact that every non cyclic subgroup of $\mathbb{R}$ is dense.

Lemma 11. *Let $\mathbf{r} = (r_n)_{n\geq 0}$ be a sequence of integers such that $r_n \geq 2$ for every $n \geq 0$. Let $C_{\mathbf{r}}$ be the subgroup of $(\mathbb{R}, +)$ generated by $\{(r_0 \cdots r_n)^{-1} : n \geq 0\}$. Then*

$$(C_{\mathbf{r}})^p \cap \triangle(p,1) \cap \{v \in \mathbb{R}^p : v > 0\}$$

is dense in $\triangle(p,1)$, for every $p \geq 2$, where $(C_{\mathbf{r}})^p$ is the Cartesian product $\prod_{i=1}^p C_{\mathbf{r}}$.

Lemma 12. *Let K be an infinite dimensional metrizable Choquet simplex, and let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n \geq 0$ the integer p_n divides p_{n+1}. Then there exist an increasing subsequence $(n_i)_{i\geq 1}$ of indices and a sequence of matrices $(M_i)_{i\geq 1}$ managed by $(p_{n_i})_{i\geq 0}$ such that for every $i \geq 0$,*

$$k_{i+1} \leq \min\{M_i(l,k) : 1 \leq l \leq k_i, 1 \leq k \leq k_{i+1}\},$$

and such that K is affine homeomorphic to the inverse limit $\varprojlim_n (\triangle(k_i, p_{n_i}), M_i)$, where k_i is the number of rows of M_i, for every $i \geq 0$.

Proof. For every $n \geq 0$, let $r_n \geq 2$ be the integer such that $p_{n+1} = p_n r_n$.
Let $(A_n)_{n\geq 1}$ be the sequence of matrices given in Lemma 10. We can assume that $A_n : \triangle(n+3,1) \longrightarrow \triangle(n+2,1)$, for every $n \geq 1$. Now we define the subsequence $(n_i)_i$ by induction.

We set $n_1 = 0$.

Let $i \geq 1$ and suppose that we have defined $n_i \geq 0$. We set $\mathbf{r}^{(\mathbf{i})} = (r_n)_{n\geq n_i}$. For every $1 \leq j \leq i+3$, Lemma 11 ensures the existence of $v^{(i,j)} \in (C_{\mathbf{r}^{(\mathrm{i})}})^{i+2} \cap \triangle(i+2,1) \cap \{v \in \mathbb{R}^{i+2} : v > 0\}$ such that

$$\|v^{(i,j)} - A_i(\cdot,j)\|_1 < \frac{1}{2^i}. \tag{9}$$

Let B_i be the matrix given by

$$B_i(\cdot,j) = v^{(i,j)}, \text{ for every } 1 \leq j \leq i+3.$$

Observe that (9) implies that

$$\sum_{n\geq 1} \sup\{\|A_n v - B_n v\|_1 : v \in \triangle_{n+3}\} < \infty.$$

It follows from [6, Lemma 9] that K is affine homeomorphic $\varprojlim_n (\triangle(i+2,1), B_i)$.

Let $n_{i+1} > n_i$ be such that $r_{n_i} \cdots r_{n_{i+1}-1} v^{(i,j)}$ is an integer vector and such that $r_{n_i} \cdots r_{n_{i+1}-1} v^{(i,j)} > i+3$, for every $1 \leq j \leq i+3$.
We define

$$M_i = \frac{p_{n_{i+1}}}{p_{n_i}} B_i.$$

Thus $M_i = P_i^{-1} B_i P_{i+1}$, where P_i is the diagonal matrix given by $P_i(j,j) = p_{n_i}$ for every $1 \leq j \leq i+2$ and $i \geq 1$. This shows that $\varprojlim_n (\triangle(i+2,1), B_i)$ is affine homeomorphic to $\varprojlim_n (\triangle(i+2, p_{n_i}), M_i)$.
The proof conclude verifying that $(M_i)_{i\geq 0}$ is managed by $(p_{n_i})_{i\geq 0}$. □

7. Proof of the main theorems.

7.1. **Proof of Theorem A.** The proof of Theorem A is a corollary of previous results.

Proof of Theorem A. Let $ext(K)$ be the set of extreme points of K. If $ext(K)$ is finite, then the proof is direct from Proposition 3 and Lemma 9. If $ext(K)$ is infinite, the proof follows from Proposition 3 and Lemma 12. □

7.2. **Proof of Theorem B.** We refer to [8] for definitions and properties about Toeplitz $\mathbb{Z}$-subshifts or Toeplitz flows. See [11] and [19] for details about ordered Bratteli diagram, Kakutani-Rokhlin partitions and dimension groups associated to minimal $\mathbb{Z}$-actions on the Cantor set.
We denote by Σ a finite alphabet with at least two elements. For $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ and $n \leq m \in \mathbb{Z}$, we set $x[n,m] = x_n \cdots x_m$. In a similar way, if $w = w_0 \cdots w_{n-1}$ is a word in Σ^n, we set $w[k,l] = w_k \cdots w_l$ for every $0 \leq k \leq l < n$.
The next result follows from the proof of [17, Theorem 8].

Lemma 13. *Let $x_0 \in \Sigma^{\mathbb{Z}}$ be a Toeplitz sequence and let $(X, \sigma|_X, \mathbb{Z})$ be the associated Toeplitz $\mathbb{Z}$-subshift. There exist a period structure $(p_n)_{n\geq 0}$ of x_0 and a sequence of matrices $(A_n)_{n\geq 0}$ managed by $(p_n)_{n\geq 0}$ such that the dimension group associated to $(X, \sigma|_X, \mathbb{Z})$ is isomorphic to*

$$\mathbb{Z} \xrightarrow{A_0^T} \mathbb{Z}^{k_1} \xrightarrow{A_1^T} \mathbb{Z}^{k_2} \xrightarrow{A_2^T} \cdots .$$

Furthermore, if k_n is the number of rows of A_n and $r_n = \frac{p_{n+1}}{p_n}$, then for every $m > n > 0$ and $1 \leq k \leq k_m$,

$$\begin{aligned} &|\{1 \leq l \leq k_m : A_{n,m-1}(\cdot, l) = A_{n,m-1}(\cdot, k)\}| \\ \leq\ &(A_{n,m-1}(1,k) - r_{n+2}\cdots r_{m-1}, \cdots, A_{n,m-1}(k_n,k) - r_{n+2}\cdots r_{m-1})!, \end{aligned}$$

where $A_{n,m-1} = A_n \cdots A_{m-1}$.

Proof. In the proof of Theorem 8 in [17] the authors show there exist a period structure $(p_n)_{n\geq 1}$ of x_0 and a sequence $(\mathcal{P}_n)_{n\geq 0}$ of nested Kakutani-Rokhlin partitions of $(X, \sigma|_X, \mathbb{Z})$ such that $\mathcal{P}_0 = \{X\}$ and $\mathcal{P}_n = \{T^j(C_{n,k}) : 0 \leq j < p_n, 1 \leq k \leq k_n\}$, where

$$C_{n,k} = \{x \in X : x[0, p_n - 1] = w_{n,k}\} \text{ for every } 1 \leq k \leq k_n,$$

with $W_n = \{w_{n,1}, \cdots, w_{n,k_n}\}$ the set of the words w of x_0 of length p_n verifying $w[0, p_{n-1} - 1] = x_0[0, p_{n-1} - 1]$, for every $n \geq 1$ (with $p_0 = 1$).
Thus the dimension group with unit associated to $(X, \sigma|_X, \mathbb{Z})$ is isomorphic to

$$\lim_{\rightarrow n} (\mathbb{Z}^{k_n}, A_n^T) = \mathbb{Z} \xrightarrow{A_0^T} \mathbb{Z}^{k_1} \xrightarrow{A_1^T} \mathbb{Z}^{k_2} \xrightarrow{A_2^T} \cdots ,$$

where $A_n(i,j)$ is the number of times that the word $w_{n,i}$ appears in the word $w_{n+1,j}$, for every $1 \leq i \leq k_n$, $1 \leq j \leq k_{n+1}$ and $n \geq 1$, and the matrix A_0^T is the vector in $\mathbb{Z}^{k_1}$ whose coordinates are equal to p_1.
Since $w_{n+1,i} \neq w_{n+1,j}$ for $i \neq j$, equal columns of the matrix A_n produce different concatenations of words in W_n. This implies that for every $1 \leq k \leq k_{n+1}$, the number of columns of A_n which are equal to $A_n(\cdot, k)$ can not exceed the number of different concatenations of r_n words in W_n using exactly $A_n(j,k)$ copies of $w_{n,j}$, for every $1 \leq j \leq k_n$. This means that the number of columns which are equal to $A_n(\cdot, k)$ is smaller or equal to $(A_n(1,k), \cdots, A_n(k_n,k))!$.
Now fix $n > 0$ and take $m > n$. The coordinate (i,j) of the matrix $A_{n,m-1}$ contains the number of times that the word $w_{n,i} \in W_n$ appears in $w_{m,j} \in W_m$. Observe that every word u in W_m is a concatenation of $r_{n+2} \cdots r_{m-1}$ words in W_{n+2}. In addition, each word in W_{n+2} starts with $x_0[0, p_{n+1} - 1] \in W_{n+1}$, which is a word containing every word in W_n (we can always assume that the matrices A_n are positive). Thus there exist $0 \leq l_1 < \cdots < l_{r_{n+1}\cdots r_{m-1}} < p_m$ such that $u[l_s, l_s + p_n - 1] = w[l_s, l_s + p_n - 1] \in W_n$, for every $1 \leq s \leq r_{n+2} \cdots r_{m-1}$ and $u, w \in W_m$.

This implies that the number of all possible concatenations of words in W_n producing a word in W_m according to the column k of the matrix $A_{n,m-1}$ is smaller or equal to

$$(A_{n,m-1}(1,k) - r_{n+2}\cdots r_{m-1}, \cdots, A_{n,m-1}(k_n,k) - r_{n+2}\cdots r_{m-1})!.$$

□

Proof of Theorem B. Let $x_0 \in X$ be a Toeplitz sequence. Let $(p_n)_{n\geq 1}$ and $(A_n)_{n\geq 0}$ be the period structure of x_0 and the sequence of matrices given by Lemma 13 respectively. It is straightforward to check that Lemma 13 is also true if we take a subsequence of $(p_n)_{n\geq 0}$. Thus we can assume that for every $n \geq 1$, the matrix A_n has its coordinates strictly grater than 1 and that there exist positive integers $r_{n,1}, \cdots, r_{n,d} > 1$ such that

$$\frac{p_{n+1}}{p_n} = r_n = r_{n,1}\cdots r_{n,d}.$$

Le define $q_{n+1,i} = r_{0,i}\cdots r_{n,i}$ for every $1 \leq i \leq d$, and $\Gamma_{n+1} = \prod_{i=1}^{d} q_{n+1,i}\mathbb{Z}$, for every $n \geq 0$. We have $\Gamma_{n+1} \subseteq \Gamma_n$, $\bigcap_{n\geq 1}\Gamma_n = \{0\}$ and $|\mathbb{Z}^d/\Gamma_n| = p_n$. Let $(F_n)_{n\geq 0}$ be a Følner sequence associated to $(\Gamma_n)_{n\geq 1}$ as in Lemma 5. We denote R_n as in Section 5 (the set that defines "border").
Now, we define an increasing sequence $(n_i)_{i\geq 1}$ of integers as follows:
We set $n_1 = 1$. For $i \geq 1$, given n_i we chose $n_{i+1} > n_i + 1$ such that

$$\sum_{g\in R_{n_i}} \frac{|F_{n_{i+1}} \setminus F_{n_{i+1}} - g|}{|F_{n_{i+1}}|} < \frac{1}{|F_{n_i}| r_{n_i} r_{n_i+1}}.$$

Thus we have

$$\begin{aligned}
\frac{|F_{n_{i+1}}|}{|F_{n_i}|} - \sum_{g\in R_{n_i}} |F_{n_{i+1}} \setminus F_{n_{i+1}} - g| &> \frac{|F_{n_{i+1}}|}{|F_{n_i}|} - \frac{|F_{n_{i+1}}|}{|F_{n_i}| r_{n_i} r_{n_i+1}} \\
&= r_{n_i}\cdots r_{n_{i+1}-1} - r_{n_i+2}\cdots r_{n_{i+1}-1} \\
&> r_{n_i}\cdots r_{n_{i+1}-1} - k_{n_i} r_{n_i+2}\cdots r_{n_{i+1}-1}
\end{aligned}$$

Let $M_0 = A_0$ and $M_i = A_{n_i}\cdots A_{n_{i+1}-1}$ be for every $i \geq 1$. For every $1 \leq k \leq k_{n_{i+1}}$ we get

$$M_i(k_{n_i},k) - \sum_{g\in R_{n_i}} |F_{n_{i+1}} \setminus F_{n_{i+1}} - g| > M_i(k_{n_i},k) - r_{n_i+2}\cdots r_{n_{i+1}-1},$$

which implies that

$$(M_i(1,k), \cdots, M_i(k_{n_i}-1,k), M_i(k_{n_i},k) - \sum_{g\in R_{n_i}} |F_{n_{i+1}} \setminus F_{n_{i+1}} - g|)!$$

is grater than

$$(M_i(1,k) - r_{n_i+2}\cdots r_{n_{i+1}-1}, \cdots, M_i(k_{n_i},k) - r_{n_i+2}\cdots r_{n_{i+1}-1})!$$

Then from the previous inequality and Lemma 13 we get that the number of columns of M_i which are equal to $M_i(\cdot,k)$ is smaller than

$$(M_i(1,k), \cdots, M_i(k_{n_i}-1,k), M_i(k_{n_i},k) - \sum_{g\in R_{n_i}} |F_{n_{i+1}} \setminus F_{n_{i+1}} - g|)!$$

As in the proof of Proposition 3, we define $\tilde{M}_i$ and we call l_i and l_{i+1} the number of rows and columns of $\tilde{M}_i$ respectively, for every $i \geq 0$. According to the notations of the proof of Proposition 3, in our case M_0 corresponds to the matrix M and $\tilde{M}_0$ corresponds to the matrix $\tilde{M}$. Observe that the bound on the number of columns which are equal to $M_i(\cdot,k)$ (and then to $\tilde{M}_i(\cdot,k)$) ensures the existence of enough possibilities to fill the

coordinates of F_{n_i} in order to obtain different function $B_{i,1}\cdots,B_{i,l_i}\in\{1,\cdots,l_1\}^{F_{n_i}}$ as in the proof of Proposition 3, for every $i\geq 1$ (see Remark 2).
Lemma 7 implies that the Toeplitz $\mathbb{Z}^d$-subshift $(Y,\sigma|_Y,\mathbb{Z}^d)$ defined from $(B_{i,1},\cdots,B_{i,l_i})_{i\geq 1}$ has an ordered group $\mathcal{G}(Y,\sigma|_Y,\mathbb{Z}^d)$ isomorphic to $(H/inf(H),(H/inf(H))^+,u+inf(H))$, where (H,H^+) is given by

$$\mathbb{Z}\xrightarrow{\tilde{M}_0^T}\mathbb{Z}^{l_0}\xrightarrow{\tilde{M}_1^T}\mathbb{Z}^{l_2}\xrightarrow{\tilde{M}_2^T}\mathbb{Z}^{l_3}\xrightarrow{\tilde{M}_3^T}\cdots,$$

with $\tilde{M}_0=|F_1|(1,\cdots,1)$ and $u=[1,0]$.
Lemma 2 implies that (H,H^+,u) is isomorphic to the dimension group with unit (J,J^+,w) associated to $(X,\sigma|_X,\mathbb{Z})$. Thus $(J/inf(J),(J/inf(J))^+,w+inf(J))$, the ordered group associated to $(X,\sigma|_X,\mathbb{Z})$, is isomorphic to $\mathcal{G}(Y,\sigma|_Y,\mathbb{Z}^d)$. We conclude the proof applying Theorem 1. □

In [25], the author shows that every minimal Cantor system $(Y,T,\mathbb{Z})$ having an associated Bratteli diagram which satisfies the equal path number property, is strong orbit equivalent to a Toeplitz subshift $(X,\sigma|_X,\mathbb{Z})$. Thus the next result is inmediat.

Corollary 1. *Let $(X,T,\mathbb{Z})$ be a minimal Cantor having an associated Bratteli diagram which satisfies the equal path number property. Then for every $d\geq 1$ there exists a Toeplitz subshift $(Y,\sigma|_Y,\mathbb{Z}^d)$ which is orbit equivalent to $(X,T,\mathbb{Z})$.*

Acknowledgments. We would like to thank the referee for many valuable comments which served to improve the article.

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ON AUTOMORPHISM GROUPS OF LOW COMPLEXITY SUBSHIFTS

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Abstract. In this article we study the automorphism group $\mathrm{Aut}(X,\sigma)$ of subshifts (X,σ) of low word complexity. In particular, we prove that $\mathrm{Aut}(X,\sigma)$ is virtually $\mathbb{Z}$ for aperiodic minimal subshifts and certain transitive subshifts with non-superlinear complexity. More precisely, the quotient of this group relative to the one generated by the shift map is a finite group. In addition, we show that any finite group can be obtained in this way. The class considered includes minimal subshifts induced by substitutions, linearly recurrent subshifts and even some subshifts which simultaneously exhibit non-superlinear and superpolynomial complexity along different subsequences. The main technique in this article relies on the study of classical relations among points used in topological dynamics, in particular, asymptotic pairs. Various examples that illustrate the technique developed in this article are provided. In particular, we prove that the group of automorphisms of a d-step nilsystem is nilpotent of order d and from there we produce minimal subshifts of arbitrarily large polynomial complexity whose automorphism groups are also virtually $\mathbb{Z}$.

1. Introduction

An automorphism of a topological dynamical system (X,T), where $T\colon X\to X$ is a homeomorphism of the compact metric space X, is a homeomorphism from X to itself which commutes with T. We call $\mathrm{Aut}(X,T)$ the group of automorphisms of (X,T). There is an analogous definition of measurable automorphism for measure-preserving systems $(X,\mathcal{B},\mu,T)$, where $(X,\mathcal{B},\mu)$ is a standard probability space and $T:X\to X$ a measure-preserving transformation of this space. The group of measurable automorphisms is historically denoted by $C(T)$. This notation stands for the *centralizer group* of $(X,\mathcal{B},\mu,T)$.

The study of automorphism groups is a classical and widely considered subject in ergodic theory. The group $C(T)$ has been intensively studied for mixing measure-preserving systems of finite rank. The reader is referred to [18] for a complete survey. Let us mention some key theorems. D. Ornstein [34] proved that a mixing measure-preserving system of rank one has a trivial group of measurable automorphisms which consists of powers of T. Later, A. del Junco [14] showed that the well studied weakly mixing (but not mixing) rank one Chacon subshift also has this property. Finally, for mixing measure-preserving systems of finite rank, J. King and J.-P.

Date: July 1, 2015.

2010 *Mathematics Subject Classification.* Primary: 54H20; Secondary: 37B10.

Key words and phrases. Minimal subshifts, automorphism group, complexity.

This research was partially supported by grants Basal-CMM & Fondap 15090007, CONICYT Doctoral fellowship 21110300, ANR grants SubTile, DynA3S and FAN, and the cooperation project MathAmSud DYSTIL. The first and third authors thank the University of Picardie Jules Verne where this research was finished.

Thouvenot (see [27]) proved that $C(T)$ is virtually $\mathbb{Z}$, that is, its quotient relative to the subgroup $\langle T \rangle$ generated by T is a finite group.

In the non-weakly mixing case, B. Host and F. Parreau [25] proved that $C(T)$ is also virtually $\mathbb{Z}$ for a family of uniquely ergodic subshifts arising from constant-length substitutions and equals $\mathrm{Aut}(X, T)$. Concomitantly, M. Lemańczyk and M. Mentzen [30] proved that any finite group can be obtained as a quotient $C(T)/\langle T \rangle$ using substitution subshifts satisfying Host-Parreau's result.

In the topological setting, since the seminal work of G.A. Hedlund [20], several results have shown that the group of automorphisms for classes of subshifts in which the complexity grows quickly with word length might possess a very rich collection of subgroups. Here, by complexity we mean the increasing function $p_X \colon \mathbb{N} \to \mathbb{N}$ which counts the number of words of length $n \in \mathbb{N}$ appearing in points of the subshift (X, σ), where σ is the shift map. In particular, the automorphism group of the fullshift on two symbols contains isomorphic copies of any finite group [20] and the automorphism group of a mixing shift of finite type contains the free group on two generators, the direct sum of countably many copies of $\mathbb{Z}$ and the direct sum of every countable collection of finite groups [7, 26]. Similar richness in automorphism groups has been found in synchronized systems [19] and in multidimensional subshifts [22, 43].

In contrast, there is much evidence in the measurable and topological setting to suggest that low complexity systems ought to have a "small" automorphism group ([25, 30, 10, 33, 39]). Recently V. Salo and I. Törmä in [39] considered this problem in the context of subshifts generated by constant-length or primitive Pisot substitutions and proved that the group of automorphisms is virtually $\mathbb{Z}$. This generalizes the seminal result of E. Coven concerning constant-length substitutions on two letters [10]. In [39], the authors also asked whether or not the same result holds for subshifts constructed from primitive substitutions or, even more generally, for linearly recurrent subshifts [17].

In Theorem 3.1 of Section 3, we give a positive answer to the latter question, proving that the group of automorphisms of a transitive subshift is virtually $\mathbb{Z}$ if the subshift satisfies $\liminf_{n \to +\infty} \frac{p_X(n)}{n} < \infty$ together with a technical condition on the asymptotic pairs (which happens to be satisfied by aperiodic minimal subshifts). The class of systems satisfying this condition includes primitive substitutions, linearly recurrent subshifts and, more generally, any minimal subshift with linear complexity. Moreover, since the condition of the theorem involves a $\liminf$, Theorem 3.1 also applies to subshifts which simultaneously present non-superlinear and superpolynomial complexity along different subsequences. Explicit examples are given in Section 4. Our main tool for proving Theorem 3.1 is a detailed study of the structure of asymptotic pairs in the subshifts under consideration. These points always exist in an aperiodic subshift [3, Chapter 1]. This strategy is related to the study of *asymptotic composants* introduced by M. Barge and B. Diamond in [5]. This last notion proved to be a powerful invariant for studying one-dimensional substitution tiling spaces.

It is natural to ask which finite groups can arise as a quotient $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$ for subshifts satisfying the conditions of Theorem 3.1. As discussed above, a byproduct of the results in [25] and [30] shows that any finite group G is isomorphic to the quotient group $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$ of a constant-length substitutive minimal subshift

(X, σ). Here we provide a direct proof of this result by giving an explicit constant-length substitutive minimal subshift such that $\mathrm{Aut}(X, \sigma)$ is isomorphic to $\mathbb{Z} \oplus G$ (Theorem 3.6).

In the process of submitting this article, we became aware of a new article by V. Cyr and B. Kra [13]. While our Theorem 3.1 and Theorem 1.4 in [13] seem very close to each other, the methods and directions pursued in both articles are quite different. Our technique consists of looking at the action of automorphisms on the asymptotic pairs of a subshift. Together with studying the action of automorphisms on other interesting equivalence relations associated to special topological factors (mainly maximal equicontinuous factors and d-step nilfactors), this has enabled us to shed light on the properties of the automorphism groups of several classes of transitive subshifts which exhibit complexities with polynomial or higher growth. In comparison, the authors of [13] explore the world of systems whose complexity grows at most linearly and that are not necessarily transitive.

The automorphism group of subshifts with superlinear complexity ($\lim_{n\to+\infty} p_X(n)/n) = \infty$) seems more complicated to manage than the non-superlinear case. In [12], it was proved that the quotient of the automorphism group relative to the group generated by the shift is periodic for transitive subshifts with subquadratic complexity, meaning that any element in this group has finite order. The proof of this result was achieved by means of studying a $\mathbb{Z}^2$ coloring problem and uses a deep combinatorial result of A. Quas and L. Zamboni [37].

In this article, we also explore zero entropy subshifts with superlinear complexity in several directions. We mainly discover classes of examples where the groups of automorphisms still show a small growth rate or are abelian. Our first class of examples arises from the study of symbolic extensions of nilsystems. In Section 5, we prove that, for every integer $d \geq 1$, the groups of automorphisms of proximal extensions of d-step nilsystems are d-step nilpotent groups. This result is then used to construct subshifts with arbitrary polynomial complexity and automorphism groups virtually isomorphic to $\mathbb{Z}$ (Theorem 5.12). The main tool used to prove this result is a detailed study of the regionally proximal relation of order d for such subshifts ([24],[41]). Then, in Section 6.1 we provide a subshift with superlinear complexity whose automorphism group is isomorphic to $\mathbb{Z}^d$ for some $d \in \mathbb{N}$.

We conclude the article by asking several questions and by proposing directions for future research. In particular, we explore the *visiting time* map associated to a subshift (X, σ) as an alternative to word complexity. We propose studying the increasing function $R''_X \colon \mathbb{N} \to \mathbb{N}$ which, for every $n \in \mathbb{N}$, gives the minimum possible length of words having as subwords all words of length n that appear in points in the subshift [9]. In Proposition 6.4, we prove that any finitely generated subgroup of the automorphism group of a subshift with visiting time map of polynomial growth is virtually nilpotent. This result is somehow parallel to Theorem 1.1 in [13], but applies to subshifts with visiting time map of at most polynomial growth rather than those of linear word complexity.

2. Preliminaries, notation and background

2.1. **Topological dynamical systems.** A *topological dynamical system* (or just a system) is a homeomorphism $T \colon X \to X$, where X is a compact metric space. It is classically denoted by (X, T). Let dist be a distance in X and denote by $\mathrm{Orb}_T(x)$ the orbit $\{T^n x; n \in \mathbb{Z}\}$ of $x \in X$. A topological dynamical system is *minimal* if the orbit of every point is dense in X and is *transitive* if at least one orbit is dense in X. In a transitive system, points with dense orbits are called *transitive points*. The *ω-limit set* $\omega(x)$ of a point $x \in X$ is the set of accumulation points of the positive orbit of x, or formally $\omega(x) = \bigcap_{n\geq 0} \overline{\{T^k x; k \geq n\}}$.

Let (X,T) be a topological dynamical system. We say that $x, y \in X$ are *proximal* if there exists a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $\lim_{i\to+\infty} \mathrm{dist}(T^{n_i}x, T^{n_i}y) = 0$. A stronger condition than proximality is asymptoticity. Two points $x, y \in X$ are said to be *asymptotic* if $\lim_{n\to+\infty} \mathrm{dist}(T^n x, T^n y) = 0$. Nontrivial asymptotic pairs may not exist in an arbitrary topological dynamical system but it is well known that a nonempty aperiodic subshift always admits at least one [3, Chapter 1].

A *factor map* between the topological dynamical systems (X,T) and (Y,S) is a continuous onto map $\pi \colon X \to Y$ such that $\pi \circ T = S \circ \pi$ (T and S commute). We say that (Y,S) is a *factor* of (X,T) and that (X,T) is an *extension* of (Y,S). We use the notation $\pi \colon (X,T) \to (Y,S)$ to indicate the factor map. If in addition π is a bijective map we say that (X,T) and (Y,S) are *topologically conjugate*.

We say that (X,T) is a *proximal extension* of (Y,S) via the factor map $\pi \colon (X,T) \to (Y,S)$ (or that the factor map itself is a *proximal extension*) if for every $x, x' \in X$ the condition $\pi(x) = \pi(x')$ implies that x, x' are proximal. For minimal systems, (X,T) is an *almost one-to-one extension* of (Y,S) via the factor map $\pi : (X,T) \to (Y,S)$ (or the factor map itself is an *almost one-to-one extension*) if there exists $y \in Y$ with a unique preimage for the map π. The relation between these two notions is given by the following folklore lemma. We provide a proof for completeness.

Lemma 2.1. *If the factor map $\pi : (X,T) \to (Y,S)$ between minimal systems is an almost one-to-one extension then it is also a proximal extension.*

Proof. Let $y_0 \in Y$ be a point with a unique preimage under π and consider points $x, x' \in X$ such that $\pi(x) = \pi(x')$. By the minimality of (Y,S), there exists a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $S^{n_i}(\pi(x))$ $(= S^{n_i}(\pi(x')))$ converges to y_0 as i goes to infinity. By continuity of π and since T commutes with S, the sequences $(T^{n_i}x)_{i\in\mathbb{N}}$ and $(T^{n_i}x')_{i\in\mathbb{N}}$ converge to the same unique point in the preimage of y_0 for π. This shows that points x and x' are proximal. □

2.2. **Automorphism group.** An *automorphism* of the topological dynamical system (X,T) is a homeomorphism ϕ of the space X such that $\phi \circ T = T \circ \phi$. We denote by $\mathrm{Aut}(X,T)$ the group of automorphisms of (X,T). The subgroup of $\mathrm{Aut}(X,T)$ generated by T is denoted by $\langle T \rangle$.

We will need the following two simple facts.

Lemma 2.2. *Let (X,T) be a minimal topological dynamical system. Then the action of $\mathrm{Aut}(X,T)$ on X is free. That is, every nontrivial element in $\mathrm{Aut}(X,T)$ has no fixed points.*

Proof. Take $\phi \in \mathrm{Aut}(X,T)$ and $x \in X$ such that $\phi(x) = x$. Since ϕ commutes with T and is continuous, by minimality we deduce that $\phi(y) = y$ for all $y \in X$. Thus ϕ is the identity map. □

Lemma 2.3. *Let (X,T) be a topological dynamical system. For $x \in X$ and $\phi \in \mathrm{Aut}(X,T)$ we have,*

- *if x and $\phi(x)$ are asymptotic then ϕ restricted to $\omega(x)$ is the identity map;*
- *if (X,T) is minimal then x and $\phi(x)$ are proximal if and only if ϕ is the identity map.*

Proof. In the first part, we assume $\lim_{n\to+\infty} \operatorname{dist}(T^n x, T^n \phi(x)) = 0$. For any $y \in \omega(x)$ consider a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that $T^{n_i}x$ converges to y. We get that $\phi(y) = y$, which proves the desired result.

The proof of the nontrivial direction of the second part is similar. By definition, there exists a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $\lim_{i\to+\infty} \operatorname{dist}(T^{n_i}x, T^{n_i}\phi(x)) = 0$. We can assume that $T^{n_i}x$ converges to some $y \in X$. Therefore $\phi(y) = y$. By Lemma 2.2 ϕ is the identity map. $\square$

Let $\pi\colon (X,T) \to (Y,S)$ be a factor map between the minimal systems (X,T) and (Y,S), and let ϕ be an automorphism of (X,T). We say that π is *compatible* with ϕ if $\pi(x) = \pi(x')$ implies $\pi(\phi(x)) = \pi(\phi(x'))$ for every $x, x' \in X$. We say that π is *compatible* with $\operatorname{Aut}(X,T)$ if π is compatible with every $\phi \in \operatorname{Aut}(X,T)$.

If the factor map $\pi\colon (X,T) \to (Y,S)$ is compatible with $\operatorname{Aut}(X,T)$ we can define the projection $\widehat{\pi}(\phi) \in \operatorname{Aut}(Y,S)$ by the equation $\widehat{\pi}(\phi)(\pi(x)) = \pi(\phi(x))$ for all $x \in X$. We have that $\widehat{\pi}\colon \operatorname{Aut}(X,T) \to \operatorname{Aut}(Y,S)$ is a group morphism.

Note that $\widehat{\pi}$ might not be onto or injective. Indeed, for an irrational rotation of the circle, the group of automorphisms is the whole circle but the group of automorphisms of its Sturmian extension is $\mathbb{Z}$ [33]. We will show in Lemma 5.7 that this factor map is compatible, hence $\widehat{\pi}$ is well defined but is not onto. On the other hand, the map $\widehat{\pi}$ associated to the projection onto the trivial system cannot be injective.

In the case of a compatible proximal extension between minimal systems we have:

Lemma 2.4. *Let $\pi\colon (X,T) \to (Y,S)$ be a proximal extension between minimal systems and suppose that π is compatible with* $\operatorname{Aut}(X,T)$*. Then $\widehat{\pi}\colon \operatorname{Aut}(X,T) \to$* $\operatorname{Aut}(Y,S)$ *is injective.*

Proof. Let $\phi \in \operatorname{Aut}(X,T)$ be an automorphism such that $\widehat{\pi}(\phi)$ is the identity map of Y. It suffices to prove that ϕ is the identity map of X. For $x \in X$ we have that $\pi(\phi(x)) = \widehat{\pi}(\phi)(\pi(x)) = \pi(x)$. Since π is proximal, then x and $\phi(x)$ are proximal points. From Lemma 2.3 we conclude that ϕ is the identity map. $\square$

2.3. **Subshifts.** Let $\mathcal{A}$ be a finite set that we will call *alphabet*. Elements in $\mathcal{A}$ are called *letters* or *symbols*. The set of finite sequences or *words* of length $\ell \in \mathbb{N}$ with letters in $\mathcal{A}$ is denoted by $\mathcal{A}^\ell$, the set of onesided sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{N}}$ and the set of twosided sequences $(x_n)_{n\in\mathbb{Z}}$ in $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{Z}}$. Also, a word $w = w_1 \dots w_\ell \in \mathcal{A}^\ell$ can be seen as an element of the free monoid $\mathcal{A}^*$ endowed with the operation of concatenation. The *length* of w is denoted by $|w| = \ell$.

The *shift map* $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. To simplify notations we denote the shift map by σ independently of the alphabet, the alphabet will be clear from the context.

A *subshift* is a topological dynamical system (X,σ) where X is a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ (we consider the product topology in $\mathcal{A}^{\mathbb{Z}}$). For convenience, when we state general results about topological dynamical systems we use the notation (X,T) and to state specific results about subshifts we use (X,σ).

Let (X,σ) be a subshift. The *language* of (X,σ) is the set $\mathcal{L}(X)$ containing all words $w \in \mathcal{A}^*$ such that $w = x_m \dots x_{m+\ell-1}$ for some $(x_n)_{n\in\mathbb{Z}} \in X$, $m \in \mathbb{Z}$ and $\ell \in \mathbb{N}$. We say that w *appears* or *occurs* in the sequence $(x_n)_{n\in\mathbb{Z}} \in X$. We denote by $\mathcal{L}_\ell(X)$ the set of words of length ℓ in $\mathcal{L}(X)$.

The map $p_X : \mathbb{N} \to \mathbb{N}$ defined by $p_X(\ell) = \sharp\mathcal{L}_\ell(X)$ is called the *complexity function* of (X, σ).

We recall some notations from complexity theory. Given two functions $f, g \colon \mathbb{N} \to \mathbb{N}\backslash\{0\}$ we write $f(\ell) = O(g(\ell))$ if there exists a positive constant K such that $f(\ell) \leq Kg(\ell)$ for every large enough ℓ. We also write $f(\ell) = \Theta(g(\ell))$ if $f(\ell) = O(g(\ell))$ and $g(\ell) = O(f(\ell))$. Finally, $f(\ell) = \Omega_+(g(\ell))$ if $\limsup_{\ell\to+\infty} f(\ell)/g(\ell) > 0$.

We adopt the following terminology. We say that the complexity of the subshift:

- is *polynomial* if there exists an integer $d \geq 1$ such that $p_X(\ell) = \Theta(\ell^d)$; when $d = 1$ we say the complexity is *linear* and when $d = 2$ the subshift has *quadratic* complexity;
- has *at most polynomial growth rate* if there exists an integer $d \geq 1$ such that $p_X(\ell) = O(\ell^d)$;
- is *superlinear* if $\lim\limits_{\ell\to+\infty} p_X(\ell)/\ell = +\infty$;
- is *non-superlinear* if $\liminf\limits_{\ell\to+\infty} p_X(\ell)/\ell < +\infty$;
- is *subquadratic* if $\lim\limits_{\ell\to+\infty} p_X(\ell)/\ell^2 = 0$;
- is *superpolynomial along a subsequence* if $\limsup\limits_{\ell\to+\infty} p_X(\ell)/q(\ell) = \pm\infty$ for every polynomial q;
- is *subexponential* if $\lim\limits_{\ell\to+\infty} p_X(\ell)/\alpha^\ell = 0$ for all $\alpha > 1$.

In the proof of Theorem 3.1 we will need the following well known notion that is intimately related to the concept of asymptotic pairs. A word $w \in \mathcal{L}(X)$ is said to be *left special* if there exist at least two distinct letters a and b such that aw and bw belong to $\mathcal{L}(X)$. In the same way we define *right special* words.

Let $\phi\colon (X, \sigma) \to (Y, \sigma)$ be a factor map between subshifts. By the Curtis-Hedlund-Lyndon Theorem, ϕ is determined by a *local map* $\hat{\phi}\colon \mathcal{A}^{2\mathbf{r}+1} \to \mathcal{A}$ in such a way that $\phi(x)_n = \hat{\phi}(x_{n-\mathbf{r}} \dots x_n \dots x_{n+\mathbf{r}})$ for all $n \in \mathbb{Z}$ and $x \in X$, where $\mathbf{r} \in \mathbb{N}$ is called a *radius* of ϕ. The local map $\hat{\phi}$ naturally extends to the set of words of length at least $2\mathbf{r} + 1$, and we also denote this map by $\hat{\phi}$.

2.4. **Substitutions and substitutive subshifts.** We recall some basic definitions about substitutions and the induced subshifts. For more details see [38].

Let $\mathcal{A}$ be a finite alphabet. A *substitution* is a map $\tau\colon \mathcal{A} \to \mathcal{A}^*$ which associates to each letter $a \in \mathcal{A}$ a word $\tau(a)$ of some length in $\mathcal{A}^*$. The substitution τ can be applied to a word in $\mathcal{A}^*$ and onesided or twosided infinite sequences in $\mathcal{A}$ in the obvious way by concatenating (in the case of a twosided sequence we apply τ to positive and negative coordinates separately and we concatenate at coordinate zero the results). Then substitutions can be iterated or composed n times for any integer $n \geq 1$. Denote this composition by τ^n. To avoid trivial cases we will always assume in the definition of a substitution that the length of $\tau^n(a)$ grows to infinity for every letter $a \in \mathcal{A}$.

The substitution $\tau\colon \mathcal{A} \to \mathcal{A}^*$ is *primitive* if for some integer $p \geq 1$ and every letter $a \in \mathcal{A}$ the word $\tau^p(a)$ contains all the letters of the alphabet.

The substitution $\tau\colon \mathcal{A} \to \mathcal{A}^*$ is said to be of *constant length* $\ell > 0$ if $|\tau(a)| = \ell$ for each $a \in \mathcal{A}$. The length of a substitution is also denoted by $|\tau|$. The constant-length substitution τ is *bijective* if $\tau(a)_i \neq \tau(b)_i$ for all $a, b \in \mathcal{A}$ with $a \neq b$ and all coordinates $1 \leq i \leq |\tau|$.

The subshift induced by a substitution $\tau\colon \mathcal{A} \to \mathcal{A}^*$ is denoted by (X_τ, σ), where X_τ is the set

$$\{x \in \mathcal{A}^{\mathbb{Z}};\ \text{each finite word of } x \text{ is a subword of } \tau^n(a) \text{ for some } n \geq 1 \text{ and } a \in \mathcal{A}\}.$$

We also say that (X_τ, σ) is a *substitutive subshift*. For constant-length substitutions it is well known that (X_τ, σ) is minimal if and only if the substitution τ is *primitive*. The substitution τ is said to be *aperiodic* if X_τ is an infinite set.

2.5. **Equicontinuous systems.** A topological dynamical system (X, T) is *equicontinuous* if the family of transformations $\{T^n; n \in \mathbb{Z}\}$ is equicontinuous. Let (X, T) be an equicontinuous minimal system. It is well known that the closure of the group $\langle T \rangle$ in the set of homeomorphisms of X for the uniform topology is a compact abelian group acting transitively on X [3]. When X is a Cantor set the dynamical system (X, T) is called an *odometer*.

2.6. **Nilsystems.** The following well known class of systems will allow us to compute the automorphism group of some interesting subshifts of polynomial complexity.

Let G be a group. The *commutator* of $g, h \in G$ is defined to be $[g, h] = ghg^{-1}h^{-1}$ and for $E, F \subset G$, we let $[E, F]$ denote the group spanned by $\{[e, f]\colon e \in E, f \in F\}$. The *commutator subgroups* G_j of G are defined inductively, with $G_1 = G$ and for integers $j \geq 1$, we have $G_{j+1} = [G, G_j]$. For an integer $d \geq 1$, if G_{d+1} is the trivial subgroup then G is said to be *d-step nilpotent*. Notice that a subgroup of a d-step nilpotent group is also d-step nilpotent and any abelian group is 1-step nilpotent.

Let $d \geq 1$ be an integer, G be a d-step nilpotent Lie group and Γ be a discrete cocompact subgroup of G. Then the compact nilmanifold $X = G/\Gamma$ is a *d-step nilmanifold*. The group G acts on X by left translations and we write this action by $(g, x) \mapsto gx$. Let $T\colon X \to X$ be the transformation $x \mapsto \tau x$ for some fixed element $\tau \in G$. Then (X, T) is a *d-step nilsystem*. Thus a 1-step nilsystem is exactly a translation on a compact abelian group. Nilsystems are distal systems, meaning that there are no proximal pairs. Moreover, minimal nilsystems are uniquely ergodic. See [4] and [29] for general references.

An important subclass of nilsystems are *affine nilsystems*. Let $d \geq 1$ be an integer and consider a $d \times d$ integer matrix A such that $(A - Id)^d = 0$ (such a matrix is called *unipotent*) and a vector $\vec{\alpha} \in \mathbb{T}^d$. Define the transformation $T\colon \mathbb{T}^d \to \mathbb{T}^d$ by $x \mapsto Ax + \vec{\alpha}$ (operations are considered $\mod \mathbb{Z}^d$). Since A is unipotent, one can prove that the group G spanned by A and all the translations of $\mathbb{T}^d$ is a d-step nilpotent Lie group. The stabilizer of 0 is the subgroup Γ spanned by A. Thus we can identify $\mathbb{T}^d$ with G/Γ. The topological dynamical system $(\mathbb{T}^d, T) = (G/\Gamma, T)$ is called a *d-step affine nilsystem*. This system is minimal if and only if the projection of $\vec{\alpha}$ onto $\mathbb{T}^d/ker(A - Id)$ defines a minimal rotation [35].

3. Automorphism groups of subshifts with non-superlinear complexity

Now we shall give a positive answer to the question raised in [39]: is it true that the group of automorphisms of a linearly recurrent system is virtually isomorphic to $\mathbb{Z}$? We recall that a group G *virtually* satisfies a property `P` (*e.g.*, nilpotent, solvable, isomorphic to a given group) if there is a finite index subgroup $H \subseteq G$ satisfying property `P`.

It is known that the complexity functions of linearly recurrent subshifts have at most a linear growth rate [17]. We answer the former question by considering the much larger class of minimal subshifts with non-superlinear complexity. The main tool for answering this question is a detailed study of the asymptotic relation. More precisely the so-called *asymptotic components* introduced below. This notion is related to the *asymptotic composants* introduced by M. Barge and B. Diamond in [5]. The chief result from this work that we also need here is that there are finitely many asymptotic composants. Notice that in the substitutive case the asymptotic composants can be described combinatorially [5].

Let (X,T) be a topological dynamical system. Given $x, y \in X$ we say that orbits $\mathrm{Orb}_T(x)$ and $\mathrm{Orb}_T(y)$ are *asymptotic* if there exist points $x' \in \mathrm{Orb}_T(x)$ and $y' \in \mathrm{Orb}_T(y)$ that are asymptotic. This condition is equivalent to saying that y is asymptotic to some $T^n x$ or vice versa. Then for each $x' \in \mathrm{Orb}_T(x)$, there is a point $y' \in \mathrm{Orb}_T(y)$ asymptotic to x'. We denote this relation by $\mathrm{Orb}_T(x)\ \mathcal{A}S\ \mathrm{Orb}_T(y)$. It follows that $\mathcal{A}S$ defines an equivalence relation on the collection of orbits. When an $\mathcal{A}S$-equivalence class is not reduced to a single element we call it an *asymptotic component*. The equivalence class for $\mathcal{A}S$ of the orbit of $x \in X$ is denoted by $\mathcal{AS}_{[x]}$ and the set of all asymptotic components by $\mathcal{AS}$.

It is clear from the definition that the asymptotic relation is preserved by automorphisms of (X,T): if $x, y \in X$ are asymptotic then $\phi(x), \phi(y)$ are asymptotic for every $\phi \in \mathrm{Aut}(X,T)$. It is also not difficult to check that the orbits $\mathrm{Orb}_T(\phi(x))$ and $\mathrm{Orb}_T(\phi(y))$ are asymptotic whenever $\mathrm{Orb}_T(x)$ and $\mathrm{Orb}_T(y)$ are asymptotic. Then, the image of an asymptotic component under $\phi \in \mathrm{Aut}(X,T)$ is an asymptotic component. These properties prove that every automorphism $\phi \in \mathrm{Aut}(X,T)$ induces a permutation $j(\phi)$ of the set of asymptotic components $\mathcal{AS}$. Therefore, the following group morphism is well defined:

$$
\begin{array}{rcl}
j\colon \mathrm{Aut}(X,T) & \to & \mathrm{Per}\mathcal{AS} \\
\phi & \mapsto & \left(\mathcal{AS}_{[x]} \mapsto \mathcal{AS}_{[\phi(x)]}\right),
\end{array}
\tag{1}
$$

where $\mathrm{Per}\mathcal{AS}$ denotes the set of permutations of $\mathcal{AS}$.

Now we can state the main result of this section.

Theorem 3.1. *Let (X,σ) be a subshift such that $\liminf_{n\to+\infty} \frac{p_X(n)}{n} < +\infty$. Assume there exists a point $x_0 \in X$ with $\omega(x_0) = X$ that is asymptotic to a different point. Then,*

1. *$\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is finite.*
2. *If (X,σ) is minimal, the quotient group $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is isomorphic to a finite subgroup of permutations without fixed points and $\sharp(\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle)$ divides the number of asymptotic components of (X,σ).*

Notice that the condition on the point x_0 is automatically satisfied when the dynamical system (X,σ) is minimal. In this case we obtain Theorem 1.4 in [13].

The condition on the growth rate of the complexity function is satisfied by primitive substitutive subshifts, by linearly recurrent systems and many other subshifts. Interestingly, this condition is compatible with $\limsup_{n\to+\infty} p_X(n)/n = +\infty$. In Section 4, we construct a minimal subshift which exhibits superpolynomial complexity along a subsequence even though it satisfies the complexity hypothesis of Theorem 3.1.

We remark that Statement (2) of Theorem 3.1 does not impose any restriction on the finite groups obtained as quotients $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$. Indeed, given a finite group G, it acts on itself by left multiplication: $L_g(h) = gh$ for $g, h \in G$. Then the map L_g defines a permutation of the finite set G without any fixed points. So G can be seen as a subgroup of the permutation group of $\sharp G$ elements, which satisfies Statement (2) of the theorem. In Section 3.2, we show that for every finite group G there exists a subshift (X,σ) such that $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is isomorphic to G by giving a characterization of the automorphisms of a specific family of subshifts induced by substitutions. As mentioned in the introduction, we shall give a direct proof of this result here, but it can also be deduced by combining results in [25] and [30].

Finally, we note that Statement (2) of Theorem 3.1 enables us to perform explicit computations of automorphism groups in some easy cases. The first example of this comes from Sturmian subshifts (see [28] for a detailed exposition of these systems). It is well known that these systems have unique asymptotic components, so each automorphism is a power of the shift map. A slightly more general case is when the number of asymptotic components is a prime number p (*e.g.*, $p = 2$ for the Thue-Morse subshift). In this case the group $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is a subgroup of $\mathbb{Z}/p\mathbb{Z}$, either the trivial one or $\mathbb{Z}/p\mathbb{Z}$ itself. In particular, since the Thue-Morse subshift admits an automorphism which is not the power of the shift map (the one that flips the two letters of the alphabet), then in this case the quotient is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We point out that the hypothesis on the complexity in Theorem 3.1 is only used to prove that there are finitely many asymptotic components. So any subshift where this last property holds is a good candidate for having an automorphism group that is virtually $\mathbb{Z}$. This is the case of minimal systems, but in general this is not a theorem, and we need to check the structure of asymptotic components in greater detail. In fact, the structure of asymptotic components plays a crucial role in the computation of the automorphism groups. This motivates the second example presented in Section 4.

3.1. **Proof of Theorem 3.1.** The following lemma is a key observation that allows the growth rate of the complexity function of a subshift to be related to its asymptotic components. The proof follows some classical ideas from [38].

Lemma 3.2. *Let (X,σ) be a subshift. If $\liminf_{n\to+\infty} \frac{p_X(n)}{n} < +\infty$, then the number of asymptotic components is finite. In particular, any subshift of linear complexity has a finite number of asymptotic components.*

Proof. We observe that the last statement follows from Lemma V.22 in [38]. Here we extend this result to subshifts whose complexity functions are non-superlinear.

We claim that there exists a constant κ and an increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that $p_X(n_i+1) - p_X(n_i) \le \kappa$. If not, for every $A > 0$ and for every large enough integer n we have $p_X(n+1) - p_X(n) \ge A$. It follows that for all large enough integers $m < n$, $p_X(n) - p_X(m) = \sum_{i=m}^{n-1} p_X(i+1) - p_X(i) \ge (n-m)A$. From here we get that $\liminf_{n\to+\infty} \frac{p_X(n)}{n} \ge A$. This contradicts our hypothesis since A is arbitrary and the claim follows.

Fix κ and an increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ as above. Hence, the number of left special words of length n_i of the subshift is bounded by κ (see Section 2.3 to recall the definition).

Let $\{x_0, y_0\}, \ldots, \{x_\kappa, y_\kappa\}$ denote nontrivial asymptotic pairs. Clearly, each pair induces a pair of asymptotic orbits. Since X is a subshift, for each $j \in \{0, \ldots, \kappa\}$ there exists $\ell_j \in \mathbb{Z}$ such that all coordinates of x_j and y_j larger than or equal to ℓ_j coincide whereas the $(\ell_j - 1)^{\text{th}}$ coordinates are different. Then, for each $i \in \mathbb{N}$, the word of length n_i starting at coordinate ℓ_j in both points x_j and y_j is a left special word. Since we have proved that the number of left special words of length n_i is bounded by κ, we have that the special words associated to two different asymptotic pairs in our list coincide. But this fact holds for every $i \in \mathbb{N}$ and hence the pigeonhole principle implies that two asymptotic pairs in the list must share infinitely many of their left special words. Thus, the associated pairs of asymptotic orbits are equivalent. This proves that there are at most κ asymptotic components and the result follows. $\square$

A second ingredient needed for proving Theorem 3.1 is the following corollary of Lemma 2.3.

Corollary 3.3. *Let (X, T) be a topological dynamical system. Assume there exists a point $x_0 \in X$ with $\omega(x_0) = X$ that is asymptotic to a different point. We have the following exact sequence,*

$$\{1\} \longrightarrow \langle T \rangle \xrightarrow{\text{Id}} \operatorname{Aut}(X, T) \xrightarrow{j} \operatorname{Per}\mathcal{AS},$$

where j was defined in (1)*. More precisely, for every automorphism $\phi \in \operatorname{Aut}(X, T)$, the permutation $j(\phi)$ fixes the asymptotic component $\mathcal{AS}_{[x_0]}$ if and only if ϕ is a power of T.*

Proof. Let ϕ be an automorphism in $\operatorname{Aut}(X, T)$ and suppose that $\mathcal{AS}_{[\phi(x_0)]} = \mathcal{AS}_{[x_0]}$. This means that there exists an integer $n \in \mathbb{Z}$ such that x_0 and $T^n \circ \phi(x_0)$ are asymptotic. By Lemma 2.3, $T^n \circ \phi$ is the identity map and thus $\phi \in \langle T \rangle$ as desired. $\square$

Proof of Theorem 3.1. We concentrate on the second part of Statement (2), as this is the only facet of the theorem that does not follow directly from Lemma 3.2 and Corollary 3.3. From Corollary 3.3, no asymptotic component is fixed by a nontrivial automorphism. So, the group $\operatorname{Aut}(X, \sigma)/\langle \sigma \rangle$ acts freely on the finite set of asymptotic components $\mathcal{AS}$: the stabilizer of any point is trivial. Thus, $\mathcal{AS}$ is decomposed into disjoint $\operatorname{Aut}(X, \sigma)/\langle \sigma \rangle$-orbits, and each such orbit has the same cardinality as $\operatorname{Aut}(X, \sigma)/\langle \sigma \rangle$. $\square$

3.2. **Realization of any finite group as $\mathbf{Aut}(X, \sigma)/\langle \sigma \rangle$.** In this section we provide a constructive proof that any finite group can be obtained as a quotient $\operatorname{Aut}(X, \sigma)/\langle \sigma \rangle$, where (X, σ) is a subshift satisfying the hypothesis of Theorem 3.1. As mentioned earlier, this result can be deduced from results in [25] and [30] concerning the automorphism groups of subshifts induced by constant-length substitutions. However, we prefer to give a direct proof in order to highlight the notion of asymptotic components. We also provide a new proof of the characterization of the automorphism groups of subshifts induced by the bijective constant-length substitutions of Host and Parreau [25].

3.2.1. *Properties of asymptotic pairs of subshifts induced by constant-length substitutions.*

Lemma 3.4. *Let $\tau\colon \mathcal{A} \to \mathcal{A}^*$ be a primitive aperiodic bijective constant-length substitution. Let $x = (x_n)_{n\in\mathbb{Z}}$ and $y = (y_n)_{n\in\mathbb{Z}}$ be an asymptotic pair for (X_τ, σ) such that $x_n = y_n$ for each $n \geq 0$ and $x_{-1} \neq y_{-1}$. Then, there exist asymptotic points $x' = (x'_n)_{n\in\mathbb{Z}}$ and $y' = (y'_n)_{n\in\mathbb{Z}}$ for (X_τ, σ) with $x'_n = y'_n$ for each $n \geq 0$ and $x'_{-1} \neq y'_{-1}$ such that $\tau(x') = x$ and $\tau(y') = y$.*

Proof. Let ℓ be the length of the substitution τ. By the classical result of B. Mossé [31, 32] on recognizability, the map induced by τ on X_τ, $\tau\colon X_\tau \to \tau(X_\tau)$, is one-to-one. Moreover, the collection $\{\sigma^k\tau(X_\tau); k = 0, \dots, \ell-1\}$ is a clopen partition (formed by subsets that are simultaneously closed and open) of X_τ. Then, there exist $x' = (x'_n)_{n\in\mathbb{Z}}, y' = (y'_n)_{n\in\mathbb{Z}} \in X_\tau$ and $0 \leq k_x, k_y < \ell$ such that $\sigma^{k_x}\tau(x') = x$ and $\sigma^{k_y}\tau(y') = y$.

We claim that $k_x = k_y = 0$. Since the sequences x and y are asymptotic, there are integers $n \geq 0$ and $k' \in \{0, \dots, \ell-1\}$ such that $\sigma^n(x), \sigma^n(y) \in \sigma^{k'}(\tau(X_\tau))$. The substitution τ is of constant-length ℓ, so we have $\sigma^\ell \circ \tau = \tau \circ \sigma$. Therefore, x and y are in the same clopen set $\sigma^k(\tau(X_\tau))$ for some $k \in \{0, \dots, \ell-1\}$. This shows that $k = k_x = k_y$.

Next, let us assume that $k \geq 1$. The words $x_{-k} \dots x_0$, $y_{-k} \dots y_0$ are then prefixes of the words $\tau(x'_0)$ and $\tau(y'_0)$ respectively. Since the substitution τ is bijective and $x_0 = y_0$, we have that $x'_0 = y'_0$. In particular, we get that $x_{-1} = y_{-1}$, which is a contradiction.

To complete the proof recall that the substitution τ is bijective, so for all $n \geq 0$ we have $x'_n = y'_n$ and $x'_{-1} \neq y'_{-1}$. □

Lemma 3.5. *Let $\tau\colon \mathcal{A} \to \mathcal{A}^*$ be a primitive aperiodic bijective constant-length substitution. Then, there exists an integer $p \geq 0$ such that for all asymptotic points $x = (x_n)_{n\in\mathbb{Z}}$ and $y = (y_n)_{n\in\mathbb{Z}}$ for (X_τ, σ), the onesided infinite sequences $(x_n)_{n\geq n_0}$ and $(y_n)_{n\geq n_0}$ coincide for some $n_0 \in \mathbb{Z}$ and are fixed by τ^p.*

Proof. Since x and y are asymptotic, shifting them by the same power of the shift we can assume that $x_n = y_n$ for every integer $n \geq 0$ and $x_{-1} \neq y_{-1}$. Since τ is bijective, the map $a \mapsto \tau(a)_1$ is a permutation of the alphabet. Thus, there exists an integer $p \geq 1$ such that for each letter $a \in \mathcal{A}$ every word in the sequence $(\tau^{pn}(a))_{n\geq 1}$ starts with the same letter. Hence, the sequence $(\tau^{pn}(aa\dots))_{n\geq 1}$ converges to a onesided infinite sequence $z^{(a)}$ such that $\tau^p(z^{(a)}) = z^{(a)}$ ($z^{(a)}$ is fixed by τ^p).

Now we inductively apply Lemma 3.4 to the substitution τ^p. For each integer $i \geq 0$ we get asymptotic pairs $x^{(i)}, y^{(i)} \in \mathcal{A}^{\mathbb{Z}}$ satisfying the conclusions of the lemma and such that $\tau^p(x^{(i+1)}) = x^{(i)}$, $\tau^p(y^{(i+1)}) = y^{(i)}$, with $x^{(0)} = x$ and $y^{(0)} = y$. By the choice of p, the 0 coordinate of all points $x^{(i)}$ and $y^{(i)}$ coincide at some letter $a \in \mathcal{A}$. Then $\tau^{pn}(a)$ is a prefix of the sequence $(x_j)_{j\geq 0}$ (that is equal to $(y_j)_{j\geq 0}$) for every $n \in \mathbb{N}$. Therefore, $(x_j)_{j\geq 0} = (y_j)_{j\geq 0} = z^{(a)}$ which is fixed by τ^p as desired. This concludes the proof of the lemma. □

3.2.2. *Realization of a finite group as* $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$. A first consequence of Lemma 3.5 is the realization of any finite group as the quotient group $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ of a subshift induced by a constant-length substitution.

Theorem 3.6. *Given a finite group G, there exists a minimal substitutive subshift (X, σ) such that $\mathrm{Aut}(X, \sigma)$ is isomorphic to $\mathbb{Z} \oplus G$.*

Proof. If G is the trivial group then we can consider (X, σ) to be the Fibonacci subshift, which is also an Sturmian subshift (see also [33]). This result also follows from Theorem 3.1 since one can easily prove in this case that there exists a unique asymptotic component.

Now, we assume that the finite group G is not trivial. We choose an enumeration of its elements $G = \{g_0, g_1, \ldots, g_{q-1}\}$ with $q \geq 2$ and we set g_0 to be the identity element.

For an element $g \in G$, let $L_g \colon G \to G$ denote the bijection $h \mapsto gh$. We see G as a finite alphabet and define the substitution of constant length $\tau : G \to G^*$ by

$$\tau \colon g \mapsto L_g(g_0)L_g(g_1)\cdots L_g(g_{q-1}).$$

Since the map L_g is a bijection on G, then the substitution τ is primitive and bijective.

We claim that the subshift (X_τ, σ) is not periodic, *i.e.*, it does not reduce to a periodic orbit. To show this fact, it suffices to give an example of a nontrivial asymptotic pair. By the definition of τ the word $g_0g_1 \in \mathcal{L}(X_\tau)$. Hence the words $\tau(g_0)\tau(g_1)$ and its subword $g_{q-1}g_1$ (which is different from the word g_0g_1) also belong to $\mathcal{L}(X_\tau)$. It follows that $\tau^n(g_0)\tau^n(g_1), \tau^n(g_{q-1})\tau^n(g_1) \in \mathcal{L}(X_\tau)$ for every integer $n \geq 0$. Taking a subsequence if necessary, these words converge as n goes to infinity to two different sequences x and $y \in X_\tau$ that are asymptotic by construction.

Given an element $g \in G$ we extend the definition of the map L_g to words in G^* or infinite onesided or twosided infinite sequences by: $L_g((h_i)_{i\in I}) = (gh_i)_{i\in I}$, where I is a finite or infinite set of indexes. In particular, this defines a left continuous G-action on $G^{\mathbb{Z}}$. Moreover, each map L_g preserves the subshift X_τ. Indeed, if $x = (x_n)_{n\in\mathbb{Z}} \in X_\tau$ then for all integers $j \in \mathbb{Z}$ and $m \geq 1$ the word $x_j \ldots x_{j+m-1}$ is a subword of $\tau^N(h)$ for some $N \in \mathbb{N}$ and $h \in G$. Then, $L_g(x_j \ldots x_{j+m-1}) = gx_j \ldots gx_{j+m-1}$ is a subword of $L_g(\tau^N(h))$. But we have the relation

$$L_g(\tau(h)) = \tau(L_g(h)) \text{ for every } g, h \in G, \tag{2}$$

so $L_g(x_j \ldots x_{j+m-1})$ is a subword of $\tau^N(L_g(h))$. This implies that $L_g(x) \in X_\tau$ as desired. Thus we have a left continuous action of G on X_τ. It is clear that $L \colon g \mapsto L_g$ defines an injection of G into $\mathrm{Aut}(X_\tau, \sigma)$.

To finish the proof we need the following claim:

Claim: *The map* $\varphi \colon \mathbb{Z}\times G \to \mathrm{Aut}(X_\tau, \sigma)$, $(n, g) \mapsto \sigma^n \circ L_g$ *is a group isomorphism.*

To show the injectivity of the map φ, let us assume there exists $n \in \mathbb{Z}$ and $g \in G$ such that $L_g = \sigma^n$. We can assume that $n \geq 0$, the other case is analogous. Then, for every $x \in X_\tau$ we have that $x_{kn+m} = g^{k-1}x_m$ for all $k \in \mathbb{Z}$ and $m \in \{0, \ldots, n-1\}$. But the sequence $(g^{k-1})_{k\in\mathbb{Z}}$ is periodic, so x is periodic. This is a contradiction since τ is aperiodic.

To show φ is surjective it is enough to prove that each automorphism $\phi \in \mathrm{Aut}(X_\tau, \sigma)$ can be written as a power of the shift composed with a map of kind L_g. Assume x, y is an asymptotic pair in X_τ. By Lemma 3.5, since $\phi(x)$ and $\phi(y)$ are also asymptotic points, there exist integers $p > 0$ and $n_0, n_1 \in \mathbb{Z}$ such that $z_1 = (x_n)_{n\geq n_0} = (y_n)_{n\geq n_0}$, $z_2 = (\phi(x)_n)_{n\geq n_1} = (\phi(y)_n)_{n\geq n_1}$ and both sequences are fixed by τ^p (observe that from Lemma 3.5 we can use the same power p for every couple of asymptotic pairs). Taking $\phi_1 = \sigma^{n_0-n_1} \circ \phi$ instead of ϕ we can assume that $n_1 = n_0$.

Set $g_1 = x_{n_0}$ and $g_2 = \phi_1(x)_{n_0}$. Since z_1 and z_2 are fixed by τ^p we have that $z_1 = \lim_{n\to+\infty} \tau^{pn}(g_1g_1 \ldots)$ and $z_2 = \lim_{n\to+\infty} \tau^{pn}(g_2g_2 \ldots)$. Now, by (2), for all $n \in \mathbb{N}$ we have that $L_{g_1(g_2^{-1})}(\tau^{pn}(g_2)) = \tau^{pn}(L_{g_1(g_2^{-1})}(g_2)) = \tau^{pn}(g_1)$. Then, $L_{g_1(g_2^{-1})}(z_2) = z_1$. This proves that x and $L_{g_1(g_2^{-1})} \circ \phi_1(x)$ are asymptotic points. Therefore, by Lemma 2.3, we get $\phi_1 = (L_{g_1(g_2^{-1})})^{-1} = L_{g_2(g_1^{-1})}$. So the original ϕ is a power of the shift composed with some translation L_g. This proves the claim and thus completes the proof of Theorem 3.6. $\square$

3.2.3. *Characterization of* $\mathrm{Aut}(X_\tau, \sigma)$ *for bijective constant-length substitutions subshifts.* Thanks to Lemma 3.5 we can offer a different proof of the following result due to B. Host and F. Parreau.

Theorem 3.7. [25] *Let $\tau \colon \mathcal{A} \to \mathcal{A}^*$ be a primitive bijective constant-length substitution. Then, each automorphism of the subshift (X_τ, σ) is the composition of some power of the shift with an automorphism $\phi \in \mathrm{Aut}(X_\tau, \sigma)$ of radius 0. Moreover, its local rule $\hat{\phi} \colon \mathcal{A} \to \mathcal{A}$ satisfies*

$$\tau \circ \hat{\phi} = \hat{\phi} \circ \tau. \tag{3}$$

Observe that a local map satisfying (3) defines an automorphism of the subshift. Hence, since there is a finite number of local rules of radius 0, we have an algorithm to determine the group of automorphisms for these kinds of subshifts.

Proof. First we notice that if X_τ is finite then it is reduced to a finite orbit. Hence an automorphism is a power of the shift map. From now on, we assume τ is aperiodic.

Let $x = (x_n)_{n\in\mathbb{Z}}, y = (y_n)_{n\in\mathbb{Z}} \in X_\tau$ be two asymptotic sequences and consider $\phi \in \mathrm{Aut}(X_\tau, \sigma)$. As discussed before, $\phi(x)$ and $\phi(y)$ are also asymptotic pairs.

By Lemma 3.5, there exist integers $p \geq 0$ and $n_0, n_1 \in \mathbb{Z}$ such that $(x_n)_{n\geq n_0} = (y_n)_{n\geq n_0}$, $(\phi(x)_n)_{n\geq n_1} = (\phi(y)_n)_{n\geq n_1}$ and all sequences are fixed by τ^p (observe that from Lemma 3.5 we can use the same power p for every couple of asymptotic pairs).

After shifting we can assume that $n_0 = 0$. Also, in what follows we will consider the automorphism $\phi' = \sigma^{n_1} \circ \phi$. Thus the sequence $(\phi'(x)_n)_{n\geq 0} = (\phi(x)_n)_{n\geq n_1}$ is fixed by τ^p.

Let $\mathbf{r}$ and $\hat{\phi}'$ denote the radius and the local map of ϕ' respectively. Taking a power of τ^p if needed, we can assume that the length ℓ of substitution τ^p is greater than $2\mathbf{r} + 1$. Consider different integers $m, n \geq 0$ such that $x_n = x_m$. We have $\phi'(x)_{m\ell+\mathbf{r}} = \hat{\phi}'(x_{m\ell} \ldots x_{m\ell+2\mathbf{r}}) = \hat{\phi}'(\tau^p(x_m)_{[0,2\mathbf{r}]}) = \hat{\phi}'(\tau^p(x_n)_{[0,2\mathbf{r}]}) = \phi'(x)_{n\ell+\mathbf{r}}$, where for a word $u = u_0 \ldots u_{\ell-1}$, $u_{[0,2\mathbf{r}]}$ stands for the prefix $u_0 \ldots u_{2\mathbf{r}}$. Since $\phi'(x)_{n\ell+\mathbf{r}}$ and $\phi'(x)_{m\ell+\mathbf{r}}$ are the $(r+1)^{\text{th}}$ letters of the words $\tau^p(\phi'(x)_n)$ and $\tau^p(\phi'(x)_m)$ respectively, and the substitution τ is bijective, we obtain that $\phi'(x)_n = \phi'(x)_m$. Then the map $\hat{\psi} \colon \mathcal{A} \to \mathcal{A}$ given by $\hat{\psi}(x_n) = \phi'(x)_n$ for all $n \geq 0$ is well defined.

Let $\psi \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be the shift commuting map with local map $\hat{\psi}$. By construction, for each word $w \in \mathcal{L}(X_\tau)$ we have that $\hat{\psi}(\tau^p(w)) = \tau^p(\hat{\psi}(w))$, then $\psi(X_\tau) \subseteq X_\tau$. Since τ is bijective we also get relation (3) for $\hat{\psi}$.

In the same way, using ϕ'^{-1} instead of ϕ', we obtain that ψ is invertible. By construction, we have that $\psi^{-1}\phi'(x)$ is asymptotic to x, so by Lemma 2.3, $\psi = \phi' = \sigma^{n_1} \circ \phi$. This completes the proof of Theorem 3.7.

$\square$

4. Examples illustrating Theorem 3.1

In this section we present two examples to illustrate Theorem 3.1 and the technique behind it. We start with a minimal subshift which shows non-super linear and superpolynomial complexity along subsequences. Since it is minimal, Part (2) of Theorem 3.1 is satisfied. The second example is a transitive non-minimal substitutive subshift with superlinear complexity. It does not satisfy all the hypotheses of Theorem 3.1 but the technique of the proof applies. In fact, it has a unique asymptotic component that we are able to characterize in order to prove that its automorphism group is isomorphic to $\mathbb{Z}$.

4.1. **A minimal subshift with** $\liminf_{n\to+\infty} p_X(n)/n < +\infty$ **and** $\limsup_{n\to+\infty} p_X(n)/n = +\infty$**.** Now we present an example of a minimal subshift (X, σ) induced by a point $x \in \{0,1\}^{\mathbb{N}}$ in the following way:

$$X = \{y \in \{0,1\}^{\mathbb{Z}};\ \text{all words appearing in } y \text{ also appear in } x\}.$$

The point x is chosen in order to have the following properties:
(i) x is uniformly recurrent: for any $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that every word of length N that appears in x contains all words of length n in x;
(ii) the complexity of (X, σ) is non-superlinear, that is, there exists a positive constant C such that for infinitely many values of $n \in \mathbb{N}$ we have $p_X(n) \le Cn$;
(iii) for a fixed subexponential function φ (meaning that $\lim_{n\to+\infty} \varphi(n)/\alpha^n = 0$ for every $\alpha > 1$), the complexity $p_X(n)$ is $\Omega_+(\varphi(n))$.

It is clear from (i) that (X, σ) is minimal. This property and (ii) says that (X, σ) satisfies the hypotheses of Theorem 3.1. Then its automorphism group is virtually $\mathbb{Z}$. Property (iii) illustrates that the hypothesis of Theorem 3.1 is compatible with high complexities along subsequences, in particular any polynomial complexity.

We will need the following lemmas whose simple proofs are left to the reader. Also, we will denote by $p_z(n)$ the number of words of length $n \in \mathbb{N}$ occurring in a onesided or twosided sequence z on the alphabet $\{0,1\}$.

Lemma 4.1. *Let $\xi : \{0,1\} \to \{0,1\}^*$ be a substitution of constant length L and $\tau : \{0,1\} \to \{0,1\}^*$ be a substitution such that all the words of length two in the alphabet $\{0,1\}$ appear as subwords of $\tau(0)$ and $\tau(1)$. Then for every $x \in \{0,1\}^{\mathbb{N}}$ having occurrences of all words of length two in the alphabet $\{0,1\}$, $y \in \{0,1\}^{\mathbb{N}}$ and $0 < l \le L$ we have $p_{\xi(x)}(l) = p_{\xi\circ\tau(y)}(l)$.*

In what follows $\rho : \{0,1\} \to \{0,1\}^*$ is the *Morse substitution*: $\rho(0) = 01$ and $\rho(1) = 10$. Notice that it is a bijective constant-length substitution and the words $\rho^3(0)$ and $\rho^3(1)$ contain all the words of length 2.

Lemma 4.2. *Let $\xi : \{0,1\} \to \{0,1\}^*$ be a substitution of constant length L and consider a point $x \in \{0,1\}^{\mathbb{N}}$. We have $p_{\xi\circ\rho^3(x)}(2L) \le 6L$.*

Fix a subexponential function φ. The sequence x is built recursively. We are going to construct two increasing sequences of integers $(\ell_i)_{i\ge1}$ and $(m_i)_{i\ge1}$ and a sequence of substitutions $(\tau_i : \{0,1\} \to \{0,1\}^*)_{i\ge1}$ such that:

1. $x = \lim_{i\to+\infty} \rho^3\tau_1 \ldots \rho^3\tau_i(0110^\infty)$, where $0^\infty = 00\ldots$;
2. $\ell_1 < m_1 < \ell_2 < m_2 < \ldots$;
3. $p_x(\ell_i) \le 3\ell_i$ for every integer $i \ge 1$;
4. $p_x(m_i) \ge \varphi(m_i)$ for every integer $i \ge 1$.

We separate the construction into different steps. Since there are many technical issues, we describe steps 1 and 2 before stating the recursive step in order to simplify understanding the construction.

Step 1: Set $\ell_1 = 2$ and $x^{(1)} = \rho^3(0110^\infty)$. Then, $p_{x^{(1)}}(\ell_1) = 4 \leq 3\ell_1$.

Let k_1 be a positive integer such that $2^{k_1} \geq \varphi(k_1|\rho^3|)$ (this choice is always possible since φ has subexponential growth). Let $\tau_1 : \{0,1\} \to \{0,1\}^*$ be a bijective substitution of constant length such that $\tau_1(0)$ and $\tau_1(1)$ start with 0 and the number of words of length k_1 in $\tau_1(0)$ and $\tau_1(1)$ is 2^{k_1}. The existence of such a substitution can be seen from the fact that De Bruijn graphs are Eulerian.

Now define $m_1 = k_1|\rho^3|$ and $y^{(1)} = \rho^3\tau_1(0110^\infty)$. Since $\tau_1(0)$ contains 2^{k_1} different subwords of length k_1 and ρ^3 is bijective, then $p_{y^{(1)}}(m_1) \geq 2^{k_1} \geq \varphi(m_1)$. Moreover, from Lemma 4.1 we have that $p_{x^{(1)}}(l) = p_{y^{(1)}}(l)$ for all $0 < l \leq |\rho^3|$. So, $p_{y^{(1)}}(\ell_1) \leq 3\ell_1$ and $p_{y^{(1)}}(m_1) \geq \varphi(m_1)$.

Step 2: Set $x^{(2)} = \rho^3\tau_1\rho^3(0110^\infty)$. By Lemma 4.2 we have that $p_{x^{(2)}}(2|\rho^3\tau_1|) \leq 6|\rho^3\tau_1|$. Setting $\ell_2 = 2|\rho^3\tau_1|$ one gets that $p_{x^{(2)}}(\ell_2) \leq 3\ell_2$.

Let $k_2 \geq k_1$ be an integer such that $2^{k_2} \geq \varphi(k_2|\rho^3\tau_1\rho^3|)$ and $\tau_2 : \{0,1\} \to \{0,1\}^*$ be a bijective substitution of constant length such that $\tau_2(0)$ and $\tau_2(1)$ start with 0 and the number of words of length k_2 in $\tau_2(0)$ and $\tau_2(1)$ is 2^{k_2}.

We set $m_2 = k_2|\rho^3\tau_1\rho^3|$ and $y^{(2)} = \rho^3\tau_1\rho^3\tau_2(0110^\infty)$. As in step 1, we deduce that $p_{y^{(2)}}(m_2) \geq 2^{k_2} \geq \varphi(m_2)$. Moreover, by using Lemma 4.1 in two different ways together with the results of step 1, we have that

$$\begin{aligned} p_{y^{(2)}}(l) &= p_{x^{(2)}}(l) \quad \text{for all } 0 < l \leq |\rho^3\tau_1\rho^3|, \\ p_{x^{(2)}}(l) &= p_{y^{(1)}}(l) \quad \text{for all } 0 < l \leq |\rho^3\tau_1|, \\ p_{y^{(1)}}(l) &= p_{x^{(1)}}(l) \quad \text{for all } 0 < l \leq |\rho^3|. \end{aligned}$$

Thus, if the length of τ_1 is taken large enough, we can deduce that $p_{y^{(2)}}(\ell_1) \leq 3\ell_1$, $p_{y^{(2)}}(m_1) \geq \varphi(m_1)$, $p_{y^{(2)}}(\ell_2) \leq 3\ell_2$ and $p_{y^{(2)}}(m_2) \geq \varphi(m_2)$.

General step: going from n to $n+1$. The general procedure follows what we did in step 2 almost identically. The situation after finishing step $n \geq 2$ is as follows:

(1) we have an increasing sequence of integers $k_1 \leq \ldots \leq k_n$ and for every $1 \leq i \leq n$, we have constructed a bijective substitution $\tau_i : \{0,1\} \to \{0,1\}^*$ of constant length such that $\tau_i(0)$ and $\tau_i(1)$ start with 0 and the number of words of length k_i in $\tau_i(0)$ and $\tau_i(1)$ is 2^{k_i};
(2) for every $1 \leq i \leq n$ we have that $2^{k_i} \geq \varphi(k_i|\rho^3\tau_1 \ldots \rho^3\tau_{i-1}\rho^3|)$;
(3) for every $1 \leq i \leq n$ we have defined points $x^{(i)} = \rho^3\tau_1 \ldots \rho^3\tau_{i-1}\rho^3(0110^\infty)$ and $y^{(i)} = \rho^3\tau_1 \ldots \rho^3\tau_i(0110^\infty)$;
(4) $p_{x^{(i)}}(l) = p_{y^{(i)}}(l)$ for all $0 < l \leq |\rho^3\tau_1 \ldots \rho^3\tau_{i-1}\rho^3|$ and $1 \leq i \leq n$;
(5) $p_{y^{(i)}}(l) = p_{x^{(i+1)}}(l)$ for all $0 < l \leq |\rho^3\tau_1 \ldots \rho^3\tau_i|$ and $1 \leq i \leq n-1$;
(6) we produced a sequence of integers $\ell_1 < m_1 < \ell_2 < \ldots < \ell_n < m_n$ such that for every $1 \leq i \leq n$: $\ell_i = 2|\rho^3\tau_1 \ldots \rho^3\tau_{i-1}|$, $m_i = k_i|\rho^3\tau_1 \ldots \rho^3\tau_{i-1}\rho^3|$, $p_{y^{(n)}}(\ell_i) \leq 3\ell_i$ and $p_{y^{(n)}}(m_i) \geq \varphi(m_i)$.

Repeating what we did in step 2, to pass to step $n+1$ first we set $x^{(n+1)} = \rho^3\tau_1 \ldots \rho^3\tau_n\rho^3(0110^\infty)$. Then from Lemma 4.2 we get that

$$p_{x^{(n+1)}}(2|\rho^3\tau_1 \ldots \rho^3\tau_n|) \leq 6|\rho^3\tau_1 \ldots \rho^3\tau_n|.$$

Putting $\ell_{n+1} = 2|\rho^3\tau_1 \ldots \rho^3\tau_n|$ one deduces that $p_{x^{(n+1)}}(\ell_{n+1}) \leq 3\ell_{n+1}$.

Let $k_{n+1} \geq k_n$ be an integer such that $2^{k_{n+1}} \geq \varphi(k_{n+1}|\rho^3\tau_1 \ldots \rho^3\tau_n\rho^3|)$ and $\tau_{n+1} : \{0,1\} \to \{0,1\}^*$ be a bijective substitution of constant length such that $\tau_{n+1}(0)$ and $\tau_{n+1}(1)$ start with 0 and the number of words of length k_{n+1} in $\tau_{n+1}(0)$ and $\tau_{n+1}(1)$ is $2^{k_{n+1}}$. We set $m_{n+1} = k_{n+1}|\rho^3\tau_1 \ldots \rho^3\tau_n\rho^3|$ and $y^{(n+1)} = \rho^3\tau_1 \ldots \rho^3\tau_n\rho^3\tau_{n+1}(0110^\infty)$. Then $p_{y^{(n+1)}}(m_{n+1}) \geq 2^{k_{n+1}} \geq \varphi(b_{n+1})$. Moreover, up to a modification in the length of τ_{n+1}, by Lemma 4.1 and the recurrence procedure, we have that

$$p_{x^{(i)}}(l) = p_{y^{(i)}}(l) \quad \text{for all } 0 < l \leq |\rho^3\tau_1 \ldots \rho^3\tau_{i-1}\rho^3| \text{ and } 1 \leq i \leq n+1;$$
$$p_{y^{(i)}}(l) = p_{x^{(i+1)}}(l) \quad \text{for all } 0 < l \leq |\rho^3\tau_1 \ldots \rho^3\tau_i| \text{ and } 1 \leq i \leq n.$$

Thus, an appropriate choice of parameters and the recurrence allow us to deduce that $p_{y^{(n+1)}}(\ell_i) \leq 3\ell_i$ for every $1 \leq i \leq n+1$ and $p_{y^{(n+1)}}(m_i) \geq \varphi(m_i)$ for every $1 \leq i \leq n+1$. We have proved that properties (1) to (6) hold at the end of step $n+1$. This finishes the recurrence procedure.

To conclude, observe that $(y^{(n)})_{n\geq 1}$ converges to the desired point x. Indeed, convergence follows from the fact that $\rho^3\tau_1 \ldots \rho^3\tau_n(0)$ is a prefix of $y^{(n)}$ and $y^{(n+1)}$ for all $n \geq 1$. In addition, since $\lim_{n\to+\infty} y^{(n)} = x$, then given $i \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $p_x(\ell_i) = p_{y^{(n)}}(\ell_i)$ and $p_x(m_i) = p_{y^{(n)}}(m_i)$. This proves that $p_x(\ell_i) \leq 3\ell_i$ and $p_x(m_i) \geq \varphi(m_i)$ for all $i \in \mathbb{N}$.

We are left to prove that x is a uniformly recurrent point. This follows from the fact that all words of a given length appearing in x are contained in $\rho^3\tau_1 \ldots \rho^3\tau_N(01)$ for some $N \in \mathbb{N}$.

4.2. **A substitutive subshift with superlinear complexity.** It is known that $p_{X_\tau}(n) = \Theta(\varphi(n))$ with $\varphi(n) \in \{n, n\log\log n, n\log n, n^2\}$ for any substitution $\tau : \mathcal{A} \to \mathcal{A}^*$ (see [36]). Clearly, if $\varphi(n) \neq n$, *i.e.*, the subshift has superlinear complexity, then the hypothesis on the complexity of Theorem 3.1 is not satisfied. However, the structure of the asymptotic components might be quite simple, allowing its automorphism group to be computed using the same technique developed to prove Theorem 3.1.

The next example is a transitive non-minimal substitutive subshift with $p_{X_\tau}(n) = \Theta(n\log\log n)$. Moreover, it has a unique asymptotic component. This, in addition to the particular form of the unique asymptotic component, will suffice to conclude that the automorphism group is isomorphic to $\mathbb{Z}$. We remark that it is also possible to construct examples of the same kind with $p_{X_\tau}(n) = \Theta(n^2)$ [36].

Let $\mathcal{A} = \{0,1\}$ and consider the substitution $\tau\colon \mathcal{A} \to \mathcal{A}^*$ defined by

$$\tau(0) = 010 \text{ and } \tau(1) = 11.$$

It is not difficult to check that (X_τ, σ) is a non-minimal transitive subshift. Moreover, $p_{X_\tau}(n) = \Theta(n\log\log n)$ (see Section 4.4 in [8] for details).

4.2.1. *Basic properties of τ and some notation.* We will need some specific notation. For a sequence $x \in \{0,1\}^{\mathbb{Z}}$ we write $x = x^-.x^+$ where $x^- = \ldots x_{-2}x_{-1}$ and $x^+ = x_0x_1\ldots$. For any $a \in \{0,1\}$ we set $a^{+\infty} = aaa\cdots$ and $a^{-\infty} = \cdots aaa$. Thus the sequence $\cdots aaa.aaa\cdots \in \{0,1\}^{\mathbb{Z}}$ can be written as $a^{-\infty}.a^{+\infty}$. We also write $\tau^{+\infty}(a) = \lim_{n\to+\infty}\tau^n(a^{+\infty})$ and $\tau^{-\infty}(a) = \lim_{n\to+\infty}\tau^n(a^{-\infty})$ when the limits exist.

We list some easy properties that the subshift (X_τ, σ) satisfies. Being simple, the proofs are left to the reader.

Recall that $w \in \mathcal{L}(X_\tau)$ if and only if there exists $a \in \{0,1\}$ and $N \in \mathbb{N}$ such that w is a subword of $\tau^N(a)$. Then, by definition of τ, any word $w \in \mathcal{L}(X_\tau)$ containing the symbol 0 must be a subword of some $\tau^N(0)$. From here we easily deduce that: (i) $00, 1010, 11011 \notin \mathcal{L}(X_\tau)$, (ii) 010 is always preceded and followed by 11 in a word of $\mathcal{L}(X_\tau)$ and (iii) two consecutive occurrences of 010 in $w \in \mathcal{L}(X_\tau)$ are separated by an even number of 1's.

These properties allow a recognizability property for τ to be proved.

Lemma 4.3. *For any $x \in X_\tau$ there exists a unique $x' \in X_\tau$ such that $\tau(x') = \sigma^\ell(x)$ for some $\ell \in \{0,1,2\}$.*

Proof. First we prove that any point $x \in X_\tau \setminus \{1^{-\infty}.1^{+\infty}\}$ can be decomposed in a unique way as a concatenation of words 010 and 11. By (i), every 0 in x appears in the word 101 and, by (i) and (ii), this word is contained in 1101011. We therefore have a unique way of determining 010. This property and (iii) enable 11 to be uniquely localised and the desired decomposition follows. Then there exists a unique point $x' \in \{0,1\}^{\mathbb{Z}}$ such that $\tau(x') = \sigma^\ell(x)$ for some $\ell \in \{0,1,2\}$. It is constructed by replacing the 010's by 0's and the 11's by 1's in the previous decomposition and then shifting to recenter on coordinate 0. It is clear that $x' \in X_\tau$.

To finish we just remark that $\tau(1^{-\infty}.1^{+\infty}) = 1^{-\infty}.1^{+\infty}$. □

4.2.2. *Automorphism group of τ.* We will prove that (X_τ, σ) has a unique asymptotic component. Then we will describe it explicitly in order to compute the automorphism group. For this, first we show that asymptotic points should end with $1^{+\infty}$.

Let $x, y \in X_\tau$ be two asymptotic points. After shifting we can assume that $x_{-1} = 0$, $y_{-1} = 1$ and $x^+ = y^+$. Since $00 \notin \mathcal{L}(X_\tau)$, then $x_0 = y_0 = 1$. Also, $x_1 = y_1 = 1$. If not, by (ii) $x_2x_3 = 11$ and thus $y_{-1}y_0y_1y_2y_3 = 11011$ which is not in $\mathcal{L}(X_\tau)$ by (i).

Now suppose that x^+ starts with $1^{2n+1}0$ for some $n \geq 1$. Then, property (i) implies that $x_{-3} \dots x_{2n+3} = 0101^{2n+1}010$ which contradicts (iii). Thus, x^+ either starts with $1^{2n}0$ for some integer $n \geq 1$ or it is equal to $1^{+\infty}$.

To finish we need to discard the first case. We prove this fact by contradiction, so assume x^+ (and thus y^+) starts with $1^{2n_1}0$ for some integer $n_1 \geq 1$.

By Lemma 4.3 together with a detailed analysis of the decomposition given by this lemma, there exist unique sequences $x^{(1)} = \cdots 0.1^{n_1}010 \cdots$ and $y^{(1)} = \cdots 1.1^{n_1}010 \cdots$ in X_τ such that $x = \tau(x^{(1)})$ and $y = \tau(y^{(1)})$ (the dot indicates the position just before coordinate 0). Clearly $x^{(1)}$ and $y^{(1)}$ are asymptotic. By the same argument developed earlier, if n_1 is odd then points $x^{(1)}, y^{(1)} \notin X_\tau$ which is a contradiction. If n_1 is even we can proceed as before to get another pair of asymptotic points $x^{(2)} = \cdots 0.1^{n_2}010 \cdots$ and $y^{(2)} = \cdots 1.1^{n_2}010 \cdots$, for some integer $n_2 \geq 1$. As before, either n_2 is odd, and we get a contradiction, or n_2 is even, and we can continue recursively producing asymptotic points $x^{(i)} = \cdots 0.1^{n_i}010 \cdots$ and $y^{(i)} = \cdots 1.1^{n_i}010 \cdots$ in X_τ for all $1 \leq i \leq m$, where $n_1 = 2n_2 = 2^2 n_3 = \ldots = 2^{m-1} n_m$ and $m \leq \log_2(n_1)$, until we get a contradiction as before or we stop with $n_m = 1$. In this last case $x^{(m)} = \cdots 0.1010 \cdots$ and $y^{(m)} = \cdots 1.1010 \cdots$. But (i) tells us that $01010 \notin \mathcal{L}(X_\tau)$, so we also get a contradiction.

We have proved that $x^+ = 1^{+\infty}$ and then (X_τ, σ) has a unique asymptotic component.

Furthermore, it can be proved using the same kind of arguments as above that $x = x^{-}.1^{+\infty} \in X_\tau \setminus \{1^{-\infty}.1^{+\infty}\}$ if and only if $x^{-} = \tau^{-\infty}(0)1^n$ for some integer $n \geq 1$. Hence, if $x, y \in X_\tau$ are asymptotic then they belong to

$$\{1^{-\infty}.1^{+\infty}, \sigma^n(\tau^{-\infty}(0).1^{+\infty}); n \in \mathbb{Z}\}.$$

We finish this section by proving that the automorphism group of (X_τ, σ) is isomorphic to $\mathbb{Z}$. Observe that (X_τ, σ) is a subshift of subquadratic growth, then the main result of [12] gives that $\mathrm{Aut}(X_\tau, \sigma)/\langle\sigma\rangle$ is a periodic group.

Lemma 4.4. $\mathrm{Aut}(X_\tau, \sigma) = \langle\sigma\rangle$.

Proof. Let $\bar{x} = \tau^{-\infty}(0).1^{+\infty}$. As discussed above, if $x, y \in X_\tau$ are asymptotic then they belong to $\{1^{-\infty}.1^{+\infty}, \sigma^n(\bar{x}) \; ; \; n \in \mathbb{Z}\}$.

Consider $\phi \in \mathrm{Aut}(X_\tau, \sigma)$. Since $1^{-\infty}.1^{+\infty}$ is the unique fixed point for σ in X_τ, then $\phi(1^{-\infty}.1^{+\infty}) = 1^{-\infty}.1^{+\infty}$. Also, since ϕ maps asymptotic points to asymptotic points, then $\bar{x}$ should be mapped to $\sigma^n(\bar{x})$ for some $n \in \mathbb{Z}$. But the orbit of $\bar{x}$ is dense in X_τ, hence $\phi = \sigma^n$. This finishes the proof. $\square$

5. The group of automorphisms of nilsystems and some associated subshifts

The purpose of this section is two fold. First we prove that the group of automorphisms of a proximal extension of an inverse limit of a minimal d-step nilsystem (and thus of a minimal d-step nilsystem) is d-step nilpotent. Then, we use this result to construct subshifts of arbitrary polynomial complexity whose group of automorphism is virtually $\mathbb{Z}$. Another important motivation of this section is to illustrate how the understanding of special topological factors of a subshift allows the computation of its automorphism group.

We will need some preliminary results to enable dealing with d-step nilsystems and their inverse limits.

5.1. **Dynamical cubes, regionally proximal relation of order d and nilfactors.** We recall the machinery and terminology introduced in [24] to study nilsystems in topological dynamics.

Let (X, T) be a topological dynamical system and consider an integer $d \geq 1$. Let $X^{[d]}$ denote the set X^{2^d}. We index the coordinates of a point in $X^{[d]}$ using the natural correspondence with points in $\{0,1\}^d$ and we usually denote these points in bold letters. For example, a point $\mathbf{x}$ in $X^{[2]}$ is written as $(\mathbf{x}_{00}, \mathbf{x}_{10}, \mathbf{x}_{01}, \mathbf{x}_{11})$. We denote by $x^{[d]}$ the special point $(x, x, \dots, x)$ (2^d times), where $x \in X$. The space of *cubes of order d*, denoted by $\mathbf{Q}^{[d]}(X)$, is the closure in $X^{[d]}$ of the set $\{(T^{\vec{n}\cdot\epsilon}x)_{\epsilon=(\epsilon_1,\dots,\epsilon_d)\in\{0,1\}^d} \in X^{[d]}; x \in X, \vec{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d\}$, where $\vec{n} \cdot \epsilon = \sum_{i=1}^d n_i \cdot \epsilon_i$. As an example, $\mathbf{Q}^{[3]}(X)$ is the closure in X^8 of the set of points

$$(x, T^{n_1}x, T^{n_2}x, T^{n_1+n_2}x, T^{n_3}x, T^{n_1+n_3}x, T^{n_2+n_3}x, T^{n_1+n_2+n_3}x),$$

where $x \in X$ and $(n_1, n_2, n_3) \in \mathbb{Z}^3$ (see Section 3 of [24] for further details). We say that points $x, y \in X$ are *regionally proximal of order d* if for any $\delta > 0$ there exist $x', y' \in X$ and $\vec{n} \in \mathbb{Z}^d$ such that $\mathrm{dist}(x, x') < \delta$, $\mathrm{dist}(y, y') < \delta$ and $\mathrm{dist}(T^{\vec{n}\cdot\epsilon}x', T^{\vec{n}\cdot\epsilon}y') < \delta$ for every $\epsilon \in \{0,1\}^d \setminus \{(0, \dots, 0)\}$. The set of regionally proximal pairs of order d of (X, T) is denoted by $\mathbf{RP}^{[d]}(X)$. In [24] for distal systems and then in [41] for general minimal systems, it was proved that $\mathbf{RP}^{[d]}(X)$ is an equivalence relation. Clearly $\mathbf{RP}^{[d+1]}(X) \subseteq \mathbf{RP}^{[d]}(X)$.

The following theorem relates the regionally proximal relation of order d with the space of cubes of order $d+1$.

Theorem 5.1 ([24],[41])**.** *Let (X,T) be a minimal topological dynamical system. For every integer $d \geq 1$, the following statements are equivalent:*

(1) $(x,y) \in \mathbf{RP}^{[d]}(X)$*;*
(2) $(x,y,\ldots,y) \in \mathbf{Q}^{[d+1]}(X)$
(3) $(x,x,\ldots,x,y) \in \mathbf{Q}^{[d+1]}(X)$*;*
(4) There exists a sequence $(\vec{n}_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}^{d+1}$ such that $T^{\vec{n}_i\cdot\epsilon}x$ converges to y as i goes to infinity for every $\epsilon \in \{0,1\}^{d+1} \setminus \{(0,\ldots,0)\}$.

From Theorem 5.1 it is clear that T preserves the equivalence classes of $\mathbf{RP}^{[d]}(X)$. Then, it induces a map T_d on the quotient space $Z_d(X) = X/\mathbf{RP}^{[d]}(X)$. Moreover, the natural projection $\pi_d : (X,T) \to (Z_d(X),T_d)$ defines a topological factor map. The following theorem describes the topological structure of $(Z_d(X),T_d)$.

Theorem 5.2 ([24])**.** *Let (X,T) be a minimal topological dynamical system. For each integer $d \geq 1$, $(Z_d(X),T_d)$ is topologically conjugate to an inverse limit of minimal d-step nilsystems. Moreover, it is the maximal factor of (X,T) with this property, that is, any other factor of (X,T) which is an inverse limit of minimal d-step nilsystems factorizes through $(Z_d(X),T_d)$ (in particular, it is a factor of $(Z_d(X),T_d)$).*

The system $(Z_d(X),T_d)$ is called the *maximal d-step nilfactor* of (X,T). We notice that the bonding maps in the inverse limit $(Z_d(X),T_d)$ are topological factors between minimal d-step nilsystems. These kind of inverse limits are also called *systems of order d* in [24].

Some direct consequences of Theorem 5.2 are: (1) $(Z_1(X),T_1)$ is the maximal equicontinuous factor of (X,T) (see [3]) and (2) condition $\mathbf{RP}^{[d]}(X) = \Delta_X$ (the diagonal of $X\times X$) characterizes topological conjugacy with the inverse limits of d-step nilsystems. It follows from $\mathbf{RP}^{[d+1]}(X) \subseteq \mathbf{RP}^{[d]}(X)$ and (2) that the maximal $d+1$-step nilfactor of an inverse limit of d-step nilsystems is the system itself.

Let $\pi\colon (X,T) \to (Y,S)$ be a factor map between minimal systems. For an integer $d \geq 1$, $\pi_d\colon (X,T) \to (Z_d(X),T_d)$ and $\widetilde{\pi}_d\colon (Y,S) \to (Z_d(Y),S_d)$ are the factor maps induced by the regionally proximal relations of order d in each system. Since $(Z_d(X),T_d)$ is the maximal d-step nilfactor of (X,T) and $(Z_d(Y),S_d)$ is an inverse limit of minimal d-step nilsystems which is a factor of (X,T), then by Theorem 5.2 there exists a unique factor map $\varphi_d\colon (Z_d(X),T_d) \to (Z_d(Y),S_d)$ such that $\varphi_d \circ \pi_d = \widetilde{\pi}_d \circ \pi$.

Lemma 5.3. *Let $\pi\colon (X,T) \to (Y,S)$ be an almost one-to-one extension between minimal systems. Then, for any integer $d \geq 1$ the canonical induced factor map $\varphi_d : (Z_d(X),T_d) \to (Z_d(Y),S_d)$ is a topological conjugacy (equivalently, maximal d-step nilfactors of (X,T) and (Y,S) coincide).*

Proof. Recall $\pi_d\colon X \to Z_d(X)$ and $\widetilde{\pi}_d\colon Y \to Z_d(Y)$ denote the quotient maps described above. First we prove that $\varphi_d : (Z_d(X),T_d) \to (Z_d(Y),S_d)$ is an almost one-to-one extension. This fact will imply the result.

Let $x \in X$ be such that $\pi^{-1}\{\pi(x)\} = \{x\}$. We claim that $\varphi_d^{-1}\{\varphi_d(\pi_d(x))\} = \{\pi_d(x)\}$. Let $x' \in X$ be such that $\varphi_d(\pi_d(x)) = \varphi_d(\pi_d(x'))$, so we get $\widetilde{\pi}_d(\pi(x)) = \widetilde{\pi}_d(\pi(x'))$ and thus $(\pi(x),\pi(x')) \in \mathbf{RP}^{[d]}(Y)$. By Theorem 5.1, there exists a sequence $(\vec{n}_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}^{d+1}$ such that $S^{\vec{n}_i\cdot\epsilon}\pi(x')$ converges to $\pi(x)$ for every $\epsilon\in\{0,1\}^{d+1}\setminus\{(0,\dots,0)\}$. Taking a subsequence we can assume that $T^{\vec{n}_i\cdot\epsilon}x'$ converges to x, the unique point in $\pi^{-1}\{\pi(x)\}$, for every $\epsilon\in\{0,1\}^{d+1}\setminus\{(0,\dots,0)\}$. Then, again by Theorem 5.1, we have that $(x,x')\in\mathbf{RP}^{[d]}(X)$. This implies that $\pi_d(x)=\pi_d(x')$ and then φ_d is an almost one-to-one extension.

Finally, by Lemma 2.1, φ_d is a proximal extension. But $(Z_d(X),T_d)$ is a distal system, so there are no proximal pairs. This proves that φ_d is a topological conjugacy. □

As an application of the previous results we obtain the following corollary.

Corollary 5.4. *Let $\pi\colon(X,T)\to(Y,S)$ be an almost one-to-one extension between minimal systems. If (Y,S) is an inverse limit of minimal d-step nilsystems then it is the maximal d-step nilfactor of (X,T).*

For instance, since any Sturmian subshift is an almost one-to-one extension of an irrational rotation on the circle (see [28]), this rotation is its maximal 1-step nilsystem or more classically its maximal equicontinuous factor. Similarly, Toeplitz subshifts are symbolic almost one-to-one extensions of odometers (see [16]), hence odometers are their maximal 1-step nilsystems.

5.2. **The group of automorphisms of a nilsystem.** The following is the main result of this section.

Theorem 5.5. *Let (X,T) be an inverse limit of minimal d-step nilsystems for some integer $d\geq 1$. Then its group of automorphisms $\mathrm{Aut}(X,T)$ is d-step nilpotent.*

To prove the theorem we need to introduce some further notation. Given a function $\phi\colon X\to X$ and an integer $d\geq 1$, for each $k\in\{1,\dots,d\}$ we define the *k-face transformation* $\phi^{[d],k}:X^{[d]}\to X^{[d]}$ by:

$$(\phi^{[d],k}(\mathbf{x}))_\epsilon=\begin{cases}\phi x_\epsilon & \text{if } \epsilon_k=1\\ x_\epsilon & \text{if } \epsilon_k=0\end{cases},$$

for every $\mathbf{x}\in X^{[d]}$ and $\epsilon\in\{0,1\}^d$. For example, for $d=2$ the face transformations associated to $\phi\colon X\to X$ are $\phi^{[2],1}=\mathrm{id}\times\phi\times\mathrm{id}\times\phi$ and $\phi^{[2],2}=\mathrm{id}\times\mathrm{id}\times\phi\times\phi$. We remark that $\phi^{[d+1],k}=\phi^{[d],k}\times\phi^{[d],k}$ for any $k\in\{1,\dots,d\}$.

When $\phi=T$, the transformations $T^{[d],1},T^{[d],2},\dots,T^{[d],d}$ are called the *face transformations* and $\mathcal{F}_d$ denotes the group spanned by them. Also, we denote by $\mathcal{G}_d$ the group spanned by $\mathcal{F}_d$ and the diagonal transformation $T\times\cdots\times T$ (2^d times). We remark that $\mathbf{Q}^{[d]}(X)$ is invariant under $\mathcal{G}_d$. This result can be extended to face transformations associated to an automorphism.

Lemma 5.6. *Let (X,T) be a minimal topological dynamical system. Consider $\phi\in$ Aut(X, T) and an integer $d\geq 1$. For every $k\in\{1,\dots,d\}$ the face transformation $\phi^{[d],k}$ leaves invariant $\mathbf{Q}^{[d]}(X)$.*

Proof. Fix $k\in\{1,\dots,d\}$. By minimality of (X,T), for all $x\in X$ there exists a sequence $(n_i)_{i\in\mathbb{N}}$ of integers such that $T^{n_i}x$ converges to $\phi(x)$. Then, by the definition of face transformations, $(T^{[d],k})^{n_i}(x^{[d]})$ converges to $\phi^{[d],k}(x^{[d]})$ (recall that $x^{[d]}=(x,\dots,x)$). This implies that $\phi^{[d],k}(x^{[d]})\in\mathbf{Q}^{[d]}(X)$.

Let $\mathbf{x}\in\mathbf{Q}^{[d]}(X)$. By definition, there exist $x\in X$ and a sequence $(g_i)_{i\in\mathbb{N}}$ in $\mathcal{G}_d$ such that $g_i(x^{[d]})$ converges to $\mathbf{x}$. Since ϕ commutes with T we have that $\phi^{[d],k}$

commutes with each element of $\mathcal{G}_d$ and thus $\phi^{[d],k}g_i(x^{[d]}) = g_i\phi^{[d],k}(x^{[d]}) \in \mathbf{Q}^{[d]}(X)$. Taking the limit we conclude that $\phi^{[d],k}(\mathbf{x}) \in \mathbf{Q}^{[d]}(X)$. This proves that $\phi^{[d],k}$ leaves invariant $\mathbf{Q}^{[d]}(X)$. $\square$

Proof of Theorem 5.5. Let $\phi_1,\ldots,\phi_{d+1} \in \mathrm{Aut}(X,T)$. Using Lemma 5.6 we have that $\phi_i^{[d+1],i}$ leaves invariant $\mathbf{Q}^{[d+1]}(X)$ for every $i=1,\ldots,d+1$. Therefore, their iterated commutator $[[[\ldots[\phi_1^{[d+1],1},\phi_2^{[d+1],2}],\ldots],\phi_d^{[d+1],d}],\phi_{d+1}^{[d+1],d+1}]$ also leaves invariant $\mathbf{Q}^{[d+1]}(X)$. Let $h=[[[\ldots[\phi_1,\phi_2],\ldots],\phi_d],\phi_{d+1}]$ be the iterated commutator of $\phi_1,\ldots,\phi_{d+1}$. We claim that

$$\mathrm{id}\times\mathrm{id}\cdots\times\mathrm{id}\times h = [[[\ldots[\phi_1^{[d+1],1},\phi_2^{[d+1],2}],\ldots],\phi_d^{[d+1],d}],\phi_{d+1}^{[d+1],d+1}].$$

We prove this equality by induction on d. To illustrate how to deduce this fact we start showing the case $d=2$. In this case,

$$\begin{aligned}
\phi_1^{[3],1} &= \mathrm{id}\times\phi_1\times\mathrm{id}\times\phi_1\times\mathrm{id}\times\phi_1\times\mathrm{id}\times\phi_1;\\
\phi_2^{[3],2} &= \mathrm{id}\times\mathrm{id}\times\phi_2\times\phi_2\times\mathrm{id}\times\mathrm{id}\times\phi_2\times\phi_2;\\
\phi_3^{[3],3} &= \mathrm{id}\times\mathrm{id}\times\mathrm{id}\times\mathrm{id}\times\phi_3\times\phi_3\times\phi_3\times\phi_3.
\end{aligned}$$

Then, $[\phi_1^{[3],1},\phi_2^{[3],2}] = \mathrm{id}\times\mathrm{id}\times\mathrm{id}\times[\phi_1,\phi_2]\times\mathrm{id}\times\mathrm{id}\times\mathrm{id}\times[\phi_1,\phi_2]$ and

$$[[\phi_1^{[3],1},\phi_2^{[3],2}],\phi_3^{[3],3}] = \mathrm{id}\times\mathrm{id}\times\mathrm{id}\times\mathrm{id}\times\mathrm{id}\times\mathrm{id}\times\mathrm{id}\times[[\phi_1,\phi_2],\phi_3]$$

as desired.

Now suppose the equality holds for $d-1$ and let $\phi_1,\ldots,\phi_d,\phi_{d+1}\in\mathrm{Aut}(X,T)$. Let

$$h' = [[[\ldots[\phi_1,\phi_2],\ldots],\phi_{d-1}],\phi_d] \text{ and } h=[[[\ldots[\phi_1,\phi_2],\ldots],\phi_d],\phi_{d+1}] = [h',\phi_{d+1}].$$

By the induction hypothesis we have that

$$[[[\ldots[\phi_1^{[d],1},\phi_2^{[d],2}],\ldots],\phi_{d-1}^{[d],d-1}],\phi_d^{[d],d}] = \mathrm{id}\times\mathrm{id}\cdots\times\mathrm{id}\times h'.$$

Since $\phi_k^{[d+1],k} = \phi_k^{[d],k}\times\phi_k^{[d],k}$ for every $k\in\{1,\ldots,d\}$ we have

$$\begin{aligned}
&[[[\ldots[\phi_1^{[d+1],1},\phi_2^{[d+1],2}],\ldots],\phi_{d-1}^{[d+1],d-1}],\phi_d^{[d+1],d}]\\
=&\mathrm{id}\times\mathrm{id}\cdots\times\mathrm{id}\times h'\times\mathrm{id}\times\mathrm{id}\cdots\times\mathrm{id}\times h'.
\end{aligned}$$

Thus,

$$[[[\ldots[\phi_1^{[d+1],1},\phi_2^{[d+1],2}],\ldots],\phi_d^{[d+1],d}],\phi_{d+1}^{[d+1],d+1}] = \mathrm{id}\times\cdots\times\mathrm{id}\times[h',\phi_{d+1}]$$

and the claim is proved.

Therefore, we have that $\mathrm{id}\times\mathrm{id}\cdots\times\mathrm{id}\times h(x^{[d]}) = (x,x,\ldots,x,h(x))\in\mathbf{Q}^{[d+1]}(X)$ for every $x\in X$. By Theorem 5.1, we have that $(h(x),x)\in\mathbf{RP}^{[d]}(X)$ for every $x\in X$. But the system is an inverse limit of d-step nilsystems, then by Theorem 5.2 we have that $\mathbf{RP}^{[d]}(X)=\Delta_X$ and thus $h(x)=x$. We conclude that h is the identity automorphism, which proves that $\mathrm{Aut}(X,T)$ is a d-step nilpotent group. $\square$

To extend Theorem 5.5 to proximal extensions of inverse limits of minimal d-step nilsystems we need to understand the action of automorphisms on the regionally proximal relation of order d. The following lemma states this fact.

Lemma 5.7. *Let (X, T) be a minimal topological dynamical system. For all $\phi \in \mathrm{Aut}(X, T)$ and all integer $d \geq 1$ we have that $(x, y) \in \mathbf{RP}^{[d]}(X)$ if and only if $(\phi(x), \phi(y)) \in \mathbf{RP}^{[d]}(X)$. Consequently, the projection $\pi_d \colon (X, T) \to (Z_d(X), T_d)$ is compatible with $\mathrm{Aut}(X, T)$.*

Proof. We only need to prove that $(\phi(x), \phi(y)) \in \mathbf{RP}^{[d]}(X)$ whenever $(x, y) \in \mathbf{RP}^{[d]}(X)$. By Theorem 5.1, there exists a sequence $(\vec{n}_i)_{i \in \mathbb{N}}$ in $\mathbb{Z}^{d+1}$ such that $T^{\vec{n}_i \cdot \epsilon} x$ converges to y as i goes to infinity for every $\epsilon \in \{0, 1\}^{d+1} \setminus \{(0, \ldots, 0)\}$. Since ϕ is continuous and commutes with T we also have that $T^{\vec{n}_i \cdot \epsilon} \phi(x)$ converges to $\phi(y)$ as i goes to infinity for every $\epsilon \in \{0, 1\}^{d+1} \setminus \{(0, \ldots, 0)\}$ too. Then Theorem 5.1 allows us to prove our claim. □

Finally we have the following corollary of Theorem 5.5.

Corollary 5.8. *Let (X, T) be a proximal extension of an inverse limit of minimal d-step nilsystems for $d \geq 1$. Then, there is an injection from $\mathrm{Aut}(X, T)$ to $\mathrm{Aut}(Z_d(X), T_d)$. In particular, $\mathrm{Aut}(X, T)$ is a d-step nilpotent group.*

Proof. By Theorem 5.2 and the hypothesis, $\pi_d : (X, T) \to (Z_d(X), T_d)$ is also a proximal extension. Then, by Lemma 5.7, this factor is compatible with $\mathrm{Aut}(X, T)$ and thus from Lemma 2.4 we get that $\widehat{\pi_d} \colon \mathrm{Aut}(X, T) \to \mathrm{Aut}(Z_d(X), T_d)$ is injective. This proves the result since by Theorem 5.5 $\mathrm{Aut}(Z_d, T_d)$ is a d-step nilpotent group. □

Since Sturmian and Toeplitz subshifts are almost one-to-one extensions of their maximal equicontinuous factors (maximal 1-step nilfactors), then they are also proximal extensions (Lemma 2.1). We obtain from the last corollary that their automorphism groups are abelian. More precisely, Lemmas 5.7 and 2.4 together imply that their automorphism groups are subgroups of the automorphism group of their maximal equicontinuous factors, which we characterize in Lemma 5.9 below. For integers $d > 1$, it is not difficult to construct minimal subshifts that are almost one-to-one extensions of d-step nilsystems by considering codings on well chosen partitions. An example of this kind will be developed in Section 5.3.

By a byproduct of Theorem 3.1 and Corollary 5.8, it is possible to obtain coarser properties of the finite group $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ for substitutive Toeplitz subshifts. This is achieved in [11] where explicit computations of automorphism groups of constant length substitutions are given.

We finish this section with a characterization of the group of automorphisms of an equicontinuous system (or 1-step nilsystems). This result is well known but for the sake of completeness we provide a short proof here (see [2]).

Lemma 5.9. *Let (X, T) be an equicontinuous minimal system. Then $\mathrm{Aut}(X, T)$ is the closure of the group $\langle T\rangle$ in the set of homeomorphisms of X for the topology of uniform convergence. Moreover, $\mathrm{Aut}(X, T)$ is homeomorphic to X.*

Proof. Denote by G the closure in the set of homeomorphisms of X of the group $\langle T\rangle$ for the topology of uniform convergence. Clearly $G \subseteq \mathrm{Aut}(X, T)$. Moreover, by Ascoli's Theorem, it is a compact abelian group.

Now we prove that $\mathrm{Aut}(X, T) \subseteq G$. Consider a point $x \in X$ and an automorphism $\phi \in \mathrm{Aut}(X, T)$. By minimality, there exists a sequence of integers $(n_i)_{i \in \mathbb{N}}$ such that $(T^{n_i} x)_{i \in \mathbb{N}}$ converges to $\phi(x)$. Taking a subsequence, we can assume that the sequence of maps $(T^{n_i})_{i \in \mathbb{N}}$ converges uniformly to a homeomorphism g in G.

Combining both of these facts we get that $\phi(x) = g(x)$ and thus $g^{-1} \circ \phi(x) = x$. Since $g^{-1} \circ \phi \in \mathrm{Aut}(X, T)$, by Lemma 2.2 we conclude that $\phi = g$ and consequently $\phi \in G$.

To finish, we remark that Lemma 2.2 ensures that the map from G to X sending $g \in G$ to $g(x) \in X$ is a homeomorphism onto its image $Y \subseteq X$. Since Y is T invariant and T is minimal we get that $Y = X$. This proves that $\mathrm{Aut}(X, T)$ is homeomorphic to X. □

5.3. **Coding an affine nilsystem.** We introduce a class of subshifts with polynomial complexity of arbitrarily high degree whose group of automorphisms is virtually $\mathbb{Z}$. We build these systems as extensions of minimal nilsystems.

5.3.1. *Coding topological dynamical systems.* We start by recalling some general results about symbolic codifications.

Let (X, T) be a minimal topological dynamical system and let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be a finite collection of subsets of X. We say that $\mathcal{U}$ is a *cover* of X if $\bigcup_{i=1}^m U_i = X$. Clearly, finite partitions of X are covers. The *refinement* of two covers $\mathcal{U} = \{U_1, \ldots, U_m\}$ and $\mathcal{V} = \{V_1, \ldots, V_p\}$ of X is given by $U \vee V = \{U_i \cap V_j; i = 1, \ldots m,\ j = 1, \ldots p\} \setminus \{\emptyset\}$. For $N \in \mathbb{N}$ we set $\mathcal{U}_N = \bigvee_{i=-N}^N T^{-i}\mathcal{U}$.

Let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be a cover of X and set $\mathcal{A} = \{1, \ldots, m\}$. We say that $\omega = (w_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ is a *$\mathcal{U}$-name of a point* $x \in X$ if $x \in \bigcap\limits_{i \in \mathbb{Z}} T^{-i} U_{w_i}$. Define

$$X_{\mathcal{U}} = \{\omega \in A^{\mathbb{Z}}; \bigcap_{i \in \mathbb{Z}} T^{-i} U_{w_i} \neq \emptyset\} \subseteq \mathcal{A}^{\mathbb{Z}}.$$

It is easy to prove that $X_{\mathcal{U}}$ is shift invariant and closed whenever the U_i's are closed. In addition, if $\overline{\mathcal{U}}$ denotes the collection $\{\overline{U}_1, \ldots, \overline{U}_m\}$ we have that $\overline{X_{\mathcal{U}}} \subseteq X_{\overline{\mathcal{U}}}$.

We say that $\mathcal{U}$ *separates points* if every $\omega \in X_{\overline{\mathcal{U}}}$ is a $\mathcal{U}$-name of exactly one point $x \in X$. If $\mathcal{U}$ separates points we can build a factor map $\pi_{\mathcal{U}} : (\overline{X_{\mathcal{U}}}, \sigma) \to (X, T)$, where $\pi_{\mathcal{U}}(\omega)$ is defined as the unique point in $\bigcap\limits_{i \in \mathbb{Z}} T^{-i} \overline{U_{w_i}}$.

Lemma 5.10. *Let (X, T) be a minimal topological dynamical system and let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be a finite partition of X that separates points. Suppose that for every $N \in \mathbb{N}$ every atom of $\mathcal{U}_N$ has nonempty interior, then $(\overline{X_{\mathcal{U}}}, \sigma)$ is a minimal subshift.*

Proof. Take points $\omega, \omega' \in \overline{X_{\mathcal{U}}}$ and an integer $N \in \mathbb{N}$. Set $x = \pi_{\mathcal{U}}(\omega)$ and $x' = \pi_{\mathcal{U}}(\omega')$. By definition we have that $\bigcap_{-N}^N T^{-i} U_{w_i} \neq \emptyset$. Therefore, by hypothesis, it has nonempty interior. Since (X, T) is minimal there exists $n \in \mathbb{Z}$ such that $T^n x' \in \mathrm{int}(\bigcap_{-N}^N T^{-i} U_{w_i})$. This implies that $w'_{n-N} \ldots w'_{n+N} = w_{-N} \ldots w_N$. We have proved that $(\overline{X_{\mathcal{U}}}, \sigma)$ is a minimal subshift. □

5.3.2. *Automorphism groups of some symbolic extensions of nilsytems.* Now we compute automorphism groups of a family of symbolic extensions of some nilsystems. This family was studied in details in [1]. Even though we will recall many of the results we need here, we will freely make use of many results from [1].

First we recall the construction of [1]. Let $A = (a_{i,j})_{i,j \in \mathbb{N}}$ be the infinite matrix where $a_{i,j} = \binom{j}{i}$. In [1, Section 4], it was proved that A^i is well defined for all $i \in \mathbb{N}$ and

$$A^i = \begin{pmatrix} 1 & i & i^2 & i^3 & i^4 & \cdots \\ & 1 & 2i & 3i^2 & 4i^3 & \cdots \\ & & 1 & 3i & 6i^2 & \cdots \\ & & & 1 & 4i & \cdots \\ & & & & 1 & \cdots \\ & & & & \cdots & \cdots \end{pmatrix}.$$

Let $\alpha \in [0,1)$ be an irrational number. For any integer $d \geq 1$ define A_d to be the restriction of A to the upper left corner of dimension d. Notice that $A_d^i = (A^i)_d$ for every $i \in \mathbb{N}$.

Let $T_d\colon \mathbb{T}^d \to \mathbb{T}^d$ be the map that sends $(x_0, \ldots, x_{d-1}) \in \mathbb{T}^d$ to the first d coordinates of $A_{d+1}(x_0, \ldots, x_{d-1}, \alpha)^t$, where in $\mathbb{T}^d$ all operations are modulo one. For example, T_2 is the map $(x_0, x_1) \mapsto (x_0 + x_1 + \alpha, x_1 + 2\alpha)$ and T_3 is the map $(x_0, x_1, x_2) \mapsto (x_0 + x_1 + x_2 + \alpha, x_1 + 2x_2 + 3\alpha, x_2 + 3\alpha)$. So for any $x \in \mathbb{T}$ we can write $T_d(x) = A_d x + \vec{\alpha}$. This is the classical presentation of an affine nilsystem (see Section 2.6).

Next, fix an integer $d \geq 1$. For every $i, n \in \mathbb{Z}$ let $H_{i,n}$ be the affine hyperplane in $\mathbb{R}^d$ given by the equation $\sum_{k=0}^{d-1} i^k x_k + i^d \alpha = n$. It can be proved that $T_d^i H_{i,n} = H_{0,n}$. Also, for each $i \in \mathbb{Z}$ the canonical projections of the hyperplanes $(H_{i,n})_{n \in \mathbb{Z}}$ to $\mathbb{T}^d$ are the same. Call this projection $\widehat{H}_i$ and refer to it as a projected hyperplane. We remark that $\widehat{H}_0 = \{(0, x_1, \ldots, x_{d-1}); (x_1, \ldots, x_{d-1}) \in \mathbb{T}^{d-1}\}$ and that the intersection of more than $d+1$ different projected hyperplanes in $(\widehat{H}_i)_{i \in \mathbb{Z}}$ is empty. We refer to Section 5 of [1] for further details.

For each $i \in \mathbb{Z}$, since the projected hyperplane $\widehat{H}_i$ is defined from equations with integer coefficients, it naturally induces a finite partition $\mathcal{C}_i$ of $\mathbb{T}^d$ whose boundaries are defined by $\widehat{H}_i$ (the ambiguities in the choice of the boundaries are solved arbitrarily).

For each integer $n \geq 1$ we define the partition $\mathcal{V}_d = \mathcal{C}_0 \bigvee \ldots \bigvee \mathcal{C}_d$, then its atoms are the nonempty intersections of the sets induced by $\widehat{H}_0, \ldots, \widehat{H}_d$. It is proved in Lemma 9 of [1] that those atoms have convex interiors. Also, it is shown in Lemma 5 and 7 in [1] that no point in $\widehat{H}_0 \cup \ldots \cup \widehat{H}_d$ belongs to the interior of an atom. Thanks to the equality $T_d^i H_{i,n} = H_{0,n}$, we remark that the partition $T_d^{-i}\mathcal{V}_d$ is the one induced by $\widehat{H}_i, \ldots, \widehat{H}_{i+d}$ and its atoms also have a convex interior.

We claim that partition $\mathcal{V}_d$ separates points. Let x and y be different points in $\mathbb{T}^d$. Since every point in $\mathbb{T}^d$ belongs to at most d projected hyperplanes $(\widehat{H}_i)_{i \in \mathbb{Z}}$, we have that $x, y \notin \widehat{H}_i$ for all large enough $i \in \mathbb{N}$. In particular $x, y \notin \widehat{H}_i \cup \ldots \cup \widehat{H}_{i+d}$ for all large enough $i \in \mathbb{N}$, which implies that they belong to the interior of atoms of the partition $T_d^{-i}\mathcal{V}_d$. Choose $\tilde{x} = (\tilde{x}_0, \ldots, \tilde{x}_{d-1}), \tilde{y} = (\tilde{y}_0, \ldots, \tilde{y}_{d-1}) \in \mathbb{R}^d$ with $x = \tilde{x} \mod \mathbb{Z}^d$ and $y = \tilde{y} \mod \mathbb{Z}^d$. The difference in $\mathbb{R}$ between $\sum_{k=0}^{d-1} i^k \tilde{x}_k + i^d \alpha$ and $\sum_{k=0}^{d-1} i^k \tilde{y}_k + i^d \alpha$ behaves like $i^{\overline{k}}(\tilde{x}_{\overline{k}} - \tilde{y}_{\overline{k}})$, where $\overline{k} = \max\{0 \leq k < d; \tilde{x}_k \neq \tilde{y}_k\}$. Then it grows to infinity with $i \in \mathbb{N}$. Thus for a large $i \in \mathbb{N}$ we can find a point $\tilde{z} = (\tilde{z}_0, \ldots, \tilde{z}_{d-1})$ in the segment joining $\tilde{x}$ and $\tilde{y}$ such that $\sum_{k=0}^{d-1} i^k \tilde{z}_k + i^d \alpha \in \mathbb{Z}$, meaning that $\tilde{z} \mod \mathbb{Z}^d \in \widehat{H}_i$. Because no point in $\widehat{H}_i \cup \ldots \cup \widehat{H}_{i+d}$ belongs to the interior of an atom of the partition $T_d^{-i}\mathcal{V}_d$, we have that x and y are in different atoms of partition $T_d^{-i}\mathcal{V}_d$. Therefore, if i is large enough and $N \geq i$ these points also lie in different atoms of $\bigvee_{i=-N}^{N} T_d^{-i}\mathcal{V}_d$, which shows that $\mathcal{V}_d$ separates points.

We recall that $(\overline{X_{\mathcal{V}_d}},\sigma)$ is the subshift induced by $\mathcal{V}_d$. By Lemma 5.10, since $\mathcal{V}_d$ separates points and $(\mathcal{V}_d)_N$ has nonempty interior for all $N\in\mathbb{N}$, one has that $(\overline{X_{\mathcal{V}_d}},\sigma)$ is a minimal subshift and there is a factor map $\pi_d\colon(\overline{X_{\mathcal{V}_d}},\sigma)\to(\mathbb{T}^d,T_d)$. Moreover, by construction, the set of points in $\mathbb{T}^d$ with more than one preimage for π_d consists of points which fall in $F_d=\widehat{H}_0\cup\widehat{H}_1\cup\ldots\cup\widehat{H}_{d-1}$ under some power of T_d, *i.e.*, $\bigcup_{j\in\mathbb{Z}}T_d^{-j}F_d=\bigcup_{j\in\mathbb{Z}}T_d^{-j}\widehat{H}_0$. This set has zero Lebesgue measure and thus there exist points with exactly one preimage for π_d. In particular, $(\overline{X_{\mathcal{V}_d}},\sigma)$ is an almost one-to-one extension of $(\mathbb{T}^d,T_d)$. By Corollary 5.4 we get,

Lemma 5.11. *The maximal d-step nilfactor of $(\overline{X_{\mathcal{V}_d}},\sigma)$ is the affine nilsystem $(\mathbb{T}^d,T_d)$. Then $\mathbb{T}^d$ can be identified with the quotient $\overline{X_{\mathcal{V}_d}}/\mathbf{RP}^{[d]}(\overline{X_{\mathcal{V}_d}})$.*

We are ready to compute the group of automorphisms for these examples.

Theorem 5.12. *The group $\mathrm{Aut}(\overline{X_{\mathcal{V}_d}},\sigma)$ is virtually $\mathbb{Z}$.*

Proof. Let $\phi\in\mathrm{Aut}(\overline{X_{\mathcal{V}_d}},\sigma)$ and set $W=\{\omega=(w_i)_{i\in\mathbb{Z}}\in\overline{X_{\mathcal{V}_d}};\#\pi_d^{-1}\{\pi_d(\omega)\}\geq 2\}$. Then $\pi_d(W)$ is the set of points in $\mathbb{T}^d$ with more than one preimage for π_d. As discussed above $\pi_d(W)=\bigcup_{j\in\mathbb{Z}}T_d^{-j}F_d=\bigcup_{j\in\mathbb{Z}}T_d^{-j}\widehat{H}_0$.

By Lemma 5.7, ϕ preserves $\mathbf{RP}^{[d]}(\overline{X_{\mathcal{V}_d}})$. Since π_d is induced by this relation, then W is invariant under ϕ. We also get that $\widehat{\pi_d}(\phi)\in\mathrm{Aut}(\mathbb{T}^d,T_d)$ leaves invariant $\pi_d(W)=\bigcup_{j\in\mathbb{Z}}T^{-j}\widehat{H}_0$.

The affine nilsystem $(\mathbb{T}^d,T_d)$ is ergodic by construction (α is irrational) and the associated matrix has 1 as unique eigenvalue. Theorem 2 and Corollary 1 in [42] imply that $\widehat{\pi_d}(\phi)\in\mathrm{Aut}(\mathbb{T}^d,T_d)$ is an affine transformation, *i.e.*, it has the form $Bx+\vec{\beta}$, where B is an invertible integer matrix and $\vec{\beta}\in\mathbb{T}^d$ (recall that operations are taken modulo one). Hence, the image of the projected hyperplane $\widehat{H}_0$ by the affine map $\widehat{\pi_d}(\phi)$ is still a projected hyperplane. But the set $\pi_d(W)$ is invariant for $\widehat{\pi_d}(\phi)$ and so we get that the projected hyperplane $\widehat{\pi_d}(\phi)\widehat{H}_0$ is included in the union of the projected hyperplanes $(T_d^{-j}\widehat{H}_0)_{j\in\mathbb{Z}}$. By Baire's theorem and since $\widehat{\pi_d}(\phi)\widehat{H}_0$ and $T_d^{-j}\widehat{H}_0$ for $j\in\mathbb{Z}$ share the same dimension, we obtain that $\widehat{\pi_d}(\phi)\widehat{H}_0$ is equal to some $T_d^{-j}\widehat{H}_0$. Finally, the automorphism $T_d^j\widehat{\pi_d}(\phi)\in\mathrm{Aut}(\mathbb{T}^d,T_d)$ leaves $\widehat{H}_0$ invariant.

We are left to study the automorphisms of $(\mathbb{T}^d,T_d)$ which leave $\widehat{H}_0$ invariant. Let $\varphi\in\mathrm{Aut}(\mathbb{T}^d,T_d)$ be such an automorphism. As discussed before, by [42] φ has the form $\varphi(x)=Bx+\vec{\beta}\mod\mathbb{Z}^d$, where $B=(B_{i,j})_{1\leq i,j\leq d}$ is an invertible matrix with integer entries and $\vec{\beta}=(\beta_0,\ldots,\beta_{d-1})^t\in\mathbb{R}^d$. Since φ commutes with T_d we have for every $x\in\mathbb{T}^d$ that $A_dBx+A\vec{\beta}+\vec{\alpha}=BA_dx+B\vec{\alpha}+\vec{\beta}\mod\mathbb{Z}^d$. This allows us to conclude that B commutes with A_d as real matrices and that $(B-Id)\vec{\alpha}=(A_d-Id)\vec{\beta}\mod\mathbb{Z}^d$.

The map φ leaves $\widehat{H}_0$ invariant, meaning that $\varphi(0,x_1,\ldots,x_{d-1})\in\widehat{H}_0$ for any $(x_1,\ldots,x_{d-1})\in\mathbb{T}^{d-1}$. This allows us to deduce that coefficients $B_{1,2}=\ldots=B_{1,d}=0=\beta_0$. Also, since $A_d^iB=BA_d^i$ for every $i\in\mathbb{N}$, by looking at the first rows of these matrices, we deduce that for all $1\leq j\leq d$ and $i\in\mathbb{N}$

$$\sum_{k=1,k\neq j}^{d}(B_{j,k})i^{k-1}+(B_{j,j}-B_{1,1})i^{j-1}=0.$$

But the vectors $(1, i, i^2, \ldots, i^{d-1})$ are linearly independent for different values of $i \in \mathbb{N}$, so $B = B_{1,1} I_d$. Therefore, $(A_d - Id)\vec{\beta} = (B - Id)\vec{\alpha} = (B_{1,1} - 1)\vec{\alpha} \mod \mathbb{Z}^d$. Since A_d is upper triangular with ones in the diagonal, we deduce that $(B_{1,1} - 1)\alpha \in \mathbb{Q}$ and thus $B_{1,1} = 1$. We have proved that $B = Id$ and then φ is the rotation by $\vec{\beta} = (0, \beta_1, \ldots, \beta_{d-1})^t$ and $(A_d - Id)\vec{\beta} \in \mathbb{Z}^d$. This last property can be written as

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & d \\ & & & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \beta_1 \\ \vdots \\ \beta_{d-1} \end{pmatrix} \in \mathbb{Z}^d.$$

This implies that $d\beta_{d-1} \in \mathbb{Z}$ which is possible for finitely many $\beta_{d-1} \mod \mathbb{Z} \in \mathbb{T}$. Inductively, we deduce that there are finitely many rational solutions $\vec{\beta} = (0, \beta_1, \ldots, \beta_{d-1})^t \mod \mathbb{Z}^d$ in $\mathbb{T}^d$. This means that the group of automorphisms that leaves $\widehat{H}_0$ invariant is a finite group of rational rotations. Therefore, $\widehat{\pi}(\mathrm{Aut}(\overline{X_{\mathcal{V}_d}}, \sigma))$ is spanned by T_d and a finite set. We recall that the factor map $\pi_d \colon (\overline{X_{\mathcal{V}_d}}, \sigma) \to (\mathbb{T}^d, T_d)$ is almost one-to-one, so by Lemma 2.4 $\widehat{\pi} \colon \mathrm{Aut}(\overline{X_{\mathcal{V}_d}}, \sigma) \to \mathrm{Aut}(\mathbb{T}, T_d)$ is an injection. We conclude that $\mathrm{Aut}(\overline{X_{\mathcal{V}_d}}, \sigma)$ is also spanned by σ and a finite set. The result follows. $\square$

To finish this section, we mention that the main theorem in [1] (see page 2) asserts that the complexity function of $(\overline{X_{\mathcal{V}_d}}, \sigma)$ is given by

$$p(n) = \frac{1}{V(0, 1, \ldots, d-1)} \sum_{0 \leq k_1 < k_2 < \ldots < k_d \leq n+d-1} V(k_1, k_2, \ldots, k_d),$$

where $V(k_1, k_2, \ldots, k_d) = \prod_{1 \leq i < j \leq d} (k_j - k_i)$ is a Vandermonde determinant. We note that varying $d \in \mathbb{N}$ results in polynomial complexities of arbitrary degree.

Thus we have proved that particular symbolic codings of affine nilsystems produce subshifts of polynomial complexity of arbitrary degree whose automorphism groups are virtually $\mathbb{Z}$. A natural question is whether or not this is still true for symbolic extensions of general nilsystems induced by coding on well chosen partitions.

6. Final comments and open questions

In this section, we comment on some natural questions that follow from our own work together with recent work on the topic of this article.

6.1. **Realization of automorphism groups.** By the Curtis-Hedlund-Lyndon theorem the collection of automorphisms of a subshift is countable. So it is natural to ask whether any countable group can be realized as an automorphism group of a subshift. This is a complicated question and, as was mentioned in the introduction, many partial answers have been given in the case of positive entropy subshifts. In the context of this article the question we want to address is:

Question 6.1. *Given a countable group G (not necessarily finitely generated), does there exist a minimal subshift with subexponential complexity (X, σ) such that* $\mathrm{Aut}(X, \sigma)$ *is isomorphic to G ?*

We are far from solving this question. As a first step we provide subshifts whose automorphism groups are isomorphic to $\mathbb{Z}^d$ for some integer $d \geq 1$.

Proposition 6.2. *For every integer $d \geq 1$, there exists a minimal subshift (X, σ) with complexity satisfying $p_X(n) = \Theta(n^d)$ such that $\mathrm{Aut}(X, \sigma)$ is isomorphic to $\mathbb{Z}^d$.*

Thus, we remark that the statement of Theorem 3.1 is no longer valid for arbitrary polynomial complexity.

Proof. Let $\alpha_1, \dots, \alpha_d \in \mathbb{R} \setminus \mathbb{Q}$ be rationally independent numbers. For every $i \in \{1, \dots, d\}$ let $([0,1), R_{\alpha_i})$ be the rotation modulo one by angle α_i on the unit interval and let (X_i, σ_i) be the Sturmian subshift associated to it (we write σ_i to distinguish the shift in each of the systems). We recall that each Sturmian subshift is obtained from the coding of the orbits of points for R_{α_i} with respect to the partition $\{[0, 1-\alpha_i), [1-\alpha_i, 1)\}$. Since each α_i is an irrational number, there exists an almost one-to-one extension $\pi_i : (X_i, \sigma_i) \to ([0,1), R_{\alpha_i})$ and π_i is injective except for the orbit of $1 - \alpha_i$, where every point has exactly two preimages. This last fact implies that $([0,1), R_{\alpha_i})$ is its maximal equicontinuous factor and that, in (X_i, σ_i), the proximal relation is an equivalence relation.

Set $X = X_1 \times X_2 \cdots \times X_d$, $\sigma = \sigma_1 \times \sigma_2 \cdots \times \sigma_d$ and $R_{\vec{\alpha}} = R_{\alpha_1} \times \cdots \times R_{\alpha_d}$. Since the angles $\alpha_1, \dots, \alpha_d$ are rationally independent, the product system $([0,1)^d, R_{\vec{\alpha}})$ is minimal. This implies, by Theorem 7 in [3, Chapter 11], that (X, σ) is transitive. However, in each subshift (X_i, σ_i), the proximal relation is an equivalence relation and so by Theorem 9 in [3, Chapter 11] we get that (X, σ) is a minimal subshift. In addition, the product system $([0,1)^d, R_{\vec{\alpha}})$ is its maximal equicontinuous factor. The factor map $\pi = \pi_1 \times \cdots \times \pi_d : (X, \sigma) \to ([0,1)^d, R_{\vec{\alpha}})$ is almost one-to-one and each point in $[0,1)^d$ has at most 2^d preimages for π.

Recall that for each $i \in \{1, \dots, d\}$ the group $\mathrm{Aut}(X_i, \sigma_i)$ is generated by σ_i (see the comment below Theorem 3.1 or [33]). It is clear that the map $(\phi_1, \dots, \phi_d) \in \mathrm{Aut}(X_1, \sigma_1) \times \cdots \times \mathrm{Aut}(X_d, \sigma_d) \mapsto \phi_1 \times \cdots \times \phi_d \in \mathrm{Aut}(X, \sigma)$ is an embedding of the group $\mathbb{Z}^d$. We claim that this embedding is actually an isomorphism.

By Lemma 5.7 the factor $\pi : (X, \sigma) \to ([0,1)^d, R_{\vec{\alpha}})$ is compatible with $\mathrm{Aut}(X, \sigma)$, so for every $\phi \in \mathrm{Aut}(X, \sigma)$ the automorphism $\widehat{\pi}(\phi) \in \mathrm{Aut}([0,1)^d, R_{\vec{\alpha}})$ is well defined. Moreover, it preserves the set of points in $[0,1)^d$ that have a maximum number of preimages for π: namely the set $\mathrm{Orb}_{R_{\alpha_1}}(1-\alpha_1) \times \cdots \times \mathrm{Orb}_{R_{\alpha_d}}(1-\alpha_d)$. Hence there exist $n_1, \dots, n_d \in \mathbb{Z}$ such that $\widehat{\pi}(\phi)(1-\alpha_1, \dots, 1-\alpha_d) = (R_{\alpha_1}^{n_1}(1-\alpha_1), \dots, R_{\alpha_d}^{n_d}(1-\alpha_d))$. This implies that $\widehat{\pi}(\phi) = R_{\alpha_1}^{n_1} \times \cdots \times R_{\alpha_d}^{n_d} = \widehat{\pi}(\sigma_1^{n_1} \times \cdots \times \sigma_d^{n_d})$. But, by Lemma 2.4, the map $\widehat{\pi} : \mathrm{Aut}(X, \sigma) \to \mathrm{Aut}([0,1)^d, R_{\vec{\alpha}})$ is injective, thus $\phi = \sigma_1^{n_1} \times \cdots \times \sigma_d^{n_d}$. This proves our claim and $\mathrm{Aut}(X, \sigma)$ is isomorphic to $\mathbb{Z}^d$.

To finish we compute the complexity function of (X, σ). It is well known that $p_{X_i}(n) = n + 1$ for every $i \in \{1, \dots, d\}$. Thus, the complexity function of (X, σ) is $p_X(n) = (n+1)^d$. □

Another direction to explore in order to answer Question 6.1 is to analyse specific families of subshifts. In particular, Toeplitz subshifts have proved to be a very good source of inspiration for constructively solving some open problems in different branches of topological dynamics. As was stated in Corollary 5.8, the automorphism group of a Toeplitz subshift is a subgroup of its maximal equicontinuous factor which is an odometer. These systems are well understood so we may expect to explicitly describe this subgroup.

6.2. **Relation between dynamical properties and automorphisms.**

6.2.1. *Complexity versus group of automorphisms.* The results of [12, 13] and of this paper show the relation between the complexity and the growth rate of the automorphism groups of subshifts, especially for subquadratic complexities. Is it possible to extend these results to higher complexities? Inspired by the main theorem of this paper and examples in Sections 5.3 and 6.1, we ask

Question 6.3. *Let (X, σ) be a minimal or transitive subshift such that*

$$d = \inf\{\delta \in \mathbb{N}; 0 < \liminf_{n\to+\infty} p_X(n)/n^\delta < +\infty\} > 0.$$

Is the automorphism group of such a subshift virtually $\mathbb{Z}^k$ for some $k \leq d$?

6.2.2. *Recurrence and growth rate of automorphism groups.* Is it possible to give an extension of Theorem 3.1 to a class of subshifts with higher complexity? To address this question we propose exploring an alternative notion to word complexity. For a subshift (X, σ), we define the *visiting time* map by:

$$R''_X(n) := \inf\{|w|;\ w \in \mathcal{L}(X) \text{ contains each word of } X \text{ of length } n\},$$

where $n \in \mathbb{N}$. To the best of our knowledge, this concept was first introduced in [9] but without any name. We have borrowed the notation from this reference and we bestow a name on it. Clearly, this map is finite for every $n \in \mathbb{N}$ if and only if the subshift is transitive. In this case, it satisfies $R''_X(n) \geq p_X(n) + n - 1$. Moreover, for a minimal subshift $R''_X(n)$ is less than the so-called *recurrence function* $R_X(n)$ as defined in [21]. We will not comment any further on this latter function.

Some computations are known for particular subshifts. For instance, linearly recurrent subshifts, which include primitive substitutive subshifts, satisfy $R''_X(n) = O(n)$. Also, it is proved in [9] that $R''_X(n) \leq 2n$ for every Sturmian subshift.

For higher polynomial degree we obtain the following result.

Proposition 6.4. *Let (X, σ) be a subshift such that $R''_X(n) = O(n^d)$ for some integer $d \geq 1$. Then, each finitely generated subgroup of* $\mathrm{Aut}(X, \sigma)$ *is a virtually nilpotent group whose step only depends on d.*

Proof. Let $\mathcal{S} = \langle \phi_1, \ldots, \phi_\ell \rangle \subseteq \mathrm{Aut}(X, \sigma)$ be a finitely generated group. Let $\mathbf{r}$ be an upper bound of the radii of the local maps associated to all generators ϕ_i of $\mathcal{S}$ and their inverses. For $n \in \mathbb{N}$, consider

$$B_n(\mathcal{S}) = \{\phi_{i_1}^{s_1} \cdots \phi_{i_m}^{s_m}; 1 \leq m \leq n,\ i_1, \ldots, i_m \in \{1, \ldots, \ell\},\ s_1, \ldots, s_m \in \{1, -1\}\}\ .$$

Let w be a word of length $R''_X(2n\mathbf{r}+1)$ containing every word of length $(2n\mathbf{r}+1)$ of X. If $\phi, \phi' \in B_n(\mathcal{S})$ are different, then $\phi(w) \neq \phi'(w)$. Further, there is an injection from $B_n(\mathcal{S})$ into the set of words of length $R''_X(2n\mathbf{r}+1) - 2\mathbf{r}$ (the injection is just the evaluation of ϕ on w). This implies that $\sharp B_n(\mathcal{S}) \leq p_X(R''_X(2n\mathbf{r}+1) - 2\mathbf{r})$. We deduce from the hypothesis on R''_X that $\sharp B_n(\mathcal{S}) \leq n^{d^2+1}$ for all large enough integers $n \in \mathbb{N}$. The proof is completed by applying the quantitative result of Y. Shalom and T. Tao in [40] generalizing Gromov's classical result on the growth rate of groups. □

Notice that Theorem 1.8 of [40] provides and explicit value for the step of the nilpotent group appearing in the proposition. It is clear that a subshift of polynomial visiting time (meaning that $R''_X(n) = O(n^d)$ for some integer $d \geq 1$) has polynomial complexity. It is straightforward to show that the converse is false by constructing explicit counterexamples.

6.3. **Extension to higher dimensional subshifts.** A natural generalization of the topic developed in this article is to study the automorphism groups of higher dimensional subshifts and even of tiling systems.

We believe that the study of asymptotic components or the somehow analogous notion of nonexpansive directions in higher dimensions may also provide useful tools to address computations of automorphism groups in this context. For instance, in [15] such an approach allowed the authors to prove that the automorphism group of the minimal component of the Robinson subshift of finite type is trivial, *i.e.*, it is generated by the shift map.

Acknowledgements

We are very grateful to Andrew Hart for helping to revise the last version of this article. We thank both referees that participated with their reports to improve the paper's clarity, readability, and organization.

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UNIMODULAR PISOT SUBSTITUTIONS AND DOMAIN EXCHANGES

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ABSTRACT. We show that any Pisot substitution on a finite alphabet is conjugate to a primitive proper substitution (satisfying then a coincidence condition) whose incidence matrix has the same eigenvalues as the original one, with possibly 0 and 1. Then, we prove also substitutive systems sharing this property and admitting "enough" multiplicatively independent eigenvalues (like for unimodular Pisot substitutions) are measurably conjugate to domain exchanges in Euclidean spaces which factorize onto minimal translations on tori. The combination of these results generalizes a well-known result of Arnoux-Ito to any unimodular Pisot substitution.

1. INTRODUCTION

A classical way to tackle problems in geometric dynamics is to code the dynamics through a well-chosen finite partition to obtain a "nice" subshift which is easier to study (see the emblematic works [Had98] and [Mor21]). The interesting aspects of the subshift could then be lifted back to the dynamical systems.
In the seminal paper [Rau82], G. Rauzy proposed to go in the other way round: take your favorite subshift and try to give it a geometrical representation. He took what is now called the Tribonacci substitution given by

$$\tau : 1 \mapsto 12, 2 \mapsto 13 \text{ and } 3 \mapsto 1,$$

and proved that the subshift it generates is measure theoretically conjugate to a rotation on the torus $\mathbb{T}^2$. A similar result was already known for substitutions of constant length under some necessary and sufficient conditions [Dek78]. Later, in [AR91], the author show that subshifts whose block complexity is $2n+1$, and satisfy what is called the Condition (*) (which includes the subshift generated by τ), are measure theoretically conjugate to an interval exchange on 3 intervals.
The substitution τ has the specificity to be a unimodular (and irreducible) Pisot substitution, that is, its incidence matrix has determinant 1, its characteristic polynomial is irreducible and its dominant eigenvalue is a Pisot number (all its algebraic conjugates are, in modulus, strictly less than 1). These properties provide key arguments to prove the main result in [Rau82]. It naturally leads to what is now called the **Pisot conjecture** for symbolic dynamics:

Let σ be a Pisot substitution. Then, the subshift it generates has purely discrete spectrum, i.e.*, is measure theoretically conjugate to a translation on a group.*

1991 *Mathematics Subject Classification.* Primary: 54H20; Secondary: 37B20.
Key words and phrases. minimal Cantor systems, finite rank Bratteli-Vershik dynamical systems, eigenvalues, Pisot conjecture, substitutions.
Both authors acknowledge the ANR Program SubTile. Fabien Durand also acknowledges the ANR Programs DynA3S and FAN.

Many attempts have been done in this direction. The usual strategy is the same as the Rauzy's one in [Rau82]: show first that the substitutive system is measurably conjugate to a domain exchange (see Definition 4). Then prove this system is measurably conjugate to a translation on a group.
A first important rigidity result, due to Host [Hos86], is that any eigenfunction of a primitive substitution is continuous. In a widely cited, but unpublished manuscript, Host also proved that the Pisot conjecture is true for unimodular substitutions defined on two letters, provided a condition called *strong coincidence condition* holds. This combinatorial condition first appeared in [Dek78] cited above. Barge an Diamond in [BD02], show then this condition is satisfied for any unimodular Pisot substitution on two letters. So the Pisot conjecture is true in this case [HS03].
Following the Rauzy's strategy, but in a different way from the Host's approach, Arnoux and Ito in [AI01], associate a self-affine domain exchange called *Rauzy fractal* to any unimodular Pisot substitution. They proved, this system is measurably conjugate to the substitutive system provided the substitution satisfies a combinatorial condition. Few time later, Host's results were generalized by Canterini and Siegel in [CS01] to any unimodular Pisot substitution and to the non-unimodular case [Sie03, Sie04], but without avoiding the strong coincidence condition. These works led to the development of a huge number of techniques to study the Rauzy fractals (see for instance [Fog02] and references therein). Let us mention also other fruitful geometrical approaches by using tilings in [BK06, BBJK06] and more recently in [Bar14] for the one-dimensional case.
In this paper, we show a similar result to [AI01] and [CS01] but skipping the combinatorial condition: any unimodular Pisot substitution is measurably conjugate to a self-affine domain exchange. Notice the domain exchange may, a priori, be different from the usual Rauzy fractal.

Theorem 1. *Let σ be a unimodular Pisot substitution on d letters and let (Ω, S) be the associated substitutive dynamical system. Then, there exist a self-affine domain exchange transformation $(E, \mathcal{B}, \tilde{\lambda}, T)$ in $\mathbb{R}^{d-1}$ and a continuous onto map $F\colon \Omega \to E$ which is a measurable conjugacy map between the two systems.*
If $\pi\colon \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$ denotes the canonical projection, then the map $\pi \circ F$ defines, for some constant $r \geq 1$, an a.e. r-to-one factor map from (Ω, S) to the dynamical system associated with a minimal translation on the torus $\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$.

The toral translation is explicitly described in [CS01] (see also [Fog02]). To show the Pisot conjecture, one still have to show this domain exchange is conjugate to the toral translation.
We postpone to the next section the basic definitions and notions we use for dynamical systems, substitutive dynamics and Pisot substitutions. In Section 3, we prove by using the notion of return words, that any substitutive subshift is conjugate to a *proper* substitution (*i.e.*, having a nice combinatorial property implying, in particular, the strong coincidence condition). But, this new substitution may not be irreducible since the spectrum of its matrix contain the spectrum of a power of the older one but may also contain the values 0 and 1. We show then, in Section 4, that a such subshift, having enough multiplicatively independent eigenvalues (precised later), is measurably conjugate to a self-affine domain exchange. A byproduct of these two results gives us Theorem 1. The proof follows the same strategy as in [CS01]. However, here, the standard property of irreducibility of Pisot substitutions are not used. We strongly need, instead, a condition on the eigenvalues which is

precisely: the number of multiplicatively independent non trivial eigenvalues equals $\sum_{0<|\lambda|<1} \dim E_\lambda$ where E_λ denotes the eigenspace associated with the eigenvalue λ of the substitution matrix. This suggests a possible extension of these results to linearly recurrent symbolic systems like in [BJS12].

2. Basic definitions

2.1. **Words and sequences.** An *alphabet* A is a finite set of elements called *letters*. Its cardinality is $|A|$. A *word* over A is an element of the free monoid generated by A, denoted by A^*. Let $x = x_0x_1 \cdots x_{n-1}$ (with $x_i \in A$, $0 \le i \le n-1$) be a word, its *length* is n and is denoted by $|x|$. The *empty word* is denoted by ϵ, $|\epsilon| = 0$. The set of non-empty words over A is denoted by A^+. The elements of $A^{\mathbb{Z}}$ are called *sequences*. If $x = \dots x_{-1}x_0x_1 \dots$ is a sequence (with $x_i \in A$, $i \in \mathbb{Z}$) and $I = [k, l]$ an interval of $\mathbb{Z}$ we set $x_I = x_kx_{k+1} \cdots x_l$ and we say that x_I is a *factor* of x. If $k = 0$, we say that x_I is a *prefix* of x. The set of factors of length n of x is written $\mathcal{L}_n(x)$ and the set of factors of x, or the *language* of x, is denoted by $\mathcal{L}(x)$. The *occurrences* in x of a word u are the integers i such that $x_{[i,i+|u|-1]} = u$. If u has an occurrence in x, we also say that u *appears* in x. When x is a word, we use the same terminology with similar definitions.
A word u is *recurrent* in x if it appears in x infinitely many times. A sequence x is *uniformly recurrent* if it is recurrent and for each factor u, the difference of two consecutive occurrences of u in x is bounded.

2.2. **Morphisms and matrices.** Let A and B be two finite alphabets. Let σ be a *morphism* from A^* to B^*. When $\sigma(A) = B$, we say σ is a *coding*. We say σ is *non erasing* if there is no $b \in A$ such that $\sigma(b)$ is the empty word. If $\sigma(A)$ is included in B^+, it induces by concatenation a map from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$: $\sigma(\dots x_{-1}.x_0x_1 \dots) = \dots \sigma(x_{-1}).\sigma(x_0)\sigma(x_1) \dots$, also denoted by σ. With the morphism σ is naturally associated its *incidence matrix* $M_\sigma = (m_{i,j})_{i \in B, j \in A}$ where $m_{i,j}$ is the number of occurrences of i in the word $\sigma(j)$. Notice that for any positive integer n we get $M_{\sigma^n} = M_\sigma^n$.
We say that an endomorphism is *primitive* whenever its incidence matrix is primitive (*i.e.*, when it has a power with strictly positive coefficients). The Perron's theorem tells that the dominant eigenvalue is a real simple root of the characteristic polynomial and is strictly greater than the modulus of any other eigenvalue.

2.3. **Substitutions and substitutive sequences.** We say that an endomorphism $\sigma : A^* \to A^*$ is a *substitution* if there exists a letter $a \in A$ such that the word $\sigma(a)$ begins with a and $\lim_{n \to +\infty} |\sigma^n(b)| = +\infty$ for any letter $b \in A$. In this case, for any positive integer n, $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$. Since $|\sigma^n(a)|$ tends to infinity with n, the sequence $(\sigma^n(\cdots aaa \cdots))_{n \ge 0}$ converges (for the usual product topology on $A^{\mathbb{Z}}$) to a sequence denoted by $\sigma^\infty(a)$. The substitution σ being continuous for the product topology, $\sigma^\infty(a)$ is a fixed point of σ: $\sigma(\sigma^\infty(a)) = \sigma^\infty(a)$.
A substitution σ is *left proper* (resp. *right proper*) if all words $\sigma(b)$, $b \in A$, starts (resp. ends) with the same letter. For short, we say that a left and right proper substitution is *proper*.
The *language* of $\sigma : A^* \to A^*$, denoted by $\mathcal{L}(\sigma)$, is the set of words having an occurrence in $\sigma^n(b)$ for some $n \in \mathbb{N}$ and $b \in A$. Notice that we have $\mathcal{L}(\sigma^n) = \mathcal{L}(\sigma)$ for any positive integer n.

2.4. **Dynamical systems and subshifts.** A *measurable dynamical system* is a quadruple $(X, \mathcal{B}, \mu, T)$ where X is a space endowed with a σ-algebra $\mathcal{B}$, a probability measure μ and measurable map $T : X \to X$ that preserves the measure μ, *i.e.*, $\mu(T^{-1}B) = \mu(B)$ for any $B \in \mathcal{B}$. This system is called *ergodic* if any T-invariant measurable set has measure 0 or 1. Two measurable dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{B}', \nu, S)$ are *measure theoretically conjugate* if we can find invariant subsets $X_0 \subset X$, $Y_0 \subset Y$ with $\mu(X_0) = \nu(Y_0) = 1$ and a bimeasurable bijective map $\psi \colon X_0 \to Y_0$ such that $S \circ \psi = \psi \circ T$ and $\mu(\psi^{-1}B) = \nu(B)$ for any $B \in \mathcal{B}'$.
By a *topological dynamical system*, or dynamical system for short, we mean a pair (X, S) where X is a compact metric space and S a continuous map from X to itself. It is well-known that such a system endowed with the Borel σ-algebra admits a probability measure μ preserved by the map S, and then form a measurable dynamical system. If the probability measure μ is unique, the system is said *uniquely ergodic*.
A *Cantor system* is a dynamical system (X, S) where the space X is a Cantor space, *i.e.*, X has a countable basis of its topology which consists of closed and open sets and does not have isolated points. The system (X, S) is *minimal* whenever X and the empty set are the only S-invariant closed subsets of X. We say that a minimal system (X, S) is *periodic* whenever X is finite.
A dynamical system (Y, T) is called a *factor* of, or is *semi-conjugate* to, (X, S) if there is a continuous and onto map $\phi : X \to Y$ such that $\phi \circ S = T \circ \phi$. The map ϕ is a *factor map*. If ϕ is one-to-one we say that ϕ is a *conjugacy*, and, that (X, S) and (Y, T) are *conjugate*.

For a finite alphabet A, we endow $A^{\mathbb{Z}}$ with the product topology. A *subshift* on A is a pair $(X, S_{|X})$ where X is a closed S-invariant subset of $A^{\mathbb{Z}}$ $(S(X) = X)$ and S is the *shift transformation*

$$\begin{array}{cccc} S & : & A^{\mathbb{Z}} & \to & A^{\mathbb{Z}} \\ & & (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}}. \end{array}$$

We call *language* of X the set $\mathcal{L}(X) = \{x_{[i,j]}; x \in X, i \le j\}$. A set defined with two words u and v of A^* by

$$[u.v]_X = \{x \in X; x_{[-|u|,|v|-1]} = uv\}$$

is called a *cylinder set*. When u is the empty word we set $[u.v]_X = [v]_X$. The family of cylinder sets is a base of the induced topology on X. As it will not create confusion we will write $[u]$ and S instead of $[u]_X$ and $S_{|X}$.
For x a sequence on A, let $\Omega(x)$ be the set $\{y \in A^{\mathbb{N}}; y_{[i,j]} \in \mathcal{L}(x), \forall\ [i,j] \subset \mathbb{Z}\}$. It is clear that $(\Omega(x), S)$ is a subshift, it is called the *subshift generated by* x. Notice that $\Omega(x) = \overline{\{S^n x; n \in \mathbb{Z}\}}$. For a subshift (X, S) on A, the following are equivalent:

(1) (X, S) is minimal;
(2) For all $x \in X$ we have $X = \Omega(x)$;
(3) For all $x \in X$ we have $\mathcal{L}(X) = \mathcal{L}(x)$.

We also have that $(\Omega(x), S)$ is minimal if and only if x is uniformly recurrent. Note that if (Y, S) is another subshift then, $\mathcal{L}(X) = \mathcal{L}(Y)$ if and only if $X = Y$.

2.5. **Substitutive subshifts.** For primitive substitutions σ, all the fixed points are uniformly recurrent and generate the same minimal and uniquely ergodic subshift (for more details see [Que87]). We call it the *substitutive subshift generated by* σ and we denote it (Ω_σ, S).

There is another useful way to generate subshifts. For $\mathcal{L}$ a language on the alphabet A, define $X_{\mathcal{L}} \subset A^{\mathbb{Z}}$ to be the set of sequences $x = (x_n)_{n\in\mathbb{Z}}$ such that $\mathcal{L}(x) \subset \mathcal{L}$. The pair $(X_{\mathcal{L}}, T)$ is a subshift and we call it the *subshift generated by* $\mathcal{L}$. If σ is a primitive substitution, then $\Omega_\sigma = X_{\mathcal{L}_\sigma}$ where $\mathcal{L}_\sigma$ denotes the language of σ [Que87]. It follows that for any positive integer n, σ^n and σ define the same subshift, that is $\Omega_\sigma = \Omega_{\sigma^n}$.
If the set Ω_σ is not finite, the substitution σ is called *aperiodic*.
An algebraic number β is called a *Pisot-Vijayaraghan number* if all its algebraic conjugates have a modulus strictly smaller than 1.

Definition 2. *Let σ be a primitive substitution and let P_σ denote the characteristic polynomial of the incidence matrix M_σ. We say that the substitution σ is*

- *of* Pisot type *(or* Pisot *for short) if P_σ has a dominant root $\beta > 1$ and any other root β' satisfies $0 < |\beta'| < 1$;*
- *of* weakly irreducible Pisot type *(or* W. I. Pisot *for short) whenever P_σ has a real Pisot-Vijayaraghan number as dominant root, its algebraic conjugates, with possibly* 0 *or roots of the unity as other roots;*
- *an* irreducible *substitution whenever P_σ is irreducible over $\mathbb{Q}$;*
- unimodular *if* $\det M_\sigma = \pm 1$.

For instance the Fibonacci substitution $0 \mapsto 01, 1 \mapsto 0$ and the Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ are unimodular substitutions of Pisot type. Whereas the Thue-Morse substitution $0 \mapsto 01$, $1 \mapsto 10$ is a W. I. Pisot substitution. Notice that the notions of Pisot, W. I. Pisot, irreducible, unimodular depend only on the properties of the incidence matrix. So starting from a Pisot (resp. W. I. Pisot, irreducible, unimodular) substitution, we get many examples of Pisot (resp. W. I. Pisot, irreducible, unimodular) substitutions by permuting the letters of the initial one.
Standard algebraic arguments ensure that a Pisot substitution is an irreducible substitution, and of course, a Pisot substitution is of weakly irreducible Pisot type. In the following we will strongly use the fact that for any substitution of (resp. W. I. Pisot, irreducible, unimodular) Pisot type σ and for every integer $n \geq 1$, the substitutions σ^n are also of (resp. W. I. Pisot, irreducible, unimodular) Pisot type.
In [HZ98], the authors prove that the fixed point of a unimodular substitution of Pisot type is non-periodic for the shift, thus the subshift generated is a non-periodic minimal Cantor system.

2.6. **Dynamical spectrum of substitutive subshifts.** For a measurable dynamical system $(X, \mathcal{B}, \mu, T)$, a complex number λ is an *eigenvalue* of the dynamical system $(X, \mathcal{B}, \mu, T)$ with respect to μ if there exists $f \in L^2(X, \mu)$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called an *eigenfunction* (associated with λ). The value 1 is the *trivial eigenvalue* associated with a constant eigenfunction. If the system is ergodic, then every eigenvalue is of modulus 1, and every eigenfunction has a constant modulus μ-almost surely. For a topological dynamical system, if the eigenfunction f is continuous, λ is called a *continuous eigenvalue*. The collection of eigenvalues is called the *spectrum* of the system, and form a multiplicative subgroup of the circle $\mathbb{S} = \{z \in \mathbb{C};\ |z| = 1\}$.
An important result for the spectrum is due to B. Host [Hos86]. It states that any eigenvalue of a substitutive subshift is a continuous eigenvalue. The following

proposition, claimed in [Hos92] (see Proposition 7.3.29 in [Fog02] for a proof), shows that the spectrum of a unimodular substitution of Pisot type is not trivial.

Proposition 3. *Let σ be a unimodular substitution of Pisot type and let α be a frequency of a letter in any infinite word of Ω_σ. Then $\exp(2i\pi\alpha)$ is a continuous eigenvalue of the dynamical system (Ω_σ, S).*

Recall that these frequencies are the coordinates of the right normalized eigenvector associated with the dominant eigenvalue of the incidence matrix of the substitution [Que87], and moreover for a unimodular Pisot substitution they are multiplicatively independent (Proposition 3.1 in [CS01]).
Notice the converse of the proposition is also true [BK06]. For a proof, see the remark below Lemma 14 or Proposition 11 in [CDHM03]. Actually, this is a general fact for any minimal Cantor system observed in [IO07]: given any continuous eigenvalue $\exp(2i\pi\alpha)$, α belongs to the additive subgroup of $\mathbb{R}$ generated by the intersection of sets of measures of clopen subsets for all the invariant probability measures. An other proof of that can be found in [CDHM03] (Proposition 11) but it was not pointed out.

2.7. **Domain exchange.** Let us recall that a compact Euclidean set is said *regular* if it equals the closure of its interior.

Definition 4. *We call* domain exchange transformation *a measurable dynamical system $(E, \mathcal{B}, \tilde{\lambda}, T)$ where E is a compact regular subset of an Euclidean space, $\tilde{\lambda}$ denotes the normalized Lebesgue measure on E and $\mathcal{B}$ denotes the Borel σ-algebra, such that:*

- *there exist compact regular subsets $E_1, \dots, E_n$ such that $E = E_1 \cup \dots \cup E_n$.*
- *The sets E_i are disjoint in measure for the Lebesgue measure λ:*
$$\lambda(E_i \cap E_j) = 0 \qquad \text{when } i \neq j.$$
- *For any index i, the map T restricted to the set E_i, is a translation such that $T(E_i) \subset E$.*

The domain exchange is said *self-affine*, if there is a finite number of affine maps $f_1, \dots, f_\ell$ such that $E = \bigcup_{i=1}^{\ell} f_i(E)$ and sharing the same linear part.

3. Matrix eigenvalues and return substitutions

In this section, we recall the notion of return substitution introduced in [Dur98a] and that any primitive substitutive subshift is conjugate to an explicit primitive and proper substitutive subshift without changing too much the eigenvalues of the associated substitution matrix [Dur98b].

Let A be an alphabet and $x \in A^{\mathbb{Z}}$ and let u be a word of x. We call *return word* to u of x every factor $x_{[i,j-1]}$ where i and j are two successive occurrences of u in x. We denote by $\mathcal{R}_{x,u}$ the set of return words to u of x. Notice that for a return word v, vu belongs to $\mathcal{L}(x)$ and u is a prefix of the word vu. Suppose x is uniformly recurrent. It is easy to check that for any word u of x, the set $\mathcal{R}_{x,u}$ is finite. Moreover, for any sequence $y \in \Omega(x)$, we have $\mathcal{R}_{y,u} = \mathcal{R}_{x,u}$. The sequence x can be written naturally as a concatenation

$$x = \cdots m_{-1} m_0 m_1 \cdots , \qquad m_i \in \mathcal{R}_{x,u},\ i \in \mathbb{Z},$$

of return words to u, and this decomposition is unique. By enumerating the elements of $\mathcal{R}_{x,u}$ in the order of their first appearence in $(m_i)_{i\geq 0}$, we get a bijective map

$$\Theta_{x,u}\colon R_{x,u} \to \mathcal{R}_{x,u} \subset A^*,$$

where $R_{x,u} = \{1, \dots, \text{Card } (\mathcal{R}_{x,u})\}$. This map defines a morphism. We denote by $D_u(x)$ the unique sequence on the alphabet $R_{x,u}$ characterized by

$$\Theta_{x,u}(D_u(x)) = x.$$

We call it the *derived sequence of* x on u. Actually this sequence enables to code the dynamics of the induced system on the cylinder $[u]$. To be more precise, we need to introduce the following notions. A finite subset $\mathcal{R} \subset A^+$ is a *code* if every word $u \in A^+$ admits at most one decomposition in a concatenation of elements of $\mathcal{R}$.

We say that a code $\mathcal{R}$ is a *circular code* if for any words

$$w_1, \dots, w_j, w, w_1', \dots, w_k' \in \mathcal{R}; s \in A^+ \text{ and } t \in A^*$$

such that

$$w = ts \text{ and } w_1 \dots w_j = s w_1' \dots w_k' t$$

then t is the empty word. It follows that $j = k+1$, $w_{i+1} = w_{i'}'$ for $1 \leq i \leq k$ and $w_1 = s$.

Proposition 5 ([Dur98a] Proposition 6)**.** *Let x be a uniformly recurrent sequence and let u be a non empty prefix of x.*

1. *The set $\mathcal{R}_{x,u}$ is a circular code.*
2. *If v is a prefix of u, then each return word on u belongs to $\Theta_{x,v}(R_{x,v}^*)$,* i.e., *it is a concatenation of return words on v.*
3. *Let v be a nonempty prefix of $D_u(x)$ and $w = \Theta_{x,u}(v)u$ then*
   - *w is a prefix of x,*
   - *$D_v(D_u(x)) = D_w(x)$.*
   - *$\Theta_{x,u} \circ \Theta_{D_u(x),v} = \Theta_{x,w}$.*

The following proposition enables to associate to a substitution an other substitution on the alphabet $R_{x,u}$.

Proposition 6 ([Dur98a])**.** *Let $x \in A^{\mathbb{N}}$ be a fixed point of the primitive substitution σ which is not periodic for the shift and u be a nonempty prefix of x. There exists a primitive substitution σ_u, defined on the alphabet $R_{x,u}$, characterized by*

$$\Theta_{x,u} \circ \sigma_u = \sigma \circ \Theta_{x,u}.$$

Even if this proposition is not stated for bi-infinite sequences, it follows that each derived sequence $D_u(x)$, where u is a prefix of an aperiodic sequence $x \in A^{\mathbb{Z}}$ fixed by a primitive substitution σ, is a fixed point of the primitive substitution σ_u. To show this it is enough to check that

$$\Theta_{x,u} \circ \sigma_u(D_u(x)) = \sigma \circ \Theta_{x,u}(D_u(x)) = \sigma x = x = \Theta_{x,u} \circ D_u(x).$$

Since $\Theta_{x,u}(R_{x,u})$ is a circular code, we get that the sequence $D_u(x)$ is fixed by the substitution σ_u. This substitution, defined in the previous proposition, is called the *return substitution* (*to* u). Moreover, we observe that for any integer $l > 0$

$$(\sigma^l)_u = (\sigma_u)^l.$$

Furthermore the incidence matrix of the return substitution has almost the same spectrum as the initial substitution. More precisely, we have:

Proposition 7 ([Dur98b]). *Let σ be a primitive substitution and let u be a prefix of a fixed point x which is not shift periodic. The incidence matrices M_σ and M_{σ_u} have the same eigenvalues, except perhaps zero and roots of the unity.*

For instance for the Tribonacci substitution τ, the induced substitution τ_1 is the same as τ. On the other hand, if we consider the substitution

$$\sigma\colon 1 \mapsto 1123, 2 \mapsto 211, \text{ and } 3 \mapsto 21,$$

it is also a substitution of Pisot type and the incidence matrix of the induced substitution σ_{11} has 0 as eigenvalue.
With the next property we obtain that if an induced system of a subshift (X, S) is a proper substitutive subshift (Ω, S), then the system (X, S) is conjugate to a proper substitutive subshift. The system (X, S) is called an *exduction* of the system (Ω, S).

Proposition 8. *Let $y = (y_i)_{i\in\mathbb{Z}}$ be a fixed point of an aperiodic primitive substitution σ on the alphabet R. Let $\Theta : R^* \to A^+$ be a non-erasing morphism, $x = \Theta(y)$ and (X, S) be the subshift generated by x.*
Then, there exist a primitive substitution ξ on an alphabet B, an admissible fixed point z of ξ, and a map $\phi : B \to A$ such that:

(1) *$\phi(z) = x$;*
(2) *If $\Theta(R)$ is a circular code, then ϕ is a conjugacy from (Ω_ξ, S) to (X, S);*
(3) *If σ is proper (resp. right or left proper), then ξ is proper (resp. right or left proper);*
(4) *There exists a prefix $u \in B^+$ of z such that $R_{y,y_0} = R_{z,u}$ and there is an integer $l \geq 1$ such that the return substitutions $\sigma^l_{y_0}$ and ξ_u are the same.*

Actually the first three statements of this proposition, correspond to Proposition 23 in [DHS99]. The substitution ξ is explicit in the proof.

Proof. The statements 1), 2), 3), and the fact that ξ is primitive, have been proven in [DHS99]. We will just give the proof of the first statement because we need it to prove the fourth statement.
Substituting a power of σ for σ if needed, we can assume that $|\sigma(j)| \geq |\Theta(j)|$ for any $j \in R$. For all $j \in R$, let us denote $m_j = |\sigma(j)|$ and $n_j = |\Theta(j)|$. We define

- An alphabet $B := \{(j, p); j \in R, 1 \leq p \leq n_j\}$;
- A morphism $\phi\colon B^* \to A^*$ by $\phi(j, p) = (\Theta(j))_p$;
- A morphism $\psi\colon R^* \to B^*$ by $\psi(j) = (j, 1)(j, 2) \cdots (j, n_j)$.

Clearly, we have $\phi \circ \psi = \Theta$. We define a substitution ξ on B by

$$\forall j \in R,\ 1 \leq p \leq n_j;\ \xi(j, p) = \begin{cases} \psi((\sigma(j))_p) & \text{if } 1 \leq p < n_j \\ \psi((\sigma(j))_{[n_j, m_j]}) & \text{if } p = n_j. \end{cases}$$

Thus for every $j \in R$, we have $\xi(\psi(j)) = \xi(j, 1) \dots \xi(j, n_j) = \psi(\sigma(j))$, *i.e.*,

$$\xi \circ \psi = \psi \circ \sigma. \tag{3.1}$$

For $z = \psi(y)$ we obtain $\xi(z) = \psi(\sigma(y)) = \psi(y) = z$, that is z is a fixed point of ξ. Moreover $\phi(z) = \phi(\psi(y)) = \Theta(y) = x$ and we get the point (1).

Let us prove the fourth statement.
Let $u = \psi(y_0) \in B^*$ where $y = \dots y_{-1}.y_0y_1\dots$, $y_i \in B$, $i \in \mathbb{Z}$. First, notice the morphism ψ is one-to-one and then we have $\psi(\mathcal{R}_{y,y_0}) = \mathcal{R}_{\psi(y),\psi(y_0)}$. It follows that

$$R_{y,y_0} = R_{\psi(y),\psi(y_0)} = R_{z,u},$$

and

$$\psi \circ \Theta_{y,y_0} = \Theta_{\psi(y),\psi(y_0)} = \Theta_{z,u}.$$

Therefore for the return substitution σ_{y_0} to y_0, Proposition 6 and Relation (3.1) give

$$\Theta_{z,u} \circ \sigma_{y_0} = \psi \circ \Theta_{y,y_0} \circ \sigma_{y_0} = \psi \circ \sigma \circ \Theta_{y,y_0} = \xi \circ \psi \circ \Theta_{y,y_0} = \xi \circ \Theta_{z,u}.$$

Consequently, we have $\sigma_{y_0} = \xi_u$. □

As a straightforward corollary of the propositions 6, 5, 8 and 7, we get

Corollary 9. *Let σ be a primitive aperiodic substitution. Then there exists a proper primitive substitution ξ on an alphabet B, such that*

(1) *(Ω_σ, S) is conjugate to (Ω_ξ, S);*
(2) *there exists $l \geq 1$ such that the substitution matrices M_σ^l and M_ξ have the same eigenvalues, except perhaps 0 and 1.*

Proof. Let us fix a nonempty prefix u of a fixed point x of σ. Thus x is not shift periodic. Substituting a power of σ for σ if needed, we can assume that the word $\Theta_{x,u}(1)u$ is a prefix of $\sigma(u)$. By the very definition of return word, for any letter $i \in R_{x,u}$, the word $\Theta_{x,u}(i)u$ has the word u as a prefix. Then $\Theta_{x,u}(1)u$ is a prefix of the word $\sigma(\Theta_{x,u}(i)u)$. It follows from the equality in Proposition 6, that $\Theta_{x,u}(1)u$ is also a prefix of the word $\Theta_{x,u} \circ \sigma_u(i)$. The uniqueness of the coding by $\Theta_{x,u}(R_{x,u})$, implies that the word $\sigma_u(i)$ starts with 1, and the substitution σ_u is left proper.
The propositions 5 and 8 imply the existence of a left proper primitive substitution ξ' such that (Ω_σ, S) is conjugate to $(\Omega_{\xi'}, S)$, moreover by Proposition 7 there exists an integer $l > 0$ such that the incidence matrices M_σ^l and $M_{\xi'}$ share the same eigenvalues, except perhaps 0 and 1.
To obtain a proper substitution we need to modify ξ'. Let a be the letter such that for all letter b, $\xi'(b) = aw(b)$ for some word $w(b)$. Now consider the substitution ξ'' defined by $\xi'' \colon b \mapsto w(b)a$. Then, ξ' and ξ'' define the same language, so we have $\Omega_{\xi'} = \Omega_{\xi''} = \Omega_\xi$ where ξ is the composition of substitutions $\xi' \circ \xi''$ and is proper. We conclude observing that $M_\xi = M_{\xi'} M_{\xi''} = M_{\xi'}^2$. □

In terms of Pisot substitutions, Corollary 9 becomes:

Corollary 10. *Let σ be an aperiodic substitution of Pisot type, then the substitutive subshift associated with σ is conjugate to a substitutive subshift (Ω_ξ, S) where ξ is a proper primitive substitution of weakly irreducible Pisot type.*

The example after Proposition 7 shows that the use of return substitutions seems to force to deal with W. I. Pisot substitutions. In fact, it is unavoidable to consider W. I. Pisot substitution to represent a substitutive subshift by a proper substitution. For instance, consider the non-proper substitution $\sigma : 0 \mapsto 001,\ 1 \mapsto 10$. The dimension group of the associated subshift, computed in [Dur96], is of rank 3. As a consequence, any proper substitution ξ representing the subshift Ω_σ should be, at least, on 3 letters (see [DHS99] for the details). Moreover Cobham's theorem (see Theorem 14 in [Dur98c]) for minimal substitutive subshifts implies that, taking powers if needed, ξ and σ share the same dominant eigenvalue. So, the substitution ξ can not be irreducible.

4. Conjugacy with a domain exchange

In this section we give sufficient conditions on a primitive proper substitution so that the associated substitutive system is measurably conjugate to a domain exchange in an Euclidean space.

4.1. **Using Kakutani-Rohlin partitions.** In this subsection, we will assume that ξ is a primitive proper substitution on a finite alphabet A equipped with a fixed order.
First let us recall a structure property of the system (Ω_ξ, S) in terms of Kakutani-Rohlin towers.

Proposition 11 ([DHS99])**.** *Let ξ be a primitive proper substitution on a finite alphabet A. Then for every $n > 0$,*

$$\mathcal{P}_n = \{S^{-k}\xi^{n-1}([a]);\ a \in A,\ 0 \le k \le |\xi^{n-1}(a)| - 1\}$$

is a clopen partition of Ω_ξ defining a nested sequence of Kakutani-Rohlin partition of Ω_ξ, more precisely:

- *The sequence of bases $(\xi^n(\Omega_\xi))_{n\ge 0}$ is decreasing and the intersection is only one point;*
- *For every $n > 0$, $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$;*
- *The sequence $(\mathcal{P}_n)_{n>0}$ spans the topology of Ω_ξ.*

To be coherent with the notations in [BDM05], we take the conventions $\mathcal{P}_0 = \{\Omega_\xi\}$ and for an integer $n \ge 1$, $r_n(x)$ denotes the *entrance time* of a point $x \in \Omega_\xi$ in the base $\xi^{n-1}(\Omega_\xi)$, that is

$$r_n(x) = \min\{k \ge 0;\ S^k x \in \xi^{n-1}(\Omega_\xi)\}.$$

By minimality, this value is finite for any $x \in \Omega_\xi$ and the function r_n is continuous. The homeomorphism $S_{\xi(\Omega_\xi)} \colon \xi(\Omega_\xi) \ni x \mapsto S^{r_2(Sx)}(Sx) \in \xi(\Omega_\xi)$ is then the induced map of the system (X, S) on the clopen set $\xi(\Omega_\xi)$. Since we have the relation

$$\xi \circ S = S_{\xi(\Omega_\xi)} \circ \xi, \tag{4.1}$$

the induced system $(\xi(\Omega_\xi), S_{\xi(\Omega_\xi)})$ is a factor of (Ω_ξ, S) via the map ξ (and in fact a conjugacy).
Note that for any integer $n > 0$,

$$r_n(Sx) - r_n(x) = \begin{cases} -1 & \text{if } x \notin \xi^{n-1}(\Omega_\xi) \\ |\xi^{n-1}(a)| - 1 & \text{if } x \in \xi^{n-1}([a]), a \in A. \end{cases} \tag{4.2}$$

More precisely, we can relate the entrance time and the incidence matrix by the following equality (see Lemma in [BDM05]): For a primitive proper substitution ξ, we have for any $x \in \Omega_\xi$ and $n \ge 2$

$$r_n(x) = \sum_{k=1}^{n-1} \langle s_k(x), (M_\xi^t)^{k-1} H(1)\rangle \tag{4.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product, M_ξ^t is the transpose of the incidence matrix, $H(1)=(1,\cdots,1)^t$ and $s_k:\Omega_\xi\to\mathbb{Z}^{\#A}$ is a continuous function defined by

$$s_k(x)_a=\#\{r_k(x)<i\le r_{k+1}(x);\ S^ix\in\xi^{k-1}([a])\},\qquad \text{for } a\in A.$$

In other words, the vector $s_k(x)$ counts, in each coordinate $a\in A$, the number of time that the positive iterates of x meet the clopen set $\xi^{k-1}([a])$ before meeting for the first time the clopen set $\xi^k(\Omega_\xi)$ and after meeting the clopen set $\xi^{k-1}(\Omega_\xi)$.
The proof of the following lemma is direct from the definition and Proposition 11.

Lemma 12. *For ξ a primitive proper substitution, we have, for any $x\in\Omega_\xi$,*

$$s_1(\xi x)=0\qquad \textit{and}\qquad \forall k>1,\ s_k(\xi x)=s_{k-1}(x).$$

For any letter $a\in A$, $k\in\mathbb{N}^*$, we also have $s_k(x)_a\le\sup_{b\in A}|\xi(b)|$.

From the ergodic point of view, it is well-known (see [Que87]) that subshifts generated by primitive substitutions are uniquely ergodic. We call μ the unique probability shift-invariant measure of (Ω_ξ,S). We have the following relations, for any positive integer n,

$$\vec{\mu}(n)=M_\xi\vec{\mu}(n+1),\qquad \text{and}\qquad \langle H(1),\vec{\mu}(1)\rangle=1, \tag{4.4}$$

where $\vec{\mu}(n)\in\mathbb{R}^{\sharp A}$ is the vector defined by

$$\vec{\mu}(n)_a=\mu(\xi^{n-1}([a])),\qquad \text{for any letter } a\in A.$$

4.2. **On the spectrum of a substitutive subshift.** From this subsection, we assume that ξ is a primitive proper substitution on a finite alphabet A.
Taking a power of ξ if needed, from classical results of linear algebra, there are M_ξ^t-invariant $\mathbb{R}$-vectorial subspaces E^0,E^u,E^b and E^s such that

1. $\mathbb{R}^{\#A}=E^0\oplus E^s\oplus E^u\oplus E^b$,
2. $M_\xi^tv=0$ for all $v\in E^0$,
3. $\lim_{k\to+\infty}(M_\xi^t)^kv=0$, $(M_\xi^t)^nv\neq0$ for all $v\in E^s\setminus\{0\}$ and any $n\in\mathbb{N}$,
4. $\lim_{k\to+\infty}||(M_\xi^t)^kv||=+\infty$ for all $v\in E^u\setminus\{0\}$ and
5. $((M_\xi^t)^kv)_{k\in\mathbb{Z}}$ is bounded and $(M_\xi^t)^nv\neq0$ for all $v\in E^b\setminus\{0\}$ and $n\in\mathbb{N}$.

Let us apply some well-know facts to our context (see [Hos86] or [FMN96] for substitutions and [BDM05] for a wider context). Let r_n and s_n be as defined in Section 4.1.

Proposition 13. *Let ξ be a primitive proper substitution on an alphabet A. If $\lambda\in\mathbb{S}$ is an eigenvalue of the system (Ω_ξ,S), then $(\lambda^{-r_n})_{n\ge1}$ converges uniformly to a continuous eigenfunction associated with λ. Moreover, $\sum_{n\ge1}\max_{a\in A}|\lambda^{|\xi^n(a)|}-1|$ converges.*

So if $\exp(2i\pi\alpha)$ is an eigenvalue of the substitutive system (Ω_ξ,S), for any letter a of the alphabet $|\xi^n(a)|\alpha$ converges to 0 mod $\mathbb{Z}$ as n goes to infinity. In an equivalent way the vector $(M_\xi^t)^n\alpha(1,\cdots,1)^t$ tends to 0 mod $\mathbb{Z}^{\#A}$. The next lemma precises this for the usual convergence.

Lemma 14. *Let $\lambda=\exp(2i\pi\alpha)$ be an eigenvalue of a substitutive system (Ω_ξ,S) for a primitive proper substitution ξ on a finite alphabet A. Then, there exist $m\in\mathbb{N}$, $v\in\mathbb{R}^{\#A}$ and $w\in\mathbb{Z}^{\#A}$ such that*

$$\alpha H(1)=v+w,\qquad (M_\xi^t)^mw\in\mathbb{Z}^{\#A}\ \textit{and}\ (M_\xi^t)^nv\to_{n\to\infty}0,$$

where all entries of $H(1)$ are equal to 1. Moreover *i) The convergence is geometric: there exist* $0 \le \rho < 1$ *and a constant* C *such that*

$$||(M_\xi^t)^n v|| \le C\rho^n, \text{ for any } n \in \mathbb{N}.$$

ii) For any positive integer n,

$$\langle v, \vec{\mu}(n)\rangle = 0 \qquad \text{and} \qquad \alpha = \langle (M_\xi^t)^{n-1} w, \vec{\mu}(n)\rangle.$$

Proof. The first claim and item $i)$ comes from [Hos86]. We have just to show the item $ii)$. Notice that the relations (4.4) give us for any positive integer

$$\langle v, \vec{\mu}(n)\rangle = \langle v, M_\xi^p \vec{\mu}(n+p)\rangle = \langle (M_\xi^t)^p v, \vec{\mu}(n+p)\rangle \to_{p\to+\infty} 0.$$

We deduce then

$$\alpha = \alpha\langle H(1), \vec{\mu}(1)\rangle = \langle v, \vec{\mu}(1)\rangle + \langle w, \vec{\mu}(1)\rangle = \langle w, \vec{\mu}(1)\rangle = \langle (M_\xi^t)^{n-1} w, \vec{\mu}(n)\rangle.$$

□

Remark. We get by Item $ii)$ of Lemma 14, that if $\exp(2i\pi\alpha)$ is an eigenvalue of a substitutive system, then α is in the subgroup of $\mathbb{R}$ generated by the component of the vector $\vec{\mu}(n)$, that is, in the subgroup generated by the frequency of occurrences of the words. This provides a converse to Proposition 3.

If $\exp(2i\pi\alpha_1), \ldots, \exp(2i\pi\alpha_{d-1})$ are $d-1$ eigenvalues of the substitutive system (Ω_ξ, S), from Proposition 13 and Lemma 14 there exist $m \in \mathbb{N}$, $v(1), \ldots v(d-1) \in \mathbb{R}^{\sharp A}$ and $w(1), \ldots, w(d-1) \in \mathbb{Z}^{\sharp A}$ such that for all $i \in \{1, \ldots, d-1\}$:

$$\alpha_i H(1) = v(i) + w(i), \qquad (M_\xi^t)^m w(i) \in \mathbb{Z}^{\# A} \text{ and } \sum_{n\ge 1} (M_\xi^t)^n v(i) \text{ converges.} \tag{4.5}$$

Notice that up to take a power of ξ, if needed, we can assume that the constant $m = 1$ and that any $v(i)$ has no component in E^0.

Let us recall Proposition 3: a unimodular Pisot substitutive subshift on d letters admits $d-1$ non trivial eigenvalues $\exp(2i\pi\alpha_1), \ldots, \exp(2i\pi\alpha_{d-1})$ that are *multiplicatively independent*, *i.e.*, $1, \alpha_1, \ldots, \alpha_{d-1}$ are rationally independent. This motivates the next proposition that interprets the arithmetical properties of the eigenvalues in terms of the vectors $v(i)$ and $w(i)$.

Proposition 15. *If* $\exp(2i\pi\alpha_1), \ldots, \exp(2i\pi\alpha_{d-1})$ *are* $d-1$ *multiplicatively independent eigenvalues of the substitutive system* (Ω_ξ, S) *for a proper primitive substitution* ξ. *Then, both families of vectors* $\{M_\xi^t v(1), \ldots, M_\xi^t v(d-1)\}$ *and* $\{M_\xi^t H(1), M_\xi^t w(1), \ldots, M_\xi^t w(d-1)\}$ *are linearly independent.*

Notice it implies also that both family of vectors $\{v(1), \ldots, v(d-1)\}$ and $\{H(1), w(1), \ldots, w(d-1)\}$ are linearly independent.

Proof. The proof is similar to Proposition 10 in [BDM05]. We adapt it to our case. Assume there exist reals $\delta_0, \delta_1, \ldots, \delta_{d-1}$, one being different from 0, such that $\delta_0 M_\xi^t H(1) + \sum_{i=1}^{d-1} \delta_i M_\xi^t w(i) = 0$. Since all the vectors are in $\mathbb{Z}^{\sharp A}$, by an algebraic classical result, we can assume that any δ_i is an integer. Taking the inner product of this sum with the vector $\vec{\mu(2)}$, the normalization and recurrence relations of this vector (Relation (4.4)) together with the normalization with respect to each $w(i)$ in item $ii)$ of Lemma 14, give us $\delta_0 + \sum_{i=1}^{d-1} \delta_i\alpha_i = 0$. The rational independence of the numbers $1, \alpha_1, \dots, \alpha_{d-1}$ implies any $\delta_i = 0$. So the vectors $M_\xi^t H(1), M_\xi^t w(1), \dots, M_\xi^t w(d-1)$ are independent.

Now, assume that there exist real numbers λ_i such that $\sum_{i=1}^{d-1} \lambda_i M_\xi^t v(i) = 0$. We obtain $(\sum_{i=1}^{d-1} \lambda_i \alpha_i) M_\xi^t H(1) - \sum_{i=1}^{d-1} \lambda_i M_\xi^t w(i) = 0$. The independence of the vectors $M_\xi^t H(1), M_\xi^t w(1), \dots, M_\xi^t w(d-1)$ implies that $\lambda_i = 0$ for any i. So the vectors $M_\xi^t v(1), \dots, M_\xi^t v(d-1)$ are independent. □

The following property gives a bound on the number of multiplicatively independent eigenvalues for a substitutive subshift.

Proposition 16. *Let ξ be a proper primitive substitution. If the substitutive system (Ω_ξ, S) admits $d-1$ eigenvalues $\exp(2i\pi\alpha_1)$, …, $\exp(2i\pi\alpha_{d-1})$, then the vectorial space spanned by the vectors $v(i)$, $E_\xi = \mathrm{Vect}(v(1), \dots, v(d-1))$, is a subspace of E^s.*

Moreover if the eigenvalues are muliplicatively independent, then $d-1 \leq \dim E^s$.

Proof. For $i \in \{1, \dots, d-1\}$, the vector $v(i)$ can be decomposed using the $\mathbb{R}$-vectorial subspaces E^0, E^u, E^b and E^s. From Lemma 14 it has no component in E^u and E^b. From the choice we made in (4.5), it has no component in E^0. Thus $v(i)$ belongs to E^s. So we get $E_\xi \subset E^s$. The bound by the dimension is obtained with Proposition 15. □

To construct the domain exchange of a Pisot substitution we will need the following direct corollary.

Corollary 17. *Let ξ be a proper primitive substitution. If the substitutive system (Ω_ξ, S) admits $\dim E^s$ multiplicatively independent eigenvalues, then $E_\xi := \mathrm{Vect}(v(1), \dots, v(\dim E^s)) = E^s$. In particular, we have $M_\xi^t(E_\xi) = E_\xi$.*

Notice that for a unimodular Pisot substitution σ, $\dim E^s + 1$ equals the degree of the associated Pisot number, or the number of letters in the alphabet. Thus, by Proposition 3, the proper W. I. Pisot substitution ξ associated to σ in Corollary 10, fulfills the conditions of Corollary 17.

4.3. **Semi-conjugacy with the domain exchange.** We prove the main result, Theorem 1, in this section. For this, we start recalling the very hypotheses we need to get the result.

Hypotheses `P`. *Let ξ be a primitive proper substitution on a finite alphabet A such that:*

- *i) The characteristic polynomial P_ξ admits a unique root greater than one in modulus.*
- *ii) The minimal substitutive subshift (Ω_ξ, S) admits $\dim E^s = d-1$ eigenvalues $\exp(2i\pi\alpha_1)$, …, $\exp(2i\pi\alpha_{d-1})$ such that $1, \alpha_1, \dots, \alpha_{d-1}$ are rationally independent.*
- *iii) Its Perron number β satisfies $\beta |\det M_{\xi|E^s}^t| = 1$.*

For instance, all these hypotheses apply to the proper substitution ξ of Corollary 10 associated with a unimodular Pisot substitution on d letters: The statement $i)$ is obvious, the others come from the fact that the space E^s is spanned by the eigenspaces associated with the algebraic conjugates $\beta_1, \dots, \beta_{d-1}$ of the Pisot number leading eigenvalue β of M_ξ. The unimodular hypothesis implies $|\beta\beta_1 \cdots \beta_{d-1}| = 1$.

From Hypotheses P $ii)$ and by a byproduct of the formula (4.3) on the entrance time r_n, with Formula (4.5) on the vectors $v(i)$, up to consider a power of ξ, we get for any $i \in \{1, \dots, d-1\}$ and $x \in \Omega_\xi$

$$\alpha_i r_n(x) = \sum_{k=1}^{n-1} \langle s_k(x), (M_\xi^t)^{k-1} v(i) \rangle \mod \mathbb{Z}.$$

Let $F_n = \left(\sum_{k=0}^{n-1} \langle s_k, (M_\xi^t)^{k-1} v(i) \rangle \right)^t_{1 \le i \le d-1}$. The Proposition 13 and Lemma 14 ensure the sequence $(F_n)_n$ uniformly converges to a continuous function $F \colon \Omega_\xi \to \mathbb{R}^{d-1}$, explicitly defined for $x \in \Omega_\xi$ by

$$F(x) = \left(\sum_{k=1}^{+\infty} \langle s_k(x), (M_\xi^t)^{k-1} v(i) \rangle \right)^t_{1 \le i \le d-1}.$$

Let V be the matrix with rows $v(1)^t, \dots, v(d-1)^t$. Then, the map F may be written as

$$F(x) = V \sum_{k=1}^{+\infty} M_\xi^{k-1} s_k(x).$$

Lemma 18. *Assume Hypotheses P $i), ii)$. There exist a continuous map $\Delta \colon \Omega_\xi \to \mathbb{R}^{\#A}$ and a bijective linear map $N \colon \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ such that for $\alpha = (\alpha_1, \dots, \alpha_{d-1})^t$ and for any $x \in \Omega_\xi$,*

1. $F \circ S(x) = F(x) + \alpha \mod \mathbb{Z}^{d-1}$;
2. $F(x) = V\Delta(x)$;
3. $M_\xi^t V^t = V^t N$;
4. *the matrix N is conjugated to the matrix $M^t_{\xi|E^s}$ restricted to the space E^s;*
5. $F \circ \xi(x) = N^t(F(x))$.

Proof. By the approximation property of the eigenfunctions in Proposition 13 (see also Relation (4.2)), we get $F \circ S(x) = F(x) + \alpha \mod \mathbb{Z}^{d-1}$.
Let us prove Statement (2). We have

$$F_n(x) = V \left(\sum_{k=1}^{n-1} M_\xi^{k-1} s_k(x) \right) \tag{4.6}$$

$$= V Proj \left(\sum_{k=1}^{n-1} M_\xi^{k-1} s_k(x) \right), \tag{4.7}$$

where $Proj \colon \mathbb{R}^{\#A} \to E_\xi = \text{Vect}\,(v(1), \dots, v(d-1))$ denotes the orthogonal projection onto E_ξ. Recall that by Corollary 17, E_ξ has dimension $d-1$. Since $(F_n)_n$ uniformly converges (see Proposition 13 and Lemma 14), the projection $Proj(\sum_{k=1}^{n-1} M_\xi^{k-1} s_k(x))$ converges when n goes to infinity to the vector $\Delta(x)$ belonging to E_ξ for any $x \in \Omega_\xi$. Therefore, we obtain Statement (2).
Let us prove the other statements. The basic properties of $s_n \circ \xi$ (Lemma 12) give for any $x \in \Omega_\xi$ and $n > 2$,

$$F_n \circ \xi = V M_\xi \left(\sum_{k=1}^{n-2} M_\xi^{k-1} s_k \right). \tag{4.8}$$

By the $\mathbb{R}$-independence of the vectors $v(i)$ (Proposition 15), the linear map V^t from $\mathbb{R}^{d-1}$ to E_ξ is bijective and since $M_\xi^t(E_\xi) = E_\xi$ (Corollary 17), there exists a bijective linear map $N \colon \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ such that

$$M_\xi^t V^t = V^t N. \tag{4.9}$$

This shows Statement (4). Therefore, using (4.6) and (4.8), we obtain for $n > 2$,

$$F_n \circ \xi = V M_\xi \sum_{k=1}^{n-2} M_\xi^{k-1} s_k = N^t F_{n-1}.$$

Passing through the limit in n, we get (5) and this achieves the proof. □

From Lemma 19 to Proposition 21, we use the strategy developed in [CS01] to tackle the Pisot conjecture. Recall that μ denotes the unique probability shift-invariant measure of the system (Ω_ξ, S), and λ denotes the Lebesgue measure on $F(\Omega_\xi)$.

Lemma 19. *Assume Hypotheses* P *i) − iii). There exists a constant C such that for any letter $a \in A$ we have:*

(1) $\lambda(F([a])) = C\mu([a])$,
(2) *for any integer n large enough, $F([a])$ is the union of the measure theoretically disjoint sets*

$$F(S^{-k}\xi^n([b])), \text{ with } 0 \le k < |\xi^n(b)|, [a] \cap S^{-k}\xi^n([b]) \neq \emptyset,$$

(3) *for any Borel set $B \subset [a]$,*

$$\lambda(F(B)) = C\mu(B).$$

Proof. Let $G = F \circ S - F - (\alpha_1, \ldots, \alpha_{d-1})^t$. From the basic properties of the map F (Lemma 5), it takes integer values. Being continuous, it is locally constant. Hence, there exists some integer $n_0 \ge 0$ such that G is constant on each sets $S^{-k}\xi^n([b])$, with $n > n_0$, $b \in A$ and $0 \le k < |\xi^n(b)|$ (see Proposition 11).
Therefore, from Item (5) of Lemma 18, for any such b and k, there exists a vector $\delta(k, b) \in \mathbb{R}^{d-1}$ such that

$$F(S^{-k}\xi^n([b])) = \delta(k, b) + F(\xi^n([b])) = \delta(k, b) + (N^t)^n F([b]).$$

By the very hypothesis P iii), we have $|\det N^t| = 1/\beta$, so we get

$$\lambda(F(S^{-k}\xi^n([b]))) = \lambda((N^t)^n F([b])) = |\det(N^t)^n| \lambda(F([b])) = \frac{1}{\beta^n}\lambda(F([b])).$$

Let $a \in A$, the partitions of Ω_ξ in Proposition 11 provide

$$[a] = \bigcup_{\substack{0 \le k < |\xi(j)|, b \in A \\ [a] \cap S^{-k}\xi^n([b]) \neq \emptyset}} S^{-k}\xi^n([b]).$$

Consequently,

$$\lambda(F([a])) \le \sum_{\substack{k,b; 0 \le k < |\xi^n(b)|, \\ [a] \cap S^{-k}\xi^n([b]) \neq \emptyset}} \frac{1}{\beta^n}\lambda(F([b])) = \frac{1}{\beta^n}(M_\xi^n(\lambda(F([b])))^t_{b \in A})_a. \tag{4.10}$$

From the Perron's Theorem, the above inequality is an equality and $(\lambda(F([b])))^t_{b\in A}$ is a multiple of the eigenvector $(\mu([a]))^t_{a\in A} = \vec{\mu}(1)$ of the dominant eigenvalue β^n of M_ξ^n. This shows Item (1). Notice that the equality in (4.10) also implies Item (2).

To prove Item (3), it is enough to use the partitions of Ω_ξ given in Proposition 11 and the ideas in the beginning of this proof. This part is similar to the proof of Proposition 4.3 in [CS01] and we left it to the reader. □

With the next proposition, we continue to follow the approach (and the proofs) in [CS01].

Proposition 20. *Assume Hypotheses* P *$i)-iii)$. There exists a μ-negligeable measurable subset $\mathcal{N} \subset \Omega_\xi$ such that F is one-to-one on each cylinder set $[a]$: for any x and y in $[a] \setminus \mathcal{N}$ satisfying $F(x) = F(y)$, we have $x = y$.*

Proof. Let $a \in A$. From Lemma 19, the sets

$$\mathcal{N}_a^{(\ell)} = \bigcup_{\substack{(k_1,j_1)\neq(k_2,j_2);\\ 0\leq k_1<|\xi^\ell(b_1)|,[a]\cap S^{-k_1}\xi^\ell([b_1])\neq\emptyset\\ 0\leq k_2<|\xi^\ell(b_2)|,[a]\cap S^{-k_2}\xi^\ell([b_2])\neq\emptyset}} F(S^{-k_1}\xi^\ell([b_1])) \cap F(S^{-k_2}\xi^\ell([b_2]))$$

have zero λ-measure, for any $\ell \in \mathbb{N}$ big enough. Item (3) of Lemma 19, gives furthermore, the sets $\mathcal{M}_a^{(\ell)} = F^{-1}(\mathcal{N}_a^{(\ell)})$ have zero measure with respect to μ.
Let x_1 and x_2 be two distinct elements of $[a]$ such that $F(x_1) = F(x_2)$. It suffices to show that they belong to some $\mathcal{M}_a^{(\ell)}$. Considering the partitions $\{\mathcal{P}_\ell\}_{l\geq 0}$ of Proposition 11, there exist infinitely many $\ell \in \mathbb{N}$ with two distinct couples (k_1, b_1) and (k_2, b_2), such that $0 \leq k_1 < |\xi^\ell(b_1)|$, $0 \leq k_2 < |\xi^\ell(b_2)|$, $x_1 \in S^{-k_1}\xi^\ell([b_1])$ and $x_2 \in S^{-k_2}\xi^\ell([b_2])$. Then, x_1 and x_2 belong to $\mathcal{M}_a^{(\ell)}$ for infinitely many ℓ, which achieves the proof. □

Proposition 21. *Assume Hypotheses* P *$i)-iii)$. The map F is one-to-one except on a set of measure zero.*

Proof. As ξ is proper, there exists a letter a such that $\xi(\Omega_\xi)$ is included in $[a]$. Therefore, from Proposition 20, F is one-to-one on $\xi(\Omega_\xi)$ except on a set $\mathcal{N}$ of zero measure. By the basic properties of the map F (precisely Item (5) of Lemma 18), if two points $x, y \in \Omega_\xi$ have the same image through F, then $F(\xi(x)) = F(\xi(y))$, and hence $x, y \in \xi^{-1}(\mathcal{N})$.
Recall that the induced system on $\xi(\Omega_\xi)$ is a factor of (Ω_ξ, S) via the map ξ (see Relation (4.1)). This implies that the measure $\mu(\xi^{-1}(\cdot))$ is invariant for the induced system $(\xi(\Omega_\xi), S_{\xi(\Omega_\xi)})$. Since it is uniquely ergodic with respect to the induced probability measure, $\mu(\xi^{-1}(\mathcal{N}))$ is proportional to $\mu(\mathcal{N})$, so it is null. This achieves the proof. □

The following proposition is a modification of the arguments in [Kul95] Lemma 2.1.

Proposition 22. *Assume Hypotheses* P *$i), ii)$. For any clopen set c in Ω_ξ, the set $F(c)$ is regular,* i.e.,

$$\overline{\text{int } F(c)} = F(c),$$

where int A *denotes the interior of the set A for the usual Euclidean topology.*

Proof. Let us first show that int $F(\Omega_\xi) \neq \emptyset$. Since $1, \alpha_1, \ldots, \alpha_{d-1}$ are rationally independant, by Lemma 18, denoting by π the canonical projection $\mathbb{R}^{d-1} \to \mathbb{R}^{d-1}/\mathbb{Z}^{d-1} = \mathbb{T}^{d-1}$, the map $\pi \circ F \colon \Omega_\xi \to \mathbb{T}^{d-1}$ has a dense image hence is onto. It follows that for any small ϵ, there exist a finite family $\mathcal{V}$ of integer vectors such that

$$B_\epsilon(0) \subset \bigcup_{p\in\mathcal{V}} F(\Omega_\xi) + p.$$

By the Baire Category Theorem, the set $F(\Omega_\xi)$ has a non empty interior.
Now let $\Omega^* = \Omega_\xi \setminus \bigcup\{O; O$ is open and int $F(O) = \emptyset\}$. From the previous remark it is a non empty compact set. Notice that $\Omega_\xi \setminus \Omega^*$ is the union of countably many open (and then σ-compact) subsets. The image $F(\Omega_\xi \setminus \Omega^*)$ is then a countable union of compact sets each of those with an empty interior. Again by the Baire Category Theorem, $F(\Omega^*)$ is dense in $F(\Omega_\xi)$ and since Ω^* is compact, $F(\Omega^*) = F(\Omega_\xi)$.
Let us show that Ω^* is S invariant. Let O be an open set in Ω_ξ such that int $F(O)$ is empty. By Lemma 18, the function $F \circ S - F - (\alpha_1, \dots, \alpha_{d-1})\colon \Omega_\xi \to \mathbb{Z}$ is constant on a partition by clopen sets $\mathcal{P}$ of Ω_ξ. For any atom c of $\mathcal{P}$, int $F(c \cap O) = \emptyset$ and then int $F(S(c\cap O))$ is empty. We have $F(SO) = \cup_{c\in\mathcal{P}} F(S(c\cap O))$ is then a countable union of compact sets with empty interiors. Again by the Baire Category Theorem, $F(SO)$ has empty interior, and Ω^* is S-invariant.
By minimality, we get that $\Omega^* = \Omega_\xi$, so the image by F of any open set has a non empty interior.
Finally, let C be a clopen set, and assume that $A := F(C) \setminus \overline{\text{int } F(C)}$ is not empty. From the previous assertion, $F(F^{-1}(A) \cap C) = A$ contains a ball and then A intersects int $F(C)$: a contradiction. This shows the statement of the proposition. $\square$

Let $\pi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}/\mathbb{Z}^{d-1} = \mathbb{T}^{d-1}$ be the canonical projection.

Proposition 23. *Assume Hypotheses* P *$i) - iii)$. The map $Z : \Omega_\xi \to \mathbb{Z} \cup \{\infty\}$ defined by $Z(x) = \#(\pi \circ F)^{-1}(\{\pi \circ F(x)\})$ is finite and constant μ-a.e..*

Proof. We claim Z is measurable. For any $z \in \mathbb{Z}^{d-1}$, let A_z be the set $A_z = \{x \in \Omega_\xi;\ \exists y \in \Omega_\xi, F(x) = F(y) + z\}$. We have $A_z = F^{-1}(F(\Omega_\xi) + z)$, so it is a Borel set. Notice that for any integer n, $Z^{-1}(\{n\})$ is a finite intersection of such sets, so the claim is proved. By Proposition 21, the map F is a.e. one-to-one, and by compacity of the set $F(\Omega_\xi)$, the projection $\pi\colon F(\Omega_\xi) \to \mathbb{T}^{d-1}$ is finite-to-one, so the map Z is a.e. finite. It suffices to notice that Z is T-invariant, to conclude by ergodicity. $\square$

Proof of Theorem 1. Let ξ be a unimodular Pisot substitution. By Corollary 10 and Proposition 3, we can assume that ξ satisfies the hypotheses P $i) - iii)$ (Subsection 4.3). Let E be the compact set $F(\Omega_\xi)$. Proposition 11 and Lemma 18 on the properties of the map F both ensure the existence of an integer n such that the map $F \circ S - F$ is constant on any set $E_{n,a,k,} := F(S^{-k}\xi^n([a]))$ with $a \in A$ and $0 \le k < |\xi^n(a)|$. Let T be the transformation defined on $E_{n,a,k}$ by the translation of the vector $(F \circ S - F)_{|E_{n,a,k}}$. It follows from Lemma 19 and Proposition 22 that E and T define a domain exchange transformation on regular sets. Moreover, Item (5) of Lemma 18 provides it is self-affine with respect to the sets $E_{n,a,k}$ and the linear part $(N^t)^n$. Finally, Proposition 21 shows this domain exchange is measurably conjugate to the subshift (Ω_ξ, S) and Proposition 23 gives the map $\pi\circ F\colon \Omega_\xi \to \mathbb{T}^{d-1}$ is a.e. Z-to-one for some constant Z. $\square$

In the sequel, we denote by Z the constant of Proposition 23. We give here a characterization of this constant in term of the volume of the set $F(\Omega_\xi)$.

Proposition 24. *Assume Hypotheses P i) − iii). We have* $\lambda(F(\Omega_\xi)) = Z$.

Proof. The canonical projection $\pi : \mathbb{R}^{d-1} \to \mathbb{T}^{d-1}$ defines a factor map from the domain exchange to a minimal translation on the torus. So the image measure of the normalized measure $\frac{\lambda}{\lambda(F(\Omega_\xi))}$ is the Lebesgue measure on the torus. For any integrable function $f : F(\Omega_\xi) \to \mathbb{R}$, the conditional expectation $E(f|\pi^{-1}(\mathcal{B}_\mathbb{T}))$, with respect to the Borel σ-algebra of the torus $\mathcal{B}_\mathbb{T}$, is constant over any π-fiber. So it follows for a.e. points $y \in F(\Omega_\xi)$,

$$E(f|\pi^{-1}(\mathcal{B}_\mathbb{T}))(y) = \sum_{x\in F(\Omega_\xi);\ \pi(x)=\pi(y)} \gamma_{x,\pi(y)} f(x),$$

for some non negative measurable function $x \mapsto \gamma_{x,\pi(x)}$ such that

$$\sum_{x;\pi(x)=\pi(y)} \gamma_{x,\pi(y)} = 1 \quad \text{for a.e. } y. \tag{4.11}$$

Since for any integrable function $f : F(\Omega_\xi) \to \mathbb{R}$ with support in a unit square U, we have

$$\begin{aligned}\frac{1}{\lambda F(\Omega_\xi)} \int_U f d\lambda = & \int_U E(f|\pi^{-1}(\mathcal{B}_\mathbb{T}))d\lambda \\ = & \int_{U\cap F(\Omega_\xi)} \gamma_{x,\pi(x)} f(x) d\lambda(x).\end{aligned}$$

We obtain that $\gamma_{x,\pi(x)} = \frac{1}{\lambda F(\Omega_\xi)}$. We get the conclusion by the equation (4.11) □

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MINIMAL CONFIGURATIONS FOR THE FRENKEL-KONTOROVA MODEL ON A QUASICRYSTAL

BY
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Abstract. In this paper, we consider the Frenkel-Kontorova model of a one dimensional chain of atoms submitted to a potential. This potential splits into an interaction potential and a potential induced by an underlying substrate which is a quasicrystal. Under standard hypotheses, we show that every minimal configuration has a rotation number, that the rotation number varies continuously with the minimal configuration, and that every non negative real number is the rotation number of a minimal configuration. This generalizes well known results obtained by S. Aubry and P.Y. le Daeron in the case of a crystalline substrate.

Date: September 20, 2005.

1. Introduction

The Frenkel-Kontorova model [FK] describes the physical situation of a layer of a material over a substrate of other material (see for instance [BK]). In the one dimensional case, the layer of material is described by the configurations of a bi-infinite chain of particles on the real line. These configurations are parametrized by a bi-infinite non decreasing sequence $(\theta_n)_{n\in\mathbb{Z}}$ of real numbers, where θ_n represents the position of the particle labeled by n.
The potential energy of the chain reads:

$$\mathcal{E}((\theta_n)_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} U(\theta_n - \theta_{n+1}) + V(\theta_n),$$

where U describes the interaction between particles (only interactions with the nearest neighbors are considered), and V is a potential induced by the substrate and depends on its nature.
The following standard extra asumptions are made on U and V:

- *Smoothness:* the functions U and $V : \mathbb{R} \to \mathbb{R}$ are C^2;
- *Convexity:* $U''(x) > 0, \quad \forall\, x \in \mathbb{R}$;
- *Behavior at* ∞: $\displaystyle\lim_{x\to\pm\infty} \frac{U(x)}{|x|} = +\infty.$

Even if the above sum is only formal, it is possible to look for equilibrium configurations which minimize locally the energy (ground states). More precisely let us consider the function $\mathcal{H} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by:

$$\mathcal{H}(\theta, \theta') = U(\theta - \theta') + V(\theta).$$

For a configuration $(\theta_n)_{n\in\mathbb{Z}}$, let us set:

$$\mathcal{H}_p(\theta_i, \theta_{i+1}, \dots, \theta_{i+p}) = \sum_{j=0}^{j=p-1} \mathcal{H}(\theta_{i+j}, \theta_{i+j+1}).$$

We say that the *segment* $(\theta_i, \theta_{i+1}, \dots, \theta_{i+p})$ of the configuration $(\theta_n)_{n\in\mathbb{Z}}$ is *minimal* if

$$\mathcal{H}_p(\theta_i, \theta_{i+1}, \dots, \theta_{i+p}) \le \mathcal{H}_p(\theta'_i, \theta'_{i+1}, \dots, \theta'_{i+p}),$$

for any other segment $(\theta'_i, \theta'_{i+1}, \dots, \theta'_{i+p})$ such that $\theta'_i = \theta_i$ and $\theta'_{i+p} = \theta_{i+p}$. A configuration $(\theta_n)_{n\in\mathbb{Z}}$ is *minimal* if all its segments are minimal.

The substrate is a *crystal* when the configuration of the chain of atoms it is made of, is an increasing sequence $\mathcal{QC} = (s_n)_{n\in\mathbb{Z}}$ such that there exists $q \in \mathbb{Z}^+$ and $L > 0$ verifying:

$$s_{n+q} = s_n + L, \quad \forall\, n \in \mathbb{Z}.$$

In this case it is natural to consider that a potential V *associated* with the crystal $\mathcal{QC}$ is a periodic C^2-function with period L:

$$V(\theta + L) = V(\theta), \quad \forall\, \theta \in \mathbb{R}.$$

This situation when the substrate potential is periodic has been described by S. Aubry and P. Y. Le Dearon. Their seminal work [AD], together with the independent approach of J. Mather [M], gave rise to the so called *Aubry-Mather theory*, which yields in particular a good understanding of minimal configurations.

Let $\rho \in \mathbb{R}$, a configuration $(\theta_n)_{n\in\mathbb{Z}}$ has a *rotation number* equal to ρ if the limit:

$$\lim_{n\to\pm\infty} \frac{\theta_n}{n} = \rho.$$

Let us remark that the inverse of the rotation number can be interpreted as a particle density.
Aubry and le Daeron proved in particular that any minimal configuration has a rotation number, that the rotation number is a continuous function when defined on the set of minimal configurations equipped with the product topology, and that any positive real number is the rotation number for some minimal configuration[1].

The aim of this paper is to consider the case when the substrate is a quasicrystal in order to derive, in this more general setting, a similar description of the set of minimal configurations.

To fix notations and definitions, let us consider a bi-infinite substrate chain of atoms represented by its configuration $(s_n)_{n\in\mathbb{Z}}$. Two segments $(s_n, \dots, s_{n+p})$ and $(s_q, \dots, s_{q+p})$ are said *equivalent* if there exists $\tau \in \mathbb{R}$ such that:

$$s_{q+i} = s_{n+i} + \tau, \quad \forall\, i = 0, \dots, p.$$

The chain $\mathcal{QC} = (s_n)_{n\in\mathbb{Z}}$ is a *quasicrystal* if the following properties are satisfied[2](see for instance [LP]):

- *Finite local complexity*
  For any $M > 0$, the chain possesses only finitely many equivalence classes of segments with diameters smaller than M.
- *Repetitivity*
  For any segment S in the chain, there exists $R > 0$ such that any ball with radius R contains a segment equivalent to S.
- *Uniform pattern distribution*
  For any segment S in the chain, and for any point $x \in \mathbb{R}$, the quantity
  $$\frac{n(S, x, M)}{M}$$
  converges when $M \to +\infty$ uniformly in x to a limit $\nu(S)$ that does not depend on x, where $n(S, x, M)$ denotes the number of segments equivalent to S in the interval $[x, x + M]$.

Notice that a crystal (with period L) is a quasicrystal and in this particular case, for each segment S in $\mathcal{QC}$, one has:

$$\nu(S) = \frac{p(S)}{L},$$

where $p(S)$ stands for the number of segments equivalent to S in a period L.

For any $R > 0$, a function $V_{\mathcal{QC}} : \mathbb{R} \to \mathbb{R}$ is a *potential with range* R associated with a quasicrystal $\mathcal{QC}$ if for each pair of points x and y in $\mathbb{R}$ such that

$$\mathcal{QC} \cap B_R(x) - x = \mathcal{QC} \cap B_R(y) - y,$$

we have:

$$V_{\mathcal{QC}}(x) = V_{\mathcal{QC}}(y),$$

[1] Actually Aubry-Mather theory says much more about the combinatorics of minimal configurations when projected on a circle with length L.

[2] See Proposition 2.1 for a dynamical interpretation.

$$\begin{array}{rl}
L & L \\
L\,S & L\,S \\
L\,S\,L & L\,S\,L \\
L\,S\,L\,L\,S & L\,S\,L\,L\,S \\
L\,S\,L\,L\,S\,L\,S\,L & L\,S\,L\,L\,S\,L\,S\,L
\end{array}$$

FIGURE 1. Construction of the Fibonacci chain

where $B_M(z)$ stands for the ball with center z and radius M. Whenever $\mathcal{QC}$ is a crystal with period L, it is clear that a potential with range $R > 0$ associated with this crystal is a periodic potential with period L.
We call *short range potential* associated with a quasicrystal $\mathcal{QC}$ a potential with range R, for some $R > 0$.
Example: A standard example of quasicrystal is given by the Fibonacci sequence. Consider the set $\mathcal{G}$ of configurations $(s_n)_n$ such that:

- s_0 is located at 0;
- the lengths of the intervals $[s_n, s_{n+1}]$ have two possible sizes: either large and equal to L or small and equal to S.

The substitution:

$$\left\{ \begin{array}{l} L \rightarrow LS \\ S \rightarrow L \end{array} \right.$$

induces a map Ψ on $\mathcal{G}$ defined as follows:
For a sequence $(s_n)_n$ in $\mathcal{G}$, consider the sequence of lengths $(l_n)_n \in \{L, S\}^\infty$ defined by $l_n = s_{n+1} - s_n$, $\forall n \in \mathbb{Z}$. Applying to each l_n the substitution rule we get a new sequence $(l'_n)_n \in \{L, S\}^\infty$. The new configuration $(s'_n)_n = \Psi((s_n)_n)$ is obtained by setting:

- $s'_0 = 0$;
- $s'_{n+1} = s'_n + l'_n$, $\forall n \in \mathbb{Z}$.

Starting with the equidistributed configuration $(s^0_n)_n$, where $s_{n+1} - s_n = L, \forall n \in \mathbb{Z}$, it is easy to check that the sequence of configuration $(\Psi^k((s^0_n)_n))_k$ converges when $k \to +\infty$ (for the product topology) to a configuration $(s^\infty_n)_n$. This configuration is on the one hand a quasicrystal and on the other hand a periodic point with period 2 of the operator Ψ. This quasicrystal is called the *Fibonacci chain* (See Figure 1).

There are several ways to construct a short range potential associated with the Fibonacci chain. A simple one consists in choosing two real valued smooth functions, $v_{L,L}$, and $v_{S,L}$ with compact support on the interval $(-I, I)$ where $0 < 2I < S(< L)$. A potential V_{Fib} with range $2L$, can be defined as follows (see Figure 2):

- for each $n \in \mathbb{Z}$ and for each $\theta \in (s^\infty_n - I, s^\infty_n + I)$:
  - $V_{Fib}(\theta) = v_{L,L}(\theta - s^\infty_n)$ if both intervals $[s^\infty_{n-1}, s^\infty_n]$ and $[s^\infty_n, s^\infty_{n+1}]$ have the same length L;

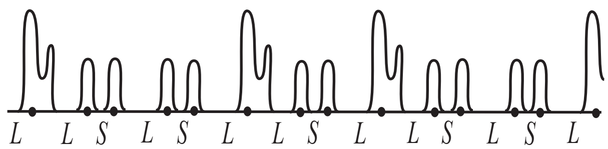

FIGURE 2. A short range potential associated with the Fibonacci chain

  – $V_{Fib}(\theta) = v_{S,L}(\theta - s_n^\infty)$ if the intervals $[s_{n-1}^\infty, s_n^\infty]$ and $[s_n^\infty, s_{n+1}^\infty]$ have different lengths.
- for $\theta \notin \cup_{n\in\mathbb{Z}}(s'_n - I, s'_n + I),\ V_{Fib}(\theta) = 0$.

The main result of this paper is the following theorem:

Theorem 1.1. [3]
For the Frenkel-Kontorova model with a short range potential associated with a quasicrystal:
- (i) *any minimal configuration has a rotation number;*
- (ii) *the rotation number is a continuous function when defined on the set of minimal configurations equipped with the product topology;*
- (iii) *for any $\rho \geq 0$, there exists a minimal configuration with rotation number ρ.*

It turns out that, once the appropriate objects have been defined, the proof of Theorem 1.1 has the same structure as the modern proof for crystals that can be found for instance in [B] or [C]. More precisely, in the crystal case, a periodic potential factorizes through a real valued function defined on a circle. In the quasicrystal case, a short range potential factorizes through a real valued function defined on a more sophisticated compact metric space called the *hull* of the quasicrystal. This hull possesses locally the product structure of an interval by a Cantor set *i.e* it is a solenoid. This solenoid can be seen as the suspension of the action of a minimal homeomorphism on the Cantor set.
Minimal homeomorphisms on the Cantor set have been extensively studied in topological dynamics and possess a powerful combinatorial description in terms of Kakutani-Rohlin towers (see for instance [GPS]). The aim of Section 2, which is devoted to the substrate, is to rephrase these well known results in our specific context, namely for a suspension, in order to see the hull as an inverse limit of one dimensional branched manifolds. These branched manifolds will play a central role in the proof.
In the crystal case, when projecting a minimal configuration on the circle, the Aubry-Mather theory shows that it wraps around the circle in a very special way, namely it is ordered as the orbit of a degree one homeomorphism of circle. In the quasicrystal case, there exists also some combinatorial obstructions, they are

[3]From a more physical point of view, it is straightforward but interesting to rephrase Theorem 1.1 in terms of particle density of minimal configurations.

described and analyzed in Section 3 which is devoted to the ground states of the overlying layer.
Section 4 is devoted to the proof of Theorem 1.1. First, as for the crystal case, we show, using the inverse limit structure of the hull given in Section 2 and the combinatorial obstructions gotten in Section 3, that minimal configurations have a rotation number (point (i)). Then we prove (again as in the crystal case) the continuity of the rotation number (point (ii)). The proof of point (iii) of Theorem 1.1 in the crystal case is done first by constructing periodic minimal configurations for any positive rational rotation number and then to use the continuity of the rotation number to get a minimal configuration for any prescribed positive rotation number. In the quasicrystal case, the scheme is exactly the same, but the set of rational numbers needs to be replaced by another dense subset of the positive reals. More precisely when the rotation number is not 0, its inverse has to be a finite linear combination with positive integer coefficients of the densities of patches of the quasicrystal.
This paper ends with two final remarks developed in Section 5, the first one concerns dynamical systems. In the case of a crystal, minimal configurations for the Frenkel-Kontorova model are orbits of a twist map on an open annulus. Similarly, in the quasicrystal case, these minimal configurations are also orbits of a dynamical system that we describe. The second one consists in giving the bases of a possible extension of the theory to quasicrystals in higher dimension.

Remark: It should be pointed out that one can find in the literature several studies on the the Frenkel-Kontorova model with a quasi-periodic potential, for instance a potential which is the sum of two periodic potentials with incommensurable periods (see for instance [EFRJ]). Such potential cannot arise naturally from an underlying one dimensional substrate. Actually, the underlying object which organizes the minimal configurations and which was a circle in the crystal case and a solenoid in the quasicrystal case, becomes a 2-torus. More precisely the real line is immersed as a line with irrational slope in the 2-torus. Actually, this is a situation more complex than the one we are dealing with in this paper which essentially uses dimension 1 objects, and this explains the lack of exact results in this quasi-periodic case.

Nota Bene: To avoid an unnecessary dichotomy and unless explicitly specified, the quasicrystals that appear in the sequel will not be crystals.

2. The hull of a quasicrystal

In this section, we recall some background results concerning quasicrystals. Most of these results are true in any dimension and they are presented here in the particular case of the dimension 1. Material for Subsections 2.1 and 2.2 can be found in [KP], [BBG] and [BG]. For Subsections 2.4 and 2.5 a discrete approach can be found in [HPS] and we refer again to [BBG], [BG] and [S] for a more geometrical point of view.

2.1. **The hull as a dynamical system.** Consider a chain of atoms whose configuration is a quasicrystal $\mathcal{QC} = (s_n)_{n\in\mathbb{Z}}$. It is clear that each translated copy $\mathcal{QC} - u = (s_n - u)_{n\in\mathbb{Z}}$, $u \in \mathbb{R}$, of $\mathcal{QC}$ is again a quasicrystal.
The set of translated copies $\mathcal{QC}+\mathbb{R}$ of a quasicrystal can be equipped with a topology that, roughly speaking, says that two quasicrystal configurations are close one to

the other if in a big ball centered at 0 in $\mathbb{R}$, the segments of both configurations inside the ball are equivalent and equal up to a small translation. Such a topology is metrizable and an associated metric can be defined as follows (see [RW] for more details):
Consider two quasicrystal configurations $\mathcal{QC} - u_1$ and $\mathcal{QC} - u_2$ in $\mathcal{QC} + \mathbb{R}$. Let A denote the set of $\epsilon \in]0, 1[$ for which there exists u with $|u| < \epsilon$, such that $\mathcal{QC} - u_1$ and $\mathcal{QC} - u_2 + u$ coincide in $B_{1/\epsilon}(0)$. Then

$$\delta(\mathcal{QC} - u_1,\, \mathcal{QC} - u_2) \,=\, \inf A \quad \text{if} \quad A \neq \emptyset$$
$$\delta(\mathcal{QC} - u_1,\, \mathcal{QC} - u_2) \,=\, 1 \quad \text{if} \quad A = \emptyset\ .$$

Hence the diameter of $\mathcal{QC} + \mathbb{R}$ is bounded by 1 and the $\mathbb{R}$-action on $\mathcal{QC} + \mathbb{R}$ is continuous. The *continuous hull* $\Omega(\mathcal{QC})$ of the quasicrystal $\mathcal{QC}$ is the completion of the metric space $(\mathcal{QC} + \mathbb{R}, \delta)$.
As a direct consequence of the finite local complexity property, it is easy to check (see for instance [RW]) that $\Omega(\mathcal{QC})$ is a compact metric space and that any element in $\Omega(\mathcal{QC})$ is a quasicrystal whose segments are equivalent to segments in $\mathcal{QC}$. The translation group $\mathbb{R}$ acts on $\Omega(\mathcal{QC})$ and the dynamical system $(\Omega(\mathcal{QC}), \mathbb{R})$ possesses (by construction) a dense orbit (namely the orbit $\mathcal{QC} + \mathbb{R}$). On the one hand, the repetitivity property is equivalent to the *minimality* of the action *i.e* all its orbits are dense, (see [KP]) and, on the other hand, the uniform pattern distribution is equivalent to the unique ergodicity *i.e* the $\mathbb{R}$-action possesses a unique invariant probability measure (see[BG]). These results yield the following proposition.

Proposition 2.1. *Let $\mathcal{QC}$ be a quasicrystal, then the dynamical system* $(\Omega(\mathcal{QC}), \mathbb{R})$ *is minimal and uniquely ergodic.*

In the sequel, we will denote by μ the unique probability measure on $\Omega(\mathcal{QC})$ which is invariant under the $\mathbb{R}$-action.

2.2. **The canonical transversal.** The *canonical transversal*, $\Omega_0(\mathcal{QC})$, of the hull $\Omega(\mathcal{QC})$ of a quasicrystal $\mathcal{QC}$ is the collection of quasicrystals in $\Omega(\mathcal{QC})$ which contain 0 (*i.e.* such that one atom in the chain is located at 0).

Proposition 2.2. *(see* [KP]*) The canonical transversal of a quasicrystal is either a finite set when $\mathcal{QC}$ is a crystal or a Cantor set when not.*

It follows that when the quasicrystal $\mathcal{QC}$ is a crystal, $\Omega(\mathcal{QC})$ is homeomorphic to a circle and when not $\Omega(\mathcal{QC})$ has a solenoidal structure, *i.e.* it is locally the product of a Cantor set by an interval.
The *return time* function $\mathcal{L} : \Omega_0(\mathcal{QC}) \to \mathbb{R}^+$ is defined by:

$$\mathcal{L}(\mathcal{T}) = \inf\{t > 0 \mid \mathcal{T} - t \in \Omega_0(\mathcal{QC})\} \quad \forall \mathcal{T} \in \Omega_0(\mathcal{QC}).$$

The *finite local complexity* implies that the function $\mathcal{L}$ is locally constant, it takes finitely many distinct values $L_1, \ldots, L_p$ and the *clopen* (closed open) sets $\mathcal{C}_i = \mathcal{L}^{-1}(L_i)$ for $i = 1, \ldots, p$ form a partition of $\Omega_0(\mathcal{QC})$[4] (see Figure 3).
The *first return map* $\tau : \Omega_0(\mathcal{QC}) \to \Omega_0(\mathcal{QC})$ is defined by:

$$\tau(\mathcal{T}) \,=\, \mathcal{T} \,-\, \mathcal{L}(\mathcal{T}) \quad \forall \mathcal{T} \,\in\, \Omega(\mathcal{QC}).$$

The unique invariant probability measure μ of the $\mathbb{R}$-action on $\Omega(\mathcal{QC})$ induces a finite measure ν on $\Omega_0(\mathcal{QC})$ which is τ-invariant.

[4]Recall that clopen sets form a countable basis for the topology of a totally disconnected set.

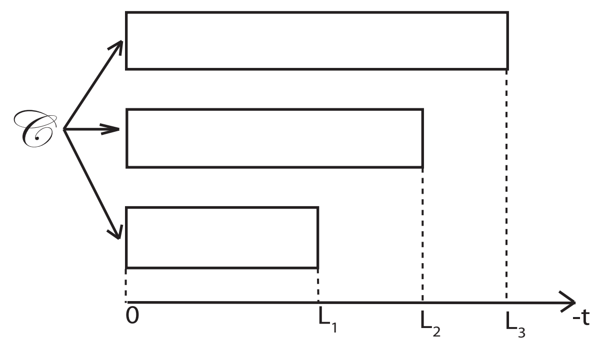

FIGURE 3. The time return function

For any $i = 1, \ldots, p$ and for any clopen set $\mathcal{C}$ in $\mathcal{C}_i$, the measure ν satisfies:

$$\nu(\mathcal{C}) \;=\; \frac{1}{L_i}\mu(\{(\mathcal{T} - u) \quad \mathcal{T} \,\in\, \mathcal{C}, \quad u \,\in\, [0, L_i]\}).$$

The subsets of $\Omega(\mathcal{QC})$ which read $\mathcal{C} - u$ where $\mathcal{C}$ is a clopen set in one of the $\mathcal{C}_i$'s and $u \in [0, L_i[$ are called *verticals*.
The following lemma is a direct byproduct of the above definition:

Lemma 2.3. *For any $S > 0$, there exists a positive constant $\epsilon_{\mathcal{QC}}(S)$ such that, for any vertical V with diameter smaller that $\epsilon_{\mathcal{QC}}(S)$ and any pair of configurations $\mathcal{QC} - x$ and $\mathcal{QC} - y$ in V, we have:*

$$\mathcal{QC} \cap B_S(x) \,-\, x = \,\mathcal{QC} \cap B_S(y) \,-\, y.$$

2.3. **Potentials on the hull.** The following result shows that a short range potential associated with a quasicrystal $\mathcal{QC}$, factorizes through a function on $\Omega(\mathcal{QC})$.

Lemma 2.4. *Let $\mathcal{QC}$ be a quasicrystal, and let $V_{\mathcal{QC}}$ be a continuous short range potential associated with $\mathcal{QC}$. Then, there exists a unique continuous function $\bar{V}_{\mathcal{QC}}$: $\Omega(\mathcal{QC}) \to \mathbb{R}$ such that:*

$$V_{\mathcal{QC}}(x) \;=\; \bar{V}_{\mathcal{QC}}(\mathcal{QC} - x), \quad \forall x \,\in\, \mathbb{R}.$$

Furthermore, when $V_{\mathcal{QC}}$ has range $R > 0$, there exists a positive constant $\epsilon_{\mathcal{QC}}(R)$ such that $\bar{V}_{\mathcal{QC}}$ is constant on each vertical with diameter smaller than $\epsilon_{\mathcal{QC}}(R)$.

Remark: Notice that when $\mathcal{QC}$ is a crystal, Lemma 2.4 simply means that for any continuous periodic function $g : \mathbb{R} \to \mathbb{R}$ with period L, there exists a continuous function $G : \mathbb{R}/L.\mathbb{Z} \to \mathbb{R}$ such that $g = G \circ \pi$, where $\pi : \mathbb{R} \to \mathbb{R}/L.\mathbb{Z}$ is the standard projection.
Proof of Lemma 2.4: Assume that $V_{\mathcal{QC}}$ is a potential with range $R > 0$. Applying Lemma 2.3, for any vertical V with diameter smaller than $\epsilon_{\mathcal{QC}}(R)$ and any pair $\mathcal{QC} - x$ and $\mathcal{QC} - y$ in V, we have:

$$\mathcal{QC} \cap B_R(x) \,-\, x = \,\mathcal{QC} \cap B_R(y) \,-\, y,$$

and thus:

$$V_{\mathcal{QC}}(x) \;=\; V_{\mathcal{QC}}(y).$$

Since the set $\mathcal{QC} + \mathbb{R} \cap V$ is dense in V, it follows that a continuous function $\bar{V}_{\mathcal{QC}}$ which satisfies $V_{\mathcal{QC}}(x) = \bar{V}_{\mathcal{QC}}(\mathcal{QC} - x)$, $\forall\, x \in \mathbb{R}$, must be constant on V and equal to $V_{\mathcal{QC}}(y)$ for any real number y such that $\mathcal{QC} - y \in V$. Conversely the function $\bar{V}_{\mathcal{QC}}$ defined this way is clearly continuous, satisfies $V_{\mathcal{QC}}(x) = \bar{V}_{\mathcal{QC}}(\mathcal{QC} - x)$, $\forall\, x \in \mathbb{R}$, and is constant on verticals with diameters smaller than $\epsilon_{\mathcal{QC}}(R)$. □

2.4. **Kakutani-Rohlin towers.** The following construction, which has been developed for the study of minimal dynamics on the Cantor set, will be useful all along this paper. It is often referred to as *Kakutani-Rohlin towers* (see [HPS]). Choose $S > 0$ and fix a clopen set $\mathcal{C}$ in one of the $\mathcal{C}_i$'s with diameter smaller than $\epsilon_{\mathcal{QC}}(S)$.
Consider the first return time function $\mathcal{L}_{\mathcal{C}}$ associated with this clopen set (which is constructed exactly as the first return time function in $\Omega_0(\mathcal{QC})$). The finite local complexity hypothesis implies that the function $\mathcal{L}_{\mathcal{C}}$ is locally constant and takes finitely many values $L_{\mathcal{C},1}, \ldots, L_{\mathcal{C},p(\mathcal{C})}$. The clopen sets $\mathcal{D}_{\mathcal{C},i} = \mathcal{L}_{\mathcal{C}}^{-1}(L_{\mathcal{C},i})$ for $i = 1, \ldots, p(\mathcal{C})$ form a partition of $\mathcal{C}$. Again because of the finite local complexity hypothesis, there exists a finite partition of $\mathcal{C}$ in clopen sets $\mathcal{E}_j$, $j = 1, \ldots r$ such that for each $j \in \{1, \ldots, r\}$, there exists $i \in \{1, \ldots p(\mathcal{C})\}$ so that the following properties are satisfied:

- $\mathcal{E}_j \subset D_{\mathcal{C},i}$;
- for each $u \in [0, L_{\mathcal{C},i}[$, $\mathcal{E}_j - u$ is a vertical with diameter smaller that $\epsilon_{\mathcal{QC}}(S)$.

For $j = 1, \ldots, r$, the set:

$$\{\mathcal{E}_j \,-\, u, \quad \forall\, u \,\in\, [0, L_{\mathcal{C},i}[\},$$

is called a *tower* with *height* $L_{\mathcal{C},i}$. The union of all these towers realizes a partition of $\Omega(\mathcal{QC})$ and the data $(\mathcal{QC}, S, \mathcal{C}, \{\mathcal{E}_j\}_{j\in\{1,\ldots,r\}})$ is called a *Kakutani-Rohlin towers system with size* S.

For $j = 1, \ldots, r$, consider the set $\mathcal{E}_j \subset D_{\mathcal{C},i}$ and for each $u \in [0, L_{\mathcal{C},i}[$, we call *floor* of the tower $\mathcal{E}_j \times [0, L_{\mathcal{C},i}[$, the vertical $\mathcal{E}_j - u$. By identifying all the points in this vertical, each tower projects on a semi-open interval and the whole hull $\Omega(\mathcal{QC})$ projects onto a smooth branched one-dimensional manifold which is a collection of r of circles $\gamma_1, \ldots, \gamma_r$ tangent at a single point. This branched manifold is called the *skeleton* of the Kakutani-Rohlin tower system $(\mathcal{QC}, S, \mathcal{C}, \{\mathcal{E}_j\}_{j\in\{1,\ldots,r\}})$. It inherits a natural orientation, a differentiable structure and a natural metric respectively issued from the orientation, the differentiable structure and the Euclidean metric of the real line $\mathbb{R}$ (see Figure 4). We denote it $\mathcal{B}$ and call $\pi : \Omega(\mathcal{QC}) \to \mathcal{B}$ the above identification.
The proof of the following lemma is plain.

Lemma 2.5. *Let $R > 0$ and $V_{\mathcal{QC}}$ a continuous potential associated with $\mathcal{QC}$ with range $R > 0$. Consider a Kakutani-Rohlin towers system with size S and let $\mathcal{B}$ be its skeleton.*
Assume that $S \geq R$, then the function $\bar{V}_{\mathcal{QC}} : \Omega(\mathcal{QC}) \to \mathbb{R}$ induced by $V_{\mathcal{QC}}$ descends to a continuous function $\hat{V}_{\mathcal{QC}} : \mathcal{B} \to \mathbb{R}$:

$$\hat{V}_{\mathcal{QC}} \circ \pi \;=\; \bar{V}_{\mathcal{QC}}.$$

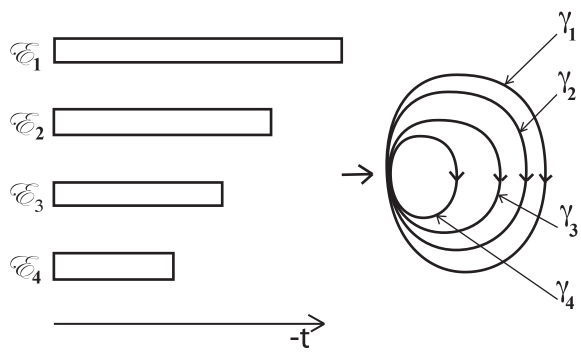

FIGURE 4. A towers system and its skeleton

Whenever the function $V_{\mathcal{QC}}$ is C^r-smooth for some $0 \leq r \leq \infty$, then the function $\hat{V}_{\mathcal{QC}}$ is also C^r-smooth.

2.5. **Inverse limits.** Let us choose an increasing sequence $(S_n)_{n\geq 0}$ going to $+\infty$ with n and let us construct inductively an infinite sequence of Kakutani-Rohlin towers system as follows (see [HPS]):

- Fix a point x_0 in $\Omega_0(\mathcal{QC})$.
- Choose a clopen set $\mathcal{C}_0$ containing x_0, with diameter smaller than $\epsilon_{\mathcal{QC}}(S_0)$ and construct a Kakutani-Rohlin towers system $(\mathcal{QC}, S_0, \mathcal{C}_0, \{\mathcal{E}_{0,j}\}_{j\in\{1,\ldots,r_0\}})$ with size S_0. Up to a renaming of the indices, we can assume that x_0 belongs to $\mathcal{E}_{0,1}$. We denote by $\mathcal{B}_0$ the corresponding skeleton and call $\pi_0 : \Omega(\mathcal{QC}) \to \mathcal{B}_0$ the standard projection.
- We choose a clopen set $\mathcal{C}_1 \subset \mathcal{E}_{0,1}$ which contains x_0 with a diameter small enough so that we can construct a Kakutani-Rohlin towers system $(\mathcal{QC}, S_1, \mathcal{C}_1, \{\mathcal{E}_{1,j}\}_{j\in\{1,\ldots,r_1\}})$ with size S_1 such that each of its towers intersects all the towers of the previous system. Up to a renaming of the indices, we can assume that x_0 belongs to $\mathcal{E}_{1,1}$. We denote by $\mathcal{B}_1$ the corresponding skeleton and call $\pi_1 : \Omega(\mathcal{QC}) \to \mathcal{B}_1$ the standard projection.
- Assume we have constructed a sequence of nested clopen sets $\mathcal{C}_n \subset \mathcal{C}_{n-1} \subset \ldots \mathcal{C}_1 \subset \mathcal{C}_0$ containing x_0 and, for each $p = 0, \ldots, n$, a Kakutani-Rohlin towers system $(\mathcal{QC}, S_p, \mathcal{C}_p, \{\mathcal{E}_{p,j}\}_{j\in\{1,\ldots,r_p\}})$ with size S_p such that each of its towers intersects all the towers of the system associated with $p-1$, and such that x_0 belongs to $\mathcal{E}_{p,1}$. We iterate the procedure by choosing a clopen set $\mathcal{C}_{n+1} \subset \mathcal{E}_{n,1}$ which contains x_0 small enough so that we can construct a Kakutani-Rohlin towers system $(\mathcal{QC}, S_{n+1}, \mathcal{C}_{n+1}, \{\mathcal{E}_{n+1,j}\}_{j\in\{1,\ldots,r_{n+1}\}})$ with size S_{n+1} such that each of its towers intersects all the towers of the system associated with n. Up to a renaming of the indices, we can assume that x_0 belongs to $\mathcal{E}_{n+1,1}$. We denote by $\mathcal{B}_{n+1}$ the corresponding skeleton and call $\pi_n : \Omega(\mathcal{QC}) \to \mathcal{B}_n$ the standard projection.

For each $n \geq 0$, fix a point y in $\mathcal{B}_{n+1}$. The set $\pi_{n+1}^{-1}(y)$ is included in a floor of a tower of the tower system $(\mathcal{QC}, S_p, \mathcal{C}_p, \{\mathcal{E}_{p,j}\}_{j\in\{1,\ldots,r_p\}})$, and thus descends through

π_n to a single point on $\mathcal{B}_n$. We have defined this way a continuous surjection:

$$\tau_n : \mathcal{B}_{n+1} \to \mathcal{B}_n.$$

The inverse limit:

$$\lim_{\leftarrow \tau_n} \mathcal{B}_n = \{(x_n)_{n\geq 0} \,|\, x_n \in \mathcal{B}_n \text{ and } \tau_n(x_{n+1}) = x_n, \ \forall n \geq 0\},$$

gives a re-interpretation of the hull $\Omega(\mathcal{QC})$:

Proposition 2.6. [BG] *When equipped with the product topology the set* $\lim\limits_{\leftarrow \tau_n} \mathcal{B}_n$ *is homeomorphic to* $\Omega(\mathcal{QC})$*.*

Notice that the map $\tau_n : \mathcal{B}_{n+1} \to \mathcal{B}_n$ induces a $p_n \times p_{n+1}$ *homology matrix* M_n whose integer coefficient $m_{n,i,j}$ is the number of times the loop $\gamma_{n+1,j}$ in $\mathcal{B}_{n+1}$ covers the loop $\gamma_{n,i}$ of $\mathcal{B}_n$ under the action of the map τ_n. We remark that the construction of the sequences of towers systems we made insures that, for all $n \geq 0$, the matrix M_n has positive coefficients. These matrices carry information about the invariant measure ν on the Cantor set through the following lemma (see for instance [GPS]):

Lemma 2.7.

$$\nu_{n,i} = \sum_{j=1}^{j=p_{n+1}} m_{n,i,j}\nu_{n+1,j}, \quad \forall i \in \{1, \dots, p(n)\},$$

where $\nu_{n,i}$ *is the measure of the clopen set* $\mathcal{E}_{n,i}$*.*

Again the following lemma is plain:

Lemma 2.8. *Let* $R > 0$ *and* $V_{\mathcal{QC}}$ *be a continuous potential associated with* $\mathcal{QC}$ *with range* $R > 0$ *and choose an increasing sequence* $(S_n)_{n\geq 0}$ *going to* $+\infty$*, such that* $R \leq S_0$*. Then, for each* $n \geq 0$*, the function* $V_{\mathcal{QC}}$ *induces on each branched manifold* $\mathcal{B}_n$ *a function* $\hat{V}_{\mathcal{QC},n}$ *which satisfies:*

$$\hat{V}_{\mathcal{QC},n} \circ \tau_n = \hat{V}_{\mathcal{QC},n+1}.$$

3. Combinatorics of minimal configurations

In this section, we consider the minimal segments for a short range potential with range R associated with $\mathcal{QC}$.

Lemma 3.1. *Let* I *and* $J = I + u$ *be two disjoint intervals in* $\mathbb{R}$ *such that for each* θ *in* I*:*

$$B_R(\theta) \cap \mathcal{QC} + u = B_R(\theta + u) \cap \mathcal{QC},$$

and let $(\theta_1, \dots, \theta_n)$ *be a minimal segment such that* $[\theta_1, \theta_n]$ *contains* I *and* J*. For any pair of consecutive atoms* θ_m *and* θ_{m+1} *in* $I \cap \mathcal{QC}$*, the interval* $[\theta_m + u, \theta_{m+1} + u]$ *contains at most two atoms of the minimal segment.*

Proof. The proof works by contradiction. Assume that there exists a pair of atoms θ_m and θ_{m+1} in $I \cap \mathcal{QC}$, such that the interval $[\theta_m + u, \theta_{m+1} + u]$ contains three consecutive atoms of the minimal segment, say θ_l, θ_{l+1}, and θ_{l+2}:

$$[\theta_l, \theta_{l+2}] \subset [\theta_m + u, \theta_{m+1} + u].$$

We consider the new segment obtained by taking the atom in position θ_{l+1} and assigning to it the new position $\theta_{l+1} - u$ (Figure 5). When $u > 0$ (what we can assume without loss of generality) this segment reads:

$$(\theta_1, \dots, \theta_i, \dots, \theta_m, \theta_{l+1} - u, \theta_{m+1}, \dots, \theta_l, \theta_{l+2}, \dots, \theta_n).$$

To get a contradiction we are going to show that the potential energy of this new segment is smaller than the potential energy of the first one. On the one hand, since $B_R(\theta_{l+1}) \cap \mathcal{QC} - u = B_R(\theta_{l+1} - u) \cap \mathcal{QC}$, the potential energy induced by the substrate on the atom that changed its position, keeps the same value:

$$V_{\mathcal{QC}}(\theta_{l+1}) = V_{\mathcal{QC}}(\theta_{l+1} - u).$$

Thus, the sum of the potential energy induced by the substrate on the whole segment is not affected by this change of position.
On the other hand, the difference of the potential energy of interaction between the new segment and the former one is given by:

$$\begin{aligned}\Delta U = &(U(\theta_m - \theta_{l+1} + u) + U(\theta_{l+1} - u - \theta_{m+1}) - U(\theta_m - \theta_{m+1})) \\ &-(U(\theta_l - \theta_{l+1}) + U(\theta_{l+1} - \theta_{l+2}) - U(\theta_l - \theta_{l+2})).\end{aligned}$$

Let us introduce the new variables:

$$X = \theta_m - \theta_{l+1} + u, \quad Y = \theta_{l+1} - u - \theta_{m+1},$$
$$X' = \theta_l - \theta_{l+1}, \quad Y' = \theta_{l+1} - \theta_{l+2}.$$

We have:

$$X \le X' < 0 \quad \text{and} \quad Y \le Y' < 0,$$

and:

$$\Delta U = (U(X) + U(Y) - U(X+Y)) - (U(X') + U(Y') - U(X'+Y')).$$

For $t \in [0,1]$, let us consider the function:

$$G(t) = U(tX + (1-t)X') + U(tY + (1-t)Y') - U(t(X+Y) + (1-t)(X'+Y')).$$

We have:

$$\Delta U = G(1) - G(0),$$

and

$$\begin{aligned} G'(t) &= U'(tX + (1-t)X')(X - X') + U'(tY + (1-t)Y')(Y - Y') \\ &\quad - U'(t(X+Y) + (1-t)(X'+Y'))(X + Y - X' - Y') \\ &= (U'(tX + (1-t)X') - U'(t(X+Y) + (1-t)(X'+Y')))(X - X') \\ &\quad + (U'(tY + (1-t)Y') - U'(t(X+Y) + (1-t)(X'+Y')))(Y - Y'). \end{aligned}$$

Observe that for $t \in [0,1]$:

$$tX + (1-t)X' \ge t(X+Y) + (1-t)(X'+Y')$$

and

$$tY + (1-t)Y' \ge t(X+Y) + (1-t)(X'+Y').$$

Using the convexity of U, more precisely the fact that U' is an increasing function we get that:

$$\Delta U \le 0,$$

and this inequality is strict as long as $\theta_m \neq \theta_l - u$ and $\theta_{m+1} \neq \theta_{l+2} - u$. In this case, we get the desired contradiction.

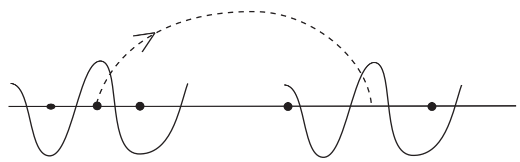

FIGURE 5. Move of a single atom in a segment

In the situation when $\theta_m = \theta_l - u$ and $\theta_{m+1} = \theta_{l+2} - u$, we remark that both segments $(\theta_m, \theta_{l+1} - u, \theta - m + 1)$ and $(\theta_{l-1}, \theta_l, \theta_{l+2})$ are not minimal and thus the new configuration we constructed is not minimal. The corresponding minimal segment (by fixing the extremities θ_1 and θ_n) has an energy which is strictly smaller, a contradiction. □

The following lemma shows that there are actually more obstructions than the ones described in Lemma 3.1.

Lemma 3.2. *With the same hypotheses and notations as in Lemma 3.1, consider two disjoint pairs of successive atoms $\theta_m < \theta_{m+1} < \theta_{m'} < \theta_{m'+1}$ in $I \cap \mathcal{QC}$, such that at least one of the four points $\theta_m + u < \theta_{m+1} + u < \theta_{m'} + u < \theta_{m'+1} + u$ does not belong to the minimal segment. Concerning the two intervals $[\theta_m + u, \theta_{m+1} + u]$ and $[\theta_{m'} + u, \theta_{m'+1} + u]$, none of the following three situations is possible (see Figure 6):*

(i) *both intervals contain two atoms of the minimal segment;*
(ii) *both intervals do not contain atoms of the minimal segment in their interiors;*
(iii) *one of the interval contains two atoms of the minimal segment and the other does not contain atoms in its interior.*

Proof. As for Lemma 3.1, we are going to reach a contradiction assuming that situation (i) occurs. The proof for the other two cases works exactly along the same lines. Let $\theta_l < \theta_{l+1} < \theta_{l'} < \theta_{l'+1}$ be atoms of the minimal segment such that:

$$[\theta_l, \theta_{l+1}] \subset [\theta_m + u, \theta_{m+1} + u]$$

and

$$[\theta_{l'}, \theta_{l'+1}] \subset [\theta_{m'} + u, \theta_{m'+1} + u].$$

Assuming again that $u > 0$, let us move some atoms of the minimal configuration to reach the following new configuration:

$$(\theta_1, \ldots, \theta_m, \theta_{l+1} - u, \ldots, \theta_{l'} - u, \theta_{m'+1}, \ldots, \theta_l, \theta_{m+1} + u, \ldots \theta_{m'} + u, \theta_{l'+1}, \ldots, \theta_n).$$

Since for each θ in I:

$$B_R(\theta) \cap \mathcal{QC} + u \;=\; B_R(\theta + u) \cap \mathcal{QC},$$

the potential energy induced by the substrate on the atoms did not change even if the atoms have changed their positions. Thus, the sum of the potential energy

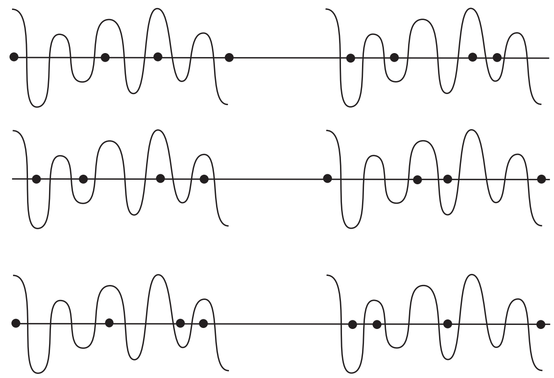

FIGURE 6. The forbidden 3 situations

induced by the substrate on the whole segment is not affected by this change of position.
On the other hand, the difference of the potential energy of interaction between the new segment and the old one is given by:

$$\Delta U = \Delta U_1 + \Delta U_2,$$

where

$$\Delta U_1 = (U(\theta_m - \theta_{l+1} + u) + U(\theta_l - \theta_{m+1} - u)) - (U(\theta_m - \theta_{m+1}) + U(\theta_l - \theta_{l+1})),$$

and

$$\Delta U_2 = (U(\theta_{l'} - u - \theta_{m'+1}) + U(\theta_{m'} + u - \theta_{l'+1})) - (U(\theta_{m'} - \theta_{m'+1}) + U(\theta_{l'} - \theta_{l'+1})).$$

Let us introduce the new variables:

$$X_0 = \theta_m \quad X_1 = \theta_{l+1} - u \quad \text{and} \quad Y_0 = \theta_l - u \quad Y_1 = \theta_{m+1}.$$

We have:

$$\Delta U_1 = (U(X_0 - X_1) + U(Y_0 - Y_1)) - (U(X_0 - Y_1) + U(Y_0 - X_1)).$$

This yields:

$$\Delta U_1 = -\int_{X_0}^{Y_0} \left(\int_{X_1}^{Y_1} U''(v-u)du \right) dv.$$

Since U is convex, $X_0 \leq Y_0$ and $X_1 \leq Y_1$ and at least one of these inequalities is strict, we get:

$$\Delta U_1 < 0,$$

and for the same reason

$$\Delta U_2 < 0.$$

This yields a contradiction. □

From the previous two lemmas, we deduce that the quantity of atoms of the minimal segments which belong to I and to $I+u$ differ by an integer smaller than 2. This is summarized in the following proposition that will be our main tool in the sequel of this paper.

Proposition 3.3. *Let $(\theta_1, \dots, \theta_n)$ be a minimal segment and let I be an interval in $[\theta_1, \theta_n]$, then there exists an integer $N \in \mathbb{Z}^+$ such that for any pair of disjoint intervals $I_1 = I + u_1$ and $I_2 = I + u_2$ in $[\theta_1, \theta_n]$ which satisfy that for each θ in I and $k = 1, 2$:*

$$B_R(\theta) \cap \mathcal{QC} + u_k \ = \ B_R(\theta + u_k) \cap \mathcal{QC},$$

each interval I_k contains either N, $N+1$ or $N+2$ atoms of the minimal segment.

4. Proof of Theorem 1.1

4.1. **Existence of a rotation number.** In this subsection, we consider a minimal configuration for a potential with range R associated with $\mathcal{QC}$. Let us consider an increasing sequence $(S_l)_{l\geq 0}$ going to $+\infty$ with l and such that $S_0 > R$ and consider also an associated sequence of Kakutani-Rohlin towers systems $(\mathcal{QC}, S_l, \mathcal{C}_l, \{\mathcal{E}_{l,j}\}_{j\in\{1,\dots,r_l\}})_{l\geq 0}$ and the corresponding sequence of skeletons $(\mathcal{B}_n)_{n\geq 0}$ as constructed in Subsection 2.5.
The identification

$$I : x \in \mathbb{R} \mapsto \mathcal{QC} - x \in \Omega(\mathcal{QC})$$

induces an immersion of the real line in $\mathcal{QC}$ and the image of a configuration $(\theta_n)_n$ through this immersion is an element $(\bar{\theta}_n)_n$ in $\Omega(\mathcal{QC})^{\mathbb{Z}}$ where $\bar{\theta}_n = \mathcal{QC} - \theta_n$, for all $n \in \mathbb{Z}$. In turn, for any $l \geq 0$, the projection $\pi_l : \Omega(X) \to \mathcal{B}_l$ transforms this sequence in an element $(\hat{\theta}^l_n)_n$ in $\mathcal{B}_l^{\mathbb{Z}}$ where $\hat{\theta}^l_n = \pi_l(\bar{\theta}_n) = \pi_l \circ I(\theta_n)$, for all $n \in \mathbb{Z}$. Furthermore we have:

$$\hat{V}_{\mathcal{QC},l}(\hat{\theta}_n) \ = \ \bar{V}_{\mathcal{QC}}(\bar{\theta}_n) \ = \ V_{\mathcal{QC}}(\theta_n).$$

The following lemma is a direct consequence of Proposition 3.3:

Lemma 4.1. *Let $(\theta_n)_n$ be a minimal configuration such that $\lim_{n\to+\infty} \theta_n = +\infty$ and $\lim_{n\to-\infty} \theta_n = -\infty$ (resp. let $(\theta_p, \dots, \theta_q)$ be a minimal segment). Then, for any $l \geq 0$ and any $j \in \{1, \dots, r_l\}$, there exists an integer $N_{l,j}$ such that for each loop $\gamma_{l,j}$ of $\mathcal{B}_l$, each connected component of $(\pi_l \circ I)^{-1}(\gamma_{l,j}) \subset \mathbb{R}$ (resp. each connected component of $(\pi_l \circ I)^{-1}(\gamma_{l,j}) \subset \mathbb{R}$ which does not intersect $(-\infty, \theta_p] \cup [\theta_q, +\infty)$) contains either $N_{l,j}$ or $N_{l,j}+1$ or $N_{l,j}+2$ atoms of the minimal configuration (resp. the minimal segment).*
In other words, when n increases, the projection of the minimal configuration (resp. the minimal segment) stays the same amount of time in a given loop up to an error of 2.

Now we can prove the existence of a non negative rotation number for any minimal configurations.

First, consider a minimal configuration $(\theta_n)_n$ such that $\lim_{n\to+\infty} \theta_n = +\infty$ and $\lim_{n\to-\infty} \theta_n = -\infty$. Let us estimate the length of the interval $[\theta_0, \theta_n]$ for $n \geq 0$. Let $n_{l,j}$ be the number of times $\pi_l \circ I([\theta_0, \theta_n])$ covers completely the loop $\gamma_{l,j}$ of $\mathcal{B}_l$. We have, for each $l \geq 0$:

$$\sum_{j=1}^{p_l} n_{l,j} L_{l,j} \,\leq\, \theta_n - \theta_0 \,\leq\, \sum_{j=1}^{p_l} n_{l,j} L_{l,j} \,+\, 2L_l,$$

where $L_{l,j}$ is the height of the tower associated with the loop $\gamma_{l,j}$ (*i.e.* the length of the loop $\gamma_{l,j}$) and

$$L_l \,=\, \max_{j\in\{1,\ldots,p_l\}} L_{l,j}.$$

On the other hand we have:

$$\sum_{j=1}^{p_l} n_{l,j} N_{l,j} \,\leq\, n \,\leq\, \sum_{j=1}^{p_l} n_{l,j}(N_{l,j}+2) \,+\, 2(N_l+2),$$

where

$$N_l \,=\, \max_{j\in\{1,\ldots,p_l\}} N_{l,j}.$$

This yields:

$$\frac{\sum\limits_{j=1}^{p_l} n_{l,j} L_{l,j}}{\sum\limits_{j=1}^{p_l} n_{l,j}(N_{l,j}+2) \,+\, 2(N_l+2)} \,\leq\, \frac{\theta_n-\theta_0}{n} \,\leq\, \frac{\sum\limits_{j=1}^{p_l} n_{l,j} L_{l,j} \,+\, 2L_l}{\sum\limits_{j=1}^{p_l} n_{l,j} N_{l,j}}.$$

When n goes to $+\infty$ the quantity:

$$\frac{n_{l,j}}{\sum\limits_{j=1}^{p_l} n_{l,j} L_{l,j}}$$

goes to the measure $\nu_{l,j}$ of the clopen set $E_{l,j}$. It follows that the sequence $(\theta_n - \theta_0)/n$ has bounded limit sup and limit inf and that any accumulation point ρ of this sequence satisfies:

$$\frac{\sum\limits_{j=1}^{p_l} \nu_{l,j} L_{l,j}}{\sum\limits_{j=1}^{p_l} \nu_{l,j}(N_{l,j}+2)} \,\leq\, \rho \,\leq\, \frac{\sum\limits_{j=1}^{p_l} \nu_{l,j} L_{l,j}}{\sum\limits_{j=1}^{p_l} \nu_{l,j} N_{l,j}}.$$

Recall that the measure ν is the transverse measure associated with an invariant probability measure on the hull $\Omega(\mathcal{QC})$ and thus:

$$\sum_{j=1}^{p_l} \nu_{l,j} L_{l,j} \,=\, 1.$$

On the other hand we have:

$$\sum_{j=1}^{p_l} \nu_{l,j} \,=\, \nu(\mathcal{C}_l).$$

We deduce that:

$$\frac{1}{\sum\limits_{j=1}^{p_l} \nu_{l,j} N_{l,j} + 2\nu(\mathcal{C}_l)} \,\leq\, \rho \,\leq\, \frac{1}{\sum\limits_{j=1}^{p_l} \nu_{l,j} N_{l,j}}.$$

Since these last inequalities are true for any $l \geq 0$, and since $\nu(\mathcal{C}_l)$ goes to 0 as l goes to $+\infty$, it follows that the sequence $(\theta_n - \theta_0)/n$ converges to the limit:

$$\lim_{l\to+\infty} \frac{1}{\sum\limits_{j=1}^{p_l} \nu_{l,j} N_{l,j}} \qquad (\star).$$

Observe that this rotation number is different from 0.

Consider now a minimal configuration which satisfies $\lim\limits_{n\to+\infty} \theta_n = M < +\infty$ or $\lim\limits_{n\to+\infty} \theta_n = m < +\infty$. The constant configuration

$$\theta_n = \theta_0, \quad \forall n \in \mathbb{Z},$$

has obviously a rotation number equal to 0. Let us assume now that the minimal configuration is not constant and satisfies $\lim\limits_{n\to+\infty} \theta_n = M < +\infty$. Let us show that we cannot have $\lim\limits_{n\to-\infty} \theta_n = -\infty$. Indeed, consider the interval $[M-2R, M+2R]$ and choose $u > 0$ such that the interval $[M-2R-u, M+2R-u]$ is disjoint from $[M-2R, M+2R]$ and such that:

$$B_{2R}(M-u) \cap \mathcal{QC} + u = B_{2R}(M) \cap \mathcal{QC}.$$

Consider now, for n large enough, the interval $[\theta_n - R, \theta_n] \subset [M-2R, M]$. The number of atoms in $[\theta_n - R, \theta_n]$ goes to $+\infty$ with n. If $\lim\limits_{n\to-\infty} \theta_n = -\infty$, it follows from Proposition 3.3 that the number of atoms in $[\theta_n - R - u, \theta_n - u]$ and thus in $[M-2R-u, M+2R-u]$, goes to $+\infty$ with n. Consequently the minimal sequence $(\theta_n)_n$ has an accumulation point in $[M-2R-u, M+2R-u]$ when n goes to $-\infty$ which is a contradiction. Thus for a minimal configuration we have:

$$\begin{aligned} \lim_{n\to+\infty} \theta_n < +\infty \quad &\Longleftrightarrow \quad \lim_{n\to-\infty} \theta_n > -\infty \\ &\Longleftrightarrow \quad (\theta_n)_n \quad \text{is bounded} \\ &\Longleftrightarrow \quad (\theta_n)_n \quad \text{has rotation number 0.} \end{aligned}$$

This ends the proof of Part (i) of Theorem 1.1.

4.2. **Continuity of the rotation number.** Consider a sequence $(\theta_{m,n})_n$ of minimal configurations with rotation numbers ρ_m which converges, in the product topology, to a minimal configuration $(\theta_n)_n$ with rotation number $\rho > 0$. We fix $l > 0$ and choose a loop $\gamma_{l,j}$ in $\mathcal{B}_l$. Consider the first time when, starting from 0 on the real line and going in the positive direction, the projection of the configuration $(\theta_n)_n$ enters in this loop. Let us do the same for the configuration $(\theta_{m,n})_n$. Since $(\theta_{m,n})_n$ converge to $(\theta_n)_n$ in the product topology, for m large enough both configurations stay the same time in the loop for their first visits. It follows from Lemma 4.1 that the minimal number of times $N_{m,l,j}$, the projections of the configurations $(\theta_{m,n})_n$ spend in the loop $\gamma_{l,j}$ of $\mathcal{B}_l$, and the minimal number of times $N_{l,j}$, the projection of the configuration $(\theta_{m,n})_n$ spends in the same loop $\gamma_{l,j}$, satisfy:

$$|N_{m,l,j} - N_{l,j}| \leq 2, \quad \forall j \in \{1, \ldots, p(l)\}.$$

The rotation number ρ_m of the configuration $(\theta_{m,n})_n$ satisfies:

$$\frac{1}{\sum\limits_{j=1}^{p_l} \nu_{l,j} N_{m,l,j} + 2\nu(\mathcal{C}_l)} \leq \rho_m \leq \frac{1}{\sum\limits_{j=1}^{p_l} \nu_{l,j} N_{m,l,j}}.$$

On the other hand

$$\frac{1}{\sum_{j=1}^{p_l} \nu_{l,j} N_{l,j} + 2\nu(\mathcal{C}_l)} \leq \rho \leq \frac{1}{\sum_{j=1}^{p_l} \nu_{l,j} N_{l,j}}.$$

This implies that for m large enough:

$$\left| \frac{1}{\rho} - \frac{1}{\rho_m} \right| \leq 8\nu(\mathcal{C}_l).$$

Considering bigger and bigger l yields:

$$\lim_{m \to +\infty} \rho_m = \rho.$$

When the rotation number $\rho = 0$, we have proved that the configuration $(\theta_n)_n$ is bounded. Let M be its upper bound and consider the loop $\gamma_{0,i}$ in $\mathcal{B}_0$ on which M descends by projection. If M falls on the singular point, we consider the loop where the $M - \epsilon$'s for $\epsilon > 0$ small enough, are falling. Fix $K > 2$, when m is big enough, the projection of the configuration $(\theta_{m,n})_n$ (whose rotation number is assumed to be different from 0) must spend at least a time K in the loop $\gamma_{0,i}$ during one of its visits and thus, thanks to Lemma 4.1 at least $K - 2$ times at each of its visits. It follows that the rotation number of $(\theta_{m,n})_n$ satisfies:

$$\rho_m \leq \frac{1}{(K-2)\nu_{0,i}},$$

and, consequently:

$$\lim_{m \to +\infty} \rho_m = 0.$$

Thus, we have proved Part (ii) of Theorem 1.1.

4.3. **Construction of minimal configurations.** Observe that a constant configuration is a minimal configuration with rotation number 0. For positive rotation numbers, we are first going to construct minimal configurations for a dense subset of rotation numbers in $\mathbb{R}^+$.

The good candidate $\mathcal{F}$ to be a dense set in $\mathbb{R}^+$ for which minimal configurations can be construct is suggested by the expression $(\star)$ obtained in the previous subsection. Again, let us consider an increasing sequence $(S_l)_{l \geq 0}$ going to $+\infty$ with l and such that $S_0 > R$. Consider also an associated sequence of Kakutani-Rohlin towers systems $(\mathcal{QC}, S_l, \mathcal{C}_l, \{\mathcal{E}_{l,j}\}_{j \in \{1,\dots,r_l\}})_{l \geq 0}$ and the corresponding sequence of skeletons $(\mathcal{B}_n)_{n \geq 0}$ as constructed in Subsection 2.5. Recalling that the $\nu_{l,j}$'s are the measures of the clopen sets $\mathcal{E}_{l,j}$, we define the set $\mathcal{F}$ as follows:

$$\mathcal{F} = \left\{ \frac{1}{\sum_{j=1}^{p_l} N_{l,j}\nu_{l,j}}, \quad \forall N_{l,j} \in \mathbb{Z}^+ \setminus \{0\}, \quad \forall j \in \{1, \dots, p_l\}, \quad \forall l \geq 0 \right\}.$$

Since the measures of the clopen sets $\mathcal{E}_{l,j}$ go to zero with l uniformly in j, we check easily that $\mathcal{F}$ is a dense subset of $\mathbb{R}^+$.

Proposition 4.2. *For any real number ρ_0 in $\mathcal{F}$, there exists a minimal configuration with rotation number ρ_0.*

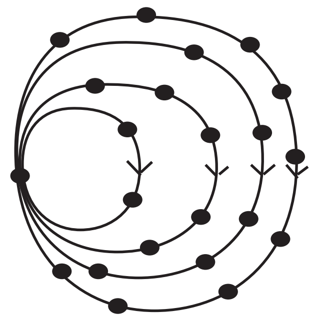

FIGURE 7. The branched manifold with its marked points

Proof. Fix $l_0 \geq 0$ and choose p_{l_0} positive integers $N_{l_0,1}, \ldots, N_{l_0,p_{l_0}}$. Consider the positive real number:

$$\rho_0 = \frac{1}{\sum_{j=1}^{p_{l_0}} N_{l_0,j}\nu_{l_0,j}} \in \mathcal{F}.$$

Let us construct a minimal configuration with rotation number ρ_0.

Step 1: For $j = 1, \ldots, p_{l_0}$, consider on the loop $\gamma_{l_0,j}$ of the oriented branched manifold $\mathcal{B}_{l_0}$, $N_{l_0,j} - 1$ points $\hat{b}_{l_0,1} < \cdots < \hat{b}_{l_0,N_{l_0,j}-1}$, disjoint from the singular point $\pi_{l_0}(x_0)$ of $\mathcal{B}_{l_0}$ (where we recall that $\cap_{l \geq 0} \mathcal{C}_l = \{x_0\}$) as shown in Figure 7. For each $j = 1, \ldots, p_{l_0}$, we consider the segment:

$$(\pi_{l_0}(x_0), \hat{b}_{l_0,1}, \ldots, \hat{b}_{l_0,N_{l_0,j}-1}, \pi_{l_0}(x_0)).$$

Thanks to Lemma 2.8, it makes sense to compute the potential energy of this segment and to consider the position of the points, $\hat{b}_{l_0,1}, \ldots, \hat{b}_{l_0,N_{l_0,j}-1}$, which minimizes this potential energy. Having done it for all loops, we denote $\hat{B}_{l_0}$ the collection of these marked points (all the $\hat{b}_{l_0,k}$'s and $\pi_{l_0}(x_0)$) on $\mathcal{B}_{l_0}$ and consider the subset of the real line $(\pi_{l_0} \circ I)^{-1}(\hat{B}_{l_0})$. It is a discrete subset that we can ordered as a bi-infinite increasing sequence $(\theta_{l_0,n})_n$. This subset of $\mathbb{R}$ contains the subset $(\pi_{l_0} \circ I)^{-1}(\pi_{l_0}(x_0))$ which is a quasicrystal. The configuration $(\theta_{l_0,n})_n$ is made with a concatenation of minimal segments whose extremities are consecutive points in $(\pi_{l_0} \circ I)^{-1}(\pi_{l_0}(x_0))$ and, there are exactly p_{l_0} different equivalence classes of segments, each of them corresponding to a minimal segment starting at the beginning and ending at the end of a loop in $\mathcal{B}_{l_0}$.

Step 2: Consider now the subset $\tau_{l_0}^{-1}(\hat{B}_{l_0})$ of the branched manifold $\mathcal{B}_{l_0+1}$. This subset contains the singular point $\pi_{l_0+1}(x_0)$ and for each $j = 1, \ldots, p_{l_0+1}$, the loop $\gamma_{l_0+1,j}$ of $\mathcal{B}_{l_0+1}$ contains $N_{l_0+1,j} - 1$ consecutive points, $\hat{b}_{l_0+1,1} < \cdots < \hat{b}_{l_0+1,N_{l_0+1,j}-1}$, distinct from the singular point $\pi_{l_0+1}(x_0)$. Actually we have:

$$N_{l_0+1,j} = \sum_{i=1}^{p_{l_0}} m_{l_0,i,j} N_{l_0,i} \qquad (\star\star)$$

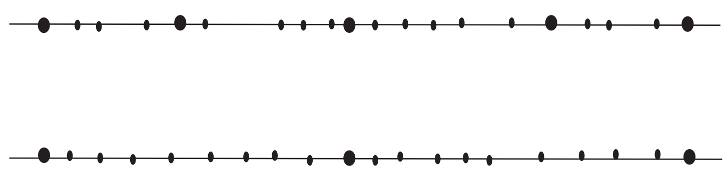

FIGURE 8. The configurations $(\theta_{l_0,n})_n$ and $(\theta_{l_0+1,n})_n$

where $m_{l_0,i,j}$ is the coefficient of the homology matrix M_{l_0}. Again, for each $j = 1, \dots, p_{l_0+1}$, we consider the segment:

$$(\pi_{l_0+1}(x_0), \hat{b}_{l_0+1,1}, \dots, \hat{b}_{l_0+1,N_{l_0+1,j}-1}, \pi_{l_0+1}(x_0)).$$

we choose the position of the points $\hat{b}_{l_0+1,1}, \dots, \hat{b}_{l_0+1,N_{l_0+1,j}-1}$ which minimizes the potential energy. Having done it for all loops, we denote $\hat{B}_{l_0+1}$ the collection of these marked points (all the $\hat{b}_{l_0+1,k}$'s and $\pi_{l_0+1}(x_0)$) on $\mathcal{B}_{l_0+1}$ and consider the subset of the real line $(\pi_{l_0+1} \circ I)^{-1}(\hat{B}_{l_0+1})$. It is a discrete subset that we can ordered as a bi-infinite increasing sequence $(\theta_{l_0+1,n})_n$. This subset of $\mathbb{R}$ contains the subset $(\pi_{l_0+1} \circ I)^{-1}(\pi_{l_0}(x_0))$ which is a quasicrystal contained in the quasicrystal $(\pi_{l_0} \circ I)^{-1}(\pi_{l_0}(x_0))$. The configuration $(\theta_{l_0+1,n})_n$ is made with a concatenation of minimal segments whose extremities are consecutive points in $(\pi_{l_0+1} \circ I)^{-1}(\pi_{l_0+1}(x_0))$ and there are exactly p_{l_0+1} equivalence classes of segments, each of them corresponding to a minimal segment starting at the beginning and ending at the end of a loop in $\mathcal{B}_{l_0+1}$ (See Figure 8).
Step 3: We iterate this procedure to get a configuration $(\theta_{l_0+m,n})_n$ for each $m \geq 0$.

Lemma 4.3. *For each $m \geq 0$, the configuration $(\theta_{l_0+m,n})_n$ has rotation number ρ_0.*

Proof. As a preliminary remark, observe that by construction:

- For any j in $\{1, \dots, p(l_0)\}$, each time the projection of the configuration $(\theta_{l_0,n})_n$ crosses the loop $\gamma_{l_0,j}$ of $\mathcal{B}_{l_0}$, it spends an amount of time $N_{l_0,j}$ in this loop.
- Similarly, for any $m \geq 0$ and for any k in $\{1, \dots, p(l_0 + m)\}$, each time the projection of the configuration $(\theta_{l_0+m,n})_n$ crosses the loop $\gamma_{l_0+m,k}$ of $\mathcal{B}_{l_0+m}$, it spends an amount of time $N_{l_0+m,k}$ in loop.
- Remark also that for any $m \geq 0$ and any k in $\{1, \dots, p(l_0 + m)\}$, each time the projection of the configuration $(\theta_{l_0,n})_n$ crosses the loop $\gamma_{l_0+m,k}$ of $\mathcal{B}_{l_0+m}$, it spends an amount of time in loop which is precisely $N_{l_0+m,k}$.

Using the same estimate as for the proof of the existence of a rotation number for a minimal configuration, we get that the configuration $(\theta_{l_0,n})_n$ has a rotation number and that this rotation number is the limit when $m \to +\infty$ of the sequence

$(\rho_m)_{m\geq 0}$, where:

$$\rho_m \;=\; \frac{1}{\sum_{j=1}^{p_{l_0+m}} \nu_{l_0+m,j} N_{l_0+m,j}} \qquad \forall\, m \geq 0.$$

Claim: *The sequence $(\rho_m)_{m\geq 0}$ is constant.*
Proof of the claim: Using the relation $(\star\star)$ we get, for each $m \geq 0$:

$$\begin{aligned}
\frac{1}{\sum_{j=1}^{p_{l_0+m+1}} \nu_{l_0+m+1,j} N_{l_0+m+1,j}} \;&=\; \frac{1}{\sum_{j=1}^{p_{l_0+m+1}} \nu_{l_0+m+1,j} \left(\sum_{i=1}^{p_{l_0+m}} m_{l_0+m,i,j} N_{l_0+m,i} \right)} \\
&=\; \frac{1}{\sum_{i=1}^{p_{l_0+m}} N_{l_0+m,i} \left(\sum_{j=1}^{p_{l_0+m+1}} m_{l_0+m+1,i,j} \nu_{l_0+m+1,j} \right)}.
\end{aligned}$$

Thanks to Lemma 2.7:

$$\nu_{l_0+m,i} \;=\; \sum_{j=1}^{p_{l_0+m+1}} m_{l_0+m+1,i,j} \nu_{l_0+m+1,j}.$$

Thus:

$$\rho_{m+1} \;=\; \rho_m, \quad \forall\, m \geq 0.$$

This proves the claim and shows that the rotation number of the configuration $(\theta_{l_0,n})_n$ is equal to ρ_0.

To conclude the proof of the lemma, we remark that a same computation yields that, for each $p \geq 0$, the configuration $(\theta_{l_0+p,n})_n$ has a rotation number and that this rotation number is the limit when $m \to +\infty$ of the sequence $(\rho_{p,m})_{m\geq 0}$, where:

$$\rho_{p,m} \;=\; \frac{1}{\sum_{j=1}^{p_{l_0+p+m}} \nu_{l_0+p+m,i} N_{l_0+p+m,j}} \qquad \forall\, m \geq 0.$$

As shown previously, the sequence $(\rho_{p,m})_{m\geq 0}$ is constant and $\rho_{p,0} = \rho_p = \rho_0$. □

Step 4:

Lemma 4.4. *There exists $M > 0$ such that:*

$$0 \leq \theta_{l_0+m,n+1} \,-\, \theta_{l_0+m,n} \;\leq\; M \qquad \forall\, m \geq 0, \quad \forall\, n \in \mathbb{Z}.$$

Proof. Notice first that because of the very construction of the configurations $(\theta_{l_0+m,n})_n$ the lemma is true if we consider only a finite subset of these sequences. Let us prove this lemma by contradiction. Let us fix $m_0 > 0$ and assume that the lemma is not true for the set of sequences $(\theta_{l_0+m,n})_n$ with $m > m_0$. Choose $M(m_0) > 0$ such that $M(m_0)$ is larger than the longest loop of $\mathcal{B}_{l_0+m_0}$. We know that there exists $m > m_0$ and $n \in \mathbb{Z}$ such that:

$$M(m_0) < \theta_{m,n+1} \,-\, \theta_{m,n}.$$

Recall that the configuration $(\theta_{l_0+m,n})_n$ is a concatenation of minimal segments whose extremities descend by projection on the singular point of $\mathcal{B}_{l_0+m_0}$. This implies that there exists a minimal segment:

$$\Theta \;=\; (\theta_{l_0+m,n_1}, \ldots, \theta_{l_0+m,n}, \theta_{l_0+m,n+1}, \ldots, \theta_{l_0+m,n_2})$$

of the configuration $(\theta_{l_0+m,n})_n$ and a loop $\gamma_{l_0+m,j}$ in $\mathcal{B}_{l_0+m_0}$ such that:

$$\pi_{l_0+m_0} \circ I(\Theta) \cap \gamma_{l_0+m_0,j} = \emptyset, \quad \text{and} \quad \gamma_{l_0+m_0,j} \subset \pi_{l_0+m_0} \circ I([\theta_{l_0+m,n}, \theta_{l_0+m,n+1}]).$$

Recall that the image $\tau_{l_0+m_0-1}(\gamma_{l_0+m_0,j})$ covers all the loops of $\mathcal{B}_{l_0+m_0-1}$. Using Lemma 4.1, we deduce that the projection of the segment Θ on $\mathcal{B}_{l_0+m_0-1}$ stays at each passage in a loop of $\mathcal{B}_{l_0+m_0-1}$ at most 3 times in this loop. It follows that the rotation number ρ_{l_0+m} of the configuration $(\theta_{l_0+m,n})_n$ satisfies:

$$\rho_{l_0+m} \geq \frac{1}{3\nu(\mathcal{C}_{l_0+m_0-1})}.$$

This inequality must be true for all $m_0 \geq 0$ and thus $\rho_0 = +\infty$, a contradiction. $\square$

Let us consider the set $\mathbb{R}^{\mathbb{Z}}$ equipped with the product topology. For $M > 0$, the set S_M of non decreasing sequences $(\xi_n)_n$ in $\mathbb{R}^{\mathbb{Z}}$ such that:

$$0 \leq \xi_n - \xi_{n-1} \leq M, \quad \forall n \in \mathbb{Z},$$

is a compact subset of $\mathbb{R}^{\mathbb{Z}}$. Thus it follows from Lemma 4.4, that the set of all the configurations $(\theta_{l_0+m,n})_n$, for $m \geq 0$ and their translated is in a compact subset of $\mathbb{R}^{\mathbb{Z}}$.

Step 5: For each $m \geq 0$, consider $u_m \in \mathbb{R}$ such that 0 belongs to the center of a minimal segment of $(\theta_{l_0+m,n} + u_m)_n$. From lemma 4.1, the sequence of configurations $(\theta_{l_0+m,n} + u_m)_n$ has an accumulation point in $\mathbb{R}^{\mathbb{Z}}$. We denote this configuration $(\theta_{\infty,n})_n$.

Lemma 4.5. *The configuration $(\theta_{\infty,n})_n$ is a minimal configuration with rotation number ρ_0.*

Proof. The fact that the configuration $(\theta_{\infty,n})_n$ is minimal is standard. Consider a segment of $(\theta_{\infty,n})_n$. By construction this segment is a limit of minimal segments and it is straightforward to show that this segment is minimal.
Let us prove now that the configuration $(\theta_{\infty,n})_n$ has rotation number ρ_0. Since the configuration is minimal, it has a rotation number ρ_∞ which is defined as the limit:

$$\lim_{l \to +\infty} \frac{1}{\sum_{j=1}^{p_l} \nu_{l,j} N_{\infty,l,j}},$$

where $N_{\infty,l,j}$ is the minimal number of times the configuration $(\theta_{\infty,n})_n$ spends in the j^{th} loop of $\mathcal{B}_l$.
We use a similar argument to the one used in the proof of the continuity of the rotation number. Fix $l_1 > l_0$, and choose a loop $\gamma_{l_1,j}$ in $\mathcal{B}_{l_1}$. Consider the first time when, starting from 0 on the real line and going in the positive direction, the configuration $(\theta_{\infty,n})_n$ enters in this loop. Let us do the same for the configuration $(\theta_{l_0+m,n} + u_m)_n$. Since a subsequence of configurations $(\theta_{l_0+m,n} + u_m)_n$ converges, when m goes $+\infty$ to the configuration $(\theta_{\infty,n})_n$, it follows that for m big enough both projections of the configurations stay the same time in the loop $\gamma_{l_1,j}$ for their first visit in this loop. It follows from Lemma 4.1 that the minimal number $N_{l_0+m,l,j}$ of times the projection of the configuration $(\theta_{l_0+m,n})_n$ spends in the loop $\gamma_{l_1,j}$ of $\mathcal{B}_{l_1}$ satisfies:

$$|N_{l_0+m,l_1,j} - N_{\infty,l_1,j}| \leq 2, \quad \forall j \in \{1, \ldots, p(l_1)\}.$$

Thus for m big enough, the rotation number ρ_0 of the configuration $(\theta_{l_0+m,n})_n$ satisfies:

$$\frac{1}{\sum_{j=1}^{p_{l_1}} \nu_{l_1,j} N_{l_0+m,l_1,j} + 2\nu(\mathcal{C}_{l_1})} \leq \rho_0 \leq \frac{1}{\sum_{j=1}^{p_{l_1}} \nu_{l_1,j} N_{l_0+m,l_1,j}}.$$

On the other hand

$$\frac{1}{\sum_{j=1}^{p_{l_1}} \nu_{l_1,j} N_{\infty,l_1,j} + 2\nu(\mathcal{C}_{l_1})} \leq \rho_\infty \leq \frac{1}{\sum_{j=1}^{p_{l_1}} \nu_{l_1,j} N_{\infty,l_1,j}}.$$

This implies:

$$\left| \frac{1}{\rho_\infty} - \frac{1}{\rho_0} \right| \leq 8\nu(\mathcal{C}_{l_1}).$$

Since this last inequality is true for all $l_1 > l_0$, we get:

$$\rho_\infty = \rho_0.$$

□

This ends the proof of Proposition 4.2. □

In order to prove Part (iii) of Theorem 1.1, we choose a positive real number ρ and consider a sequence of minimal configurations $(\theta_{m,n})_n$, $m \geq 0$, with rotation number $\rho_m \in \mathcal{F}$ such that:

$$\lim_{m \to +\infty} \rho_m = \rho.$$

A discussion completely similar to the one we used in the proof of Lemma 4.4 allows us to show that there exists $M > 0$ such that:

$$0 \leq \theta_{m,n+1} - \theta_{m,n} \leq M \qquad \forall\, m \geq 0, \quad \forall\, n \in \mathbb{Z}.$$

Consequently, the set of all the configurations $(\theta_{m,n})_n$, for $m \geq 0$ and their translated, is in a compact subset of $\mathbb{R}^{\mathbb{Z}}$ and thus, as done previously, we can exhibit a subsequence of configurations which converges to a minimal configuration $(\theta_n)_n$. Thanks to continuity property of the rotation number (Part (ii) of Theorem 1.1), we conclude that the rotation number of $(\theta_n)_n$ is ρ.

5. Final remarks

5.1. **Dynamical systems.** Minimal configurations of the Frenkel-Kontorova model obviously satisfy the variational equations:

$$U'(\theta_n - \theta_{n+1}) - U'(\theta_{n-1} - \theta_n) + V'(\theta_n) = 0, \quad \forall n \in \mathbb{Z}.$$

By introducing the new variables[5]:

$$p_n = U'(\theta_{n-1} - \theta_n), \quad \forall\, n \in \mathbb{Z},$$

we get the dynamical system defined on $\mathbb{R} \times \mathbb{R}$ by:

$$\begin{cases} p_{n+1} = p_n - V'(\theta_n) \\ \theta_{n+1} = \theta_n - (U')^{-1}(p_n - V'(\theta_n)) \end{cases} \qquad (\star\star\star)$$

[5]Recall that U' is an increasing homeomorphism of the real line.

In the crystal case, V' is a periodic function with period L, the period of the crystal. It follows that the map defined by ($\star\star\star$) descends to a map on the open annulus $\mathbb{R}/L.\mathbb{R} \times \mathbb{R}$ which is an orientation preserving diffeomorphism which preserves the standard area form. Area preserving maps of the annulus have been widely studied and Aubry-Mather theory which makes a bridge between the Frenkel-Kontorova model and dynamical systems, has been a powerful tool for both sides.

In the quasicrystal case, the dynamical system extends to an area preserving "diffeomomorphism" [6] on the solenoidal annulus $\Omega(\mathcal{QC}) \times \mathbb{R}$. The study of such maps will be the subject of a forthcoming paper.

5.2. **Quasicrystals in $\mathbb{R}^d$, $d > 1$.** As we already noticed, the construction of the hull of a quasicrystal and its interpretation as an inverse limit of branched manifolds can be done for quasicrystals in any dimension (see [BG], [BBG], [S]). On the other hand, in a recent work [KLR] , H. Koch, R de la Llave and C. Radin developed a generalization of Aubry-Mather theory for functions on lattices in $\mathbb{R}^d$. Both arguments make tempting to develop in a same way, a Aubry-Mather theory for quasicrystals in $\mathbb{R}^d$, $d > 1$.

Acknowledgments: *It is a pleasure for the authors to thank P. le Calvez for very helpful comments about Aubry-Mather theory and an unknown referee for his useful suggestions. S. P. has been supported by ECOS-Conicyt grant C03-E03.*

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Discrete weak-KAM methods for stationary uniquely ergodic setting

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November 20, 2013

Abstract

The Frenkel-Kontorova model describes how an infinite chain of atoms minimizes the total energy of the system when the energy takes into account the interaction of nearest neighbors as well as the interaction with an exterior environment. An almost-periodic environment leads to consider a family of interaction energies which is stationary with respect to a minimal topological dynamical system. We introduce, in this context, the notion of calibrated configuration (stronger than the standard minimizing condition) and, for continuous superlinear interaction energies, we prove its existence for some environment of the dynamical system. Furthermore, in one dimension, we give sufficient conditions on the family of interaction energies to ensure the existence of calibrated configurations for any environment when the underlying dynamics is uniquely ergodic. The main mathematical tools for this study are developed in the frameworks of discrete weak KAM theory, Aubry-Mather theory and spaces of Delone sets.

Keywords: almost-periodic environment, Aubry-Mather theory, calibrated configuration, Delone set, Frenkel-Kontorova model, Mañé potential, Mather set, minimizing holonomic probability, weak KAM theory

Mathematical subject classification: 37B50, 37J50, 37N20, 49L20, 49L25, 52C23

[*]supported by FAPESP 2009/17075-8, CAPES-COFECUB 661/10 and Brazilian-French Network in Mathematics

[†]supported by FAPESP 2009/17075-8 and CAPES-COFECUB 661/10

[‡]supported by FAPESP 2009/17075-8

Contents

1 Introduction

A *minimizing configuration* $\{x_k\}_{k\in\mathbb{Z}}$ for an *interaction energy* $E : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a chain of points in $\mathbb{R}^d$ arranged so that the energy of each finite segment $(x_m, x_{m+1}, \ldots, x_n)$ cannot be lowered by changing the configuration inside the segment while fixing the two boundary points. Define

$$E(x_m, x_{m+1}, \ldots, x_n) := \sum_{k=m}^{n-1} E(x_k, x_{k+1}).$$

Then $\{x_k\}_{k\in\mathbb{Z}}$ is said to be minimizing if, for all $m < n$, for all $y_m, y_{m+1}, \ldots, y_n \in \mathbb{R}^d$ satisfying $y_m = x_m$ and $y_n = x_n$, one has

$$E(x_m, x_{m+1}, \ldots, x_n) \leq E(y_m, y_{m+1}, \ldots, y_n). \tag{1}$$

If the interaction energy is C^0, *coercive* and *translation periodic*,

$$\lim_{R\to+\infty} \inf_{\|y-x\|\geq R} E(x, y) = +\infty, \tag{2}$$

$$\forall\, t \in \mathbb{Z}^d,\ \forall\, x, y \in \mathbb{R}^d, \quad E(x + t, y + t) = E(x, y), \tag{3}$$

it is easy to show (see [14]) that minimizing configurations do exist. If $d = 1$ and E is a smooth *strongly twist* translation periodic interaction energy,

$$\frac{\partial^2 E}{\partial x \partial y} \leq -\alpha < 0, \tag{4}$$

a minimizing configuration admits in addition a rotation number (see Aubry and Le Daeron [2]). The interaction energy E is supposed to model the interaction between two successive points as well as the interaction between the chain and the environment.

For environments which are aperiodic, namely, with trivial translation group, few results are known (see, for instance, [9, 13, 24]). If $d = 1$ and E is a twist interaction energy describing a quasicrystal environment, Gambaudo, Guiraud and Petite [13] showed that minimizing configurations do exist, they all have a rotation number and any prescribed real number is the rotation number of a minimizing configuration.

We shall make slightly more general assumptions on the properties of E. We say that E is *translation bounded* if

$$\forall\, R > 0, \qquad \sup_{\|y-x\|\leq R} E(x,y) < +\infty, \tag{5}$$

translation uniformly continuous if

$$\forall\, R > 0, \quad E(x,y) \text{ is uniformly continuous in } \|y-x\| \leq R, \tag{6}$$

and *superlinear* if

$$\lim_{R\to+\infty} \inf_{\|y-x\|\geq R} \frac{E(x,y)}{\|y-x\|} = +\infty. \tag{7}$$

A modification of the arguments given by Zavidovique [25, Appendix] shows that semi-infinite minimizing configurations do exist for a superlinear, translation bounded and translation uniformly continuous E. We give a short proof of this result in Appendix A, proposition 60. It is not clear that there exist bi-infinite minimizing configurations in this general context.

We call *ground energy* the lowest energy per site for all configurations

$$\bar{E} := \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n} \frac{1}{n} E(x_0,\ldots,x_n). \tag{8}$$

A configuration $\{x_n\}_{n\in\mathbb{Z}}$ is *calibrated* at the level $\bar{E}$ if, for every $k < l$,

$$\big[E(x_k,\ldots,x_l) - (l-k)\bar{E}\big] \leq \inf_{n\geq 1} \inf_{y_0=x_k,\ldots,y_n=x_l} \big[E(y_0,\ldots,y_n) - n\bar{E}\big]. \tag{9}$$

Notice that the number of sites on the right hand side is arbitrary. A calibrated configuration is obviously minimizing; the converse is false in general, as discussed in Appendix A. More generally, a configuration which is calibrated at some level c (replace $\bar{E}$ by c in (9)) is also minimizing.

If $d \geq 1$ and E is C^0, coercive and translation periodic (conditions (2) and (3)), an argument using the notion of weak KAM solutions as in [15, 11, 14] shows that there exist calibrated configurations at the level $\bar{E}$. Conversely, if $d = 1$ and E is twist translation periodic, every minimizing configuration is calibrated for some modified energy $E_\lambda(x,y) = E(x,y) - \lambda(y-x)$, $\lambda \in \mathbb{R}$, with ground energy $\bar{E}(\lambda)$. If $d = 1$ and E is arbitrary (at least translation bounded, translation uniformly continuous and superlinear), it is not known in general that a calibrated configuration does exist.

In order to give a positive answer to the question of the existence of calibrated configurations, we will consider in this paper an interaction energy which has almost periodic behavior. This leads to look at a family of interaction energies parameterized by a minimal dynamical system.

Concretely, we will assume there exists a family of interaction energies $\{E_\omega\}_\omega$ depending on an environment ω. Let Ω denote the collection of all possible environments. We assume that a chain $\{x_k + t\}_{k\in\mathbb{Z}}$ translated in the direction $t \in \mathbb{R}^d$ and interacting with the environment ω has the same local energy that $\{x_k\}_{k\in\mathbb{Z}}$ interacting with the shifted environment $\tau_t(\omega)$, where $\{\tau_t : \Omega \to \Omega\}_{t\in\mathbb{R}^d}$ is supposed to be a group of bijective maps. More precisely, each environment ω defines an interaction $E_\omega(x, y)$ which is assumed to be *topologically stationary* in the following sense

$$\forall\, \omega \in \Omega,\ \forall\, t \in \mathbb{R}^d,\ \forall\, x, y \in \mathbb{R}^d, \quad E_\omega(x + t, y + t) = E_{\tau_t(\omega)}(x, y). \tag{10}$$

In order to ensure the topological stationarity, the interaction energy will be supposed to have a *Lagrangian form.* Formally, we will use the following notations.

Notation 1. *The space of environments $\big(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}\big)$ is said to be almost periodic if Ω is a compact metric space equipped with a minimal $\mathbb{R}^d$-action $\{\tau_t\}_{t\in\mathbb{R}^d}$, that is, a family of homeomorphisms $\tau_t : \Omega \to \Omega$ satisfying the cocycle property $\tau_s \circ \tau_t = \tau_{s+t}$ for all $s, t \in \mathbb{R}^d$, and*

– *$\tau_t(\omega)$ is jointly continuous with respect to (t, ω),*
– *$\forall\, \omega \in \Omega$, $\{\tau_t(\omega)\}_{t\in\mathbb{R}^d}$ is dense in Ω.*

We say that the family of interaction energies $\{E_\omega\}_{\omega\in\Omega}$ derive from a Lagrangian if there exists a continuous function $L : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\forall\, \omega \in \Omega,\ \forall\, x, y \in \mathbb{R}^d, \quad E_\omega(x, y) := L(\tau_x(\omega), y - x). \tag{11}$$

We call the set of data $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ an almost periodic interaction model.

Notice that the expression "almost periodic" shall not be understood in the sense of H. Bohr. The almost periodicity in the Bohr sense is canonically relied to the uniform convergence. See [3] for a discussion on the different concepts of almost periodicity accordingly to the uniform topology or, for instance, the compact open topology.

Because of the particular form of $E_\omega(x, y)$, these energies are translation bounded and translation continuous uniformly in ω and in $\|y - x\| \le R$. We make precise the two notions of *coerciveness* and *superlinearity* in the Lagrangian form.

Definition 2. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model. The Lagrangian L is said to be coercive if*

$$\lim_{R\to+\infty} \inf_{\omega\in\Omega} \inf_{\|t\|\ge R} L(\omega, t) = +\infty.$$

L is said to be superlinear if

$$\lim_{R\to+\infty} \inf_{\omega\in\Omega} \inf_{\|t\|\ge R} \frac{L(\omega, t)}{\|t\|} = +\infty.$$

L is said to be ferromagnetic if, for every $\omega \in \Omega$, E_ω is of class $C^1(\mathbb{R}^d \times \mathbb{R}^d)$ and, for every $\omega \in \Omega$ and $x, y \in \mathbb{R}^d$,

$$x \in \mathbb{R}^d \mapsto \frac{\partial E_\omega}{\partial y}(x, y) \in \mathbb{R}^d \quad \textit{and} \quad y \in \mathbb{R}^d \mapsto \frac{\partial E_\omega}{\partial x}(x, y) \in \mathbb{R}^d$$

are homeomorphisms.

Note that if there is a constant $\alpha > 0$ such that $\sum_{i,j=1}^d \frac{\partial^2 E_\omega}{\partial x \partial y} v_i v_j \leq -\alpha \sum_{i=1}^d v_i^2$ for all $\omega \in \Omega, x, y \in \mathbb{R}^d$, then L is ferromagnetic and superlinear.

Let us illustrate our abstract notions by three typical examples.

Example 3. *The classical periodic one-dimensional Frenkel-Kontorova model takes into account the family of interaction energies $E_\omega(x,y) = W(y-x) + V_\omega(x)$, with $\omega \in \mathbb{S}^1$, written in Lagrangian form as*

$$L(\omega, t) = W(t) + V(\omega) = \frac{1}{2}|t-\lambda|^2 + \frac{K}{(2\pi)^2}\big(1 - \cos 2\pi\omega\big), \tag{12}$$

where λ, K are constants. Here $\Omega = \mathbb{S}^1$ and $\tau_t : \mathbb{S}^1 \to \mathbb{S}^1$ is given by $\tau_t(\omega) = \omega + t$. We observe that $\{\tau_t\}_t$ is clearly minimal.

The following example comes from [13].

Example 4. *Consider, for an irrational $\alpha \in (0,1) \setminus \mathbb{Q}$, the set*

$$\omega(\alpha) := \{n \in \mathbb{Z} : \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor = 1\},$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Notice that the distance between two consecutive elements of $\omega(\alpha)$ is $\lfloor \frac{1}{\alpha} \rfloor$ or $\lfloor \frac{1}{\alpha} \rfloor + 1$. Now let U_0 and U_1 be two real valued smooth functions with supports respectively in $(0, \lfloor \frac{1}{\alpha} \rfloor)$ and $(0, \lfloor \frac{1}{\alpha} \rfloor + 1)$. Let $V_{\omega(\alpha)}$ be the function defined by $V_{\omega(\alpha)}(x) = U_{\omega_{n+1} - \omega_n - \lfloor \frac{1}{\alpha} \rfloor}(x - w_n)$, where $\omega_n < \omega_{n+1}$ are the two consecutive elements of the set $\omega(\alpha)$ such that $\omega_n \leq x < \omega_{n+1}$. The associated interaction energy is the function

$$E_{\omega(\alpha)}(x,y) = \frac{1}{2}|x - y - \lambda|^2 + V_{\omega(\alpha)}(x). \tag{13}$$

We can directly extend the definition of $V_{\omega'}$ to any relatively dense set ω' of the real line such that the distance between two consecutive points is in $\{\lfloor \frac{1}{\alpha} \rfloor, \lfloor \frac{1}{\alpha} \rfloor + 1\}$. Let Ω' be the collection of all such sets. Then, for any $x, t \in \mathbb{R}$, we have the relation $V_{\omega'}(x+t) = V_{\omega'-t}(x)$, where $\omega' - t$ denotes the set of elements of $\omega' \in \Omega'$ translated by $-t$. In section 2, we explain how to associate a compact metric space $\Omega \subset \Omega'$, where the group of translations acts minimally, as well as a Lagrangian from which the family $\{E_\omega\}_{\omega \in \Omega}$ derives.

As we shall see in section 2, the construction given in example 4 extends to any *quasiscrystal* ω of $\mathbb{R}^d$, namely, to any set $\omega \subset \mathbb{R}^d$ which is relatively dense and uniformly discrete such that the difference set $\omega - \omega$ is discrete and any finite pattern repeats with a positive frequency (see definition 22). We will later focus on the class of environments of quasicrystal type (see definition 17). An example of almost periodic interaction model on $\mathbb{R}$ which is not of quasicrystal type can be constructed in the following way.

Example 5. *The underlying minimal flow is the irrational flow $\tau_t(\omega) = \omega + t(1, \sqrt{2})$ acting on $\Omega = \mathbb{T}^2$. The family of interaction energies E_ω derives from the Lagrangian*

$$L(\omega, t) := \frac{1}{2}|t-\lambda|^2 + \frac{K_1}{(2\pi)^2}\big(1 - \cos 2\pi\omega_1\big) + \frac{K_2}{(2\pi)^2}\big(1 - \cos 2\pi\omega_2\big), \tag{14}$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{T}^2$.

For an almost periodic interaction model, the notion of *ground energy* is given by the following definition.

Definition 6. *We call ground energy of a family of interactions $\{E_\omega\}_{\omega\in\Omega}$ of Lagrangian form $L : \Omega \times \mathbb{R}^d \to \mathbb{R}$ the quantity*

$$\bar{E} := \lim_{n\to+\infty} \inf_{\omega\in\Omega} \inf_{x_0,\dots,x_n\in\mathbb{R}^d} \frac{1}{n} E_\omega(x_0,\dots,x_n).$$

The above limit is actually a supremum by superadditivity and is finite as soon as L is assumed to be coercive. Besides, we clearly have *a priori* bounds

$$\inf_{\omega\in\Omega} \inf_{x,y\in\mathbb{R}^d} E_\omega(x,y) \le \bar{E} \le \inf_{\omega\in\Omega} \inf_{x\in\mathbb{R}^d} E_\omega(x,x). \tag{15}$$

The constant $\bar{E}$ plays the role of a drift and $E_\omega(x,y) - \bar{E}$ acts like a "signed distance". It is natural to modify the previous notion of minimizing configurations by saying that $\{x_n\}_{n\in\mathbb{Z}}$ is *calibrated at the level* $\bar{E}$ if $\sum_{k=m}^{n-1}[E(x_k,x_{k+1})-\bar{E}]$ realizes the smallest signed distance between x_m and x_n for every $m < n$. Hence, we consider the following key notions borrowed from the weak KAM theory (see, for instance, [10]).

Definition 7. *We call Mañé potential in the environment ω the function on $\mathbb{R}^d\times\mathbb{R}^d$ given by*

$$S_\omega(x,y) := \inf_{n\ge1} \inf_{x=x_0,\dots,x_n=y} \big[E_\omega(x_0,\dots,x_n) - n\bar{E}\big].$$

We say that a configuration $\{x_k\}_{k\in\mathbb{Z}}$ is calibrated for E_ω (at the level $\bar{E}$) if

$$\forall\, m < n, \quad S_\omega(x_m,x_n) = E_\omega(x_m,x_{m+1},\dots,x_n) - (n-m)\bar{E}.$$

As discussed in section 3, the Mañé potential for any almost periodic environment is always finite. More importantly, calibrated configurations always exist for some environments ω in the projection of a specific set called the *Mather set*. The Mather set, denoted Mather(L), will be introduced properly in definition 11 of this section. It is a nonempty compact set of $\Omega \times \mathbb{R}^d$ and its first projection (the projected Mather set) by $pr\colon \Omega\times\mathbb{R}^d \to \Omega$, describes the set of environments for which there exists a calibrated configuration passing through the origin of $\mathbb{R}^d$.

The next theorem extends Aubry-Mather theory of the classical periodic model. It is the first main result of this paper and will be proved in section 3.

Theorem 8. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model, with L a C^0 superlinear function. Then, for all $\omega \in pr(Mather(L))$, there exists a calibrated configuration $\{x_k\}_{k\in\mathbb{Z}}$ for E_ω such that $x_0 = 0$ and $\sup_{k\in\mathbb{Z}} \|x_{k+1}-x_k\| < +\infty$.*

This theorem states that, in the almost periodic case, there exist at least one environment and one calibrated configuration for that environment (and thus for any environment in its orbit). It may happen that the projected Mather set does not meet every orbit of the system. Indeed, in the almost periodic Frenkel-Kontorova model described in example 5 , for $\lambda = 0$, we have $\bar{E} = 0$ which is attained by taking $x_n = 0$ for every $n \in \mathbb{Z}$. In addition, it is easy to check that the Mather set is reduced to the point $(0_{\mathbb{T}^2}, 0_{\mathbb{R}})$ and in particular the projected Mather set $\{0_{\mathbb{T}^2}\}$ meets a unique orbit. We shall later show (theorem 19) that this pathology disappears for a restricted class of one-dimensional almost periodic interaction models, which generalizes example 4 and will be called *weakly twist almost periodic interaction model of quasicrystal type* (see definitions 17 and 18).

We now present the definition of the Mather set. Let $\underline{\omega} \in \Omega$ be fixed. The ground energy (in the environment $\underline{\omega}$) measures the mean energy per site of a configuration $\{x_n\}_{n\geq 0}$ which distributes in $\mathbb{R}^d$ so that $\frac{1}{n}E_{\underline{\omega}}(x_0,\ldots,x_n) \to \bar{E}$. Notice that the previous mean can be understood as an expectation of $L(\omega,t)$ with respect to a probability measure $\mu_{n,\underline{\omega}} := \frac{1}{n}\sum_{k=0}^{n-1}\delta_{(\tau_{x_k}(\underline{\omega}),\, x_{k+1}-x_k)}$:

$$\frac{1}{n}E_{\underline{\omega}}(x_0,\ldots,x_n) = \int L(\omega,t)\,\mu_{n,\underline{\omega}}(d\omega,dt). \tag{16}$$

Notice also that $\mu_{n,\underline{\omega}}$ satisfies the following property of pseudoinvariance

$$\int f(\omega)\,\mu_{n,\underline{\omega}}(d\omega,dt) - \int f(\tau_t(\omega))\,\mu_{n,\underline{\omega}}(d\omega,dt) = \frac{1}{n}\Big(f\circ\tau_{x_n}(\underline{\omega}) - f\circ\tau_{x_0}(\underline{\omega})\Big). \tag{17}$$

This suggests to consider the set of all weak* limits of $\mu_{n,\underline{\omega}}$ as $n\to+\infty$. Following [20], we call these limit measures *holonomic probabilities.*

Definition 9. *A probability measure μ on $\Omega\times\mathbb{R}^d$ is said to be holonomic if*

$$\forall\, f\in C^0(\Omega),\qquad \int f(\omega)\,\mu(d\omega,dt) = \int f(\tau_t(\omega))\,\mu(d\omega,dt).$$

Let $\mathbb{M}_{hol}$ denote the set of all holonomic probability measures.

The set $\mathbb{M}_{hol}$ is certainly not empty since it contains any $\delta_{(\omega,0)}$, $\omega\in\Omega$. It is then natural to look for holonomic measures that minimize L. We show that minimizing holonomic measures do exist and that the lowest mean value of L is the ground energy.

Proposition 10. *If L is C^0 coercive, then $\bar{E} = \inf\{\int L\,d\mu : \mu\in\mathbb{M}_{hol}\}$ and the infimun is attained by some holonomic probability measure.*

A measure that attains the previous infimum is called *minimizing.*

Definition 11. *We denote by $\mathbb{M}_{min}$ the set of minimizing measures. We call Mather set of L the set*

$$\mathrm{Mather}(L) := \cup_{\mu\in\mathbb{M}_{min}}\mathrm{supp}(\mu) \subseteq \Omega\times\mathbb{R}^d.$$

The projected Mather set is just $pr(\mathrm{Mather}(L))$, where $pr:\Omega\times\mathbb{R}^d\to\Omega$ is the first projection.

Proposition 12.

1. *If L is C^0 coercive, then*

$$\exists\,\mu\in\mathbb{M}_{min}\quad \textit{with}\quad \mathrm{Mather}(L) = \mathrm{supp}(\mu).$$

In particular, $\mathrm{Mather}(L)$ *is closed.*

2. If L is C^0 superlinear, then Mather(L) *is compact.*

The set of holonomic measures may be seen as a dual object to the set of coboundaries $\{u - u \circ \tau_t : u \in C^0(\Omega),\ t \in \mathbb{R}^d\}$. Proposition 10 admits thus a dual version that will actually be proved first.

Proposition 13 (The sup-inf formula). *If L is C^0 coercive, then*

$$\bar{E} = \sup_{u \in C^0(\Omega)} \ \inf_{\omega \in \Omega,\ t \in \mathbb{R}^d} \big[L(\omega, t) + u(\omega) - u \circ \tau_t(\omega)\big].$$

We do not know whether the above supremum is achieved in the aperiodic case (*i.e.* when any map τ_t with $t \neq 0$ has no fixed point). There is finally a third way to compute the ground energy, which says that the exact choice of the environment ω is irrelevant.

Proposition 14. *If L is C^0 coercive, then*

$$\forall\, \omega \in \Omega, \quad \bar{E} = \lim_{n \to +\infty} \ \inf_{x_0, \ldots, x_n \in \mathbb{R}^d} \frac{1}{n} E_\omega(x_0, \ldots, x_n).$$

We present now the definition of a *weakly twist interaction model of quasicrystal type* (generalizing example 4). We decided to work in a slightly more general frame than the usual one for quasicrystals (see section 2). The definition is presented only for the one-dimensional case, nevertheless the description can be done in any dimension. We begin by introducing the notions of flow boxes, transverse section, and box decomposition.

Definition 15. *Let $(\Omega, \{\tau_t\}_{t \in \mathbb{R}})$ be an almost periodic environment.*
– An open set $U \subset \Omega$ is said to be a flow box of size $R > 0$ if there exists a compact subset $\Xi \subset \Omega$, called transverse section, such that:
▪ the induced topology on Ξ admits a basis of closed and open subsets, called clopen subsets,
▪ $\tau(t, \omega) = \tau_t(\omega)$, $(t, \omega) \in \mathbb{R} \times \Xi$, is a homeomorphism from $B_R(0) \times \Xi$ onto U.
We shall later write $B_R = B_R(0)$ and $\tau_{(i)}^{-1} = \tau_{|U_i}^{-1} : U_i \to B_R \times \Xi$ for a flow box U_i.
– Two flow boxes $U_i = \tau[B_{R_i} \times \Xi_i]$ and $U_j = \tau[B_{R_j} \times \Xi_j]$ are said to be admissible if, whenever $U_i \cap U_j \neq \emptyset$, there exists $a_{i,j} \in \mathbb{R}$ such that

$$\tau_{(j)}^{-1} \circ \tau(t, \omega) = (t - a_{i,j}, \tau_{a_{i,j}}(\omega)), \quad \forall\, (t, \omega) \in \tau_{(i)}^{-1}(U_i \cap U_j).$$

– A flow box decomposition $\{U_i\}_{i \in I}$ is a cover of Ω by admissible flow boxes.

Typical examples of these structures are given by the suspensions of minimal homeomorphisms on a Cantor set with a locally constant roof functions.

The notion of transversally constant Lagrangian has been introduced in [13]. In the periodic case, equation (3) shows that the interaction energy keeps a constant value by moving the whole configuration by a distance equal to a multiple of the period. In example 4, equation (13) and the minimality of the action by an irrational rotation on the circle show that, given any finite configuration, the interaction energy keeps the same value for infinitely many translated configurations. Moreover, this set of translations is a relatively dense set in $\mathbb{R}$ depending on the configuration. We formalize this idea in the following definition.

Definition 16. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be an almost periodic interaction model admitting a flow box decomposition.*
– A flow box $\tau[B_R \times \Xi]$ is said to be compatible with respect to a flow box decomposition $\{U_i\}_{i\in I}$, where $U_i = \tau[B_{R_i} \times \Xi_i]$, when, for every $|t| < R$, there exist $i \in I$, $|t_i| < R_i$ and a clopen subset $\tilde{\Xi}_i$ of Ξ_i such that $\tau_t(\Xi) = \tau_{t_i}(\tilde{\Xi}_i)$.
– L is said to be transversally constant with respect to a flow box decomposition $\{U_i\}_{i\in I}$ if, for every flow box $\tau[B_R \times \Xi]$ compatible with respect to $\{U_i\}_{i\in I}$,

$$\forall\, \omega, \omega' \in \Xi,\ \forall\, |x|, |y| < R, \quad E_{\omega'}(x, y) = E_\omega(x, y).$$

We extend the case treated in [13] for quasicrystals to the almost periodic interaction models. Similarly to studies for the Hamilton-Jacobi equation (see, for instance, [6, 7, 8, 19]), we will consider here a stationary ergodic setting.

Definition 17. *An almost periodic interaction model $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ is said to be of quasicrystal type if the action $\{\tau_t\}_{t\in\mathbb{R}}$ is uniquely ergodic (with unique invariant probability measure λ) and L is transversally constant with respect to some flow box decomposition.*

The strongly twist property (4) is the main assumption in Aubry-Mather theory ([2, 21]). We slightly extend this property.

Definition 18. *A one-dimensional almost periodic interaction model $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ satisfies the weakly twist property if there exists a C^0 function $U : \Omega \to \mathbb{R}$ such that, for every $\omega \in \Omega$, the function $\tilde{E}_\omega(x, y) := E_\omega(x, y) + U(\tau_x(\omega)) - U(\tau_y(\omega))$ is C^2, and*

$$\forall\, x, y \in \mathbb{R}, \omega \in \Omega \quad \frac{\partial^2 \tilde{E}_\omega}{\partial x \partial y}(x, \cdot) < 0 \quad \textit{and} \quad \frac{\partial^2 \tilde{E}_\omega}{\partial x \partial y}(\cdot, y) < 0 \qquad \textit{a.e.}$$

Now we state the second main result of this paper, which says that, in the quasicrystal case, for any environment, there always exists a calibrated configuration. Its proof is detailed in section 4.

Theorem 19. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be a one-dimensional weakly twist interaction model of quasicrystal type. Then the projected Mather set meets uniformly any orbit of the flow τ_t. More precisely, for every $\omega \in \Omega$, there exists a calibrated configuration for E_ω, say $\{x_{k,\omega}\}_{k\in\mathbb{Z}}$, with bounded jumps and at a bounded distance from the origin uniformly in ω:*

$$\forall\, m < n, \quad S_\omega(x_{m,\omega}, x_{n,\omega}) = \sum_{k=m}^{n-1} E_\omega(x_{k,\omega}, x_{k+1,\omega}) - (n - m)\bar{E},$$

$$\sup_{\omega\in\Omega} \sup_{k\in\mathbb{Z}} |x_{k+1,\omega} - x_{k,\omega}| < +\infty, \quad \sup_{\omega\in\Omega} |x_{0,\omega}| < +\infty.$$

As in examples 3 and 4 as well as in the general setting described in section 2, interaction models of quasicrystal type are easily built when the interaction energy has the form $E_\omega(x, y) = W(y - x) + V_1(\tau_x(\omega)) + V_2(\tau_y(\omega))$, where W is superlinear weakly convex and V_1 and V_2 are locally transversally constant and smooth along the flow.

Definition 20. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be an almost periodic interaction model. A function $V : \Omega \to \mathbb{R}$ is said to be locally transversally constant on a flow box decomposition $\{U_i\}_{i\in I}$, where $U_i = \tau(B_{R_i} \times \Xi_i)$, if*

$$\forall\, i \in I,\ \forall\, \omega, \omega' \in \Xi_i,\ \forall\, |x| < R_i, \quad V(\tau_x(\omega)) = V(\tau_x(\omega')).$$

Notice that, in example 5, the locally transversally constant property is not verified.

Corollary 21. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be a one-dimensional almost periodic interaction model. Assume that $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ is uniquely ergodic. Let $V_1, V_2 : \Omega \to \mathbb{R}$ be C^0 locally transversally constant functions on the same flow box decomposition that are C^2 along the flow (namely, for all ω, the function $t \in \mathbb{R} \mapsto V_i(\tau_t(\omega))$ is C^2, $i = 1, 2$). Let $W : \mathbb{R} \to \mathbb{R}$ be a C^2 superlinear weakly convex function (namely, $W''(t) > 0$ a.e. and $|W'(t)| \to +\infty$ as $|t| \to +\infty$). Define*

$$L(\omega, t) = W(t) + V_1(\omega) + V_2(\tau_t(\omega)).$$

Then L is C^0, superlinear and transversally constant, $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ is a one-dimensional weakly twist interaction model of quasicrystal type and all conclusions of theorem 19 apply.

2 Backgrounds on quasicrystals

In this section, we recall the basic definitions and properties concerning Delone sets and specially quasicrystals. More details on Delone sets can be found, for instance, in [4, 17, 18]. Associated to Delone sets, we will consider strongly equivariant functions. We recall their main properties here and we refer the reader to [13, 16] for the proofs.

Definition of quasicrystal. A Delone set ω is a discrete subset of the Euclidean space $\mathbb{R}^d$ for which there exist two positive real numbers r_ω and R_ω satisfying the following properties:

- *uniform discreteness*: each open ball of radius r_ω in $\mathbb{R}^d$ contains at most one point of ω.
- *relative density*: each closed ball of radius R_ω in $\mathbb{R}^d$ contains at least one point of ω.

If precision is required, we will say that ω is r_ω-uniformly discrete and R_ω-relatively dense.

For $R > R_\omega$, we say that a subset $\mathtt{P}$ of a Delone set ω is a *R-patch* (or a *pattern* for short) of ω if, for some $y \in \omega$, one has

$$\mathtt{P} = \omega \cap \overline{B_R(y)},$$

where $B_R(y)$ denotes the open ball of a radius R centered at y. We will say that the patch is centered at y (notice that the center may not be unique). The collection of return vectors associated to the patch $\mathtt{P}$ is the set

$$\mathcal{R}_{\mathtt{P}}(\omega) = \{v \in \mathbb{R}^d : \mathtt{P} + v \text{ is a patch of } \omega\},$$

where $\mathtt{P} + v$ denotes the translation of all the points of $\mathtt{P}$ by the vector v. The set of occurrences of $\mathtt{P}$ is defined as $\omega_{\mathtt{P}} := x_{\mathtt{P}} + \mathcal{R}_{\mathtt{P}}(\omega)$.

Definition 22. *A Delone set $\omega \subset \mathbb{R}^d$ is* repetitive *if it satisfies all the two following properties:*

- finite local complexity*: for any real $R > 0$, the Delone set ω has just a finite number of R-patches up to translations;*
- repetitivity*: for each $R > 0$, there is a real number $M(R) > 0$ such that any closed ball of radius $M(R)$ contains at least one occurrence of every R-patch of ω.*

A repetitive Delone set $\omega \subset \mathbb{R}^d$ is a quasicrystal *if in addition it satisfies*

- uniform pattern distribution*: for any pattern P of ω, uniformly in $x \in \mathbb{R}^d$, the following limit exists*

$$\lim_{N \to +\infty} \frac{\#\left(\{z \in \mathbb{R}^d : z \textit{ is an occurrence of } P\} \cap B_N(x)\right)}{Leb(B_N(x))} = \nu(P) > 0.$$

Notice that the finite local complexity is equivalent to the property that the intersection of the difference set $\omega - \omega$ with any bounded set is finite (see [18]).

Basic examples of quasicrystals are derived from Beatty sequences: for a real number $\alpha \in (0, 1)$, the associated the set is $\omega(\alpha) := \{n \in \mathbb{Z} : \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor = 1\}$. For details, we refer to [18].

Observe that, when ω is a repetitive Delone set (respectively, a quasicrystal), then $\omega + v$, obtained by translating any point of ω by $v \in \mathbb{R}^d$, is also a repetitive Delone set (respectively, a quasicrystal). A Delone set is said to be *aperiodic* if $\omega + v = \omega$ implies $v = 0$, and *periodic* if its stabilizers contains a cocompact subgroup of $\mathbb{R}^d$. In the former example, it is simple to check that the quasicrystal $\omega(\alpha)$ is aperiodic if, and only if, α is irrational, as in example 4.

We introduce now a combinatorial background. For a Delone set ω and a real number $R > 0$, the *R-atlas* $\mathcal{A}_\omega(R)$ of ω is the collection of all the R-patches centered at a point of ω and translated to the origin. More precisely, we set

$$\mathcal{A}_\omega(R) := \{\omega \cap \overline{B_R(x)} - x : \ x \in \omega\}.$$

Notice that ω has finite local complexity if, and only if, $\mathcal{A}_\omega(R)$ is finite for every R. For a quasicrystal ω and a patch $\mathtt{P}$, it is plain to check that the collection of return vectors $\mathcal{R}_{\mathtt{P}}(\omega)$ is also a quasicrystal. Hence $\omega_{\mathtt{P}}$, the set of all the occurrences of $\mathtt{P}$, is also a quasicrystal.

In order to avoid an unnecessary dichotomy, we will mainly focus on aperiodic quasicrystals. The following lemma is well-known and its proof is plain by contradiction.

Lemma 23. *If ω is an aperiodic quasicrystal, then, given $S > 0$, there exists a constant $R_S > 0$ such that, for any $R \geq R_S$ and any R-patch $\mathtt{P}$ of ω, the quasicrystal $\omega_{\mathtt{P}}$ is S-uniformly discrete.*

Hull of a quasicrystal. As we already mentioned, a translation of a repetitive Delone set ω_* is also a repetitive Delone set. We will equipped the set $\omega_* + \mathbb{R}^d$ of all the translations of ω_* with a topology that reflects its combinatorial properties: the Gromov-Hausdorff topology. Roughly speaking, two Delone sets in this set will be close whenever they have the same pattern in a large neighborhood of the origin, up to a small translation.

Such a topology is metrizable and an associated metric can be defined as follows (for details, see [4, 16]): given ω and $\underline{\omega}$ two translations of ω_*, their distance is

$$D(\omega, \underline{\omega}) := \inf \Big\{ \frac{1}{r+1} : \exists\, |v|, |\underline{v}| < \frac{1}{r} \ \text{ s.t. } \ (\omega + v) \cap B_r(0) = (\underline{\omega} + \underline{v}) \cap B_r(0) \Big\}.$$

The *continuous hull* $\Omega(\omega_*)$ of the Delone set ω_* is the completion of such a metric space. The finite local complexity hypothesis implies that $\Omega(\omega_*)$ is a compact metric space and that any element $\omega \in \Omega(\omega_*)$ is a Delone set which has, up to translations, the same patterns as ω_*, namely, $\mathcal{A}_\omega(R) = \mathcal{A}_{\omega_*}(R)$ for any $R > 0$ (see [17, 4]). Moreover, $\Omega(\omega_*)$ is equipped with a continuous $\mathbb{R}^d$-action given by the homeomorphisms

$$\tau_v \colon \omega \mapsto \omega - v \qquad \text{for } \omega \in \Omega(\omega_*).$$

Given $\omega \in \Omega(\omega_*)$ and $S > 0$ such that $\omega \cap B_S(0) \neq \emptyset$, the associated *cylinder set* is defined as

$$\Xi_{\omega,S} := \{\underline{\omega} \in \Omega(\omega_*) : \omega \cap \overline{B_S(0)} = \underline{\omega} \cap \overline{B_S(0)}\}.$$

The translations of cylinder sets,

$$U_{\omega,S,\epsilon} := \{\underline{\omega} + v : v \in B_\epsilon(0),\ \underline{\omega} \in \Xi_{\omega,S}\}, \qquad \text{for } \epsilon > 0,\ S > 0,\ \omega \in \Omega(\omega_*),$$

form a base for the topology of $\Omega(\omega_*)$.

The dynamical system $(\Omega(\omega_*), \mathbb{R}^d)$ has a dense orbit (namely, the orbit of ω_*). Actually, the repetitivity hypothesis is equivalent to the *minimality* of the action, and so any orbit is dense. The uniform pattern distribution is equivalent to the *unique ergodicity*: the $\mathbb{R}^d$-action has a unique invariant probability measure. For details on these properties, we refer the reader to [17, 4]. We summarize these facts in the following proposition.

Proposition 24 ([17, 4])**.** *Let ω_* be a quasicrystal of $\mathbb{R}^d$. Then the dynamical system $(\Omega(\omega_*), \mathbb{R}^d)$ is minimal and uniquely ergodic.*

The *canonical transversal* $\Xi_0(\omega_*)$ of the hull $\Omega(\omega_*)$ of a quasicrystal is the set of quasicrystals ω in $\Omega(\omega_*)$ such that the origin 0 belongs to ω. The designation of transversal comes from the obvious fact that the set $\Xi_0(\omega_*)$ is transverse to the action: for any vector v smaller than the uniform discreteness constant, clearly $\tau_v(\omega) \notin \Xi_0(\omega_*)$ for any $\omega \in \Xi_0(\omega_*)$. This gives a Poincaré section.

Proposition 25 ([17])**.** *The canonical transversal $\Xi_0(\omega_*)$ and the cylinder sets $\Xi_{\omega,S}$ of an aperiodic quasicrystal ω_* are Cantor sets. If ω_* is a periodic quasicrystal, these sets are finite.*

It follows, in one dimension, that the hull admits a flow box decomposition. This can be generalized straightforwardly in any dimension.

Lemma 26. *Let ω_* be an aperiodic repetitive Delone set of $\mathbb{R}$ with constant of relative denseness R_{ω_*}. Then, for any large enough $R > 0$, there exist elements $\omega_1, \ldots, \omega_n \in \Xi_0(\omega_*)$ such that the collection of open sets $\{U_{\omega_i,R,R_{\omega_*}}\}_{i=1}^n$ is a flow box decomposition of the almost periodic environment $(\Omega(\omega_*), \{\tau_t\}_{t\in\mathbb{R}})$.*

Proof. By lemma 23, for all large enough R and for any R-patch $\mathtt{P}$ of ω_*, the discreteness constant $r_{\omega_\mathtt{P}}$ of the occurrence set $\omega_\mathtt{P}$, is greater than $4R_{\omega*}$. Notice that, by the definition of the constant R_{ω_*}, for all $S > 0$, the collection $\{U_{\omega,S,R_{\omega_*}}\}_{\omega\in\Xi_0(\omega_*)}$ is a cover of the hull $\Omega(\omega_*)$. Moreover, the choice of the constant R implies that, for any $\omega \in \Xi_0(\omega_*)$, the map $\tau\colon B_{R_{\omega_*}}(0) \times \Xi_{\omega,R+2R_{\omega_*}} \to U_{\omega,R+2R_{\omega_*},R_{\omega_*}}$ is an homeomorphism. This choice also implies that, for any $\omega_1, \omega_2 \in \Xi_0(\omega_*)$, there is at most one vector $a \in B_{2R_{\omega*}}(0)$ such that $\tau_a\Xi_{\omega_1,R+2R_{\omega_*}} \cap \Xi_{\omega_2,R+2R_{\omega_*}} \neq \emptyset$. Indeed, if there are $a, a' \in B_{2R_{\omega*}}(0)$ and $\omega, \omega' \in \Xi_{\omega_1,R+2R_{\omega_*}}$ such that

$$\tau_a\omega \cap B_{R+2R_{\omega_*}}(0) = \tau_{a'}\omega' \cap B_{R+2R_{\omega_*}}(0),$$

we have in particular

$$\omega \cap B_R(a) - a = \omega' \cap B_R(a') - a',$$

which means that $a-a'$ is an occurrence of an R-patch, and then $a = a'$ by the choice of R. Therefore, if two open sets $U_{\omega_1,R+2R_{\omega_*},R_{\omega_*}}$ and $U_{\omega_2,R+2R_{\omega_*},R_{\omega_*}}$ are intersecting, there are $t, t' \in B_{R_{\omega_*}}(0)$ such that $\tau_t(\Xi_{\omega_1,R+2R_{\omega_*}})$ intersects $\tau_{t'}(\Xi_{\omega_2,R+2R_{\omega_*}})$. It follows that the vector $t - t'$ is unique, and the two open sets $U_{\omega_1,R+2R_{\omega_*},R_{\omega_*}}$ and $U_{\omega_2,R+2R_{\omega_*},R_{\omega_*}}$ are admissible. Thus, any finite subcover of the collection $\{U_{\omega,R,R_{\omega_*}}\}_{\omega\in\Xi_0(\omega_*)}$ is a flow box decomposition. $\square$

For a more dynamical description of the hull in one dimension, we consider the *return time* function $\varrho : \Xi_0(\omega_*) \to \mathbb{R}^+$ given by

$$\varrho(\omega) := \inf\{t > 0 : \tau_t(\omega) \in \Xi_0(\omega_*)\}, \quad \forall\, \omega \in \Xi_0(\omega_*).$$

The finite local complexity implies that this function is locally constant. The *first return map* $T\colon \Xi_0(\omega_*) \to \Xi_0(\omega_*)$ is then given by

$$T(\omega) := \tau_{\varrho(\omega)}(\omega), \quad \forall\, \omega \in \Xi_0(\omega_*).$$

It is straightforward to check that, for a repetitive Delone set ω_*, the dynamical system $(\Omega(\omega_*), \mathbb{R})$ is conjugate to the suspension of the map T on the set $\Xi_0(\omega_*)$ with the time map given by the function ϱ. Thus, when ω_* is periodic, the continuous hull $\Omega(\omega_*)$ is homeomorphic to a circle. Otherwise, $\Omega(\omega_*)$ has a laminated structure: it is locally the product of a Cantor set by an interval. For the quasicrystal case, the unique invariant probability measure on $\Omega(\omega_*)$ induces a finite measure on $\Xi_0(\omega_*)$ that is T-invariant (see [13]).

Associated to a repetitive Delone set ω of $\mathbb{R}^d$, we will mainly consider strongly ω-equivariant functions as introduced in [16].

Definition 27. *Let ω be a repetitive Delone set of $\mathbb{R}^d$. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is strongly ω-equivariant with range $R > 0$ if, for $x, y \in \mathbb{R}^d$,*

$$\big(B_R(x) \cap \omega\big) - x = \big(B_R(y) \cap \omega\big) - y \quad \Rightarrow \quad f(x) = f(y).$$

In example 4, the function $V_{\omega(\alpha)}$ is strongly $\omega(\alpha)$-equivariant with range $\lfloor \frac{1}{\alpha} \rfloor + 1$. Let us mention another example from [16], which holds for any repetitive Delone set ω_*. Let $\delta := \sum_{x\in\omega_*} \delta_x$ be the Dirac comb supported on the points of a quasicrystal ω_* and let $g\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support. Then, one may check that the convolution product $\delta * g$ is a smooth strongly ω_*-equivariant function. Actually, any strongly ω-equivariant function can be defined by a similar procedure [16].

The following lemma shows that strongly ω_*-equivariant functions arise from functions on the space $\Omega(\omega_*)$ that are constant on the cylinder sets.

Lemma 28 ([13, 16])**.** *Given a repetitive Delone set ω_* of $\mathbb{R}^d$, let $V_{\omega_*}\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous strongly ω_*-equivariant function with range R. Then, there exists a unique continuous function $V\colon \Omega(\omega_*) \to \mathbb{R}$ such that*

$$V_{\omega_*}(x) = V \circ \tau_x(\omega_*), \quad \forall\, x \in \mathbb{R}^d.$$

Moreover, there exists $S > R$ such that V is constant on any cylinder set $\Xi_{\omega,S}$, $\omega \in \Omega(\omega_*)$. In addition, if V_{ω_*} is C^2, then V is C^2 along the flow (that is, for all ω, the function $x \in \mathbb{R}^d \mapsto V(\tau_x(\omega))$ is C^2).

Thus, for $d = 1$ and with the notation of the former lemma, we get that, for any large enough constant $R' > S + R_{\omega_*}$, the function $V\colon \Omega(\omega_*) \to \mathbb{R}$ is transversally constant on a flow box decomposition $\{U_{\omega_i,R',R_{\omega_*}}\}_{i=1}^n$ given by lemma 26. This comes from the fact that $\tau_x(\omega') \in \Xi_{\tau_x(\omega),S}$ whenever $x \in B_{R_{\omega_*}}(0)$, $\omega, \omega' \in \Xi_{\omega_i,S+R_{\omega_*}}$, and V is constant on such cylinder sets.

3 Mather set

The Mather set describes the set of environments for which there exist calibrated configurations. The Mather set is defined in terms of holonomic minimizing measures. Before proving propositions 10, 13 and 14, we note temporarily

$$\bar{E}_\omega = \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n} E_\omega(x_0,\ldots,x_n), \quad \bar{L} := \inf\Big\{ \int L\,d\mu : \mu \in \mathbb{M}_{hol} \Big\},$$
$$\text{and} \quad \bar{K} := \sup_{u\in C^0(\Omega)} \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d} \big[L(\omega,t) + u(\omega) - u\circ\tau_t(\omega)\big].$$

We first show that the infimum is attained in proposition 10.

Proof of proposition 10. We shall prove later that $\bar{L} = \bar{E}$. We prove now that the infimum is attained in $\bar{L} := \inf\{\int L\,d\mu : \mu \in \mathbb{M}_{hol}\}$. Let

$$C := \sup_{\omega\in\Omega} L(\omega, 0) \geq \bar{L} \quad \text{and} \quad \mathbb{M}_{hol,C} := \Big\{\mu \in \mathbb{M}_{hol} : \int L\,d\mu \leq C\Big\}.$$

We equip the set of probability measures on $\Omega \times \mathbb{R}^d$ with the weak topology (convergence of sequence of measures by integration against compactly supported continuous test functions). By coerciveness, for every $\epsilon > 0$ and $M > \inf L$ such that $\epsilon > (C - \inf L)/(M - \inf L)$, there exists $R(\epsilon) > 0$ with $\inf_{\omega\in\Omega, \|t\|\geq R(\epsilon)} L(\omega, t) \geq M$. By integrating $L - \inf L$, we get

$$\forall \mu \in \mathbb{M}_{hol,C}, \quad \mu\big(\Omega \times \{t : \|t\| \geq R(\epsilon)\}\big) \leq \int \frac{L - \inf L}{M - \inf L}\,d\mu \leq \frac{C - \inf L}{M - \inf L} < \epsilon.$$

We have just proved that the set $\mathbb{M}_{hol,C}$ is tight. Let $(\mu_n)_{n\geq 0} \subset \mathbb{M}_{hol,C}$ be a sequence of holonomic measures such that $\int L\,d\mu_n \to \bar{L}$. By tightness, we may assume that $\mu_n \to \mu_\infty$ with respect to the strong topology (convergence of sequence of measures by integration against bounded continuous test functions). In particular, μ_∞ is holonomic. Moreover, for every $\phi \in C^0(\Omega, [0, 1])$, with compact support,

$$0 \leq \int (L - \bar{L})\phi\,d\mu_\infty = \lim_{n\to+\infty} \int (L - \bar{L})\phi\,d\mu_n \leq \liminf_{n\to+\infty} \int (L - \bar{L})\,d\mu_n = 0.$$

Therefore, μ_∞ is minimizing. □

We next show that there is no need to take the closure in the definition of the Mather set. We will show later that it is compact.

Proof of proposition 12 – Item 1. We show that $\text{Mather}(L) = \text{supp}(\mu)$ for some minimizing measure μ. Let $\{V_i\}_{i\in\mathbb{N}}$ be a countable basis of the topology of $\Omega \times \mathbb{R}^d$ and let

$$I := \{i \in \mathbb{N} : V_i \cap \text{supp}(\nu) \neq \emptyset \text{ for some } \nu \in \mathbb{M}_{min}\}.$$

We reindex $I = \{i_1, i_2, \ldots\}$ and choose for every $k \geq 1$ a minimizing measure μ_k so that $V_{i_k} \cap \text{supp}(\mu_k) \neq \emptyset$ or equivalently $\mu_k(V_{i_k}) > 0$. Let $\mu := \sum_{k\geq 1} \frac{1}{2^k}\mu_k$. Then μ is minimizing. Suppose some V_i is disjoint from the support of μ. Then $\mu(V_i) = 0$ and, for every $k \geq 1$, $\mu_k(V_i) = 0$. Suppose by contradiction that $V_i \cap \text{supp}(\nu) \neq \emptyset$ for some $\nu \in \mathbb{M}_{min}$, then $i = i_k$ for some $k \geq 1$ and, by the choice of μ_k, $\mu_k(V_i) > 0$, which is not possible. Therefore, V_i is disjoint from the Mather set and we have just proved $\text{Mather}(L) \subseteq \text{supp}(\mu)$ or $\text{Mather}(L) = \text{supp}(\mu)$. □

Item 2 of proposition 12 will be proved later. We shall need the fact $\Phi = L - \bar{L}$ on the Mather set, that will be proved in lemma 36.

The two formulas given in propositions 10 and 13 are two different ways to compute $\bar{E}$. It is not an easy task to show that the two values are equal. It is the purpose of lemma 29 to give a direct proof of this fact. We also give a second proof using the minimax formula (see remark 31).

Since we do not yet know that $\bar{E}_\omega = \bar{L} = \bar{K} = \bar{E}$, we first prove the following result.

Lemma 29. *If L is C^0 coercive, then $\bar L = \bar K$ and there exists $\mu \in \mathbb{M}_{hol}$ such that $\bar L = \int L\, d\mu$.*

Proof. Part 1. We show that $\bar L \geq \bar K$. Indeed, for any holonomic measure μ and any function $u \in C^0(\Omega)$,

$$\begin{aligned}\int L\, d\mu &= \int [L(\omega,t) + u(\omega) - u\circ\tau_t(\omega)]\,\mu(d\omega,dt)\\ &\geq \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d} \big[L(\omega,t) + u(\omega) - u\circ\tau_t(\omega)\big].\end{aligned}$$

We conclude by taking the supremum on u and the infimum on μ.

Part 2. We show that $\bar K \geq \bar L$. Let $X := C^0_b(\Omega\times\mathbb{R}^d)$ be the vector space of bounded continuous functions equipped with the uniform norm. A coboundary is a function f of the form $f = u\circ\tau - u$ or $f(\omega,t) = u\circ\tau_t(\omega) - u(\omega)$ for some $u \in C^0(\Omega)$. Let

$$\begin{aligned}A &:= \{(f,s)\in X\times\mathbb{R} :\ f \text{ is a coboundary and } s \geq \bar K\} \quad \text{and}\\ B &:= \{(f,s)\in X\times\mathbb{R} : \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d}(L-f)(\omega,t) > s\}.\end{aligned}$$

Then A and B are nonempty convex subsets of $X\times\mathbb{R}$. They are disjoint by the definition of $\bar K$ and B is open because L is coercive. By Hahn-Banach theorem, there exists a nonzero continuous linear form Λ on $X\times\mathbb{R}$ which separates A and B. The linear form Λ is given by $\lambda\otimes\alpha$, where λ is a continuous linear form on X and $\alpha\in\mathbb{R}$. The linear form λ is, in particular, continuous on $C^0_0(\Omega\times\mathbb{R}^d)$ and, by Riesz-Markov theorem,

$$\forall\, f \in C^0_0(\Omega\times\mathbb{R}^d), \quad \lambda(f) = \int f\, d\mu,$$

for some signed measure μ. By separation, we have

$$\lambda(f) + \alpha s \leq \lambda(u - u\circ\tau) + \alpha s',$$

for $u\in C^0(\Omega)$, $f\in X$ and $s,s'\in\mathbb{R}$ such that $\inf_{\Omega\times\mathbb{R}^d}(L-f) > s$ and $s'\geq\bar K$. By multiplying u by an arbitrary constant, one obtains

$$\forall\, u\in C^0(\Omega), \quad \lambda(u - u\circ\tau) = 0.$$

The case $\alpha = 0$ is not admissible, since otherwise $\lambda(f)\leq 0$ for every $f\in X$ and λ would be the null form, which is not possible. The case $\alpha<0$ is not admissible either, since otherwise one would obtain a contradiction by taking $f = 0$ and $s\to-\infty$. By dividing by $\alpha>0$ and changing λ/α to λ (as well as μ/α to μ), one obtains

$$\forall\, f\in X, \quad \lambda(f) + \inf_{\Omega\times\mathbb{R}^d}(L-f) \leq \bar K.$$

By taking $f = c\mathbb{1}$, one obtains $c(\lambda(\mathbb{1}) - 1) \leq \bar K - \inf_{\Omega\times\mathbb{R}^d} L$ for every $c\in\mathbb{R}$, and thus $\lambda(\mathbb{1}) = 1$. By taking $-f$ instead of f, one obtains $\lambda(f) \geq \inf_{\Omega\times\mathbb{R}^d} L - \bar K$ for every $f \geq 0$, which (again arguing by contradiction) yields $\lambda(f) \geq 0$. In particular, μ is a probability measure. We claim that

$$\forall\, u \in C^0(\Omega), \qquad \int (u - u \circ \tau)\, d\mu = 0.$$

Indeed, given $R > 0$, consider a continuous function $0 \leq \phi_R \leq 1$, with compact support on $\Omega \times B_{R+1}(0)$, such that $\phi_R \equiv 1$ on $\Omega \times B_R(0)$. Then

$$u - u \circ \tau \geq (u - u \circ \tau)\phi_R + \min_{\Omega \times \mathbb{R}^d} (u - u \circ \tau)(1 - \phi_R).$$

Since λ and μ coincide on $C^0_0(\Omega \times \mathbb{R}^d) + \mathbb{R}\mathbb{1}$, one obtains

$$0 = \lambda(u - u \circ \tau) \geq \int (u - u \circ \tau)\phi_R \, d\mu + \min_{\Omega \times \mathbb{R}^d} (u - u \circ \tau) \int (1 - \phi_R)\, d\mu.$$

By letting $R \to +\infty$, it follows that $\int (u - u \circ \tau)\, d\mu \leq 0$ and the claim is proved by changing u to $-u$. In particular, μ is holonomic. We claim that

$$\forall\, f \in X, \qquad \int f \, d\mu + \inf_{\Omega \times \mathbb{R}^d} (L - f) \leq \bar{K}.$$

Indeed, we first notice that the left hand side does not change by adding a constant to f. Moreover, if $f \geq 0$ and $0 \leq f_R \leq f$ is any continuous function with compact support on $\Omega \times B_{R+1}(0)$ which is identical to f on $\Omega \times B_R(0)$, the claim follows by letting $R \to +\infty$ in

$$\int f_R \, d\mu + \inf_{\Omega \times \mathbb{R}^d} (L - f) \leq \lambda(f_R) + \inf_{\Omega \times \mathbb{R}^d} (L - f_R) \leq \bar{K}.$$

We finally prove the opposite inequality $\bar{L} \leq \bar{K}$. Given $R > 0$, denote $L_R = \min(L, R)$. Since L is coercive, $L_R \in X$. Then $L - L_R \geq 0$ and $\int L_R \, d\mu \leq \bar{K}$. By letting $R \to +\infty$, one obtains $\int L \, d\mu \leq \bar{K}$ for some holonomic measure μ. □

Remark 30. *The existence of a minimizing holonomic probability may be also obtained from basic properties of Kantorovich-Rubinstein topology on the set of probabilities measures on a Polish space (X, d). Given a point $x_0 \in X$, let us consider the set of probability measures on the Borel sets of X that admit a finite first moment,* i.e.,

$$\mathcal{P}^1(X) = \big\{\mu : \int_X d(x_0, x)\, d\mu(x) < +\infty\big\}.$$

Notice that this set does not depend on the choice of the point x_0. The Kantorovitch-Rubinstein distance on $\mathcal{P}^1(X)$ is defined for $\mu, \nu \in \mathcal{P}^1(X)$ by

$$D(\mu, \nu) := \inf \big\{ \int_{X \times X} d(x, y)\, d\gamma(x, y) : \ \gamma \in \Gamma(\mu, \nu)\big\},$$

where $\Gamma(\mu, \nu)$ denotes the set of all the probability measures γ on $X \times X$ with marginals μ and ν on the first and second factors, respectively.

Recall that a continuous function $L\colon X \to \mathbb{R}$ is said to be superlinear on a Polish space X if the map defined by $x \in X \mapsto L(x)/\big(1+d(x,x_0)\big) \in \mathbb{R}$ is proper. Notice that this definition is also independent of the choice of x_0 and, by considering the distance $\hat{d} := \min(d,1)$ on X, any proper function is superlinear for $\hat{d}$. The following well known property gives us a sufficient condition for the relative compactness in $\mathcal{P}^1(X)$ (for a detailed discussion, we refer the reader to [1]).

Property. *If L is a superlinear continuous function on a Polish space X, then the map $\mu \mapsto \int L\,d\mu$ is lower semi-continuous and proper, namely, for all $c \in \mathbb{R}$, the set $\{\mu \in \mathcal{P}^1(X) : \int L\,d\mu \le c\}$ is compact for the Kantorovich-Rubinstein topology.*

Applying this result to $X = \Omega \times \mathbb{R}^d$, one may guarantee the existence of minimizing holonomic probabilities for C^0 superlinear Lagrangians, since it is plain to check that the set of holonomic measures is a closed subset of $\mathcal{P}^1(\Omega \times \mathbb{R}^d)$ for the Kantorovich-Rubinstein topology.

Remark 31 (A second proof for the sup-inf formula)**.** *Notice that*

$$\begin{aligned}
\min_{\omega\in\Omega} L(\omega,0) &= \min_{\omega\in\Omega} \int (L+u-u\circ\tau)\,d\delta_{(\omega,0)} \qquad \forall\, u\in C^0(\Omega)\\
&\ge \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d} \int (L+u-u\circ\tau)\,d\delta_{(\omega,t)}\\
&\ge \inf_{\mu\in\mathcal{P}^1(\Omega\times\mathbb{R}^d)} \int (L+u-u\circ\tau)\,d\mu\\
&\ge \min_{\omega\in\Omega,\ t\in\mathbb{R}^d} (L+u-u\circ\tau)(\omega,t)
\end{aligned}$$

clearly implies

$$\bar{K} = \sup_{u\in C^0(\Omega)} \inf_{\mu\in\mathcal{P}^1(\Omega\times\mathbb{R}^d)} \int (L+u-u\circ\tau)\,d\mu.$$

Besides, for a positive integer ℓ, we have the equality

$$\bar{K}_\ell := \sup_{\substack{u\in C^0(\Omega)\\ \|u\|_\infty\le\ell}} \inf_{\mu\in\mathcal{P}^1(\Omega\times\mathbb{R}^d)} \int (L+u-u\circ\tau)\,d\mu = \sup_{\substack{u\in C^0(\Omega)\\ \|u\|_\infty\le\ell}} \inf_{\mu\in C_\ell} \int (L+u-u\circ\tau)\,d\mu, \tag{18}$$

where the nonempty convex subset

$$C_\ell := \big\{\mu\in\mathcal{P}^1(\Omega\times\mathbb{R}^d) : \int L\,d\mu \le \min_{\omega\in\Omega} L(\omega,0) + 2\ell\big\} \tag{19}$$

is closed thanks to the property highlighted in the previous remark. Obviously, it follows that $\bar{K}_\ell \uparrow \bar{K} \le \min_{\omega\in\Omega} L(\omega,0)$.

We will use now a topological minimax theorem which is a generalization of Sion's classical result [22]. For a recent review on such a subject, see [23]. We state a particular case of theorem 5.7 there.

Topological Minimax Theorem. *Let X, Y be Hausdorff topological spaces, and $C \subset X$, $D \subset Y$ be nonempty closed subsets. Let $F(x,y)$ be a real-valued function on $C \times D$ for which*

– *there exists a real number* $\alpha^* > \sup_{y\in D}\inf_{x\in C} F(x,y)$ *such that, for every* $\alpha \in (\sup_{y\in D}\inf_{x\in C} F(x,y), \alpha^*)$,

- for every finite set $\emptyset \neq H \subset D$, *the set* $\cap_{y\in H}\{x \in C : F(x,y) \le \alpha\}$ *is either empty or connected,*

- for every set $K \subset C$, *the set* $\cap_{x\in K}\{y \in D : F(x,y) > \alpha\}$ *is either empty or connected;*

– *for any* $y \in D$ *and* $x \in C$, $F(x,y)$ *is lower semi-continuous in* x *and upper semi-continuous in* y*;*

– *there exists* $y_0 \in D$ *such that* $x \mapsto F(x,y_0)$ *is proper.*

Then,

$$\inf_{x\in C}\sup_{y\in D} F(x,y) = \sup_{y\in D}\inf_{x\in C} F(x,y).$$

In order to apply such a result, we take then into account here the function $F : (\mu,u) \in \mathcal{P}^1(\Omega\times\mathbb{R}^d)\times C^0(\Omega) \mapsto \int (L + u - u\circ\tau)\,d\mu$ *and we consider the closed sets* C_ℓ *given in (19) and* $D_\ell := \{u \in C^0(\Omega) : \|u\|_\infty \le \ell\}$. *Since* F *is affine in both variables, it satisfies the first point of the above theorem. The property stated in the previous remark shows that* F *also verifies the second and the third points. Thus, from equation (18), we get by the topological minimax theorem*

$$\bar K_\ell = \inf_{\mu\in C_\ell}\sup_{u\in D_\ell}\int (L + u - u\circ\tau)\,d\mu. \tag{20}$$

If $\mu_0 \in C_{\ell_0}$ *is not a holonomic probability, there exists a function* $u_0 \in C^0(\Omega)$ *such that* $\int \big(u_0(\omega) - u_0(\tau_t(\omega))\big)\,d\mu_0(\omega,t) > 0$. *Moreover, up to a multiplication by a scalar, we can suppose that* $\int (u_0 - u_0\circ\tau)\,d\mu_0 > \min_{\omega\in\Omega} L(\omega,0) - \inf_{\Omega\times\mathbb{R}^d} L$. *Thus,* μ_0 *may be disregarded in the infimum in (20) whenever* $\ell \ge \ell_0 + \|u_0\|_\infty$. *Since* μ_0 *is any non-holonomic probability with respect to which* L *is integrable, we finally conclude that*

$$\bar K = \lim_{\ell\to\infty}\inf_{\mu\in C_\ell}\sup_{u\in D_\ell}\int (L + u - u\circ\tau)\,d\mu = \inf_{\mu\in\mathbb{M}_{hol}}\int L\,d\mu = \bar L.$$

The holonomic condition shall not be confused with invariance in the usual sense of dynamical systems. We may nevertheless introduce a larger space than $\Omega\times\mathbb{R}^d$ and a suitable dynamics on such a space. We will apply Birkhoff ergodic theorem with respect to that dynamical system to prove that $\bar L \ge \bar E$.

Notation 32. *Consider* $\hat\Omega := \Omega\times(\mathbb{R}^d)^{\mathbb{N}}$ *equipped with the product topology and the Borel sigma-algebra.* $\hat\Omega$ *becomes a complete separable metric space. Any probability measure* μ *on* $\Omega\times\mathbb{R}^d$ *admits a unique disintegration along the the first projection* $pr : \Omega\times\mathbb{R}^d \to \Omega$,

$$\mu(d\omega,dt) := pr_*(\mu)(d\omega)P(\omega,dt),$$

where $\{P(\omega,dt)\}_{\omega\in\Omega}$ *is a measurable family of probability measures on* $\mathbb{R}^d$. *Let* $\hat\mu$ *be the Markov measure with initial distribution* $pr_*(\mu)$ *and transition probabilities* $P(\omega,dt)$. *For Borel bounded functions of the form* $f(\omega,t_0,\dots,t_n)$, *we have*

$$\hat\mu(d\omega,d\underline{t}) = pr_*(d\omega)P(\omega,dt_0)P(\tau_{t_0}(\omega),dt_1)\cdots P(\tau_{t_0+\cdots+t_{n-1}}(\omega),dt_n).$$

If μ is holonomic, then $\hat\mu$ is invariant with respect to the shift map

$$\hat\tau : (\omega, t_0, t_1, \ldots) \mapsto (\tau_{t_0}(\omega), t_1, t_2, \ldots).$$

We will call $\hat\mu$ the Markov extension of μ. Conversely, the projection of any $\hat\tau$-invariant probability measure $\tilde\mu$ on $\Omega \times \mathbb{R}^d$ is holonomic. This gives a fourth way to compute $\bar E$

$$\bar E = \inf \Big\{ \int \hat L \, d\tilde\mu : \tilde\mu \text{ is a } \hat\tau\text{-invariant probability measure on } \hat\Omega \Big\},$$

where $\hat L(\omega, t_0, t_1, \ldots) := L(\omega, t_0)$ is the natural extension of L on $\hat\Omega$.

Proof of propositions 10, 13 and 14.

– Part 1: We know that $\bar K = \bar L$ by lemma 29.

– Part 2: We claim that $\bar E_\omega = \bar E$ for all $\omega \in \Omega$. By the topological stationarity (10) of E_ω and by the minimality of τ_t, for any $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\inf_{x_0,\ldots,x_n \in \mathbb{R}^d} E_\omega(x_0, \ldots, x_n) &= \inf_{x_0,\ldots,x_n \in \mathbb{R}^d} \inf_{t \in \mathbb{R}^d} E_\omega(x_0 + t, \ldots, x_n + t) \\
&= \inf_{x_0,\ldots,x_n \in \mathbb{R}^d} \inf_{t \in \mathbb{R}^d} E_{\tau_t(\omega)}(x_0, \ldots, x_n) \\
&= \inf_{x_0,\ldots,x_n \in \mathbb{R}^d} \inf_{\omega \in \Omega} E_\omega(x_0, \ldots, x_n),
\end{aligned}$$

which clearly yields $\bar E_\omega = \bar E$ for every $\omega \in \Omega$.

– Part 3: We claim that $\bar E \geq \bar K$. Indeed, given $c < \bar K$, let $u \in C^0(\mathbb{R}^d)$ be such that, for every $\omega \in \Omega$ and any $t \in \mathbb{R}^d$, $u(\tau_t(\omega)) - u(\omega) \leq L(\omega, t) - c$. Define $u_\omega(x) = u(\tau_x(\omega))$. Then,

$$\forall\, x, y \in \mathbb{R}^d, \quad u_\omega(y) - u_\omega(x) \leq E_\omega(x, y) - c,$$

which implies $\bar E \geq c$ for every $c < \bar K$, and therefore $\bar E \geq \bar K$.

– Part 4: We claim that $\bar L \geq \bar E$. Let μ be a minimizing holonomic probability measure with Markov extension $\hat\mu$ (see notation 32). If $(\omega, \underline{t}) \in \hat\Omega$, then

$$\sum_{k=0}^{n-1} \hat L \circ \hat\tau^k(\omega, \underline{t}) = E_\omega(x_0, \ldots, x_n) \quad \text{with} \quad x_0 = 0 \text{ and } x_k = t_0 + \cdots + t_{k-1},$$

and, by Birkhoff ergodic theorem,

$$\bar E \leq \int \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat L \circ \hat\tau^k \, d\hat\mu = \int L \, d\mu = \bar L.$$

□

A calibrated sub-action u as given by the Lax-Oleinik operator (see section 5) is not available in general for an almost periodic interaction energy E. The purpose of such a sub-action is to calibrate the energy in the following way

$$E_{\omega,u}(x, y) := E_\omega(x, y) - \big[u \circ \tau_y(\omega) - u \circ \tau_x(\omega)\big] - \bar E. \tag{21}$$

Actually, $E_{\omega,u}(x,y)$ is nonnegative and, depending whether u is forward or backward calibrated, if one of the variables x or y is fixed, the other one can be chosen so that the interaction becomes null. Notice that $U(\omega,t) := u \circ \tau_t(\omega) - u(\omega)$ is a cocycle, namely, it satisfies

$$\forall\, \omega \in \Omega,\ \forall\, s,t \in \mathbb{R}^d, \quad U(\omega, s+t) = U(\omega,s) + U(\tau_s(\omega),t). \tag{22}$$

An important ingredient of the proof of theorem 8 is the notion of Mañé subadditive cocycle.

Definition 33. *Let L be a coercive Lagrangian. We call Mañé subadditive cocycle associated to L the function defined on $\Omega \times \mathbb{R}^d$ by*

$$\Phi(\omega,t) := \inf_{n\geq 1} \inf_{0=x_0,x_1,\dots,x_n=t} \sum_{k=0}^{n-1} \big[L(\tau_{x_k}(\omega), x_{k+1}-x_k) - \bar{E}\big].$$

We call Mañé potential in the environment ω the function on $\mathbb{R}^d \times \mathbb{R}^d$ given by

$$S_\omega(x,y) := \Phi(\tau_x(\omega), y-x) = \inf_{n\geq 1} \inf_{x=x_0,\dots,x_n=y} \big[E_\omega(x_0,\dots,x_n) - n\bar{E}\big].$$

The very definitions of Φ and $\bar{E}$ show that

$$\forall\, \omega \in \Omega,\ \forall\, t \in \mathbb{R}^d, \quad \Phi(\omega,0) \geq 0 \ \text{ and } \ \Phi(\omega,t) \leq L(\omega,t) - \bar{E}. \tag{23}$$

(The sequence $\{\bar{E}_n(\omega,0) := \inf_{x_1,\dots,x_{n-1}} E_\omega(0,x_1,\dots,x_{n-1},0)\}_n$ is subadditive in n and $\bar{E} \leq \lim_{n\to\infty} \frac{1}{n}\bar{E}_n(\omega,0)$.) Moreover, Φ is upper semi-continuous (lemma 36) and a subadditive cocycle:

$$\forall\, \omega \in \Omega,\ \forall\, s,t \in \mathbb{R}, \quad \Phi(\omega,s+t) \leq \Phi(\omega,s) + \Phi(\tau_s(\omega),t). \tag{24}$$

This shows in particular that $\Phi(\omega,t) \geq \bar{E} - L(\tau_t(\omega),-t)$ and thus $\Phi(\omega,t)$ takes always real values. The nontrivial part is to prove that Φ is Mather-calibrated.

Definition 34. *A measurable function $U : \Omega \times \mathbb{R}^d \to [-\infty,+\infty[$ is called a Mather-calibrated subadditive cocycle if the following properties are satisfied:*

– $\forall\, \omega \in \Omega,\ \forall\, s,t \in \mathbb{R}^d, \quad U(\omega,s+t) \leq U(\omega,s) + U(\tau_s(\omega),t),$

– $\forall\, \omega \in \Omega,\ \forall\, s,t \in \mathbb{R}^d, \quad U(\omega,t) \leq L(\omega,t) - \bar{L}$ *and* $U(\omega,0) \geq 0,$

– $\forall\, \mu \in \mathbb{M}_{hol}$ *with* $\int L\,d\mu < +\infty \ \Rightarrow\ \int U(\omega, \sum_{k=0}^{n-1} t_k)\, \hat{\mu}(d\omega, d\underline{t}) \geq 0,\ \forall\, n \geq 1,$ *where $\hat{\mu}$ is the Markov extension of μ.*

Notice that, provided we know in advance that U is finite, $U(\omega,0) \geq 0$ by replacing $s=t=0$ in the subadditive cocycle inequality.

Lemma 35. *A Mather-calibrated subadditive cocycle U satisfies in addition*

– *$U(\omega,t)$ is finite everywhere,*

– $\sup_{\omega\in\Omega, t\in\mathbb{R}^d} |U(\omega,t)|/(1+\|t\|) < +\infty,$

– $\forall\, \mu \in \mathbb{M}_{min},\ \forall\, n \geq 1, \quad U(\omega, \sum_{k=0}^{n-1} t_k) = \sum_{k=0}^{n-1} [\hat{L} - \bar{L}] \circ \hat{\tau}^k(\omega, \underline{t}) \quad \hat{\mu}$ *a.e.*

Proof. Part 1. We show that U is sublinear. Let $K := \sup_{\omega\in\Omega,\ \|t\|\leq 1}[L(\omega,t) - \bar{L}]$. Fix $t \in \mathbb{R}^d$ and choose the unique integer n such that $n-1 \leq \|t\| < n$. Let $t_k = \frac{k}{n}t$ for $k = 0,\dots,n-1$. Then the subadditive cocycle property implies, on the one hand,

$$\forall\,\omega \in \Omega,\ \forall\,t \in \mathbb{R}^d,\quad U(\omega,t) \leq \sum_{k=0}^{n-1} U(\tau_{t_k}(\omega), t_{k+1} - t_k) \leq nK \leq (1+\|t\|)K.$$

On the other hand, thanks to the hypothesis $U(\omega,0) \geq 0$, we get the opposite inequality

$$\forall\,\omega \in \Omega,\ \forall\,t \in \mathbb{R}^d,\quad U(\omega,t) \geq U(\omega,0) - U(\tau_t(\omega),-t) \geq -(1+\|t\|)K.$$

We also have shown that U is finite everywhere.

Part 2. Suppose μ is minimizing. Since

$$\forall\,\omega \in \Omega,\ \forall\,t_0,\dots,t_{n-1} \in \mathbb{R}^d,\quad \sum_{k=0}^{n-1}\big[\hat{L} - \bar{L}\big]\circ\hat{\tau}^k(\omega,\underline{t}) \geq U\Big(\omega,\sum_{k=0}^{n-1} t_k\Big),$$

by integrating with respect to $\hat{\mu}$, the left hand side has a null integral whereas the right hand side has a nonnegative integral. The previous inequality is thus an equality that holds almost everywhere. □

Lemma 36. *If L is C^0 coercive, then the Mañé subadditive cocycle Φ is upper semi-continuous and Mather-calibrated. In particular, $\Phi = L - \bar{L}$ on* Mather(L), *or more precisely*

$$\forall\,\mu \in \mathbb{M}_{min},\ \forall\,(\omega,\underline{t}) \in \mathrm{supp}(\hat{\mu}),\ \forall\,i<j,$$
$$\Phi\Big(\tau_{\sum_{k=0}^{i-1} t_k}(\omega),\sum_{k=i}^{j-1} t_k\Big) = \sum_{k=i}^{j-1}\big[L - \bar{L}\big]\circ\hat{\tau}^k(\omega,\underline{t})$$

(or in an equivalent manner, if $x_0 = 0$ and $x_{k+1} = x_k + t_k$, $\forall\,k \geq 0$, the semi-infinite configuration $\{x_k\}_{k\geq 0}$ is calibrated for E_ω as in definition 7).

Proof. Part 1. We first show the existence of a measurable Mather-calibrated subadditive cocycle $U(\omega,t)$. From the sup-inf formula (proposition 13), for every $p \geq 1$, there exists $u_p \in C^0(\Omega)$ such that

$$\forall\,\omega \in \Omega,\ \forall\,t \in \mathbb{R}^d,\quad u_p\circ\tau_t(\omega) - u_p(\omega) \leq L(\omega,t) - \bar{L} + 1/p.$$

Let $U_p(\omega,t) := u_p\circ\tau_t(\omega) - u_p(\omega)$ and $U := \limsup_{p\to+\infty} U_p$. Then U is clearly a subadditive cocycle and satisfies $U(\omega,0) = 0$. Besides, U is finite everywhere, since $0 = U(\omega,0) \leq U(\omega,t) + U(\tau_t(\omega),-t)$ and $U(\omega,t) \leq L(\omega,t) - \bar{L}$. We just verify the last property in definition 34. Let $\mu \in \mathbb{M}_{hol}$ be such that $\int L\,d\mu < +\infty$. For $n \geq 1$, let

$$\hat{S}_{n,p}(\omega,\underline{t}) := \sum_{k=0}^{n-1}\Big[\hat{L} - \bar{L} + \frac{1}{p}\Big]\circ\hat{\tau}^k(\omega,\underline{t}) - U_p\Big(\omega,\sum_{k=0}^{n-1} t_k\Big) \geq 0.$$

By integrating with respect to $\hat\mu$, we obtain

$$0 \le \int \inf_{p\ge q} \hat S_{n,p}\, d\hat\mu \le \inf_{p\ge q} \int \hat S_{n,p}(\omega,\underline{t})\, d\hat\mu \le n \int \Big[L-\bar L+\frac{1}{q}\Big]\, d\mu.$$

By Lebesgue's monotone convergence theorem, we obtain

$$\int \Big[n(\hat L-\bar L) - U\Big(\omega, \sum_{k=0}^{n-1} t_k\Big)\Big] d\hat\mu \le \int n[L-\bar L]\, d\mu \quad \text{and}$$

$$\int U\Big(\omega, \sum_{k=0}^{n-1} t_k\Big)\, \hat\mu(d\omega, d\underline{t}) \ge 0.$$

Part 2. We next show that Φ is Mather-calibrated. We have already noticed that Φ satisfies the subadditive cocycle property, besides $\Phi \le L-\bar L$ by definition. We point out that $\Phi(\omega,0)\ge 0$, since, for $x_0=0, x_1,\dots,x_{n-1},x_n=0$, denoting $y_{\ell n+i}=x_i$, $\forall\, \ell=0,\dots,k$, $\forall\, i=0,\dots,n-1$, we have

$$kE_\omega(x_0,\dots,x_n) = E_\omega(y_0,\dots,y_{kn}) \ge \inf_{0=y_0,\dots,y_{kn}=0} E_\omega(y_0,\dots,y_{kn}),$$

which, thanks to proposition 14, implies

$$\inf_{0=x_0,\dots,x_n=0} \frac{1}{n} E_\omega(x_0,\dots,x_n) \ge \inf_{0=y_0,\dots,y_{kn}=0} \frac{1}{nk} E_\omega(y_0,\dots,y_{kn}) \xrightarrow{k\to\infty} \bar E.$$

In particular, $\Phi(\omega,t)$ is finite everywhere. Moreover, $\Phi(\omega,t)\ge U(\omega,t)$ and the third property of definition 34 is thus automatic.

Part 3. We show that Φ is upper semi-continuous. For $n\ge 1$, let

$$S_n(\omega,t) := \inf\{E_\omega(x_0,\dots,x_n) : x_0=0,\ x_n=t\}.$$

Then $\Phi = \inf_{n\ge1}(S_n - n\bar E)$ is upper semi-continuous if we prove that $S_n(\omega,t)$ is continuous whenever $\omega\in\Omega$ and $\|t\|\le D$. Let $c_0 := \inf_{\omega,x,y} E_\omega(x,y)$ and $K := \sup_{\omega\in\Omega,\ \|t\|\le D} E_\omega(0,\dots,0,t)$. By coerciveness, there exists $R>0$ such that

$$\forall\, x,y\in\mathbb{R}^d, \quad \|y-x\|>R \Rightarrow\ \forall\,\omega\in\Omega,\ E_\omega(x,y) > K-(n-1)c_0.$$

Suppose $\omega, x_0,\dots,x_n$ are such that $E_\omega(x_0,\dots,x_n)\le K$. Suppose by contradiction that $\|x_{k+1}-x_k\|>R$. Thus

$$K \ge E_\omega(x_0,\dots,x_n) \ge (n-1)c_0 + E_\omega(x_k,x_{k+1}) > K,$$

which is impossible. We have proved that the infimum in the definition of $S_n(\omega,t)$, for every $\omega\in\Omega$ and $\|t\|\le D$, can be realized by some points $\|x_k\|\le kR$. By the uniform continuity of $E_\omega(x_0,\dots,x_n)$ on the product space $\Omega\times\Pi_k\{\|x_k\|\le kR\}$, we obtain that S_n is continuous on $\Omega\times\{\|t\|\le D\}$.

Part 4. Let μ be a minimizing measure with Markov extension $\hat\mu$. We show that every $(\omega,\underline{t})$ in the support of $\hat\mu$ is calibrated. Let

$$\hat\Sigma := \Big\{(\omega,\underline{t})\in\Omega\times(\mathbb{R}^d)^{\mathbb{N}} : \forall\, n\ge1,\ \Phi\Big(\omega,\sum_{k=0}^{n-1} t_k\Big) \ge \sum_{k=0}^{n-1}\big[L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t})\Big\}.$$

The set $\hat\Sigma$ is closed, since Φ is upper semi-continuous. By lemma 35, $\hat\Sigma$ has full $\hat\mu$-measure and therefore contains $\mathrm{supp}(\hat\mu)$. Thanks to the subadditive cocycle property of Φ and the $\hat\tau$-invariance of $\mathrm{supp}(\hat\mu)$, we obtain the calibration property

$$\forall\,(\omega,\underline{t})\in\hat\Sigma,\ \forall\,0\le i<j,\quad \Phi\Big(\tau_{x_i}(\omega),\sum_{k=i}^{j-1}t_k\Big)=\sum_{k=i}^{j-1}\big[L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t}). \qquad \square$$

Proof of proposition 12 – Item 2. We now assume that L is superlinear. From lemma 35, the Mañé subadditive cocycle is at most linear. There exists $R>0$ such that

$$\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad |\Phi(\omega,t)|\le R(1+\|t\|).$$

By superlinearity, there exists $B>0$ such that

$$\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad L(\omega,t)\ge 2R\|t\|-B.$$

Let μ be a minimizing measure. Since $\Phi=L-\bar L$ μ a.e. (lemma 35), we obtain

$$\|t\|\le (R+B+|\bar L|)/R,\quad \mu(d\omega,dt)\ \text{a.e.}$$

We have proved that the support of every minimizing measure is compact. In particular, the Mather set is compact. $\square$

Proof of theorem 8. We show that, for every environment ω in the projected Mather set, there exists a calibrated configuration for E_ω passing through the origin. Let μ be a minimizing measure such that $\mathrm{supp}(\mu)=\mathrm{Mather}(L)$. Let $\hat\mu$ denote its Markov extension. For $n\ge1$, consider

$$\hat\Omega_n:=\Big\{(\omega,\underline{t})\in\Omega\times(\mathbb{R}^d)^{\mathbb{N}}:\Phi\Big(\omega,\sum_{k=0}^{2n-1}t_k\Big)\ge\sum_{k=0}^{2n-1}\big[L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t})\Big\}.$$

From lemma 36, $\mathrm{supp}(\hat\mu)\subseteq\hat\Omega_n$. From the upper semi-continuity of Φ, $\hat\Omega_n$ is closed. To simplify the notations, for every $\underline{t}$, we define a configuration $(x_0,x_1,\ldots)$ by

$$x_0=0,\ x_{k+1}=x_k+t_k\quad\text{so that}\quad \hat\tau^k(\omega,\underline{t})=(\tau_{x_k}(\omega),(t_k,t_{k+1},\ldots)).$$

Notice that, if $(\omega,\underline{t})\in\hat\Omega_n$, thanks to the subadditive cocycle property of Φ and the fact that $\Phi\le L-\bar L$, the finite configuration $(x_0,\ldots,x_{2n})$ is calibrated in the environment ω, that is,

$$\forall\,0\le i<j\le 2n,\quad \Phi\Big(\tau_{x_i}(\omega),\sum_{k=i}^{j-1}t_k\Big)=\sum_{k=i}^{j-1}\big[L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t}),$$

or written using the family of interaction energies E_ω,

$$\forall\,0\le i<j\le 2n,\quad S_\omega(x_i,x_j)=E_\omega(x_i,\ldots,x_j)-(j-i)\bar E.$$

Thanks to the sublinearity of S_ω, there exists a constant $R>0$ such that, uniformly in $\omega\in\Omega$ and $x,y\in\mathbb{R}^d$, we have $|S_\omega(x,y)|\le R(1+\|y-x\|)$. Besides, thanks to the superlinearity of E_ω, there exists a constant $B > 0$ such that $E_\omega(x,y) \geq 2R\|y-x\| - B$. Since $S_\omega(x_k, x_{k+1}) = E_\omega(x_k, x_{k+1}) - \bar{E}$, we thus obtain a uniform upper bound $D := (R + B + |\bar{E}|)/R$ on the jumps of calibrated configurations:

$$\forall\, (\omega, \underline{t}) \in \hat{\Omega}_n,\ \forall\, 0 \leq k < 2n, \quad \|x_{k+1} - x_k\| \leq D.$$

Let $\hat{\Omega}'_n = \hat{\tau}^n(\hat{\Omega}_n)$. Thanks to the uniform bounds on the jumps, $\hat{\Omega}'_n$ is again closed. Since $\hat{\mu}(\hat{\Omega}_n) = 1$, $\hat{\mu}(\hat{\Omega}'_n) = 1$ by invariance of $\hat{\tau}$. Let $\nu := pr_*(\mu)$ be the projected measure on Ω. Then $\mathrm{supp}(\nu) = pr(\mathrm{Mather}(L))$. By the definition of $\hat{\Omega}'_n$, we have

$$\begin{aligned} pr(\hat{\Omega}'_n) = \{\omega \in \Omega : \exists\, (x_{-n}, \ldots, x_n) \in \mathbb{R}^d \ \text{ s.t. } \ x_0 = 0 \ \text{ and} \\ S_\omega(x_{-n}, x_n) \geq E_\omega(x_{-n}, \ldots, x_n) - 2n\bar{E}\}. \end{aligned}$$

Again by compactness of the jumps, $pr(\hat{\Omega}'_n)$ is closed and has full ν-measure. Thus, $pr(\hat{\Omega}'_n) \supseteq pr(\mathrm{Mather}(L))$. By a diagonal extraction procedure, we obtain, for every $\omega \in \mathrm{Mather}(L)$, a bi-infinite calibrated configuration with uniformly bounded jumps passing through the origin. □

4 Calibrated configurations for quasicrystals

This section is devoted to the proof of the second main result of this paper: theorem 19. We first collect elementary results on flow boxes in lemma 37. The notions of flow boxes and flow box decomposition have been introduced in definition 15. In general, a minimal flow does not possess a cover of flow boxes. Flow boxes are open sets obtained by taking the union of every orbits of size R starting from any point belonging to a closed transverse Poincaré section. The restricted topology on a transverse section must be special: it must admit a basis of clopen sets. We then explain in lemma 38 how to build a transversally constant Lagrangian from a locally transversally constant potential. It is indeed easy to built such a potential in the context of Delone sets as explained in section 2. We show in lemma 40 how to construct a suspension with locally constant return maps that we call Kakutani-Rohlin tower. We then assume the flow to be uniquely ergodic and recall in lemma 41 the construction of a unique transverse measure associated to each transverse section.

Supposing $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ to be weakly twist (definition 18), the fundamental Aubry crossing property is explained in lemma 43. Examples of weakly twist Lagrangian are given in corollary 21. We collect in lemmas 44, 46 and 47 several intermediate results, that are consequences of the weakly twist property, about the order of the points composing a minimizing configuration. We assume moreover L to be transversally constant. Our first nontrivial result is stated in proposition 48: a finite configuration $(x_0^n, \ldots, x_n^n)$ which realizes the minimum of the energy among all configurations of the same length must be strictly monotone, and must have uniformly bounded jumps $|x_k^n - x_{k-1}^n| \leq R$. If $E_\omega(x,x) = \bar{E}$ for some $\omega \in \Omega$ and $x \in \mathbb{R}$, the proof of theorem 19 is obvious. We thus suppose $E_\omega(x,x) > \bar{E}$ for every ω and x. Our second key result shows then that $\liminf_{n\to+\infty} \frac{1}{n}|x_n^n - x_0^n| > 0$: the frequency of points x_k^n in a flow box of sufficiently large size is positive. We finally conclude this section with the proof of theorem 19.

Lemma 37. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be an almost periodic $\mathbb{R}$-action. Assume that the action is not periodic ($t \in \mathbb{R} \mapsto \tau_t(\omega) \in \Omega$ is injective for every $\omega \in \Omega$). Then*

1. *If $\tau[B_R \times \Xi]$ is a flow box, then there exists R' such that*
$$\Omega = \tau[B_{R'} \times \Xi] = \{\tau_t(\omega) : |t| < R' \text{ and } \omega \in \Xi\}.$$
2. *If $\tau[B_R \times \Xi]$ is a flow box, then $\tau : \mathbb{R} \times \Xi \to \Omega$ is open and $\tau[B_R \times \Xi']$ is again a flow box for every clopen subset $\Xi' \subset \Xi$.*
3. *If $\tau[B_R \times \Xi]$ is a flow box, then, for every $R' > 0$ and $\omega \in \Xi$, there exists a clopen set $\Xi' \subset \Xi$ containing ω such that $\tau[B_{R'} \times \Xi']$ is again a flow box.*
4. *If $U = \tau[B_R \times \Xi]$ and $U' = \tau[B_{R'} \times \Xi']$ are two admissible flow boxes, if $\tau[B_{2R+2R'} \times \Xi]$ and $\tau[B_{2R+2R'} \times \Xi']$ are also flow boxes, then*
$$U \cap U' = \tau(\tilde{B} \times \tilde{\Xi}) = \tau(\tilde{B}' \times \tilde{\Xi}')$$
for some clopen sets $\tilde{\Xi}$, $\tilde{\Xi}'$ and some open convex subsets $\tilde{B} \subset B_R$, $\tilde{B}' \subset B_{R'}$.
5. *If $\{U_i\}_{i\in I}$ is a flow box decomposition, then, for every $\omega \in \Omega$ and $R > 0$, there exits a flow box $\tau[B_R \times \Xi]$, with a transverse section Ξ containing ω, that is compatible with respect to $\{U_i\}_{i\in I}$.*

Proof. Let $\theta_s : \mathbb{R} \times \Xi \to \mathbb{R} \times \Xi$ be the translation $(t, \omega) \mapsto (t + s, \omega)$. We observe the trivial conjugacy $\tau_s \circ \tau = \tau \circ \theta_s$ and note that both $\tau_s : \Omega \to \Omega$ and $\theta_s : \mathbb{R} \to \mathbb{R}$ are homeomorphisms.

Item 1. Let $U = \tau[B_R \times \Xi]$. The set $\cup_{t\in\mathbb{R}}\tau_t(U)$ is invariant, open, and therefore equal to Ω. By compactness $\Omega = \tau_{t_1}(U) \cup \ldots \cup \tau_{t_r}(U) = \tau[B_{R'} \times \Xi]$, with $R' = R + \max_i |t_i|$.

Item 2. Let V be an open subset of $\mathbb{R} \times \Xi$. Given $(t, \omega) \in V$, there exist $0 < \epsilon < R$ and a clopen set $\Xi' \subset \Xi$ containing ω such that $B_\epsilon(t) \times \Xi' \subset V$. Then
$$\tau(B_\epsilon(t) \times \Xi') = \tau \circ \theta_t(B_\epsilon(0) \times \Xi') = \tau_t \circ \tau(B_\epsilon(0) \times \Xi') \text{ is open in } \Omega.$$
If $\Xi' \subset \Xi$ is a clopen set, then $B_R(0) \times \Xi'$ is open in $B_R(0) \times \Xi$ and $\tau[B_R \times \Xi']$ is open in Ω.

Item 3. We may clearly assume $R' \geq R$. For every $\frac{3}{4}R \leq |s| \leq 2R'$, by aperiodicity, there exists a clopen set $\Xi_s \subset \Xi$ containing ω such that τ is injective on $[B_{R/4}(0) \cup B_{R/4}(s)] \times \Xi_s$. Furthermore, for every $|s| \leq \frac{3}{4}R$, τ is injective on $[B_{R/4}(0) \cup B_{R/4}(s)] \times \Xi$ by the definition of a flow box. Let $\{B_{R/4}(s_i)\}_i$ be a finite cover of $\overline{B_{2R'}(0)}$ so that τ is injective on each $[B_{R/4}(0) \cup B_{R/4}(s_i)] \times \Xi'$, where $\Xi' = \cap_i \Xi_{s_i}$. Then there exists $\epsilon >$ such that τ is injective on $[B_\epsilon(0) \cup B_\epsilon(s)] \times \Xi'$, for every $|s| \leq 2R'$. By conjugacy, τ is injective on $[B_\epsilon(s) \cup B_\epsilon(s')] \times \Xi'$, for every $|s|, |s'| \leq R'$. We thus have obtained that $\tau : B_{R'}(0) \times \Xi' \to \Omega$ is injective. Moreover, τ is open on $B_{R'}(0) \times \Xi'$ by *item 2*.

Item 4. Assume $U \cap U' \neq \emptyset$. There exists $a \in \mathbb{R}$ such that, if $\omega \in \Xi$, $\omega' \in \Xi'$, $|t| < R$, $|t'| < R$, then $\tau_t(\omega) = \tau_{t'}(\omega')$ if, and only if, $t' = t - a$ and $\omega' = \tau_a(\omega)$. In particular, a belongs to $B_R - B_{R'}$ and is unique. Then $\tilde\Xi := \Xi \cap \tau_a^{-1}(\Xi')$ is a clopen subset of Ξ and $\tilde B := B_R \cap (a + B_{R'})$ is an open convex subset of B_R.

Item 5. Let $\{U_i = \tau(B_{R_i} \times \Xi_i)\}$ be a flow box decomposition. Consider $\omega \in \Omega$ and $R > 0$. For every $|x| \le R$, $\tau_x(\omega) \in U_i$ for some box U_i. Then $x \in B_{R_i}(t_i)$ for some t_i such that $\omega_i := \tau_{t_i}(\omega) \in \Xi_i$. By compactness, one can find a finite set of indices I such that $\cup_{i\in I} B_{R_i}(t_i)$ covers $B_R(0)$. Let $i_0 \in I$ be such that $0 \in B_{R_{i_0}}(t_{i_0})$ and $\omega_{i_0} = \tau_{t_{i_0}}(\omega) \in \Xi_{i_0}$. We claim that, for every $i \in I$, there exists a clopen subset $\Xi_{i_0}^i \subset \Xi_{i_0}$ containing ω_{i_0} such that $\tau_{t_i - t_{i_0}}(\Xi_{i_0}^i)$ is a clopen subset of Ξ_i.

Assuming the claim is true, we denote $\Xi := \tau_{-t_{i_0}}(\cap_{i\in I} \Xi_{i_0}^i)$ and, by taking $\Xi_{i_0}^i$'s smaller if necessary, we choose Ξ sufficiently small so that $\tau(B_R \times \Xi)$ is a flow box. If $|x| < R$, $x \in B_{R_i}(t_i)$ for some index $i \in I$. Then $\tilde\Xi_i := \tau_{t_i - t_0}(\cap_{j\in I} \Xi_{i_0}^j)$ is a clopen subset of Ξ_i and

$$\tau_x(\Xi) = \tau_{x-t_i}(\tau_{t_i}(\Xi)) = \tau_{x-t_i}(\tilde\Xi_i).$$

We now prove the claim. We may assume that every $B_{R_i}(t_i)$ has a nonempty intersection with $B_{R'}(0)$. Let $i \in I$ and $x \in B_{R_i}(t_i) \cap B_{R'}(0)$. The segment $[0,x]$ can be split into successive segments $[x_{k-1}, x_k]$, $k = 1, \ldots, n$, each one included in a ball $B_{R_{i_k}}(t_{i_k})$ for some index i_k. The last index satisfies $i_n = i$. We construct by induction clopen subsets $\Xi_{i_0}^{(k)}$ of Ξ_{i_0} containing ω_{i_0} such that $\tau_{t_{i_k} - t_{i_0}}(\Xi_{i_0}^{(k)})$ is a clopen subset of Ξ_{i_k} containing ω_{i_k}. Let $\Xi_{i_0}^{(0)} = \Xi_{i_0}$. Since x_k belongs to both $B_{R_{i_k}}(t_{i_k})$ and $B_{R_{i_{k+1}}}(t_{i_{k+1}})$, we have

$$\begin{gathered}
\tau_{(i_k)}(x_k - t_{i_k}, \omega_{i_k}) = \tau_{(i_{k+1})}(x_k - t_{i_{k+1}}, \omega_{i_{k+1}}), \\
\omega_{i_k} \in \Xi_{i_k}, \quad \omega_{i_{k+1}} \in \Xi_{i_{k+1}}, \\
a_k := t_{i_{k+1}} - t_{i_k},\ \omega_{i_{k+1}} = \tau_{a_k}(\omega_{i_k}),\ x_k - t_{i_{k+1}} = x_k - t_{i_k} - a_k.
\end{gathered}$$

By admissability of the two flow boxes U_{i_k} and $U_{i_{k+1}}$, there exists a clopen subset Ξ'_{i_k} of $\tau_{t_{i_k} - t_{i_0}}(\Xi_{i_0}^{(k)})$ containing ω_{i_k} such that $\tau_{a_k}(\Xi'_{i_k}) \subset \Xi_{i_{k+1}}$. We have proved that $\Xi_{i_0}^{(k+1)} := \tau_{t_{i_0} - t_{i_k}}(\Xi'_{i_k})$ is a clopen subset of Ξ_{i_0} containing ω_{i_0} and that $\tau_{t_{i_{k+1}} - t_{i_0}}(\Xi_{i_0}^{(k+1)})$ is a clopen subset of $\Xi_{i_{k+1}}$. □

An interaction model does not possess a canonical notion of vertical section. Such a notion naturally exists whenever the model admits a flow box decomposition (definition 15). We prove in the next lemma that locally transversally constant functions $V_1, V_2 : \Omega \to \mathbb{R}$ (a set of conditions checked on boxes of size R) enable to construct a transversally constant Lagrangian $L(\omega, t) = W(t) + V_1(\omega) + V_2(\tau_t(\omega))$ (a set of conditions checked on every sufficiently thin flow box). Corollary 21 follows from this lemma.

Lemma 38. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be an almost periodic interaction model admitting a flow box decomposition. Let $V_1, V_2 : \Omega \to \mathbb{R}$ be two locally transversally constant functions on the same flow box decomposition (definition 20), and $W = \mathbb{R} \to \mathbb{R}$ be any function. Define $L(\omega, t) = W(t) + V_1(\omega) + V_2(\tau_t(\omega))$. Then L is transversally constant (definition 16).*

Proof. Assume V_1 and V_2 are locally transversally constant on a flow box decomposition $\{U_i\}_{i\in I}$. Let $\tau[B_R \times \Xi]$ be a flow box which is compatible with respect to $\{U_i\}_{i\in I}$. If $|x|, |y| < R$ and $\omega, \omega' \in \Xi$, then

$$E_\omega(x,y) = W(y-x) + V_{1,\omega}(x) + V_{2,\omega}(y).$$

There exist $i \in I$, $|t_i| < R_i$ and $\tilde{\Xi}_i$ a clopen subset of Ξ_i such that $\tau_x(\Xi) = \tau_{t_i}(\tilde{\Xi}_i)$. Then $\tau_x(\omega) = \tau_{t_i}(\omega_i)$ and $\tau_x(\omega') = \tau_{t_i}(\omega_i')$ for some $\omega_i, \omega_i' \in \tilde{\Xi}_i$. We have

$$V_{1,\omega}(x) = V_{1,\omega_i}(t_i) = V_{1,\omega_i'}(t_i) = V_{1,\omega'}(x).$$

Similarly $V_{2,\omega}(y) = V_{2,\omega'}(y)$. We have thus proved $E_{\omega'}(x,y) = E_\omega(x,y)$. □

The existence of a flow box decomposition (definition 15) enables us to build a global transverse section of the flow with locally constant return times. We extend for an almost periodic interaction model what has been done for quasicrystals in [13]. We first define the notion of Kakutani-Rohlin tower and show that an interaction model possessing a flow box decomposition admits a Kakutani-Rohlin tower.

Definition 39. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be a one-dimensional almost periodic interaction model possessing a flow box decomposition $\{U_i\}_{i\in I}$. We call Kakutani-Rohlin tower a partition $\{F_\alpha\}_{\alpha\in A}$ of Ω of the form*

$$F_\alpha = \tau\big([0,H_\alpha) \times \Sigma_\alpha\big) = \cup_{0\le t<H_\alpha} \tau_t(\Sigma_\alpha)$$

for some some height $H_\alpha > 0$ and some transverse section Σ_α (closed set admitting a basis of clopen subsets), where $\tau\big((0,H_\alpha)\times\Sigma_\alpha\big)$ is a flow box (open and homeomorphic to $(0,H_\alpha)\times\Sigma_\alpha$), and $\cup_{\alpha\in A}\tau(\{H_\alpha\}\times\Sigma_\alpha) = \cup_{\alpha\in A}\tau(\{0\}\times\Sigma_\alpha) = \cup_{\alpha\in A}\Sigma_\alpha$. Moreover, we say that a Kakutani-Rohlin tower is compatible with respect to $\{U_i\}_{i\in I}$ if, for every $\alpha \in A$, there exist $i\in I$, $t_i \in \mathbb{R}$ and a clopen subset $\tilde{\Xi}_i \subset \Xi_i$ such that $\Sigma_\alpha = \tau_{t_i}(\tilde{\Xi}_i)$ and $[t_i, t_i + H_\alpha) \subset [-R_i, R_i)$.

Lemma 40. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be a one-dimensional almost periodic $\mathbb{R}$-action possessing a flow box decomposition $\{U_i\}_{i\in I}$. Then there exists a Kakutani-Rohlin tower $\{F_\alpha\}_{\alpha\in A}$ which is compatible with respect to $\{U_i\}_{i\in I}$.*

Proof. Let $\{U_i\}_{i=1}^n$ be a flow box decomposition, where $U_i = \tau[B_{R_i} \times \Xi_i]$. By definition, U_i is an open set of Ω. We denote $V_i := \tau\big([-R_i, R_i) \times \Xi_i\big)$. We shall build by induction on $i = 1, \ldots, n$ a collection of flow boxes $\{\tau\big((0,H_{i,j})\times\Sigma_{i,j}\big)\}_j$ such that

– the sets $F_{i,j} := \tau\big([0,H_{i,j})\times\Sigma_{i,j}\big)$ are pairwise disjoint,
– $V_i \setminus \cup_{k<i} V_k = \cup_j \tau\big([0,H_{i,j})\times\Sigma_{i,j}\big) = \cup_j F_{i,j}$,
– $\tau(\{-R_i\}\times\Xi_i) \setminus \cup_{k<i} V_k \subset \cup_j \tau(\{0\}\times\Sigma_{i,j})$,
– $\cup_{k<i}\tau(\{R_k\}\times\Xi_k) \cap (V_i \setminus \cup_{k<i}V_k) \subset \cup_j \tau(\{0\}\times\Sigma_{i,j})$,
– $\tau(\{H_{i,j}\}\times\Sigma_{i,j}) \cap \cup_{k<i} V_k \subset \cup_{k<i}\cup_j \tau(\{0\}\times\Sigma_{k,j})$,
– $\tau(\{H_{i,j}\}\times\Sigma_{i,j}) \setminus \cup_{k<i}V_k \subset \tau(\{R_i\}\times\Xi_i) \setminus \cup_{k<i}V_k$.

For $i = 1$, we choose $H_{1,1} = 2R_1$ and $\Sigma_{1,1} = \tau_{-R_1}(\Xi_1)$. Assume that we have built the sets $\tau\big([0,H_{k,j})\times\Sigma_{k,j}\big)$ for every $k < i$ and j. Thanks to the admissibility of the flow boxes $\{U_i\}_{i\in I}$, the set $V_i \cap V_k$, if nonempty, is of the form $\tau(J_{i,k} \times \Xi_{i,k})$, where $J_{i,k} = [a_{i,k}, b_{i,k})$, with $-R_i \le a_{i,k} < b_{i,k} \le R_i$, and $\Xi_{i,k}$ is a clopen set of Ξ_i. The complement $V_i \setminus V_k$ is the union of sets of the form

$$\tau\big([-R_i, a_{i,k}) \times \Xi_{i,k}\big), \quad \tau\big([b_{i,k}, R_i) \times \Xi_{i,k}\big) \quad \text{or} \quad \tau\big([-R_i, R_i) \times (\Xi_i \setminus \Xi_{i,k})\big).$$

Hence, $V_i \setminus \cup_{k<i} V_k$ is obtained as a disjoint union of sets $\tau\big([c_\alpha, d_\alpha) \times \tilde\Sigma_\alpha\big)$, where $\tilde\Sigma_\alpha$ is any clopen set of the form $\cap_{k<i} S_k$, with either $S_k = \Xi_{i,k}$ or $S_k = \Xi_i \setminus \Xi_{i,k}$, and $[c_\alpha, d_\alpha)$ corresponds to any connected component of $[-R_i, R_i) \setminus \cup_{k<i} J_{i,k}$. We next rewrite $\tau\big([c_\alpha, d_\alpha) \times \tilde\Sigma_\alpha\big)$ as $\tau\big([0, H_{i,j}) \times \Sigma_{i,j}\big)$, with $j = j(\alpha)$, where $\Sigma_{i,j} = \tau_{c_\alpha}(\tilde\Sigma_\alpha)$ and $H_{i,j} = d_\alpha - c_\alpha$. By construction, for all $k < i$ with $V_i \cap \overline{V_k} \ne \emptyset$, $\tau(\{R_k\} \times \Xi_k) \cap V_i = \tau(\{b_{i,k}\} \times \Xi_{i,k})$ and its part which is not in $\cup_{l<i} V_l$ is included into $\cup_j \tau(\{0\} \times \Sigma_{i,j})$. Furthermore, $\tau(\{H_{i,j}\} \times \Sigma_{i,j})$ either is included into $\tau(\{R_i\} \times \Xi_i)$ or intersects V_k for some $k < i$ and therefore is included into $\cup_{k<i} \cup_j \tau(\{0\} \times \Sigma_{k,j})$. $\square$

When a Kakutani-Rohlin tower is built, we obtain a global transverse section $\cup_{\alpha\in A} \Sigma_\alpha$ with a return time constant on each Σ_α and equal to H_α. We can induce on a particular section Σ_{α_0} and build a second Kakutani-Rohlin tower with larger heights. We explain in the next paragraph the notations that will be used for these successive towers.

If $\{F^0_\alpha\}_{\alpha\in A^0}$ is a Kakutani-Rohlin tower of order 0, denote $F^0_\alpha := \tau\big([0, H^0_\alpha) \times \Sigma^0_\alpha\big)$. We say that $\Sigma^0 := \cup_\alpha \Sigma^0_\alpha$ is the basis of the tower. Let ω_* be a reference point of the base Σ^0. Consider α_0 such that $\omega_* \in \Sigma^0_{\alpha_0}$. The construction of the tower of order 1 is done by inducing the flow on $\Sigma^1 := \Sigma^0_{\alpha_0}$. We obtain a partition of Σ^1 given by $\{\Sigma^1_\beta\}_{\beta\in A^1}$, where $\beta = (\alpha_0, \ldots, \alpha_p)$, $p \ge 1$, $\alpha_p = \alpha_0$, $\alpha_i \ne \alpha_0$ for $i = 1, \ldots, p-1$,

$$\Sigma^1_\beta = \Sigma^0_{\alpha_0} \cap \tau^{-1}_{H^0_{\alpha_0}}(\Sigma^0_{\alpha_1}) \cap \ldots \cap \tau^{-1}_{H^0_{\alpha_0}+\ldots+H^0_{\alpha_{p-1}}}(\Sigma^0_{\alpha_p}).$$

By minimality, there is a finite collection of such nonempty sets Σ^1_β. Define then

$$H^1_\beta := H^0_{\alpha_0} + \ldots + H^0_{\alpha_{p-1}},$$
$$F^1_\beta := \tau\big([0, H^1_\beta) \times \Sigma^1_\beta\big) = \bigcup_{i=0}^{p-1} \tau\big([t_i, t_i + H^0_{\alpha_i}) \times \Sigma^0_{\alpha_i}\big), \text{ with } t_i = \sum_{j=0}^{i-1} H^0_{\alpha_j}. \tag{25}$$

We have just obtained a new Kakutani-Rohlin tower $\{F^1_\beta\}_{\beta\in A^1}$ of basis $\Sigma^0_{\alpha_0}$. We induced again on the section $\Sigma^1_{\beta_0}$ that contains ω_* and build the tower of order 2. We shall write $\{F^l_\alpha\}_{\alpha\in A^l}$ for the successive towers that are built using this procedure and F^l_* for the tower of height H^l_* whose basis Σ^l_* contains ω_* . The preceding construction gives $\min_{\alpha\in A^{l+1}} H^{l+1}_\alpha \ge H^l_*$ and in particular $H^{l+1}_* \ge H^l_*$. It may happen that $H^l_* = H^{l+1}_* = H^{l+2}_* = \ldots$ In that case, the flow is a suspension over Σ^l_* of constant return time H^l_* (and Ω is isomorphic to $\Sigma^l_* \times S^1$). In order to exclude this situation, we split the basis $\Sigma^l_{\alpha_0}$ which contains ω_* into two disjoint clopen sets $\Sigma^l_{\alpha_0} = \Sigma^l_{\alpha'_0} \cup \Sigma^l_{\alpha''_0}$. We obtain again a Kakutani-Rohlin tower and we induce as before on the subset which contains ω_*. If $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ is not periodic, we may choose the splitting so that $H^{l+1}_* > H^l_*$ at each step of the construction.

We now assume the flow $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ to be uniquely ergodic. Let λ be the unique ergodic invariant probability measure. The average frequency of return vectors to a transverse section of a flow box measures the thickness of the section. The next lemma gives a precise definition of a family of transverse measures $\{\nu_\Xi\}_\Xi$ parameterized by every transverse section Ξ.

Lemma 41. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be an almost periodic and uniquely ergodic $\mathbb{R}$-action. Given Ξ a transverse section, let $\mathcal{R}_\Xi(\omega)$ be the set of return times to Ξ,*

$$\mathcal{R}_\Xi(\omega) := \{t \in \mathbb{R} : \tau_t(\omega) \in \Xi\}, \quad \forall\, \omega \in \Omega.$$

Then, for every nonempty clopen set $\Xi' \subset \Xi$, the following limit exists uniformly with respect to $\omega \in \Omega$ and is positive:

$$\nu_\Xi(\Xi') := \lim_{T\to+\infty} \frac{\#(\mathcal{R}_{\Xi'}(\omega) \cap B_T(0))}{\mathrm{Leb}(B_T(0))} > 0.$$

Moreover, ν_Ξ extends to a finite and nonnegative measure on Ξ, called transverse measure to Ξ, and, for every flow box $U = \tau[B_R \times \Xi]$,

$$\lambda(\tau(B' \times \Xi')) = \mathrm{Leb}(B')\nu_\Xi(\Xi'), \quad \forall\, B' \subset B_R(0),\ \forall\, \Xi' \subset \Xi \quad (\textit{Borel sets}).$$

Proof. Let $U = \tau[B_R \times \Xi]$ be a flow box. Let $t_1 \neq t_2$ be two return times of $\mathcal{R}_\Xi(\omega)$. Since τ is injective on $B_R(0) \times \Xi$, it is straightforward that $B_R(t_1) \cap B_R(t_2) = \emptyset$. For $\omega \in \Omega$ and $T > 0$, consider

$$\mu_{T,\omega}(U') = \frac{1}{\mathrm{Leb}(B_T(0))} \int_{B_T(0)} \mathbf{1}_{U'}(\tau_s(\omega))\, ds, \quad \forall\, U' \subset \Omega \quad \text{(Borel set)}.$$

The unique ergodicity of the action implies that, for all $\phi \in C^0(\Omega)$, $\mu_{T,\omega}(\phi)$ converges uniformly in ω to $\lambda(\phi)$ as $T \to +\infty$. Let $B' \subset B_R(0)$ be a Borel set and $\Xi' \subset \Xi$ be a nonempty clopen set. For $U' = \tau(B' \times \Xi')$, notice then that

$$\{s \in \mathbb{R}^d : \tau_s(\omega) \in U'\} = \bigcup_{t\in\mathcal{R}_{\Xi'}(\omega)} t + B', \ \ \mu_{T,\omega}(U') = \sum_{t\in\mathcal{R}_{\Xi'}(\omega)} \frac{\mathrm{Leb}(B_T(0) \cap (t + B'))}{\mathrm{Leb}(B_T(0))},$$

and, whenever $T > 2R$,

$$\mathrm{Leb}(B')\frac{\#(B_{T-R}(0) \cap \mathcal{R}_{\Xi'}(\omega))}{\mathrm{Leb}(B_T(0))} \le \mu_{T,\omega}(U') \le \mathrm{Leb}(B')\frac{\#(B_{T+R}(0) \cap \mathcal{R}_{\Xi'}(\omega))}{\mathrm{Leb}(B_T(0))}.$$

Moreover, clearly $\#(B_T(0)\cap\mathcal{R}_{\Xi'}(\omega)) \le \frac{\mathrm{Leb}(B_{T+R}(0))}{\mathrm{Leb}(B_R(0))}$ and $\lim_{T\to+\infty} \frac{\mathrm{Leb}(B_{T+R}(0))}{\mathrm{Leb}(B_T(0))} = 1$. Thus, if B' is open in $B_R(0)$, then U' is open in Ω and

$$\lambda(U') \le \liminf_{T\to+\infty} \mu_{T,\omega}(U') \le \frac{\mathrm{Leb}(B')}{\mathrm{Leb}(B_{2R}(0))}.$$

In particular, if B' is negligible, thanks to the regularity of Leb, $\lambda(U') = 0$. If B' is open, $\overline{B'} \subset B_R(0)$ and $\partial B'$ is negligible, then, for every $\epsilon > 0$, there exist nonnegative continuous functions $\phi \le \psi$ such that

$$\phi \le \mathbf{1}_{\tau(B'\times\Xi)} \le \mathbf{1}_{\tau(\overline{B'}\times\Xi)} \le \psi \quad \text{and} \quad \lambda(\psi - \phi) < \epsilon.$$

Therefore, $\mu_{T,\omega}(\tau(B' \times \Xi'))$ converges uniformly in ω to $\lambda(\tau(B' \times \Xi))$ as $T \to +\infty$.

On the one hand, for all clopen set $\Xi' \subset \Xi$, $\tau(B_R(0) \times \Xi')$ is a flow box and

$$\lim_{T\to+\infty} \frac{\#(B_T(0) \cap \mathcal{R}_{\Xi'}(\omega))}{\mathrm{Leb}(B_T(0))} := \nu_\Xi(\Xi') \quad \text{(exists uniformly in } \omega).$$

On the other hand, for every $B' = B_{R'}(s')$, $s' \in B_R(0)$, $\|s'\| + R' < R$,

$$\lambda(\tau(B' \times \Xi')) = \lim_{T\to+\infty} \mu_{T,\omega}(\tau(B' \times \Xi')) = \mathrm{Leb}(B')\nu_\Xi(\Xi').$$

Hence, ν_Ξ extends to a measure on the Borel sets of Ξ and by the monotone class theorem $\lambda(\tau(B' \times \Xi')) = \mathrm{Leb}(B')\nu_\Xi(\Xi')$ for every Borel sets $B' \subset B_R(0)$ and $\Xi' \subset \Xi$.

We finally remark that $\nu_\Xi(\Xi') > 0$ for every nonempty clopen set $\Xi' \subset \Xi$, since otherwise there would exist an open set of Ω of λ-measure zero. ☐

We come back to Kakutani-Rohlin towers of flows. Let $\{F^l_\alpha\}_{\alpha\in A^l}$ be such a tower of order l and $\{F^{l+1}_\beta\}_{\beta\in A^{l+1}}$ be the subsequent tower as introduced in (25). We recall the definition of the homology matrix as explained in lemma 2.7 of [13]. For every $\alpha \in A^l$ and $\beta \in A^{l+1}$, $\beta = (\alpha_0, \ldots, \alpha_p)$, $\alpha_0 = \alpha_p$, $\alpha_i \neq \alpha_0$ for $i = 1, \ldots, p-1$, we denote

$$M^l_{\alpha,\beta} := \#\{0 \le k \le p-1 : \alpha_k = \alpha\}.$$

A flow box of order $l+1$, $\tau\big([0, H^{l+1}_\beta) \times \Sigma^{l+1}_\beta\big)$, is obtained as a disjoint union of flow boxes of order l of the type $\tau\big([t_i, t_i + H^l_{\alpha_i}) \times \Sigma^l_{\alpha_i}\big)$. The integer $M^l_{\alpha,\beta}$ counts the number of times a flow box of order $l+1$ indexed by β cuts a flow box of order l indexed by α. The main result that we shall need is given by the following lemma.

Lemma 42. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be a one-dimensional almost periodic and uniquely ergodic $\mathbb{R}$-action. Let $\{F^l_\alpha\}_{\alpha\in A^l}$ be a sequence of Kakutani-Rohlin towers built as in (25). Let ν^l be the transverse measure associated to the transverse section $\cup_{\alpha\in A^l}\Sigma^l_\alpha$. If $\nu^l_\alpha := \nu^l(\Sigma^l_\alpha)$, then*

$$\nu^l_\alpha = \sum_{\beta\in A^{l+1}} M^l_{\alpha,\beta}\nu^{l+1}_\beta.$$

Proof. Let $\Xi = \cup_{\beta\in A^{l+1}}\Sigma^{l+1}_\beta$. For $\omega \in \Xi$, let $0 = t_0, t_1, t_2, \ldots$ be its successive return times to Ξ. We introduce as in lemma 41 the set of return times to the transverse section Σ^l_α, say, $\mathcal{R}^l_\alpha(\omega) := \{t \in \mathbb{R} : \tau_t(\omega) \in \Sigma^l_\alpha\}$. The set $\mathcal{R}^{l+1}_\beta(\omega)$ is defined similarly. Since

$$\#\big(\mathcal{R}^l_\alpha(\omega) \cap [0, t_n)\big) = \sum_{\beta\in A^{l+1}} M^l_{\alpha,\beta}\ \#\big(\mathcal{R}^{l+1}_\beta(\omega) \cap [0, t_n)\big),$$

we divide by t_n and apply lemma 41 to conclude. ☐

The main property used in one-dimensional Aubry theory [2] is the twist property. It will not be used in the infinitesimal form. The following lemma is an easy consequence of definition 18. It shows that the energy of a configuration can be lower by exchanging the positions.

Lemma 43 **(Aubry crossing lemma).** *If L satisfies the weakly twist property, then, for every $\omega \in \Omega$, for every $x_0, x_1, y_0, y_1 \in \mathbb{R}$ verifying $(y_0 - x_0)(y_1 - x_1) < 0$,*

$$\big[E_\omega(x_0, x_1) + E_\omega(y_0, y_1)\big] - \big[E_\omega(x_0, y_1) + E_\omega(y_0, x_1)\big] = \alpha(y_0 - x_0)(y_1 - x_1) > 0,$$

with $\alpha = \frac{1}{(y_0-x_0)(y_1-x_1)} \int_{x_0}^{y_0} \int_{x_1}^{y_1} \frac{\partial^2 \tilde{E}_\omega}{\partial x \partial y}(x, y)\, dydx < 0$ and $\tilde{E}_\omega$ as in definition 18.

Proof. The inequality is obtained by integrating the function $\frac{\partial^2}{\partial x \partial y} \tilde{E}_\omega$ on the domain $[\min(x_0, y_0), \max(x_0, y_0)] \times [\min(x_1, y_1), \max(x_1, y_1)]$. □

The first consequence of Aubry crossing lemma is that minimizing configurations shall be strictly ordered. We begin by an intermediate lemma.

Lemma 44. *Let L be a weakly twist Lagrangian, $\omega \in \Omega$, $n \geq 2$, and $x_0, \ldots, x_n \in \mathbb{R}$ be a nonmonotone sequence (that is, a sequence which does not satisfy $x_0 \leq \ldots \leq x_n$ nor $x_0 \geq \ldots \geq x_n$).*

– If $x_0 = x_n$, then $E_\omega(x_0, \ldots, x_n) > \sum_{i=0}^{n-1} E_\omega(x_i, x_i)$.

– If $x_0 \neq x_n$, then there exists a subset $\{i_0, i_1, \ldots, i_r\}$ of $\{0, \ldots, n\}$, with $i_0 = 0$ and $i_r = n$, such that $(x_{i_0}, x_{i_1}, \ldots, x_{i_r})$ is strictly monotone and

$$E_\omega(x_0, \ldots, x_n) > E_\omega(x_{i_0}, \ldots, x_{i_r}) + \sum_{i \notin \{i_0, \ldots, i_r\}} E_\omega(x_i, x_i).$$

Proof. We prove the lemma by induction.

Let $x_0, x_1, x_2 \in \mathbb{R}$ be a nonmonotone sequence. Then x_0, x_1, x_2 are three distinct points. Thus, $x_0 < x_1$ implies $x_2 < x_1$ and $x_1 < x_0$ implies $x_1 < x_2$. In both cases, lemma 43 tells us that

$$E_\omega(x_0, x_1) + E_\omega(x_1, x_2) > E_\omega(x_0, x_2) + E_\omega(x_1, x_1).$$

Let $(x_0, \ldots, x_{n+1})$ be a nonmonotone sequence. We have two cases: either $x_0 \leq x_n$ or $x_0 \geq x_n$. We shall only give the proof for the case $x_0 \leq x_n$.

Case $x_0 = x_n$. Then $(x_0, \ldots, x_n)$ is nonmonotone and by induction

$$\begin{aligned} E_\omega(x_0, \ldots, x_{n+1}) &> E_\omega(x_n, x_{n+1}) + \sum_{i=0}^{n-1} E_\omega(x_i, x_i) \\ &= E_\omega(x_0, x_{n+1}) + \sum_{i=1}^{n} E_\omega(x_i, x_i). \end{aligned}$$

Case $x_0 < x_n$. Whether $(x_0, \ldots, x_n)$ is monotone or not, we may choose a subset of indices $\{i_0, \ldots, i_r\}$ such that $i_0 = 0$, $i_r = n$, $x_{i_0} < x_{i_1} < \ldots < x_{i_r}$ and

$$E_\omega(x_0, \ldots, x_{n+1}) \geq \Big(E_\omega(x_{i_0}, \ldots, x_{i_r}) + \sum_{i \notin \{i_0, \ldots, i_r\}} E_\omega(x_i, x_i)\Big) + E_\omega(x_n, x_{n+1}).$$

If $x_n \leq x_{n+1}$, then $(x_0, \ldots, x_n)$ is necessarily nonmonotone and the previous inequality is strict. If $x_n = x_{n+1}$, the lemma is proved by modifying $i_r = n + 1$. If $x_n < x_{n+1}$, the lemma is proved by choosing $r + 1$ indices and $i_{r+1} = n + 1$.

If $x_{n+1} < x_n = x_{i_r}$, by applying lemma 43, one obtains

$$E_\omega(x_{i_{r-1}}, x_{i_r}) + E_\omega(x_n, x_{n+1}) > E_\omega(x_n, x_{i_r}) + E_\omega(x_{i_{r-1}}, x_{n+1}),$$
$$E_\omega(x_0, \dots, x_{n+1}) > E_\omega(x_{i_0}, \dots, x_{i_{r-1}}, x_{n+1}) + \sum_{i \notin \{i_0, \dots, i_r\}} E_\omega(x_i, x_i) + E_\omega(x_n, x_n).$$

If $x_{i_{r-1}} < x_{n+1}$, the lemma is proved by choosing $i_r = n+1$. If $x_{i_{r-1}} = x_{n+1}$, the lemma is proved by choosing $r-1$ indices and $i_{r-1} = n+1$. If $x_{n+1} < x_{i_{r-1}}$, we apply again lemma 43 until there exists a largest $s \in \{0, \dots, r\}$ such that $x_s < x_{n+1}$ or $x_{n+1} \le x_0$. In the former case, the lemma is proved by choosing $s+1$ indices and by modifying $i_{s+1} = n+1$. In the latter case, namely, when $x_{n+1} \le x_0 < x_n$, we have

$$E_\omega(x_0, \dots, x_{n+1}) > E_\omega(x_0, x_{n+1}) + \sum_{i=1}^{n} E_\omega(x_i, x_i)$$

and the lemma is proved whether $x_{n+1} = x_0$ or $x_{n+1} < x_0$. □

The Mañé subadditive cocycle $\Phi(\omega, t)$ (definition 33) is obtained by minimizing a normalized energy $E_\omega(x_0, \dots, x_n) - n\bar{E}$ on all the configurations satisfying $x_0 = 0$ and $x_n = t$. The following lemma shows that it is enough to minimize on strictly monotone configurations (unless $t = 0$).

Corollary 45. *If L satisfies the weakly twist property, then, for every $\omega \in \Omega$, the Mañé subadditive cocycle $\Phi(\omega, t)$ satisfies:*
– *if $t = 0$, $\Phi(\omega, 0) = E_\omega(0, 0) - \bar{E}$,*
– *if $t > 0$, $\Phi(\omega, t) = \inf_{n \ge 1} \inf_{0 = x_0 < x_1 < \dots < x_n = t} [E_\omega(x_0, \dots, x_n) - n\bar{E}]$,*
– *if $t < 0$, $\Phi(\omega, t) = \inf_{n \ge 1} \inf_{0 = x_0 > x_1 > \dots > x_n = t} [E_\omega(x_0, \dots, x_n) - n\bar{E}]$.*

Proof. Lemma 44 tells us that we can minimize the energy of $E_\omega(x_0, \dots, x_n) - n\bar{E}$ by the sum of two terms:
– either $x_n = x_0$, then

$$E_\omega(x_0, \dots, x_n) - n\bar{E} \ge \big[E_\omega(x_0, x_0) - \bar{E}\big] + \sum_{i \notin \{0, n\}} \big[E_\omega(x_i, x_i) - \bar{E}\big];$$

– or $x_n \ne x_0$, then for some $(x_{i_0}, \dots, x_{i_r})$ strictly monotone, with $i_0 = 0$ and $i_r = n$,

$$E_\omega(x_0, \dots, x_n) - n\bar{E} \ge \big[E_\omega(x_{i_0}, \dots, x_{i_r}) - r\bar{E}\big] + \sum_{i \notin \{i_0, \dots, i_r\}} \big[E_\omega(x_i, x_i) - \bar{E}\big].$$

We conclude the proof by noticing that $\bar{E} \le \inf_{x \in \mathbb{R}} E_\omega(x, x)$. □

We recall that a finite configuration $(x_0, x_1, \dots, x_n)$ is said to be minimizing in the environment ω if $E_\omega(x_0, x_1, \dots, x_n) \le E_\omega(y_0, y_1, \dots, y_n)$ whenever $x_0 = y_0$ and $x_n = y_n$. The following lemmas show that, under certain conditions, a minimizing configuration is strictly monotone.

Lemma 46. *Suppose that L satisfies the weakly twist property. For every $\omega \in \Omega$, if $(x_0, \dots, x_n)$ is a minimizing configuration, with $x_0 \ne x_n$, such that x_i is strictly between x_0 and x_n for every $0 < i < n-1$, then $(x_0, \dots, x_n)$ is strictly monotone.*

Proof. Let $(x_0, \ldots, x_n)$ be such a minimizing sequence. We show, in part 1, it is monotone, and, in part 2, it is strictly monotone.

Part 1. Assume by contradiction that $(x_0, \ldots, x_n)$ is not monotone. According to lemma 44, one can find a subset of indices $\{i_0, \ldots, i_r\}$ of $\{0, \ldots, n\}$, with $i_0 = 0$ and $i_r = n$, such that $(x_{i_0}, \ldots, x_{i_r})$ is strictly monotone and

$$E_\omega(x_0, \ldots, x_n) > E_\omega(x_{i_0}, \ldots, x_{i_r}) + \sum_{i \notin \{i_0, \ldots, i_r\}} E_\omega(x_i, x_i).$$

We choose the largest integer r with the above property. Since $(x_0, \ldots, x_n)$ is not monotone, we have necessarily $r < n$. Since $(x_0, \ldots, x_n)$ is minimizing, one can find $i \notin \{i_0, \ldots, i_r\}$ such that $x_i \notin \{x_{i_0}, \ldots, x_{i_r}\}$. Let s be one of the indices of $\{0, \ldots, r\}$ such that x_i is between x_{i_s} and $x_{i_{s+1}}$. Then, by lemma 43,

$$E_\omega(x_{i_s}, x_{i_{s+1}}) + E_\omega(x_i, x_i) > E_\omega(x_{i_s}, x_i) + E_\omega(x_i, x_{i_{s+1}}).$$

We have just contradicted the maximality of r. Therefore, $(x_0, \ldots, x_n)$ must be monotone.

Part 2. Assume by contradiction that $(x_0, \ldots, x_n)$ is not strictly monotone. Then $(x_0, \ldots, x_n)$ contains a subsequence of the form $(x_{i-1}, x_i, \ldots, x_{i+r}, x_{i+r+1})$ with $r \geq 1$ and $x_{i-1} \neq x_i = \ldots = x_{i+r} \neq x_{i+r+1}$. To simplify the proof, we assume $x_{i-1} < x_{i+r+1}$. We want built a configuration $(x'_{i-1}, x'_i, \ldots, x'_{i+r}, x'_{i+r+1})$ so that $x'_{i-1} = x_{i-1}$, $x'_{i+r+1} = x_{i+r+1}$ and

$$E_\omega(x_{i-1}, x_i, \ldots, x_{i+r}, x_{i+r+1}) > E_\omega(x'_{i-1}, x'_i, \ldots, x'_{i+r}, x'_{i+r+1}).$$

By changing by a coboundary as in definition 18, we may assume that $E_\omega(x, y)$ is C^2 in x and y. Indeed, since $(x_{i-1}, \ldots, x_{i+r+1})$ is minimizing, we have

$$E_\omega(x_{i-1}, \ldots, x_{i+r+1}) = E_\omega(x_{i-1}, x_i + \epsilon, x_{i+1} - \epsilon, \ldots, x_{i+r} - \epsilon, x_{i+r+1}) + o(\epsilon^2).$$

Let

$$\alpha = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} \frac{\partial^2 E_\omega}{\partial x \partial y}(x, x_i)\, dx < 0,$$

$$\beta = \frac{1}{x_{i+r+1} - x_{i+r}} \int_{x_{i+r}}^{x_{i+r+1}} \frac{\partial^2 E_\omega}{\partial x \partial y}(x_{i+r}, y)\, dy < 0.$$

By Aubry crossing lemma,

$$\begin{aligned} &E_\omega(x_{i-1}, x_i + \epsilon) + E_\omega(x_i + \epsilon, x_{i+1} - \epsilon) \\ &\qquad = E_\omega(x_{i-1}, x_{i+1} - \epsilon) + E_\omega(x_i + \epsilon, x_i + \epsilon) - 2\epsilon(x_i - x_{i-1})\alpha + o(\epsilon). \end{aligned}$$

Since $x_i = x_{i+r}$, obviously $E_\omega(x_i + \epsilon, x_i + \epsilon) = E_\omega(x_{i+r} + \epsilon, x_{i+r} + \epsilon)$. Again by Aubry crossing lemma,

$$\begin{aligned} &E_\omega(x_{i+r} + \epsilon, x_{i+r} + \epsilon) + E_\omega(x_{i+r} - \epsilon, x_{i+r+1}) \\ &\qquad = E_\omega(x_{i+r} - \epsilon, x_{i+r} + \epsilon) + E_\omega(x_{i+r} + \epsilon, x_{i+r+1}) - 2\epsilon(x_{i+r+1} - x_{i+r})\beta + o(\epsilon). \end{aligned}$$

Then, for ϵ small enough, we have

$$E_\omega(x_{i-1},\dots,x_{i+r+1}) > E_\omega(x_{i-1},x_i-\epsilon,\dots,x_{i-r-1}-\epsilon,x_{i+r}+\epsilon,x_{i+r+1}),$$

which contradicts that $(x_{i-1},\dots,x_{i+r+1})$ is minimizing. We have thus proved that $(x_0,\dots,x_n)$ is strictly monotone. $\square$

Lemma 47. *Let L be a weakly twist transversally constant Lagrangian. Then, there exists $R>0$ such that the fact $(x_0,\dots,x_n)\in\mathbb{R}$ is a minimizing configuration for an arbitrary environment $\omega\in\Omega$ and verifies $|x_n-x_0|\geq R$ implies that $(x_0,\dots,x_n)$ is strictly monotone.*

Proof. Let $\{U_i=\tau[B_{R_i}\times\Xi_i]\}_{i\in I}$ be a flow box decomposition with respect to which L is transversally constant. Since $\{U_i\}_{i\in I}$ is a finite cover, we may choose R large enough so that every orbit of size R meets every box entirely: for every ω, for every $|y-x|\geq R$, for every $i\in I$, there exists $t_i\in\mathbb{R}$ such that $(t_i-R_i,t_i+R_i)\subset[x,y]$ and $\tau_{t_i}(\omega)\in\Xi_i$.

We first show that there cannot exist $r\geq 0$ and $0<k<n-r$ such that

$$x_k<x_{k-1},\quad x_k=\ldots=x_{k+r}\quad\text{and}\quad x_k<x_{k+r+1}.$$

Otherwise, Aubry crossing lemma implies that

$$E_\omega(x_{k-1},x_k)+E_\omega(x_k,x_{k+r+1}) > E_\omega(x_{k-1},x_{k+r+1})+E_\omega(x_k,x_k).$$

We rewrite the configuration $(x_0,\dots,x_{k-1},x_{k+r+1},\dots,x_n)$ as $(y_0,\dots,y_{n-r-1})$. Let U_i be a flow box containing $\tau_{x_k}(\omega)$. There exists $|s|<R_i$ and $\omega'\in\Xi_i$ such that $\tau_{x_k}(\omega)=\tau_s(\omega')$. By the choice of R, there exists t such that $(t-R_i,t+R_i)\subset[x_0,x_n]$ and $\tau_t(\omega)\in\Xi_i$. Let $z_0=\ldots=z_r:=t+s$ and $1\leq l\leq n-r-1$ be such that $y_{l-1}<z_0\leq y_l$. Using the fact that L is transversally constant on U_i, we have

$$E_\omega(x_k,x_k)=E_{\omega'}(s,s)=E_{\tau_t(\omega)}(s,s)=E_\omega(z_0,z_0).$$

By applying again Aubry crossing lemma, we obtain

$$E_\omega(y_{l-1},y_l)+E_\omega(z_0,z_0)\geq E_\omega(y_{l-1},z_0)+E_\omega(z_0,y_l),$$

with a strict inequality if $z_0<y_l$. We have just obtained a new configuration $(y_0,\dots,y_{l-1},z_0,\dots,z_r,y_l,\dots,y_{n-r-1})$ of n points with a strictly lower energy, which contradicts the fact that $(x_0,\dots,x_n)$ is minimizing.

There cannot exist similarly $r\geq 0$ and $0<k<n-r$ such that

$$x_k>x_{k-1},\quad x_k=\ldots=x_{k+r}\quad\text{and}\quad x_k>x_{k+r+1}.$$

There cannot exists either a sub-configuration $(x_{k-1},x_k,\dots,x_{k+r},x_{k+r+1})$, $r\geq 1$, of the form $x_{k-1}\neq x_{k+r+1}$ and $x_k=\ldots=x_{k+r}$ strictly between x_{k-1} and x_{k+r+1} thanks to lemma 46. We are thus left to a configuration of the form

$$x_0=\ldots=x_r<\ldots<x_{n-r'}=\ldots=x_n\quad\text{or}\quad x_0=\ldots=x_r>\ldots>x_{n-r'}=\ldots=x_n$$

for some $r, r' \geq 0$. Assume by contradiction that $x_0 = x_1$ (the case $x_{n-1} = x_n$ is done similarly). As before, there exist U_i containing $\tau_{x_0}(\omega)$, $|s| < R_i$ and $\omega' \in \Xi_i$ such that $\tau_{x_0}(\omega) = \tau_s(\omega')$, as well as there exists $t \in \mathbb{R}$ such that $(t - R_i, t + R_i) \subset [\min\{x_0, x_n\}, \max\{x_0, x_n\}]$ and $\tau_t(\omega) \in \Xi_i$. One can show in an analogous way that, whenever $z := t + s$ belongs to $(\min\{x_{l-1}, x_l\}, \max\{x_{l-1}, x_l\}]$ for $2 \leq l \leq n$, $E(x_0, x_1, \ldots, x_n) \geq E(x_1, \ldots, x_{l-1}, z, x_l, \ldots, x_n)$, with strict inequality if $z < \max\{x_{l-1}, x_l\}$. Since $(x_0, x_1, \ldots, x_n)$ is as minimizing configuration, this implies that $z = \max\{x_{l-1}, x_l\}$ and thus $(x_1, \ldots, x_{l-1}, z, x_l, \ldots, x_n)$ is a minimizing configuration. The first part of this proof shows that this cannot happen.

The proof that $(x_0, \ldots, x_n)$ is strictly monotone is complete. □

Proposition 48. *Let L be a weakly twist transversally constant Lagrangian. Then, there exists $R > 0$ such that, for $\omega \in \Omega$, $n \geq 2$, and $(x_0, \ldots, x_n)$ with $E(x_0, \ldots, x_n) = \min_{(y_0, \ldots, y_n)} E_\omega(y_0, \ldots, y_n)$, the inequality $\operatorname{diam}(\{x_k : 0 \leq k \leq n\}) \geq R$ implies that $(x_0, \ldots, x_n)$ is strictly monotone and $\sup_{1 \leq k \leq n} |x_k - x_{k-1}| \leq R$.*

Proof. Consider $\omega \in \Omega$, $n \geq 2$, and $(x_0, \ldots, x_n)$ realizing the minimum of the energy among all configurations of length n in the environment ω.

Part 1. We show there exists $R' > 0$ (independent from ω and n) such that $|x_1 - x_0| \leq R'$ and $|x_2 - x_1| \leq R'$. Indeed, we have

$$E_\omega(x_0, x_1) \leq E_\omega(x_1, x_1) \quad \text{and} \quad E_\omega(x_0, x_1, x_2) \leq E_\omega(x_2, x_2, x_2),$$

which implies

$$E_\omega(x_0, x_1) \leq \sup_{x \in \mathbb{R}} E_\omega(x, x) \quad \text{and} \quad E_\omega(x_1, x_2) \leq 2 \sup_{x \in \mathbb{R}} E_\omega(x, x) - \inf_{x, y \in \mathbb{R}} E_\omega(x, y).$$

The existence of R' follows then from the coerciveness of L, which is uniform with respect to ω. Similarly, we have $|x_{n-1} - x_{n-2}| \leq R'$ and $|x_n - x_{n-1}| \leq R'$.

Part 2. We show there exists $R'' > 0$ such that, if $(x_0, \ldots, x_m)$ is strictly monotone, then $|x_i - x_{i-1}| \leq R''$ for every $1 \leq i \leq m$. It is clear from the definition that, if L is transversally constant with respect to a particular flow box decomposition $\{\tau[B_{r_i} \times \Xi_i]\}$, then L is transversally constant for any flow box decomposition such that its flow boxes are compatible with respect to $\{\tau[B_{r_i} \times \Xi_i]\}$. Therefore, let $\{U_i = \tau[B_{R'} \times \Xi'_i]\}$ be a finite cover of Ω by flow boxes such that $\tau[B_{2R'} \times \Xi'_i]$ is again a flow box and L is transversally constant with respect to $\{\tau[B_{2R'} \times \Xi'_i]\}$. We choose $R'' > 0$ large enough so that every orbit of length R'' meets entirely each $\tau[B_{2R'} \times \Xi'_i]$. Let U_i be a flow box containing $\tau_{x_1}(\omega)$: there exist $|s_1| < R'$ and $\omega' \in \Xi'_i$ such that $\tau_{x_1}(\omega) = \tau_{s_1}(\omega')$. From part 1, we deduce that $\tau[B_{2R'} \times \Xi'_i]$ contains $\{\tau_{x_0}(\omega), \tau_{x_1}(\omega), \tau_{x_2}(\omega)\}$: there exist $|s_0|, |s_2| < 2R'$ such that $\tau_{x_0}(\omega) = \tau_{s_0}(\omega')$ and $\tau_{x_2}(\omega) = \tau_{s_2}(\omega')$. Assume by contradiction $|x_i - x_{i-1}| > R''$. Then, there exists $t \in \mathbb{R}$ such that $(t - 2R', t + 2R') \subset [\min\{x_{i-1}, x_i\}, \max\{x_{i-1}, x_i\}]$ and $\tau_t(\omega) \in \Xi'_i$. Let $z_0 = t + s_0$, $z_1 = t + s_1$ and $z_2 = t + s_2$. Notice that (x_{i-1}, x_i) and (z_0, z_1, z_2) are ordered in the same way. As L is transversally constant on $\tau[B_{2R'} \times \Xi'_i]$, we obtain

$$E_\omega(x_0, x_1, x_2) = E_{\omega'}(s_0, s_1, s_2) = E_{\tau_t(\omega)}(s_0, s_1, s_2) = E_\omega(z_0, z_1, z_2).$$

Aubry crossing lemma applied twice gives

$$\begin{aligned} E_\omega(x_{i-1},x_i)+E_\omega(z_0,z_1,z_2) &> E_\omega(x_{i-1},z_1)+E_\omega(z_0,x_i)+E_\omega(z_1,z_2), \\ &> E_\omega(x_{i-1},z_1,x_i)+E_\omega(z_0,z_2). \end{aligned}$$

As L is transversally constant, $E_\omega(z_0,z_2)=E_\omega(x_0,x_2)$ as above and we obtain

$$E_\omega(x_{i-1},x_i)+E_\omega(x_0,x_1,x_2) > E_\omega(x_{i-1},z_1,x_i)+E_\omega(x_0,x_2).$$

The configuration $(x_0,x_2,\ldots,x_{i-1},z_1,x_i,\ldots,x_m)$ has a strictly lower energy, which contradicts the fact that $(x_0,\ldots,x_m)$ is minimizing. We obtain similarly that, if $(x_m,\ldots,x_n)$ is strictly monotone, then $|x_{i-1}-x_i|\le R''$ for every $m+1\le i\le n$.

Part 3. Let R''' be the constant given by lemma 47. Take $R>2R''+4R'''$. If $|x_n-x_0|>R'''$, then $(x_0,\ldots,x_n)$ is strictly monotone by lemma 47 and the jumps $|x_i-x_{i-1}|$ are uniformly bounded by R''. The proof is finished.

Assume by contradiction that $|x_n-x_0|\le R'''$. Let $a=\min_{0\le k\le n}x_k$ and $b=\max_{0\le k\le n}x_k$. Since $\operatorname{diam}(\{x_k:0\le k\le n\})\ge R$, one of the two inequalities $|a-x_0|>R/2$ or $|b-x_0|>R/2$ must be satisfied. Assume to simplify $|b-x_0|>R/2$ (the case $|a-x_0|>R/2$ is done similarly). Hence, $b=x_m$ for some $0<m<n$. Since $(x_0,\ldots,x_m)$ and $(x_m,\ldots,x_n)$ are minimizing and satisfy $|x_m-x_0|>R'''$ and $|x_m-x_n|>R'''$, these two configurations are strictly monotone. Then, part 2 tells us that the jumps $|x_i-x_{i-1}|$ are uniformly bounded by R''. In particular, $|x_{m+1}-x_m|\le R''$. The configuration $(x_0,\ldots,x_{m+1})$ is minimizing and, since $|x_m-x_0|>R''+2R'''$, it satisfies $|x_{m+1}-x_0|>R'''$. By lemma 47, it must be strictly monotone. Thus, $(x_0,\ldots,x_n)$ is strictly monotone and $|x_n-x_0|>|x_{m+1}-x_0|>R'''$, which is a contradiction. ☐

The proof of the fact that $|x_k-x_{k-1}|$ is uniformly bounded uses the same ideas as in lemma 3.1 of [13]. The fact that L is transversally constant enables us to translate subconfigurations without modifying the total energy. For a minimizing and strictly monotone configuration, by minimality of the energy, two consecutive points cannot enclose a translated subconfiguration of three points. More precisely, we have the following lemma that extends lemma 3.2 of [13].

Lemma 49. *Let L be a weakly twist Lagrangian which is transversally constant for a flow box decomposition $\{U_i\}_{i\in I}$. Suppose that the flow box $\tau[B_R\times\Xi]$ is compatible with respect to $\{U_i\}_{i\in I}$. Let $(x_0,\ldots,x_n)$ be a strictly monotone minimizing configuration for some environment $\omega\in\Omega$. Let $(a-R,a+R)$ and $(b-R,b+R)$ be two disjoint intervals such that $\tau_a(\omega)\in\Xi$ and $\tau_b(\omega)\in\Xi$. Assume that $(a-R,a+R)$ is a subset of $[x_0,x_n]$. Let A be the number of sites $0\le k\le n$ such that x_k belongs to $(a-R,a+R)$ and let B be defined similarly. Then $B\le A+2$. In particular, if $(b-R,b+R)\subset[x_0,x_n]$, then $|A-B|\le 2$.*

Proof. To simplify we assume that $(x_0,\ldots,x_n)$ is strictly increasing. The proof is done by contradiction by assuming $B\ge A+3$. Denote

$$\begin{aligned} \{y_1,\ldots,y_A\} &:= \{x_0,\ldots,x_n\}\cap(a-R,a+R) \quad \text{and} \\ \{y'_1,\ldots,y'_B\} &:= \{x_0,\ldots,x_n\}\cap(b-R,b+R). \end{aligned}$$

Let y_0 be the greatest $x_k \le a - R$ and y_{A+1} be the smallest $x_k \ge a + R$. We write $s_k := y'_k - b$ and $z_k := a + s_k$ for $k = 1, \ldots, B$. The partition into $A + 1$ disjoint intervals $\cup_{k=1}^{A+1} (y_{k-1}, y_k]$ must contain $A+3$ distinct points $\{z_1, \ldots, z_{A+3}\}$. We have therefore to consider two cases.

Case 1. Either some interval $(y_{k-1}, y_k]$ contains three points (z_{i-1}, z_i, z_{i+1}). By Aubry crossing lemma,

$$\begin{aligned} E_\omega(y_{k-1}, y_k) + E_\omega(z_{i-1}, z_i) &> E_\omega(y_{k-1}, z_i) + E_\omega(z_{i-1}, y_k), \\ E_\omega(z_{i-1}, y_k) + E_\omega(z_i, z_{i+1}) &\ge E_\omega(z_{i-1}, z_{i+1}) + E_\omega(z_i, y_k). \end{aligned}$$

Since L is transversally constant on $\tau[B_R \times \Xi]$, we obtain

$$\begin{aligned} E_\omega(y'_{i-1}, y'_i, y'_{i+1}) + E_\omega(y_{k-1}, y_k) &= E_\omega(z_{i-1}, z_i, z_{i+1}) + E_\omega(y_{k-1}, y_k) \\ &> E_\omega(z_{i-1}, z_{i+1}) + E_\omega(y_{k-1}, z_i, y_k) \\ &= E_\omega(y'_{i-1}, y'_{i+1}) + E_\omega(y_{k-1}, z_i, y_k). \end{aligned}$$

We have obtained a configuration $(\ldots, y'_{i-1}, y'_{i+1}, \ldots, y_{k-1}, z_i, y_k, \ldots)$ with strictly lower energy, which contradicts the fact that $(x_0, \ldots, x_n)$ is minimizing.

Case 2. Or there exist two distinct intervals $(y_{k-1}, y_k]$ and $(y_{l-1}, y_l]$, $k < l$, that contain each two points (z_{i-1}, z_i) and (z_{j-1}, z_j), respectively. Notice that we may have $y_k = y_{l-1}$, but we must have $z_i < z_{j-1}$, $z_{i+1} \in (a - R, a + R)$, and possibly $z_{i+1} = z_{j-1}$. We want to obtain a contradiction by showing that one can decrease the sum of energies $E_\omega(y'_{i-1}, \ldots, y'_j) + E_\omega(y_{k-1}, \ldots, y_l)$ while fixing the four boundary points. By changing by a coboundary as in definition 18, we may assume that $E_\omega(x, y)$ is C^2 in x and y.

We perturb the point z_i slightly by a small quantity ϵ and allow an increase of the energy of order ϵ^2. Since (z_{i-1}, z_i, z_{i+1}) is minimizing, we have

$$E_\omega(z_{i-1}, z_i, z_{i+1}) = E_\omega(z_{i-1}, z_i - \epsilon, z_{i+1}) + o(\epsilon^2).$$

By Aubry crossing lemma,

$$\begin{aligned} &E_\omega(y_{k-1}, y_k) + E_\omega(z_{i-1}, z_i - \epsilon) \\ &\qquad = E_\omega(y_{k-1}, z_i - \epsilon) + E_\omega(z_{i-1}, y_k) - \epsilon(z_{i-1} - y_{k-1})\alpha + o(\epsilon), \end{aligned}$$

with $\alpha = \frac{1}{z_{i-1} - y_{k-1}} \int_{y_{k-1}}^{z_{i-1}} \frac{\partial^2 E_\omega}{\partial x \partial y}(x, y_k)\, dx < 0$. Again by Aubry crossing lemma,

$$E_\omega(y_{l-1}, y_l) + E_\omega(z_{j-1}, z_j) \ge E_\omega(y_{l-1}, z_j) + E_\omega(z_{j-1}, y_l),$$

with equality if $z_j = y_l$. Since L is transversally constant, we obtain

$$\begin{aligned} &E_\omega(y'_{i-1}, \ldots, y'_j) + E_\omega(y_{k-1}, \ldots, y_l) \\ &\qquad = E_\omega(z_{i-1}, \ldots, z_j) + E_\omega(y_{k-1}, \ldots, y_l) \\ &\qquad > E_\omega(z_{i-1}, y_k, \ldots, y_{l-1}, z_j) + E_\omega(y_{k-1}, z_i - \epsilon, z_{i+1}, \ldots, z_{j-1}, y_l) \\ &\qquad = E_\omega(y'_{i-1}, w_k, \ldots, w_{l-1}, y'_j) + E_\omega(y_{k-1}, z_i - \epsilon, z_{i+1}, \ldots, z_{j-1}, y_l) \end{aligned}$$

with $t_k := y_k - a$, $w_k := b + t_k$,$\ldots$,$t_{l-1} := y_{l-1} - a$, $w_{l-1} := b + t_{l-1}$. We have obtained a configuration $(\ldots, y'_{i-1}, w_k, \ldots, w_{l-1}, y'_j, \ldots, y_{k-1}, z_i - \epsilon, z_{i+1}, \ldots, z_{j-1}, y_l, \ldots)$ with strictly lower energy, which contradicts the fact that $(x_0, \ldots, x_n)$ is minimizing. □

It may happen that $E_\omega(x,x) = \bar{E}$ for some $\omega \in \Omega$ and $x \in \mathbb{R}$. Let $x_n = x$ for every n. Then $(x_n)_{n\in\mathbb{Z}}$ is a calibrated configuration in the environment ω and $\delta_{(\tau_x(\omega),0)}$ is a minimizing measure. If L is transversally constant on a flow box $\tau[B_R \times \Xi]$ such that $\tau_x(\omega) \in \Xi$, then $\delta_{(\omega',0)}$ is a minimizing measure for every $\omega' \in \Xi$. The projected Mather set contains Ξ and theorem 19 is proved. We are thus left to understand the case $\inf_{\omega\in\Omega,\ x\in\mathbb{R}} E_\omega(x,x) > \bar{E}$.

Lemma 50. *Let L be a weakly twist Lagrangian for which*

$$\inf_{\omega\in\Omega,\ x\in\mathbb{R}} E_\omega(x,x) > \bar{E}.$$

For $\omega \in \Omega$ and for every n, let $(x_0^n, \ldots, x_n^n)$ be a configuration realizing the minimum $E_\omega(x_0^n, \ldots, x_n^n) = \min_{x_0,\ldots,x_n\in\mathbb{R}} E_\omega(x_0, \ldots, x_n)$. Then $\lim_{n\to+\infty} |x_n^n - x_0^n| = +\infty$.

Proof. The proof is done by contradiction. Let $\omega \in \Omega$ and $R > 0$. Assume there exist infinitely many n's for which every configuration $(x_0^n, \ldots, x_n^n)$ realizing the minimum of $E_\omega(x_0, \ldots, x_n)$ satisfies $|x_n^n - x_0^n| \le R$. Thanks to lemma 44, we can find distinct indices $\{i_0, \ldots, i_r\}$ of $\{0, \ldots, n\}$ such that $i_0 = 0$, $i_r = n$, $(x_{i_0}^n, \ldots, x_{i_r}^n)$ is monotone (possibly not strictly monotone) and

$$E_\omega(x_0^n, \ldots, x_n^n) \ge E_\omega(x_{i_0}^n, \ldots, x_{i_r}^n) + \sum_{i\notin\{i_0,\ldots,i_r\}} E_\omega(x_i^n, x_i^n).$$

Let $\epsilon > 0$ be chosen so that $E_\omega(x,y) \ge \bar{E} + \epsilon$ for every $|y - x| \le \epsilon$. Thus, if θ_n denotes the number of indices $1 \le k \le r$ such that $|x_{i_k}^n - x_{i_{k-1}}^n| > \epsilon$, it is clear that $\theta_n \le R/\epsilon$. Since

$$n\bar{E} \ge E_\omega(x_0^n, \ldots, x_n^n) \ge (n - \theta_n)(\bar{E} + \epsilon) + \theta_n \inf_{x,y\in\mathbb{R}} E_\omega(x,y),$$

we obtain a contradiction by letting $n \to +\infty$. □

We now assume that L is transversally constant. We show in the following proposition that a sequence of configurations $(x_0^n, \ldots, x_n^n)$ realizing the minimum of the energy $E_\omega(x_0, \ldots, x_n)$ among all configurations of length n admits a weak rotation number in the sense that

$$\liminf_{n\to+\infty} \frac{|x_n^n - x_0^n|}{n} > 0. \tag{26}$$

The existence of a rotation number for an infinite minimizing configuration $(x_k)_{k\in\mathbb{Z}}$ has been established in [13]. The following proposition extends partially this result in two directions: the interaction model is more general; we compute the rotation number of a sequence of configurations of increasing length and not the rotation number of a unique infinite configuration.

Proposition 51. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be a one-dimensional weakly twist quasicrystal interaction model. Assume that*

$$\inf_{\omega\in\Omega,\ x\in\mathbb{R}} E_\omega(x,x) > \bar{E}.$$

For $\omega \in \Omega$ and for every n, let $(x_0^n, \ldots, x_n^n)$ be a configuration realizing the minimum of the energy among all configurations of length n:

$$E_\omega(x_0^n, \ldots, x_n^n) = \min_{x_0,\ldots,x_n} E_\omega(x_0, \ldots, x_n).$$

Then,

- $\bar{E} = \lim_{n\to+\infty} \frac{1}{n} E_\omega(x_0^n, \ldots, x_n^n) = \sup_{n\geq 1} \frac{1}{n} E_\omega(x_0^n, \ldots, x_n^n)$,
- *for n sufficiently large, $(x_0^n, \ldots, x_n^n)$ is strictly monotone,*
- *there is $R > 0$ (independent of ω) such that* $\sup_{n\geq1} \sup_{1\leq k\leq n} |x_k^n - x_{k-1}^n| \leq R$,
- $\liminf_{n\to+\infty} \frac{1}{n} |x_n^n - x_0^n| > 0$.

Proof. We shall assume that the flow $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ is not periodic.

Step 1. The first item has been proved in proposition 14; the limit can be obtained as a supremum because of superadditivity. Moreover, from lemma 50, $|x_n^n - x_0^n| \to +\infty$. From proposition 48, the configuration $(x_0^n, \ldots, x_n^n)$ must be strictly monotone and have uniformly bounded jumps R. We are left to prove the last item of the proposition.

Step 2. By definition of a quasicrystal, L is transversally constant with respect to some flow box decomposition $\{U_i\}_{i\in I}$ (definition 15). Let $\{F_\alpha\}_{\alpha\in A}$ be a Kakutani-Rohlin tower that is compatible with respect to $\{U_i\}_{i\in I}$ (definition 39) and let $\Sigma = \cup_{\alpha\in A}\Sigma_\alpha$ be its basis. We may assume that $\min_{\alpha\in A} H_\alpha$ is as large as we want and, in particular, larger than R (see the construction (25)). We also assume that n is sufficiently large so that every tower F_α of basis Σ_α is completely cut by the trajectory $\tau_t(\omega)$ for $t \in (\min\{x_0^n, x_n^n\}, \max\{x_0^n, x_n^n\})$. We consider ν the transverse measure to Σ (as defined in lemma 41) and we denote $\nu_\alpha := \nu(\Sigma_\alpha)$.

Step 3. Let $S^n < T^n$ be the two return times to Σ (namely, $\tau_{S^n}(\omega) \in \Sigma$ and $\tau_{T^n}(\omega) \in \Sigma$) that are chosen so that $[S^n, T^n)$ is the smallest interval containing the sequence $(x_k^n)_{k=0}^n$. From the definition of a Kakutani-Rohlin tower, $[S^n, T^n)$ can be written as a disjoint union of intervals of type $I_{\alpha,i} := [t_{\alpha,i}, t_{\alpha,i} + H_\alpha)$, where the list $\{t_{\alpha,i}\}_i$, $i = 1, \ldots, C_\alpha^n$, denotes the successive return times to Σ_α between S^n and T^n. We distinguish two exceptional intervals among this list: the two intervals which contain x_0^n and x_n^n. If $x_0^n < x_n^n$, then $N_{\alpha,i}^n$ denotes the number of points $(x_k^n)_{k=1}^n$ belonging to $I_{\alpha,i}$ and N_α^n denotes the maximum of $N_{\alpha,i}^n$. If $x_n^n < x_0^n$, then $N_{\alpha,i}^n$ and N_α^n are defined similarly by considering in this case $(x_k^n)_{k=0}^{n-1}$. From lemma 49, we obtain $N_\alpha^n - 2 \leq N_{\alpha,i}^n \leq N_\alpha^n$ for every nonexceptional interval $I_{\alpha,i}$. We show that $\sup_{n\geq1} N_\alpha^n < +\infty$ for every $\alpha \in A$. The proof is done by contradiction.

Let $E_{\alpha,i}^n$ be the energy of the configuration localized in $I_{\alpha,i}$. More precisely, assume first $x_0^n < x_n^n$; index the part of $(x_k^n)_{k=1}^n$ in $I_{\alpha,i}$ by $(x_{k,\alpha,i}^n)_{k=1}^N$ with $N = N_{\alpha,i}^n$; denote by $x_{0,\alpha,i}^n$ the nearest point strictly smaller than $x_{1,\alpha,i}^n$ and define the partial energy $E_{\alpha,i}^n := E_\omega(x_{0,\alpha,i}^n, \ldots, x_{N,\alpha,i}^n)$. If $x_n^n < x_0^n$, the part of $(x_k^n)_{k=0}^{n-1}$ in $I_{\alpha,i}$ is indexed by $(x_{k,\alpha,i}^n)_{k=0}^{N-1}$ with $N = N_{\alpha,i}^n$; denote by $x_{N,\alpha,i}^n$ the nearest point strictly larger than $x_{N-1,\alpha,i}^n$ and define $E_{\alpha,i}^n$ similarly.

Thanks to the hypothesis $\inf_{x\in\mathbb{R}} E_\omega(x,x) > \bar{E}$, one can choose $\epsilon > 0$ such that $E_\omega(x,y) \geq \bar{E} + \epsilon$ as soon as $|y - x| \leq \epsilon$. Let $\bar{H} := \max_{\alpha\in A} H_\alpha$. Then, if $\theta_{\alpha,i}^n$ denotes the number of consecutive points $x_{k,\alpha,i}^n$ in $I_{\alpha,i}$ satisfying $|x_{k,\alpha,i}^n - x_{k-1,\alpha,i}^n| > \epsilon$, obviously $\theta^n_{\alpha,i} \leq \bar H/\epsilon$. Thus, since $n = \sum_{\alpha\in A}\ \sum_{1\leq i\leq C^n_\alpha} N^n_{\alpha,i}$, we have that

$$\begin{aligned} n\bar E \geq E_\omega(x^n_0,\ldots,x^n_n) &= \sum_{\alpha\in A}\ \sum_{1\leq i\leq C^n_\alpha} E^n_{\alpha,i} \\ &\geq \sum_{\alpha\in A}\ \sum_{1\leq i\leq C^n_\alpha} \Big[\theta^n_{\alpha,i} \inf_{x,y\in\mathbb{R}} E_\omega(x,y) + \big(N^n_{\alpha,i}-\theta^n_{\alpha,i}\big)(\bar E+\epsilon)\Big] \\ &= n(\bar E+\epsilon) + \sum_{\alpha\in A}\sum_{1\leq i\leq C^n_\alpha} \theta^n_{\alpha,i}\underline{E} \ \geq\ n(\bar E+\epsilon) + \sum_{\alpha\in A} C^n_\alpha \frac{\bar H}{\epsilon}\underline{E}, \end{aligned}$$

where $\underline{E} := (\inf_{x,y\in\mathbb{R}} E_\omega(x,y) - \bar E - \epsilon) < 0$. Among the intervals $(I_{\alpha,i})_i$, $i = 1,\ldots,C^n_\alpha$, at most two of them are exceptional; the other intervals satisfy $N^n_{\alpha,i} \geq N^n_\alpha - 2$. We thus get $n \geq \sum_{\alpha\in A}(C^n_\alpha-2)(N^n_\alpha-2)$. For n sufficiently large, we have

$$\frac{C^n_\alpha}{T^n-S^n} \leq (1+\epsilon)\nu_\alpha, \quad \frac{C^n_\alpha-2}{T^n-S^n} \geq (1-\epsilon)\nu_\alpha \quad \text{and}$$
$$\frac{1}{n}\sum_{\alpha\in A} C^n_\alpha \leq \frac{(1+\epsilon)\sum_{\alpha\in A}\nu_\alpha}{(1-\epsilon)\sum_{\alpha\in A}\nu_\alpha(N^n_\alpha-2)}.$$

If $N^n_\alpha \to +\infty$ for some α and a subsequence $n\to+\infty$, then $\frac{1}{n}\sum_{\alpha\in A} C^n_\alpha \to 0$ and we obtain a contradiction with the previous inequality.

Step 4. For every α, $I_{\alpha,i} \subset [x^n_0, x^n_n]$ except maybe for at most two of them. Then

$$\frac{|x^n_n - x^n_0|}{n} \geq \frac{\sum_{\alpha\in A}(C^n_\alpha-2)H_\alpha}{\sum_{\alpha\in A} C^n_\alpha N^n_\alpha}.$$

Denote $\bar N_\alpha := \limsup_{n\to+\infty} N^n_\alpha$. From step 3 we know that $\bar N_\alpha < +\infty$. By dividing by $(T^n - S^n)$ and by letting $n\to+\infty$, we obtain

$$\liminf_{n\to+\infty} \frac{|x^n_n-x^n_0|}{n} \geq \frac{\sum_{\alpha\in A}\nu_\alpha H_\alpha}{\sum_{\alpha\in A}\nu_\alpha \bar N_\alpha} = \frac{1}{\sum_{\alpha\in A}\nu_\alpha\bar N_\alpha} > 0.$$

□

Proof of theorem 19. We assume that $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ is a one-dimensional weakly twist quasicrystal interaction model. We discuss two cases.

Case 1. Either $\inf_{\omega\in\Omega}\inf_{x\in\mathbb{R}} E_\omega(x,x) = \bar E$. Then $E_{\omega_*}(x_*,x_*) = \bar E$ for some ω_* and x_*. By hypothesis, L is transversally constant with respect to a flow box decomposition $\{U_i = \tau[B_{R_i}\times\Xi_i]\}_{i\in I}$. Let $i\in I$ be such that $\tau_{x_*}(\omega_*) \in U_i$. Let be $|t_i| < R_i$ and $\omega_i \in \Xi_i$ such that $\tau_{x_*}(\omega_*) = \tau_{t_i}(\omega_i)$. Then

$$\bar E = E_{\omega_*}(x_*,x_*) = E_{\omega_i}(t_i,t_i) = E_\omega(t_i,t_i), \quad \forall \omega\in\Xi_i.$$

We have just proved that $\delta_{(\tau_{t_i}(\omega),0)}$ is a minimizing measure for every $\omega\in\Xi_i$. The projected Mather set contains $\tau_{t_i}(\Xi_i)$. By minimality of the flow, we have $\Omega = \tau[B_R\times\Xi_i]$ thanks to item 1 of lemma 37. The projected Mather set thus meets every sufficiently long orbit of the flow.

Case 2. Or $\inf_{\omega\in\Omega}\inf_{x\in\mathbb{R}} E_\omega(x,x) > \bar{E}$. Proposition 51 shows that, if $\omega_* \in \Omega$ has been fixed, if for every $n \ge 1$ a sequence $(x_k^n)_{0\le k<n}$ of points of $\mathbb{R}$ realizing the minimum $E_{\omega_*}(x_0^n,\dots,x_n^n) = \min_{x_0,\dots,x_n} E_{\omega_*}(x_0,\dots,x_n)$ has been fixed, then
– $\bar{E} = \lim_{n\to+\infty} \frac{1}{n} E_{\omega_*}(x_0^n,\dots,x_n^n)$,
– $(x_k^n)_{0\le k<n}$ is strictly monotone for n large enough,
– there is $R > 0$ (independent of ω_*) such that $\sup_{n\ge1}\ \sup_{1\le k\le n} |x_k^n - x_{k-1}^n| < 2R$,
– $\rho := \liminf_{n\to+\infty} \frac{1}{n}|x_n^n - x_0^n| > 0$.

Let μ_{n,ω_*} be the probability measure on $\Omega\times\mathbb{R}$ defined by

$$\mu_{n,\omega_*} := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{(\tau_{x_k^n}(\omega_*),\, x_{k+1}^n - x_k^n)}.$$

Notice that $\int L\, d\mu_{n,\omega_*} = \frac{1}{n} E_{\omega_*}(x_0^n,\dots,x_n^n)$. Since the consecutive jumps of x_k^n are uniformly bounded, the sequence of measures $\{\mu_{n,\omega_*}\}_{n\ge1}$ is tight. By taking a subsequence, we may assume that $\mu_{n,\omega_*} \to \mu_\infty$ with respect to the weak topology. Moreover, μ_∞ is holonomic and minimizing. Let $\Xi \subset \Omega$ be a transverse section of a flow box $\tau[B_R\times\Xi]$. Let $\mathcal{R}_\Xi(\omega_*)$ be the set of return times to Ξ as defined in lemma 41. Let $pr^1 : \Omega\times\mathbb{R}\to\Omega$ be the first projection. Then

$$\begin{aligned} pr^1_*(\mu_{n,\omega_*})(\tau[B_R\times\Xi]) &= \frac{1}{n}\#\big\{k : x_k^n \in \cup_{t\in\mathcal{R}_\Xi(\omega_*)} B_R(t)\big\} \\ &\ge \frac{1}{n}\#(B_{T_n}(c_n)\cap\mathcal{R}_\Xi(\omega_*)), \end{aligned}$$

with $T_n := \frac{1}{2}|x_n^n - x_0^n|$ and $c_n := \frac{1}{2}(x_0^n + x_n^n)$. The previous inequality comes from the fact that the intervals $B_R(t)$ are disjoints and contain at least one x_k^n. Then

$$pr^1_*(\mu_{n,\omega_*})(\tau[B_R\times\Xi]) \ge \frac{2T_n}{n}\ \frac{\#(B_{T_n}(0)\cap\mathcal{R}_\Xi(\tau_{c_n}(\omega_*))}{\mathrm{Leb}(B_{T_n}(0))}.$$

By taking the limit as $n\to+\infty$, one obtains $pr^1_*(\mu_\infty)(\overline{\tau[B_R\times\Xi]}) \ge \rho\nu_\Xi(\Xi) > 0$. Therefore, since Ξ is arbitrary, every orbit of the flow of length $2R$ meets the projected Mather set. □

5 Lax-Oleinik operators

The Lax-Oleinik operator is a tool used in PDE's to solve Hamilton-Jacobi equations. The Frenkel-Kontorova model appears naturally by discretization in time of these equations. The solutions of the Lax-Oleinik operator are called viscosity solutions or weak KAM solutions in the continuous time setting. We will call them here sub-actions.

Definition 52. *Let $(\Omega,\{\tau_t\}_{t\in\mathbb{R}^d},L)$ be an almost periodic interaction model. We call backward Lax-Oleinik operator the (nonlinear) operator acting on the space of Borel measurable functions by*

$$T_-[u](\omega) := \inf_{t\in\mathbb{R}^d}\big[u\circ\tau_{-t}(\omega) + L(\tau_{-t}(\omega),t)\big].$$

Similarly, we call forward Lax-Oleinik operator the operator

$$T_+[u](\omega) := \sup_{t\in\mathbb{R}^d} \big[u\circ\tau_t(\omega) - L(\omega,t)\big].$$

We will see that these Lax-Oleinik operators are less regularizing than the usual operators used in discrete weak KAM theory [14] (or in discrete dynamic programming [15]), when they are defined for a specific choice of an environment. For the usual definition of $T_\pm$, for a particular choice of E, see Appendix A, definition 59. From now on, we denote by $\mathcal{L}^\infty(X)$ the space of bounded Borel measurable functions on a topological space X.

Definition 53. *A measurable function u is called a sub-action (at the level $\bar L = \bar E$) if one of the following conditions is satisfied*

$$\begin{gathered}\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad u\circ\tau_t(\omega)\le u(\omega)+L(\omega,t)-\bar L\\ \iff\ u+\bar L\le T_-[u]\ \iff\ u-\bar L\ge T_+[u].\end{gathered}$$

There are then two possibilities for calibration: a sub-action u is said to be

$$\begin{aligned}&\textit{backward calibrated}\quad \textit{if}\quad T_-[u]=u+\bar L,\\ &\textit{forward calibrated}\quad \textit{if}\quad T_+[u]=u-\bar L.\end{aligned}$$

Continuous calibrated sub-actions do exist in the periodic setting. The main problem we are facing is that bounded measurable sub-actions may not exist in the almost periodic setting. We recall that $\bar L = \bar E$ may be computed using four formulas, given by definition 6, and propositions 10, 13 and 14.

As in definition 59, one may introduce two Lax-Oleinik operators $T_{\omega\pm}$, associated to the interaction E_ω for any $\omega\in\Omega$, each one acting on measurable functions as follows

$$T_{\omega-}[u](y) := \inf_{x\in\mathbb{R}^d}\big[u(x)+E_\omega(x,y)\big], \tag{27}$$

$$T_{\omega+}[u](x) := \sup_{y\in\mathbb{R}^d}\big[u(y)-E_\omega(x,y)\big]. \tag{28}$$

Notice that, if u is a solution of $T_-[u]=u+\bar L$ or $T_+[u]=u-\bar L$, then, for every $\omega\in\Omega$, $u_\omega(x):=u\circ\tau_x(\omega)$ is a solution of $T_{\omega-}[u_\omega]=u_\omega+\bar E$ or $T_{\omega+}[u_\omega]=u_\omega-\bar E$.

The main result in this section is about the existence of a bounded calibrated sub-action provided an obvious obstruction is removed. The following result is similar to Gottschalk-Hedlund theorem. We denote by $C_b^{usc}(\Omega)$ and $C_b^{lsc}(\Omega)$ the spaces of bounded upper semi-continuous and bounded lower semi-continuous functions, respectively.

Theorem 54. *Let $(\Omega,\{\tau_t\}_{t\in\mathbb{R}^d},L)$ be an almost periodic interaction model. Assume that L is C^0 coercive. Then, the following conditions are equivalent:*

1. $\exists\,u\in C_b^{lsc}(\Omega),\quad T_-[u]=u+\bar L,$
2. $\exists\,u\in C_b^{usc}(\Omega),\quad T_+[u]=u-\bar L,$

3. $\forall\, \omega \in \Omega,\ \sup_{n\geq 0} |T_-^n[0](\omega) - n\bar L| < +\infty$,

4. $\forall\, \omega \in \Omega,\ \sup_{n\geq 0} |T_+^n[0](\omega) + n\bar L| < +\infty$,

5. $\exists\, \omega \in \Omega,\ \exists\, u \in \mathcal{L}^\infty(\mathbb{R}^d),\quad T_{\omega-}[u] = u + \bar E$,

6. $\exists\, \omega \in \Omega,\ \exists\, u \in \mathcal{L}^\infty(\mathbb{R}^d),\quad T_{\omega+}[u] = u - \bar E$.

(As usual, $T_\pm^n$ denotes the n^{th} iterate of $T_\pm$.) Moreover, any bounded measurable solution of $T_{\omega-}[u] = u + \bar E$ or $T_{\omega+}[u] = u - \bar E$ is actually uniformly continuous.

The backward and forward calibrated solutions are two very different objects obtained by reversing the group action. Define

$$\check\tau_t := \tau_{-t}, \quad \rho(\omega, t) = (\tau_{-t}(\omega), t), \quad \text{and} \quad \check L := L \circ \rho. \tag{29}$$

The family of interactions associated to $\check L$ reads

$$\check E_\omega(x, y) := \check L(\check\tau_x(\omega), y - x) = E_\omega(-y, -x). \tag{30}$$

Notice that coerciveness and superlinearity are preserved by changing L to $\check L$. For every probability measure μ, we associate the reversed measure

$$\check\mu := \rho_*^{-1}(\mu). \tag{31}$$

Then μ is holonomic for $\{\tau_t\}_t$ if, and only if, $\check\mu$ is holonomic for $\{\check\tau_t\}_t$, and μ is minimizing for L if, and only if, $\check\mu$ is minimizing for $\check L$. In particular, L and $\check L$ have the same ground energy. For every measurable function u, we associate the reversed function

$$\check u := -u, \quad \text{then} \quad T_+[u] = -\check T_-[\check u]. \tag{32}$$

This duality between T_- and $\check T_+$ implies readily

$$u + \bar L \leq T_-[u] \iff u - \bar L \geq T_+[u] \iff \check u + \bar L \leq \check T_-[\check u], \tag{33}$$
$$u - \bar L = T_+[u] \iff \check u + \bar L = \check T_-[\check u]. \tag{34}$$

The second equivalence means that u is forward calibrated for L if, and only if, $\check u$ is backward calibrated for $\check L$.

We will use the following regularity along every orbit of the action.

Definition 55. *A function $u \in \mathcal{L}^\infty(\Omega)$ is said to be equicontinuous along the group action if*

$$\lim_{\epsilon\to 0+} \sup_{\omega\in\Omega} \sup_{\|t\|\leq\epsilon} |u \circ \tau_t(\omega) - u(\omega)| = 0.$$

Lemma 56. *Assume that L is C^0 coercive.*

1. *If u is lower semi-continuous and finite everywhere, then $T_-[u] \in \mathcal{L}^\infty(\Omega)$. If u is upper semi-continuous and finite everywhere, then $T_+[u] \in \mathcal{L}^\infty(\Omega)$. If u is a finite everywhere sub-action which is either lower semi-continuous or upper semi-continuous, then $u \in \mathcal{L}^\infty(\Omega)$.*

2. *If $u \in C^0(\Omega)$, then $T_-[u] \in C^0(\Omega)$.*

3. *If $u \in \mathcal{L}^\infty(\Omega)$, then $T_-[u] \in \mathcal{L}^\infty(\Omega)$ and is equicontinuous along the group action. Moreover, the modulus of equicontinuity is uniform over $\|u\|_\infty \leq R$, that is,*

$$\forall\, R > 0, \qquad \lim_{\epsilon \to 0+} \sup_{\|u\|_\infty \leq R} \sup_{\omega \in \Omega} \sup_{\|t\| \leq \epsilon} |T_-[u] \circ \tau_t(\omega) - T_-[u](\omega)| = 0.$$

4. *If $\{u_n\}_{n \geq 0}$ is a nondecreasing sequence of lower semi-continuous functions such that $\sup_{n \geq 0} \|u_n\|_\infty < +\infty$, then*

$$\sup_{n \geq 0} T_-[u_n] = T_-\big[\sup_{n \geq 0} u_n\big].$$

If $\{u_n\}_{n \geq 0}$ is any sequence of measurable functions, then

$$\inf_{n \geq 0} T_-[u_n] = T_-\big[\inf_{n \geq 0} u_n\big].$$

5. *If $u \in C_b^{lsc}(\Omega)$, then $T_-[u] \in C_b^{lsc}(\Omega)$. If $u \in C_b^{usc}(\Omega)$, then $T_-[u] \in C_b^{usc}(\Omega)$.*

Proof. Part 1. Let $F_N := \{\omega \in \Omega : u(\omega) \leq N\}$. As u is lower semi-continuous, F_N is closed; as u is finite everywhere, $\Omega = \cup_{N \in \mathbb{Z}} F_n$. By Baire's theorem, there exists $N(u)$ such that $F_{N(u)}$ has nonempty interior. By minimality, on may find $D > 0$ such that, for every $\omega \in \Omega$, there exists $\|t\| \leq D$ with $\tau_{-t}(\omega) \in F_{N(u)}$. By the definition of the backward Lax-Oleinik operator, $T_-[u](\omega) \leq u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t)$. We obtain the uniform upper bound:

$$\sup_{\omega \in \Omega} T_-[u](\omega) \leq N(u) + \sup_{\omega \in \Omega,\ \|t\| \leq D} L(\tau_{-t}(\omega), t).$$

By the lower semi-continuity of u, we obtain the following uniform lower bound

$$\inf_{\omega \in \Omega} T_-[u](\omega) \geq \inf_{\omega \in \Omega} u(\omega) + \inf_{\omega \in \Omega,\ t \in \mathbb{R}^d} L(\omega, t).$$

We have just proved that $T_-[u]$ is bounded. If u is upper semi-continuous, $\check{u}$ is lower semi-continuous and $T_+[u] = -\check{T}_-[\check{u}]$ is bounded by the previous proof.

If u is a lower semi-continuous and finite everywhere sub-action, then $u \leq T_-[u] - \bar{L}$. As $T_-|u]$ is bounded, u is bounded from above, being bounded from bellow by semi-continuity. Similarly, from upper semi-continuity and $u \geq T_+[u] + \bar{L}$, one obtains that $u \in \mathcal{L}^\infty(\Omega)$.

Part 2. We first notice that, if $u \in \mathcal{L}^\infty(\Omega)$, then an optimal translation $t \in \mathbb{R}^d$ given in the definition of $T_-[u]$ is uniformly bounded from above by a constant $D > 0$, which is obtained from the coerciveness of L:

$$\inf_{\omega \in \Omega,\ \|t\| \geq D} \big[u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t)\big] > \sup_{\omega \in \Omega} \big[u(\omega) + L(\omega, 0)\big].$$

The family of continuous functions $\big\{\omega \in \Omega \mapsto u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t)\big\}_{\|t\| \leq D}$ is equicontinuous and, by the compactness of Ω, the infimum $T_-[u]$ is also continuous.

Part 3. For $R > 0$, choose as in part 2 a constant $D_R > 0$ so that, for every $\|u\|_\infty \leq R$,

$$\forall\, \omega \in \Omega, \quad T_-[u](\omega) = \inf_{\|t\| \leq D_R} \big[u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t)\big].$$

(Notice that we can choose $\|t\| \leq D_R$ uniformly over the set $\{u : \|u\|_\infty \leq R\}$ for every R.) Then, given $\eta > 0$, there exists $\|t\| \leq D_R$ such that, for all $\omega \in \Omega$ and $s \in \mathbb{R}^d$,

$$\begin{aligned} &T_-[u](\tau_s(\omega)) - T_-[u](\omega) \leq \\ &\quad \leq \big[u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t+s)\big] - \big[u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t)\big] + \eta \leq \\ &\qquad\qquad \leq L(\tau_{-t}(\omega), t+s) - L(\tau_{-t}(\omega), t) + \eta. \end{aligned}$$

Taking first suprema and letting then $\eta \to 0$, one obtains

$$\sup_{\omega \in \Omega,\ \|s\| \leq \epsilon} \big|T_-[u](\tau_s(\omega)) - T_-[u](\omega)\big| \leq \sup_{\omega \in \Omega,\ \|t\| \leq D_R,\ \|s\| \leq \epsilon} |L(\omega, t+s) - L(\omega, t)|.$$

The right hand side goes to 0 as $\epsilon \to 0$ by the uniform continuity of L on compact sets. We have proved that $\{T_-[u]\}_{\|u\|_\infty \leq R}$ is equicontinuous along the group action.

Part 4. Since the set $\{u_n\}_n$ is uniformly bounded in $\mathcal{L}^\infty(\Omega)$, the infimum on t in the definition of $T_-[u_n]$ can be realized over $\|t\| \leq D_R$, for some $D_R > 0$, uniformly in ω and $n \geq 0$. Define

$$f_n(\omega, t) := u_n \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t).$$

Then $f_n : \Omega \times \{\|t\| \leq D_R\} \to \mathbb{R}$ is lower semi-continuous and nondecreasing in n. The following lemma 57 shows that, for every ω fixed,

$$\sup_{n \geq 0} \inf_{\|t\| \leq D_R} f_n(\omega, t) = \inf_{\|t\| \leq D_R} \sup_{n \geq 0} f_n(\omega, t) \ \Leftrightarrow\ \sup_{n \geq 0} T_-[u_n](\omega) = T_-\Big[\sup_{n \geq 0} u_n\Big](\omega).$$

For any sequence $\{u_n\}_n$, the property $\inf_n T_-[u_n] = T_-[\inf_n u_n]$ is obtained by simply permuting the two infima.

Part 5. Let $u \in C_b^{lsc}(\Omega)$. There exists a nondecreasing sequence of continuous functions u_n such that $\sup_{n \geq 0} u_n = u$. Part 4 implies that $\sup_{n \geq 0} T_-[u_n] = T_-[u]$. Moreover, $T_-[u_n]$ is continuous by part 2, which shows that $T_-[u]$ is lower semi-continuous. Besides, $T_-[u]$ is bounded by part 3. If $u \in C_b^{usc}(\Omega)$, then there exists a nonincreasing sequence of continuous functions u_n such that $u = \inf_n u_n$. One gets by part 4 that $\inf_n T_-[u_n] = T_-[u]$ is upper semi-continuous and by part 3 that $T_-[u]$ is bounded. □

We have used the following basic lemma.

Lemma 57. *Let X be a compact metric space and $u_n : X \to \mathbb{R}$ be a nondecreasing sequence of lower semi-continuous functions. Suppose that* $\sup_n u_n(x) < +\infty$ *for every $x \in X$. Then* $\sup_n \inf_{x \in X} u_n(x) = \inf_{x \in X} \sup_n u_n(x)$.

Proof. On the one hand, it is clear that

$$\inf_{x\in X}\sup_{n\geq 0}u_n(x)\geq \sup_{n\geq 0}\inf_{x\in X}u_n(x).$$

On the other hand, since u_n is lower semi-continuous, the minimum of every u_n is attained: let $x_n\in X$ be such that $\inf_{x\in X}u_n(x)=u_n(x_n)$. By compactness of X, let x_∞ be an accumulation point of $\{x_n\}_n$. Let $u=\sup_n u_n$, which is finite by assumption. For $\epsilon>0$, choose N such that $u_N(x_\infty)>u(x_\infty)-\epsilon$. Since u_N is lower semi-continuous, choose a neighborhood U of x_∞ so that $u_N(x)>u(x_\infty)-2\epsilon$ for every $x\in U$. Since $\{u_n\}_n$ is nondecreasing, we have that, for $n\geq N$ sufficiently large, $x_n\in U$ and $u_n(x_n)\geq u_N(x_n)>u(x_\infty)-2\epsilon$, from which we obtain $\sup_{n\geq 0}\inf_{x\in X}u_n(x)>u(x_\infty)-2\epsilon$. Letting $\epsilon\to 0$, we have just proved that $\sup_{n\geq 0}\inf_{x\in X}u_n(x)\geq\inf_{x\in X}\sup_{n\geq 0}u_n(x)$. □

We will also need to recall the notions of lower semi-continuous envelope u_{lsc} and upper semi-continuous envelope u_{usc} of a bounded function u, namely,

$$\forall\,\omega\in\Omega,\quad u_{lsc}(\omega):=\sup\{\phi(\omega):\phi\leq u \text{ and } \phi\in C^0(\Omega)\},\tag{35}$$

$$\forall\,\omega\in\Omega,\quad u_{usc}(\omega):=\inf\{\phi(\omega):u\leq\phi \text{ and } \phi\in C^0(\Omega)\}.\tag{36}$$

We have then a key lemma.

Lemma 58. *Let $u\in\mathcal{L}^\infty(\Omega)$.*

1. *If $v:=T_-[u]$, then $v_{lsc}=T_-[u_{lsc}]$ and $v_{usc}\leq T_-[u_{usc}]$.*
2. *If $v:=T_+[u]$, then $v_{usc}=T_+[u_{usc}]$ and $v_{lsc}\geq T_+[u_{lsc}]$.*
3. *If $u+\bar L\leq T_-[u]$, then $u_{lsc}+\bar L\leq T_-[u_{lsc}]$ and $u_{usc}+\bar L\leq T_-[u_{usc}]$.*
4. *If $u-\bar L\geq T_+[u]$, then $u_{lsc}-\bar L\geq T_+[u_{lsc}]$ and $u_{usc}-\bar L\geq T_+[u_{usc}]$.*
5. *If $u+\bar L=T_-[u]$, then $u_{lsc}+\bar L=T_-[u_{lsc}]$.*
6. *If $u-\bar L=T_+[u]$, then $u_{usc}-\bar L=T_+[u_{usc}]$.*

Proof. Even items may be derived immediately from respective odd items simply by reversing the group action and using, in particular, relation (32). So we only prove the odd items of the lemma.

Part 1. Let $\phi\in C^0(\Omega)$ be such that $\phi\leq v$. Then, for all ω and t, $\phi(\tau_t(\omega))\leq u(\omega)+L(\omega,t)-\bar L$. For a fixed t, $\phi(\tau_t(\omega))-L(\omega,t)+\bar L$ is continuous in ω. By definition of the envelope, $\phi(\tau_t(\omega))\leq u_{lsc}(\omega)+L(\omega,t)-\bar L$ for all ω and t. By taking the supremum on ϕ, we obtain $v_{lsc}(\tau_t(\omega))\leq u_{lsc}(\omega)+L(\omega,t)-\bar L$ or $v_{lsc}\leq T_-[u_{lsc}]$. Conversely, $u_{lsc}\leq u$ implies $T_-[u_{lsc}]\leq T_-[u]$. By lemma 56, part 5, $T_-[u_{lsc}]$ is lower semi-continuous. We thus obtain that $T_-[u_{lsc}]\leq v_{lsc}$. Hence, $T_-[u_{lsc}]=v_{lsc}$.

Let $\{\phi_n\}_n\subset C^0(\Omega)$ be a nonincreasing sequence such that $\inf_n\phi_n=u_{usc}$. By lemma 56, part 4, $T_-[u_{usc}]=\inf_n T_-[\phi_n]\geq T_-[u]=v$. By lemma 56, part 5, $T_-[u_{usc}]$ is upper semi-continuous. We have obtained that $T_-[u_{usc}]\geq v_{usc}$.

Part 3. If $u + \bar{L} \le T_-[u]$, by taking the semi-continuous envelope of both parts of the inequality and by using the first part of this lemma, we obtain $u_{lsc} + \bar{L} \le T_-[u_{lsc}]$. Moreover, $u + \bar{L} \le T_-[u_{usc}]$. By lemma 56, part 5, $T_-[u_{usc}]$ is upper semi-continuous. In particular, $u_{usc} + \bar{L} \le T_-[u_{usc}]$.

Part 5. If $u + \bar{L} = T_-[u]$, then $u_{lsc} + \bar{L} \le T_-[u_{lsc}]$ by part 3. Let $\{\phi_n\}_n$ be a nondecreasing sequence of continuous functions such that $u_{lsc} = \sup_n \phi_n$. Then $\phi_n \le u$, $T_-[\phi_n] \le T_-[u] = u + \bar{L}$, $T_-[\phi_n]$ is continuous, $T_-[\phi_n] \le u_{lsc} + \bar{L}$, and, by lemma 56, part 4, we obtain $T_-[u_{lsc}] \le u_{lsc} + \bar{L}$. Thus, $T_-[u_{lsc}] = u_{lsc} + \bar{L}$. □

Proof of theorem 54. It is clear by reversing the direction of the group action as in (29) and (30) that *item 1* ⇔ *item 2*, *item 3* ⇔ *item 4*, and *item 5* ⇔ *item 6*. It is also clear that *item 1* ⇒ *item 5* using lemma 56 (item 3) to show that $u_\omega \in C^0_b(\mathbb{R}^d)$.

Part 1. We prove that *item 5* ⇒ *item 3*. Notice first that

$$T^n_-[0](\bar{\omega}) = \inf \big\{ E_{\bar{\omega}}(x_{-n}, \ldots, x_{-1}, x_0) : x_0 = 0 \text{ and } x_{-k} \in \mathbb{R}^d \big\}, \quad \forall \bar{\omega} \in \Omega.$$

By assumption, there exist $\omega \in \Omega$ and $u \in \mathcal{L}^\infty(\mathbb{R}^d)$ such that

$$\forall y \in \mathbb{R}^d, \quad u(y) = \inf_{x \in \mathbb{R}^d} \big\{ u(x) + E_\omega(x, y) - \bar{E} \big\}.$$

On the one hand, we have that

$$\forall t \in \mathbb{R}^d, \ \forall x_{-n}, \ldots, x_0 \in \mathbb{R}^d, \quad E_{\tau_t(\omega)}(x_{-n}, \ldots, x_0) \ge u(x_0 + t) - u(x_{-n} + t) + n\bar{E}.$$

Since $\bar{E} = \bar{L}$, by minimality of the interaction model, we obtain thus

$$\inf_{\bar{\omega} \in \Omega} \inf_{n \ge 0} \big[T^n_-[0](\bar{\omega}) - n\bar{L} \big] \ge -2\|u\|_\infty.$$

On the other hand, for all $t \in \mathbb{R}^d$, there are $x^t_{-n}, \ldots, x^t_0 \in \mathbb{R}^d$, with $x^t_0 = 0$, such that

$$E_{\tau_t(\omega)}(x^t_{-n}, \ldots, x^t_0) \le u(x^t_0 + t) - u(x^t_{-n} + t) + n\bar{E} + \sum_{k=0}^{n-1} \frac{1}{2^k},$$

which yields

$$\forall n \ge 1, \ \forall t \in \mathbb{R}^d, \quad T^n_-[0](\tau_t(\omega)) - n\bar{L} \le 2(\|u\|_\infty + 1),$$

and an upper bound also follows from the minimality of the action.

Part 2. We prove that *item 3* ⇒ *item 1*. We claim that it is enough to show the existence of $v_0 \in \mathcal{L}^\infty(\Omega)$ such that

$$v_0 + \bar{L} \le T_-[v_0] \quad \text{and} \quad \sup_{n \ge 0} \big\| T^n_-[v_0] - n\bar{L} \big\|_\infty < +\infty. \tag{37}$$

Indeed, we may first assume that $v_0 \in C^{lsc}_b(\Omega)$ since by lemma 58, part 3,

$$(v_0)_{lsc} + \bar{L} \le T_-[(v_0)_{lsc}],$$
$$-\|v_0\|_\infty \le (v_0)_{lsc} \le T^n_-[(v_0)_{lsc}] - n\bar{L} \le T^n_-[v_0] - n\bar{L} \le \|T^n_-[v_0] - n\bar{L}\|_\infty.$$

From now on, suppose that v_0 is lower semi-continuous and bounded. Let $v_n := T_-^n[v_0] - n\bar{L}$. Then v_n is lower semi-continuous by lemma 56, part 5, $v_{n+1} \geq v_n$ by the sub-action property, $\sup_n \|v_n\|_\infty < +\infty$ by the claim, and $T_-[v_n] = v_{n+1} + \bar{L}$ by construction. By lemma 56, part 4, if $v = \sup_n v_n$, then

$$v \in C_b^{lsc}(\Omega) \quad \text{and} \quad T_-[v] = T_-[\lim_{n\to+\infty} v_n] = \lim_{n\to+\infty} T_-[v_n] = v + \bar{L}.$$

It remains just to prove the existence of $v_0 \in \mathcal{L}^\infty(\Omega)$ verifying (37). Define then $v_0 := \inf_{k\geq 0}[T_-^k[0] - k\bar{L}]$. Notice that v_0 is finite everywhere by assumption and satisfies $v_0 + \bar{L} \leq T_-[v_0]$ by the following inequalities

$$\begin{gathered} v_0(\omega) = \inf_{n\geq 0} \inf_{x_{-n},\ldots,x_{-1},x_0=0} \big[E_\omega(x_{-n},\ldots,x_0) - n\bar{L}\big], \\ \forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d, \quad v_0(\tau_t(\omega)) \leq v_0(\omega) + E_\omega(0,t) - \bar{L}. \end{gathered}$$

Moreover, v_0 is upper semi-continuous and, by lemma 56, part 1, v_0 is bounded. Notice that

$$v_n := T_-^n[v_0] - n\bar{L} = \inf_{k\geq n}[T_-^k[0] - k\bar{L}].$$

is a nondecreasing sequence. Define

$$u_n := \sup_{k\geq n}[T_-^k[0] - k\bar{L}].$$

Then u_0 is finite everywhere by hypothesis, and lower semi-continuous. By lemma 56, part 1, $T_-[u_0]$ is bounded. Since

$$T_-[u_0] - \bar{L} \geq u_1 \geq u_n \geq v_n \geq v_1 \geq v_0,$$

we finally obtain that $\sup_n \|v_n\|_\infty = \sup_n \|T_-^n[v_0] - n\bar{L}\|_\infty < +\infty$.

□

Appendices

A Minimizing configurations for general interaction

The existence of a semi-infinite minimizing configuration without asking it to be calibrated at the level $\bar{E}$ is easier to guarantee and requires few hypothesis. We consider, in the first part of this appendix, a unique interaction energy $E(x,y)$ that will be supposed to be superlinear (7), translation bounded (5) and translation uniformly continuous (6). By adapting a point of view proposed by Zavidovique [25, Appendix], we will show that there always exists a semi-infinite minimizing configuration $\{x_n\}_{n=-\infty}^0$ with bounded jumps. The configuration will actually be calibrated at some level $\bar{c}$, which has no reason to be equal to $\bar{E}$. We consider, in the second part of this appendix, an almost periodic interaction model and show the existence of a bi-infinite calibrated configuration for some $E_{\bar{\omega}}$. We do not describe the set of such environments $\bar{\omega}$.

The main problem for a general interaction energy is to obtain an *a priori* bound on the jumps $\|x_{n+1} - x_n\|$ of any finite minimizing configuration. The main tool is to construct a discrete weak KAM solution (or a calibrated sub-action as in [14]). We will say that $u : \mathbb{R}^d \to \mathbb{R}$ is *Lipschitz in the large* if

$$\sup_{x,y\in\mathbb{R}^d} \frac{|u(y) - u(x)|}{\|y - x\| + 1} < +\infty. \tag{38}$$

Definition 59. *We call backward Lax-Oleinik operator the (nonlinear) operator T_- acting on continuous functions $u : \mathbb{R}^d \to \mathbb{R}$ by*

$$\forall\, y \in \mathbb{R}^d, \quad T_-[u](y) := \inf\{u(x) + E(x, y) : x \in \mathbb{R}^d\}.$$

We say that u is a calibrated sub-action for E at the level $c \in \mathbb{R}$ if $T_-[u] = u + c$.

For translation periodic interaction energy E, it was shown in [14] that the interaction energy $E_\lambda(x, y) = E(x, y) - \langle\lambda, y - x\rangle$ admits a periodic calibrated sub-action u_λ at the level $\bar{E}(\lambda)$. Notice then that $u(x) := u_\lambda(x) + \langle\lambda, x\rangle$ becomes calibrated for $E = E_0$ at the level $\bar{E}(\lambda)$. It was also shown there that $\lambda \mapsto -\bar{E}(\lambda)$ is convex and superlinear. These two simple observations implies that the equation $T_-[u] = u + c$ admits a solution Lipschitz in the large for all values c in $(-\infty, \sup_\lambda \bar{E}(\lambda)]$.

For general interaction energies as discussed in this appendix, we do not have an *a priori* growth on calibrated sub-actions. An important observation in [25] is that translation boundedness implies Lipschitz in the large and superlinearity implies sublinearity and compactness. Let

$$\bar{c} := \sup_{u\in C^0(\mathbb{R}^d)} \inf_{x,y\in\mathbb{R}^d} [E(x, y) + u(x) - u(y)]. \tag{39}$$

Proposition 60. *Let $E : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^0 superlinear, translation bounded and translation uniformly continuous interaction energy. Then there exists a uniformly continuous function $\bar{u} : \mathbb{R}^d \to \mathbb{R}$ which solves the Lax-Oleinik equation $T_-[\bar{u}] = \bar{u} + \bar{c}$. In particular, there exists a backward calibrated configuration $\{x_{-k}\}_{k=0}^{+\infty}$ at the level $\bar{c}$ with uniformly bounded jumps $\sup_{k\geq 1} \|x_{-k+1} - x_{-k}\| < +\infty$.*

Proposition 61. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model. Suppose L is superlinear. Then*

$$\bar{c} := \sup_{u\in C^0(\mathbb{R}^d)} \inf_{x,y\in\mathbb{R}^d} [E_\omega(x, y) + u(x) - u(y)]$$

is independent of ω and, for a certain $\bar{\omega} \in \Omega$, there exists a (bi-infinite) calibrated configuration for $E_{\bar{\omega}}$ at the level $\bar{c}$.

As we noticed above, the constant $\bar{c}$ may not be equal to $\bar{E}$ if we do not assume any growth at infinity on u. It is not clear that calibrated configurations exist for any environment ω.

The first two lemmas exhibit *a priori* compactness for the Lax-Oleinik operator. Let

$$c_0 := \inf_{x,y\in\mathbb{R}^d} E(x, y) \quad \text{and} \quad K_0 := \sup_{\|y-x\|\leq 1} E(x, y) - c_0. \tag{40}$$

Notice that $c_0 \leq \bar{c} \leq \sup_x E(x, x)$ and that $K_0 < +\infty$ thanks to the translation boundedness. Then, we have the following lemma.

Lemma 62. *Let $c_0 \le c \le \bar{c}$ and $u \in C^0(\mathbb{R}^d)$ be such that $u(y) - u(x) \le E(x,y) - c$ for every $x, y \in \mathbb{R}^d$. Then u is Lipschitz in the large with constant K_0,*

$$\forall\, x, y \in \mathbb{R}^d, \quad |u(y) - u(x)| \le K_0\big(\|y - x\| + 1\big).$$

Proof. Let $n \ge 1$ be the unique integer satisfying $n - 1 < \|y - x\| \le n$. Define $x_k := x + \frac{k}{n}(y - x)$, for $k = 0, \ldots, n$. Then

$$u(x_{k+1}) - u(x_k) \le E(x_k, x_{k+1}) - c, \quad |u(x_{k+1}) - u(x_k)| \le K_0,$$
$$|u(y) - u(x)| \le nK_0 \le K_0\big(\|y - x\| + 1\big). \qquad \square$$

Notice that T_- is a monotone operator, $u \le v \Rightarrow T_-[u] \le T_-[v]$, commutes with the constants, $T_-[u + \lambda] = u + \lambda$, $\forall\, \lambda \in \mathbb{R}$, and is concave, $T_-[\lambda u + (1 - \lambda)v] \ge \lambda T_-[u] + (1 - \lambda)T_-[v]$, $\forall\, \lambda \in [0, 1]$. Notice also that $u + c \le T_-[u]$ is equivalent to $u(y) - u(x) \le E(x, y) - c$, $\forall\, x, y \in \mathbb{R}^d$. Define the semi-norm

$$\|u\|_{Lip} := \sup_{0 < \|y - x\| \le R_0} \frac{|u(y) - u(x)|}{\|y - x\| + \sup\limits_{\|x - z\| \vee \|y - z\| \le 2R_0} |E(z, y) - E(z, x)|}, \tag{41}$$

where $R_0 > 0$ is a constant chosen *a priori* and given explicitly by the formula

$$R_0 := \frac{1}{K_0}\Big(K_0 + B_0 + \sup_{x \in \mathbb{R}^d} E(x, x)\Big), \tag{42}$$

with $B_0 > 0$ defined by the superlinearity:

$$\forall\, x, y \in \mathbb{R}^d, \quad E(x, y) \ge 2K_0\|y - x\| - B_0. \tag{43}$$

We equip $C^0(\mathbb{R}^d)$ with the topology of the uniform convergence on any compact sets. Then $C^0(\mathbb{R}^d)$ becomes a Frechet space. Let

$$\mathcal{H}_c := \big\{u \in C^0(\mathbb{R}^d) : u(0) = 0,\ u + c \le T_-[u] \text{ and } \|u\|_{Lip} \le 1\big\}. \tag{44}$$

Define $\tilde{T}_-[u] := T_-[u] - T_-[u](0)$. Notice that the case $c_0 = \bar{c}$ occurs if, and only if, $u \equiv 0$ satisfies the inequality $u + \bar{c} \le T_-[u]$. For the general situation, we point out the following lemma.

Lemma 63. *For every $c_0 < c < \bar{c}$, $\mathcal{H}_c$ is a nonempty compact convex set of $C^0(\mathbb{R}^d)$, $\tilde{T}_-[\mathcal{H}_c] \subseteq \mathcal{H}_c$, and $\tilde{T}_-$ is a continuous map restricted to $\mathcal{H}_c$.*

Proof. Define

$$\tilde{\mathcal{H}}_c := \big\{u \in C^0(\mathbb{R}^d) : u(0) = 0 \text{ and } u + c \le T_-[u]\big\}.$$

Because of the monotonicity and concavity of T_-, $\tilde{\mathcal{H}}_c$ is a closed convex subset of $C^0(\mathbb{R}^d)$ invariant by $\tilde{T}_-$. By the choice of c, $\tilde{\mathcal{H}}_c$ is nonempty. By Ascoli theorem, $\mathcal{H}_c$ is compact in $C^0(\mathbb{R}^d)$. We prove that $\tilde{T}_-[\tilde{\mathcal{H}}_c] \subseteq \mathcal{H}_c$ and that $\tilde{T}_- : \tilde{\mathcal{H}}_c \to C^0(\mathbb{R}^d)$ is continuous.

We first prove that $\|T_-[u]\|_{Lip} \leq 1$ for every $u \in \tilde{\mathcal{H}}_c$. We claim that an optimal point x_{opt} in the definition of $T_-[u](x)$ is at a uniform distance from x. Indeed, notice that we have $T_-[u](x) = u(x_{opt}) + E(x_{opt}, x) \leq u(x) + E(x, x)$, and then

$$\begin{aligned} 2K_0\|x - x_{opt}\| - B_0 \leq E(x_{opt}, x) &\leq u(x) - u(x_{opt}) + E(x, x) \\ &\leq K_0(\|x - x_{opt}\| + 1) + E(x, x), \end{aligned}$$

from which it follows that

$$\|x - x_{opt}\| \leq R_0.$$

We show now that $\|T_-[u]\|_{Lip} \leq 1$. For $\|y - x\| \leq R_0$, we obtain that

$$\begin{aligned} T_-[u](x) &= u(x_{opt}) + E(x_{opt}, x), \\ T_-[u](y) &\leq u(x_{opt}) + E(x_{opt}, y), \\ T_-[u](y) - T_-[u](x) &\leq \sup_{\|x-z\| \vee \|y-z\| \leq 2R_0} |E(z, y) - E(z, x)|, \\ \|T_-[u]\|_{Lip} &\leq 1. \end{aligned}$$

We next show the T_- restricted to $\tilde{\mathcal{H}}_c$ is continuous. For $u, v \in \tilde{\mathcal{H}}_c$ and $R > 0$, notice that

$$\begin{aligned} T_-[u](x) &= u(x_{opt}) + E(x_{opt}, x), \\ T_-[v](x) &\leq v(x_{opt}) + E(x_{opt}, x), \\ \sup_{\|x\| \leq R} |T_-[v](x) - T_-[u](x)| &\leq \sup_{\|x\| \leq R + K_0} |v(x) - u(x)|. \end{aligned}$$

Then T_- and therefore $\tilde{T}_-$ are continuous for the topology of the uniform convergence on compact sets. ☐

Proof of proposition 60. The set $\mathcal{H}_{\bar{c}} = \cap_{c_0 < c < \bar{c}} \mathcal{H}_c$ is a nonempty compact convex subset of the Hausdorff topological vector space $C^0(\mathbb{R}^d)$ and $\tilde{T}_- : \mathcal{H}_{\bar{c}} \to \mathcal{H}_{\bar{c}}$ is a continuous map. By Schauder theorem (see [5] for a recent reference), $\tilde{T}_-$ admits a fixed point $\bar{u} \in \mathcal{H}_{\bar{c}}$. Let $\underline{c} := \tilde{T}_-[\bar{u}](0)$, then $T_-[\bar{u}] = \bar{u} + \underline{c}$. Since $\bar{u} \in \mathcal{H}_{\bar{c}}$, we have, on the one hand, $\bar{u} + \bar{c} \leq T_-[\bar{u}] = \bar{u} + \underline{c}$ and therefore $\bar{c} \leq \underline{c}$. On the other hand,

$$\bar{c} \geq \inf_{x,y} \big[E(x, y) + \bar{u}(x) - \bar{u}(y)\big] = \inf_{y} \big[T_-[\bar{u}](y) - \bar{u}(y)\big] = \underline{c}.$$

We have just shown that there exists $\bar{u} \in C^0(\mathbb{R}^d)$, uniformly Lipschitz in the large, with $\|\bar{u}\|_{Lip} \leq 1$, such that $T_-[\bar{u}] = \bar{u} + \bar{c}$, where $\bar{c}$ is given by (39). We construct by induction a backward calibrated configuration using the identity

$$\forall\, k \geq 1, \quad \bar{u}(x_{-k+1}) = \bar{u}(x_{-k}) + E(x_{-k}, x_{-k+1}) - \bar{c}.$$ ☐

Proof of proposition 61. Let

$$\bar{c}(\omega) := \sup_{u \in C^0(\mathbb{R}^d)} \inf_{x,y \in \mathbb{R}^d} [E_\omega(x, y) + u(x) - u(y)].$$

The conclusion of the proof of proposition 60 asserts that the supremum in $\bar c(\omega)$ can be realized on a smaller space which may be defined independently of ω. Let

$$C^0_{Lip}(\mathbb{R}^d) := \Big\{u \in C^0(\mathbb{R}^d) \ : u(0) = 0,\ \|u\|_{Lip} \leq 1\Big\},$$

where the new semi-norm $\|u\|_{Lip}$ is given by

$$\|u\|_{Lip} := \sup_{\|y-x\|\geq \bar R} \frac{|u(y)-u(x)|}{2\bar K\|y-x\|} \bigvee \\ \bigvee \sup_{0<\|y-x\|\leq \bar R} \ \inf_{\substack{\|x-z\|\leq 2\bar R \\ \|y-z\|\leq 2\bar R}} \frac{|u(y)-u(x)|}{\|y-x\| + \sup_{\omega\in\Omega} |E_\omega(z,y)-E_\omega(z,x)|},$$

with $\bar K$, $\bar R$ given as in (40), (42) and (43):

$$\bar K := \sup_{\omega\in\Omega,\ \|y-x\|\leq 1} E_\omega(x,y) - \inf_{\omega\in\Omega,\ x,y\in\mathbb{R}^d} E_\omega(x,y),$$
$$\bar R := \frac{1}{\bar K}\Big(\bar K + \bar B + \sup_{\omega\in\Omega,\ x\in\mathbb{R}^d} E_\omega(x,x)\Big),$$
$$\forall\, x,y\in\mathbb{R}^d, \quad \inf_{\omega\in\Omega} E_\omega(x,y) \geq 2\bar K\|y-x\| - \bar B.$$

Then

$$\bar c(\omega) := \max_{u\in C^0_{Lip}(\mathbb{R}^d)} \ \inf_{x,y\in\mathbb{R}^d} [E_\omega(x,y) + u(x) - u(y)].$$

For every $u \in C^0_{Lip}(\mathbb{R}^d)$, the infimum is a continuous function of ω thanks to the uniform superlinearity of E_ω. In particular, $\omega \mapsto \bar c(\omega)$ is lower semi-continuous. By the topological stationarity (10) of E_ω, $\omega \mapsto \bar c(\omega)$ is constant along any orbit $\{\tau_t(\omega)\}_{t\in\mathbb{R}^d}$. The set $\{\omega : \bar c(\omega) \leq \inf \bar c\}$ is closed, nonempty, and invariant. By minimality, $\bar c$ is a constant function.

We now prove the existence of a calibrated configuration at the level $\bar c$. Let $\omega\in\Omega$ be fixed. By proposition 60, there exists $u_\omega \in C^0_{Lip}(\mathbb{R}^d)$ such that

$$\forall\, y\in\mathbb{R}^d, \quad u_\omega(y) = \min_{x\in\mathbb{R}^d,\ \|y-x\|\leq\bar R} \big[u_\omega(x) + E_\omega(x,y) - \bar c\big].$$

Let $n\geq 1$. We construct by induction a backward configuration $\{x_{-k}\}_{k=0}^{k=2n}$ starting at $x_0 = 0$ and satisfying

$$\forall\, 1\leq k\leq 2n, \quad u_\omega(x_{-k+1}) = u_\omega(x_{-k}) + E_\omega(x_{-k}, x_{-k+1}) - \bar c.$$

By shifting by the same amount the environment $\omega_n = \tau_{x_{-n}}(\omega)$ and the configuration $x^n_k := x_{k-n} - x_{-n}$, we obtain a finite configuration $\{x^n_k\}_{k=-n}^n$ centered at the origin $x^n_0 = 0$ and calibrated for E_{ω_n} at the level $\bar c$. Thanks to the fact that the successive jumps are uniformly bounded, by a diagonal extraction procedure, one can find a subsequence of integers $\{n'\}$, $\bar\omega\in\Omega$, and a bi-infinite configuration $\{\bar x_k\}_{k=-\infty}^{+\infty}$ so that $\omega_n \to \bar\omega$ and $x^n_k \to \bar x_k$ for every $k\in\mathbb{Z}$ along the subsequence $\{n'\}$. Since the calibration property passes to the limit, $\{\bar x_k\}_{k=-\infty}^{+\infty}$ is a calibrated configuration for $E_{\bar\omega}$ at the level $\bar c$. $\square$

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C^*-ALGEBRAS OF PENROSE HYPERBOLIC TILINGS

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ABSTRACT. Penrose hyperbolic tilings are tilings of the hyperbolic plane which admit, up to affine transformations a finite number of prototiles. In this paper, we give a complete description of the C^*-algebras and of the K-theory for such tilings. Since the continuous hull of these tilings have no transversally invariant measure, these C^*-algebras are traceless. Nevertheless, harmonic currents give rise to 3-cyclic cocycles and we discuss in this setting a higher-order version of the gap-labelling.

Keywords: *Hyperbolic aperiodic tilings, C^*-algebras of dynamical systems, K-theory, Cyclic cohomology*
2000 Mathematics Subject Classification: *37A55, 37B051, 46L55, 46L80*
Subject Classification: *Noncommutative topology and geometry, Dynamical systems*

1. INTRODUCTION

The non-commutative geometry of a quasi-periodic tiling studies an appropriate C^*-algebra of a dynamical system (X, G), for a compact metric space X, called *the hull*, endowed with a continuous Lie group G action. This C^*-algebra is of relevance to study the space of leaves which is pathological in any topological sense. The hull owns also a geometrical structure of *lamination* or *foliated space*, the transverse structure being just metric [**?**]. The C^*-algebras and the non-commutative tools provide then topological and geometrical invariants for the tiling or the lamination. Moreover, some K-theoretical invariants of Euclidean tilings have a physical interpretation. In particular, when the tiling represents a quasi-crystal, the image of the K-theory under the canonical trace labels the gaps in the spectrum of the Schrödinger operator associated with the quasi-crystal [**?**].

For an Euclidean tiling, the group G is $\mathbb{R}^d$ and $\mathbb{R}^d$-invariant ergodic probability measures on the hull are in one-to-one correspondence with ergodic transversal invariant measures and also with extremal traces on the C^*-algebra [**?**]. These algebras are well studied and this leads, for instance, to give distinct proofs of the *gap labelling conjecture* [**?**, **?**, **?**], i.e. for minimal $\mathbb{R}^d$-action, the image of the K-theory under a trace is the countable subgroup of $\mathbb{R}$ generated by the images under the corresponding transversal invariant measure of the compact-open subsets of the (Cantor) canonical transversal.

For a hyperbolic quasi-periodic tiling, the situation is quite distinct. The group of affine transformations acts on the hull and since this group is not unimodular, there is no transversally invariant measure [**?**]. A new phenomena shows up for the C^*-algebra of the tiling: it has no trace. Nevertheless, the affine group is amenable, so the hull admits at least one invariant probability measure. These measures are actually in one-to-one correspondence with harmonic currents [**?**], and they provide 3-cyclic cocycles on the smooth algebra of the tiling.

The present paper is devoted to give a complete description of the C^*-algebra and the K-theory of a specific family of hyperbolic tilings derivated from the example given by Penrose in [**?**]. The dynamic of the hulls under investigation, have a structure of double suspension (this make sens in term of groupoids as we shall see in section **??**) which enables to make explicit computations. This suggests that the pairing with the 3-cyclic cocycle is closely related to the one-dimension gap-labelling for a subshift associated with the tiling. But the right setting to state an analogue of the gap-labelling seems to be Frechet algebras and a natural question is whether this bring in new computable invariants.

Background on tiling spaces is given in the next section and we construct examples of hyperbolic quasi-periodic tilings in the third section. A description of the considered hulls is given in section 4. In section 5, we recall the background on the groupoids and their C^*-algebras. Sections 6 and 7 are devoted to the complete description of the C^*-algebras of the examples and their K-theory groups in terms of generetors are given in section 8. For readers interested in topological invariants of the hull, we compute its K-theory and its Cěch cohomology and we relate these computations to the former one. In the last section we construct 3-cyclic cocycles associated to these tilings and we discuss an odd version of the gap-labelling.

2. Background on tilings

Let $\mathbb{H}_2$ be the real hyperbolic 2-space, identified with the upper half complex plane: $\{(x,y) \in \mathbb{R}^2 \,|y>0\}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. We denote by G the group of *affine transformations* of this space: i.e. the isometries of $\mathbb{H}_2$ of the kind $z \mapsto az+b$ with a, b reals and $a>0$.

A tiling $T = \{t_1, \ldots, t_n, \ldots\}$ of $\mathbb{H}_2$, is a collection of convex compact polygons t_i with geodesic borders, called *tiles*, such that their union is the whole space $\mathbb{H}_2$, their interiors are pairwise disjoint and they meet full edge to full edge. For instance, when F is a fundamental domain of a co-compact lattice Γ of isometries of $\mathbb{H}_2$, then $\{\gamma(F),\ \gamma \in \Gamma\}$ is a tiling of $\mathbb{H}_2$. However the set of tilings is much richer than the one given by lattices as we should see later on. Similarly to the Euclidian case, a tiling is said of *G-finite type* or *finite affine type*, if there exists a finite number of polygons $\{p_1, \ldots, p_n\}$ called *prototiles* such that each t_i is the image of one of these polygons by an element of G. Besides its famous Euclidean tiling, Penrose in [**?**] constructs a finite affine type tiling made with a single prototile which is not stable for any Fuchsian group. The construction goes as follows.

2.1. **Hyperbolic Penrose's tiling.** Let P be the convex polygon with vertices A_p with affix $(p-1)/2+\imath$ for $1 \le p \le 3$ and $A_4 : 2\imath+1$ and $A_5 : 2\imath$ P is a polygon with 5 geodesic edges. Consider the two maps:

$$R : z \mapsto 2z \text{ and } S : z \mapsto z+1.$$

The hyperbolic Penrose's tiling is defined by $\mathcal{P} = \{R^k \circ S^n P|\ n,k \in \mathbb{Z}\}$ (see figure **??**). This is an example of finite affine type tiling of $\mathbb{H}_2$.

This tiling is stable under no co-compact group of hyperbolic isometries. The proof is homological: we associate with the edge A_4A_5 a positive charge and two negative charges with edges A_1A_2, A_2A_3. If $\mathcal{P}$ was stable for a Fuchsian group, then P would tile a compact surface. Since the edge A_4A_5 can meet only the edges

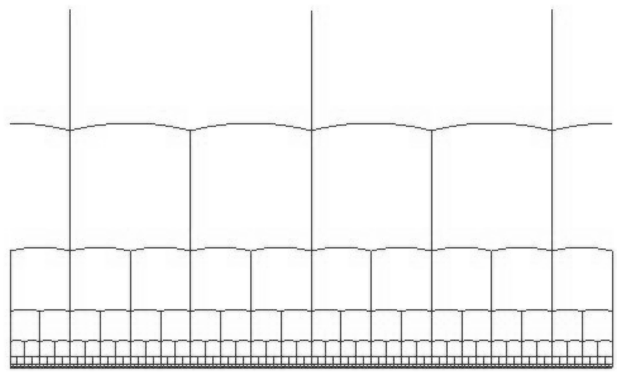

FIGURE 1. The hyperbolic Penrose's tiling

A_1A_2 or A_2A_3, the surface has a neutral charge. This is in contradiction with the fact P is negatively charged.
G. Margulis and S. Mozes [**?**] have generalized this construction to build a family of prototiles which cannot be used to tile a compact surface. Notice the group of isometries which preserves $\mathcal{P}$ is not trivial and is generated by the transformation R. In order to break this symmetry, it is possible, by a standard way, to decorate prototiles to get a new finite affine type tiling which is stable under no non-trivial isometry (we say in this case that the tiling is *aperiodic*).

3. Background on tiling spaces

In this section, we recall some basic definitions and properties on dynamical systems associated with tilings. We refer to [**?**], [**?**] and [**?**] for the proofs. We give then a description of the dynamical system associated to the hyperbolic Penrose's tiling.

3.1. **Action on tilings space.** First, note that the group G acts transitively, freely (without a fixed point) and preserving the orientation of the surface $\mathbb{H}_2$, thus G is a Lie group homeomorphic to $\mathbb{H}_2$. The metric on $\mathbb{H}_2$ gives a left multiplicative invariant metric on G. We fix the point O in $\mathbb{H}_2$ with affix i that we call *origin*.
For a tiling T of G finite type and an isometry p in G, the image of T by p is again a tiling of $\mathbb{H}_2$ of G finite type. We denote by $G.T$ the set of tilings which are image of T by isometries in G. The group G acts on this set by the left action:

$$\begin{array}{ccc} G \times G.T & \longrightarrow & G.T \\ (p, T') & \longmapsto & p.T' = p(T'). \end{array}$$

We equip $G.T$ with a metrizable topology, so that the action becomes continuous. A base of neighborhoods is defined as follows: two tilings are close one to the other if they agree, on a big ball of $\mathbb{H}_2$ centered at the origin, up to an isometry in G close

to the identity. This topology can be generated by the metric δ on $G.T$ defined by (see [**?**]):
For T and T' be two tilings of $G.T$, let

$$A = \{\epsilon \in (0, \frac{1}{\sqrt{2}}] \mid \exists\, g \in B_\epsilon(Id) \subset G \text{ s.t. } g.T \cap B_{1/\epsilon} = T' \cap B_{1/\epsilon}\}$$

where $B_{1/\epsilon}$ is the set of points $x \in \mathbb{H}_2 \cong G$ such that $d(x, O) < 1/\epsilon$.
We define:

$$\delta(T, T') = \inf A \text{ if } A \neq \emptyset$$
$$\delta(T, T') = \frac{1}{\sqrt{2}} \text{ else.}$$

The *continuous hull* of the tiling T, is the metric completion of $G.T$ for the metric δ. We denote it by X_T^G. Actually this space is a set of tilings of $\mathbb{H}_2$ of G-finite type. A *patch* of a tiling T is a finite set of tiles of T. It is straightforward to show that patches of tilings in X_T^G are copies of patches of T. The set X_T^G is then a compact metric set and the action of G on $G \cdot T$ can be extended to a continuous left action on this space. The dynamical system (X_T^G, G) has a dense orbit: the orbit of T. Some combinatorial properties can be interpreted in a dynamical way like, for instance, the following.

Definition 3.1. *A tiling T satisfies the repetitivity condition if for each patch P, there exists a real $R(P)$ such that every ball of $\mathbb{H}_2$ with radius $R(P)$ intersected with the tiling T contains a translated by an element G of the patch P.*

This definition can be interpreted from a dynamical point of view (for a proof see for instance [**?**]).

Proposition 3.2 (Gottschalk)**.** *The dynamical system (X_T^G, G) is minimal (any orbit is dense) if and only if the tiling T satisfies the repetitivity condition.*

We call a tiling *aperiodic* if the action of G on X_T^G is free: for all $p \neq Id$ of G and all tilings T' of X_T^G we have $p.T' \neq T'$.
As we have seen in the former section the hyperbolic Penrose's tiling is not aperiodic, however, using this example, we shall construct in section **??** uncountably many examples of repetitive and aperiodic affine finite type tilings.

When the tiling T is aperiodic and repetitive, the hull X_T^G has also a geometric structure of a specific lamination called a G-*solenoid* (see [**?**]). Locally at any point x, there exists a *vertical germ* which is a Cantor set included in X_T^G, transverse to the local G-action and which is defined independently of the neighborhood of the point x. This implies that X_T^G is locally homeomorphic to the Cartesian product of a Cantor set with an open subset (called a *slice*) of the Lie group G. The connected component of the slices that intersect is called a *leaf* and has a manifold structure. Globally, X_T^G is a disjoint union of uncountably many leaves, and it turns out that each leaf is a G-orbit. Since the action is free, each leaf is homeomorphic to $\mathbb{H}_2$.

In the aperiodic case, the G-action is *expansive*: There exists a positive real ϵ such that for every points T_1 and T_2 in the same vertical in X_T^G, if $\delta(T_1.g, T_2.g) < \epsilon$ for every $g \in G$, then $T_1 = T_2$.
Furthermore this action has locally constant return times: if an orbit (or a leaf) intersects two verticals V and V' at points v and $v.g$ where $g \in G$, then for any point w of V close enough to v, $w.g$ belongs to V'.

3.2. **Structure of the hull of the Penrose Hyperbolic tilings.** First recall the notion of suspension action for X a compact metric space and $f : X \to X$ a homeomorphim. The group $\mathbb{Z}$ acts diagonally on the product space $X \times \mathbb{R}$ by the following homeomorphism denoted $\mathcal{A}_f$

$$\begin{aligned} \mathcal{A}_f :\ & X \times \mathbb{R} \to X \times \mathbb{R} \\ & (x,t) \mapsto (f(x), t-1) \end{aligned}$$

The quotient space of $(X \times \mathbb{R})/\mathcal{A}_f$, where two points are identified if they belong to the same orbit, is a compact set for the product topology and is called *the suspension* of (X, f). The group $\mathbb{R}$ acts also diagonally on $X \times \mathbb{R}$: trivially on X and by translation on $\mathbb{R}$. Since this action commutes with $\mathcal{A}_f$, this induces a continuous $\mathbb{R}$ action on the suspension space $(X \times \mathbb{R})/\mathcal{A}_f$ that we call: the *suspension action* of the system (X, f) and we denote it by $\big((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{R}\big)$.

We recall here, the construction of the dyadic completion of the integers. On the set of integers $\mathbb{Z}$, we consider the dyadic norm defined by

$$|n|_2 = 2^{-\sup\{p\in\mathbb{N},\ 2^p \text{ divides } |n|\}} \qquad n \in \mathbb{Z}.$$

Let Ω be the completion of the set $\mathbb{Z}$ for the metric given by $|.|_2$. The set Ω has a commutative group structure where $\mathbb{Z}$ is a dense subgroup, and Ω is a Cantor set. The continuous action given by the map $o :\ x \mapsto x + 1$ on Ω is called *adding-machine* or *odometer* and is known to be minimal and equicontinuous. We denote by $((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R})$ the suspension action of this homeomorphism.

Recall that a conjugacy map between two dynamical systems is a homeomophism which commutes with the actions. Let $\mathcal{N}$ be the group of transformations $\{z \mapsto z + t,\ t \in \mathbb{R}\}$ isomorphic to $\mathbb{R}$.

Proposition 3.3. *Let $X_{\mathcal{P}}^{\mathcal{N}}$ be the closure (for the tiling topology) of the orbit $\mathcal{N}.\mathcal{P} \subset X_{\mathcal{P}}^{G}$. Then the dynamical system $(X_{\mathcal{P}}^{\mathcal{N}}, \mathcal{N})$ is conjugate to the suspension action of the odometer $((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R})$.*

Proof. Let $\phi : \mathcal{N}.\mathcal{P} \to (\Omega\times\mathbb{R})/\mathcal{A}_o$ be the map defined by $\phi(\mathcal{P}+t) = [0, t]$ where $[0, t]$ is the $\mathcal{A}_o$-class of $(0, t) \in \Omega \times \mathbb{R}$. Since the tiling is invariant under no translations, the application ϕ is well defined. It is straightforward to check that ϕ is continuous for the tiling topology and for the topology on $(\Omega \times \mathbb{R})/\mathcal{A}_o$ arising from the dyadic topology on Ω. So the map ϕ extends by continuity to $X_{\mathcal{P}}^{\mathcal{N}}$. Let us check that ϕ is a homeomorphism by constructing its inverse. Let $\psi : \mathbb{Z} \times \mathbb{R} \to X_{\mathcal{P}}^{\mathcal{N}}$ defined by $\psi(n, t) = \mathcal{P} + n + t$. This application is continuous for the dyadic topology, so it extends by continuity to $\Omega \times \mathbb{R}$. Notice $\psi(n, t)$ is constant along the orbits of the $\mathcal{A}_o$ action on $\mathbb{Z}\times\mathbb{R}$ which is dense in $\Omega\times\mathbb{R}$. Thus ψ is constant along the $\mathcal{A}_o$-orbits in $\Omega \times \mathbb{R}$ and ψ factorizes through a map $\overline{\psi}$ from the suspension $(\Omega \times \mathbb{R})/\mathcal{A}_o$ to $X_{\mathcal{P}}^{\mathcal{N}}$. It is plain to check that $\overline{\psi} \circ \phi = Id$ on the dense set $\mathcal{N}.\mathcal{P}$ and that $\phi \circ \overline{\psi} = Id$ on the dense set $\pi(\mathbb{Z} \times \mathbb{R})$ where $\pi : \Omega \times \mathbb{R} \to (\Omega \times \mathbb{R})/\mathcal{A}_o$ denotes the canonical projection. Hence ϕ is an homeomorphism from $X_{\mathcal{P}}^{\mathcal{N}}$ onto $(\Omega\times\mathbb{R})/\mathcal{A}_o$. It is obvious that ϕ commutes with the $\mathbb{R}$-actions. $\square$

4. Examples

We construct in this section a family of tilings of $\mathbb{H}_2$ of finite affine type, indexed by sequences on a finite alphabet. For uncountably many of them, the tilings will be aperiodic and repetitive, the action on the associated hull will be free and minimal. A description of these actions in terms of double-suspension is given.

4.1. **Construction of the examples.** To construct such tilings we will use the hyperbolic Penrose's tiling described in section **??**, so we will keep the notations of this section. Recall that its stabilizer group under the action of G, is the group $\langle\, R \,\rangle$ generated by the affine transformations R. The main idea is to "decorate" this tiling in order to break its symmetry, the decoration will be coded by a sequence on a finite alphabet. By a decoration, we mean that we will substitute to each tile t the same polygon t equipped with a color. We take the convention that two colored polygons are the same if and only if the polygons are the same up to an affine map and they share the same color. By substituting each tile by a colored tile, we obtain a new tiling of finite affine type with a bigger number of prototiles. Notice that the coloration is not canonical. It also possible to do the same by substituting to a tile t, an unique finite family of convex tiles $\{t_i\}_i$, like triangles, such that the union of the t_i is t and the tiles t_i overlaps only on their borders. We choose the coloration only for presentation reasons.

Let r be an integer bigger than 1. We associate to each element of $\{1, \dots, r\}$ an unique color. Let P be the polygon defined in section **??** to construct the Penrose's tiling. For an element i of $\{1, \dots, r\}$, we denote by P_i the prototile P colored in the color i. To a sequence $w = (w_k)_k \in \{1, \dots, \mathrm{r}\}^{\mathbb{Z}}$, we associate the G-finite-type tiling $\mathcal{P}(w)$ built with the prototiles P_i for i in $\{1, \dots, r\}$ and defined by:

$$\mathcal{P}(w) = \{R^q \circ S^n(P_{w_{-q}}),\ n, q \in \mathbb{Z}\}.$$

Notice that the stabilizer of this tiling is a subgroup of $\langle\, R \,\rangle$.

The set of sequences on $\{1, \dots, r\}$ is the product space $\{1, \dots, r\}^{\mathbb{Z}}$ which is a Cantor set for the product topology. There exists a natural homeomorphism on it called the *shift*. To a sequence $(w_n)_{n\in\mathbb{Z}}$ the shift σ associates the sequence $(w_{n+1})_{n\in\mathbb{Z}}$. Let Z_w denote the closure of the orbit of w by the action of the shift σ: $Z_w = \overline{\{\sigma^n(w),\ n \in \mathbb{Z}\}}$. The set Z_w is a compact metric space stable under the action of σ.

Remark 4.1. *The map*

$$Z_\omega \to X^G_{\mathcal{P}(w)};\ \omega' \mapsto \mathcal{P}(w')$$

is continuous.

Since $R.\mathcal{P}(w)$ denotes the tiling image of $\mathcal{P}(w)$ by R, we get the relation

$$R.\mathcal{P}(w) = \mathcal{P}(\sigma(w)). \tag{4.1}$$

Thanks to this, we obtain the following property:

Lemma 4.2.

- *The sequence w is aperiodic for the shift-action, if and only if $\mathcal{P}(w)$ is stable under no non-trivial affine map.*
- *The dynamical system (Z_w, σ) is minimal, if and only if $(X^G_{\mathcal{P}(w)}, G)$ is minimal.*

Proof. The first point comes from the relation **??** and from the fact that the stabilizer of $\mathcal{P}(w)$ is a subgroup of $\langle\, R \,\rangle$. The last point comes from the characterization of minimal sequences: (Z_w, σ) is minimal if and only if each words in w appears infinitely many times with uniformly bounded gap [**?**]. This condition is equivalent to the repetitivity of $\mathcal{P}(w)$. □

Recall that we have defined the group $\mathcal{N} = \{z \mapsto z + t,\ t \in \mathbb{R}\}$ and that $X_{\mathcal{P}}^{\mathcal{N}}$ stands for the closure (for the tiling topology) of the $\mathcal{N}$-orbit $\mathcal{N}.\mathcal{P}$ in $X_{\mathcal{P}}^{G}$ of the uncolored tiling $\mathcal{P}$. Notice that the continuous action of R on $X_{\mathcal{P}}^{G}$ preserves the orbit $\mathcal{N}.\mathcal{P}$ so we get an homeomorphism of $X_{\mathcal{P}}^{\mathcal{N}}$ that we denote also by R. We consider on the space $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ equipped with the product topology, the homeomorphism $\mathcal{R}$ defined by $\mathcal{R}(\mathcal{T}, w', t) = (R.\mathcal{T}, \sigma(w'), t/2)$. The quotient space $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$, where the points in the same $\mathcal{R}$ orbit are identified, is a compact space.

The affine group G also acts on the left on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$; where the action of an element $g : z \mapsto az + b$ is given by the homeomorphism

$$(\mathcal{T}, w', t) \mapsto (\mathcal{T} + \frac{b}{at}, w', at) = g.(\mathcal{T}, w', t).$$

It is straightforward to check that the application $\mathcal{R}$ commutes with this action, so this defines a G-continuous action on the quotient space $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$.

Proposition 4.3. *Let w be an element of $\{1, \dots, r\}^{\mathbb{Z}}$. Then the map*

$$\begin{array}{rcl} \Psi : G.\mathcal{P}(w) & \to & (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R} \\ g.\mathcal{P}(w) & \mapsto & [g.(\mathcal{P}, w, 1)] \end{array}$$

where $[x]$ denotes the $\mathcal{R}$-class of x, extends to a conjugacy map between $(X_{\mathcal{P}(w)}^{G}, G)$ and $\big((X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R},\ G\big)$.

Proof. Let Φ be the transformation $\mathcal{N}.\mathcal{P} \times Z_w \times \mathbb{R}_+^* \to X_{\mathcal{P}(w)}^{G}$ defined by

$$\Phi(\mathcal{P} + \tau, w', t) = R_t.(\mathcal{P}(w') + \tau)$$

where R_t denotes the map $z \mapsto tz$. According to remark **??**, the application Φ is continuous for the tiling topology on $\mathcal{N}.\mathcal{P}$, so it extends by continuity to a continuous map from $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ to $X_{\mathcal{P}(w)}^{G}$. Thanks to relation (**??**), we get $\Phi \circ \mathcal{R} = \Phi$ on the dense subset $\mathcal{N}.\mathcal{P} \times Z_w \times \mathbb{R}_+^*$. Therefore the map Φ factorizes throught a continuous map $\overline{\Phi} : (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R} \to X_{\mathcal{P}(w)}^{G}$. Since the stabilizer of the tiling $\mathcal{P}(w)$ is a subgroup of the one generated by the transformation R, and by relation (**??**), the map

$$\Psi : G \cdot \mathcal{P}(w) \to \big(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*\big)/\mathcal{R};\ R_t \cdot \mathcal{P}(w) + \tau \mapsto [\mathcal{P} + \tau/t, \omega, t]$$

is well defined. It is straightforward to check that Ψ is continuous, so it extends to a continuous map from $X_{\mathcal{P}(w)}^{G}$ to $\big(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*\big)/\mathcal{R}$ that we denote again Ψ. Furthermore we have $\overline{\Phi} \circ \Psi = Id$ on $G \cdot \mathcal{P}(w)$ and $\Psi \circ \overline{\Phi} = Id$ on the dense set $\pi(\mathcal{N}.\mathcal{P} \times Z_w \times \mathbb{R}_+^*)$ where π denotes the canonical projection $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \to \big(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*\big)/\mathcal{R}$. Hence the map $\overline{\Phi}$ is an homeomorphism. The homeomorphism Ψ obviously commutes with the action. □

Notice that, $X_{\mathcal{P}}^{\mathcal{N}}$ is locally the Cartesian product of a Cantor set by an interval of $\mathbb{R}$. For $w \in \{1, \dots, r\}^{\mathbb{Z}}$, $X_{\mathcal{P}(w)}^{G}$ is locally homeomorphic the product of a Cantor set by an open subset (a slice) of $\mathbb{R}_+^* \times \mathbb{R}$ since the Cartesian product of two Cantor sets is again a Cantor set. The G-action maps slices onto slices.

4.2. **Ergodic properties of Penrose's tilings.** For a metric space X and a continuous action of a group Γ on it, a Γ-*invariant measure* is a measure μ defined on the Borel σ-algebra of X which is invariant under the action of Γ i.e.: For any measurable set $B \subset X$ and any $g \in \Gamma$, $\mu(B.g) = \mu(B)$. For instance, any group Γ acts on itself by right multiplication, there exists (up to a scalar) only one measure invariant for this action: it is called the *Haar* measure.

Any action of an amenable group Γ (like $\mathbb{Z}$, $\mathbb{R}$ and all their extensions) on a compact metric space X admits a finite invariant measure and in particular, any homeomorphism f of X preserves a probability measure. An *ergodic* invariant measure μ is such that every measurable functions constant along the orbits are μ almost surely constant. Every invariant measure is the sum of ergodic invariant measures [**?**]. A conjugacy map sends the invariant measure to invariant measure and the ergodic measures to the ergodic measures.

In our case, the group of affine transformations G, is the extension of two groups isomorphic to $\mathbb{R}$, hence is amenable. It is well known that the only invariant measures for the suspension action $\big((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{R}\big)$ are locally the images through the canonical projection $\pi : X \times \mathbb{R} \to \big(X \times \mathbb{R}\big)/\mathcal{A}_f$ of the measures $\mu \otimes \lambda$ where μ is a f-invariant measure on X and λ denotes the Lebesgue measure of $\mathbb{R}$. The proof is actually contained in property **??**.

It is well known also that the map $o : x \mapsto x + 1$ on the dyadic set of integers Ω, admits only one invariant probability measure: the Haar probability measure on Ω. Hence the suspension of this action $\big((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R}\big)$ admits only one invariant probability measure. By proposition **??**, $X_{\mathcal{P}}^{\mathcal{N}}$ has only one invariant probability measure ν. Notice that the map R preserves $X_{\mathcal{P}}^{\mathcal{N}}$, and since $R\mathcal{N}R^{-1} = \mathcal{N}$, the probability $R_*\nu$ is $\mathcal{N}$-invariant and hence R preserves ν. Nevertheless, the local product decomposition of ν is not invariant by R, because R divides by 2 the length of the intervals of the $\mathcal{N}$-orbit. So R has to inflate the Haar measure on Ω by a factor 2.

Proposition 4.4. *If w is an element in $\{1, \ldots, r\}^{\mathbb{Z}}$, then any finite invariant measure of $\big((X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R},\ G\big)$ is locally the image through the projection $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \to (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$ of a measure $\nu \otimes \mu \otimes m$ where*

- *ν is the only invariant probability measure of $(X_{\mathcal{P}}^{\mathcal{N}}, \mathbb{R})$;*
- *m is the Haar measure of $(\mathbb{R}_+^*, .)$;*
- *μ is a finite invariant measure of (Z_w, σ).*

Proof. It is enough to prove this for an ergodic finite G-invariant measure $\overline{\theta}$ on the suspension $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$. Since $\mathcal{R}$ acts cocompactly on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$, $\overline{\theta}$ defines a finite measure on a fundamental domain of $\mathcal{R}$, and the sum of all the images of this measure by iterates of $\mathcal{R}$ and $\mathcal{R}^{-1}$ defines a σ-finite measure θ on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ which is G and $\mathcal{R}$-invariant.
Let $\pi_2 : X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \to Z_w$ be the projection to the second coordinate, then $\pi_{2*}\theta$ is a shift invariant measure on Z_w. The measure θ can be disintegrated over $\pi_{2*}\theta = \mu$ by a family of measures $(\lambda_{w'})_{w' \in Z_w}$ defined for μ-almost every $w' \in Z_w$ on $X_{\mathcal{P}}^{\mathcal{N}} \times \{w'\} \times \mathbb{R}_+^*$ such that

$$\theta(B \times C) = \int_C \lambda_{w'}(B) d\mu(w'),$$

for any Borel sets $B \subset X_{\mathcal{P}}^{\mathcal{N}} \times \mathbb{R}_+^*$ and $C \subset Z_w$.

The G-invariance of θ implies that the measures $\lambda_{w'}$ are G-invariant for almost all w'. The projection to the first coordinate $\pi_1 : X_{\mathcal{P}}^{\mathcal{N}} \times \{w'\} \times \mathbb{R}_+^* \to X_{\mathcal{P}}^{\mathcal{N}}$ is $\mathcal{N}$-equivariant. The measures $\pi_{1*}\lambda_{w'}$ are then $\mathcal{N}$-invariant measures, hence are proportional to ν. Each measure $\lambda_{w'}$ can be disintegrated over ν by a family of measures $(m_{x,w'})_{x\in X_{\mathcal{P}}^{\mathcal{N}},\, w'\in Z_w}$ on $\mathbb{R}_+^*$ defined for ν-almost every $x \in X_{\mathcal{P}}^{\mathcal{N}}$, so that

$$\lambda_{w'}(B \times \{w'\} \times I) = \int_B \int_I m_{x,w'} d\nu(x),$$

for any Borel sets $B \subset X_{\mathcal{P}}^{\mathcal{N}}$ and $I \subset \mathbb{R}_+^*$. Each measure $\lambda_{w'}$ is invariant under the action of transformations of the kind $z \mapsto az$ for $a \in \mathbb{R}_+^*$. It is then straightforward to check that the measures $m_{x,w'}$ are multiplication-invariant for almost every x. By unicity of the Haar measure, there exists a measurable positive function $(x, w') \mapsto h(x, w')$ defined almost everywhere so that $m_{x,w'} = h(x, w')m$. The $\mathcal{N}$-invariance of the measures $\lambda_{w'}$ implies that the map h is almost surely constant along the $\mathcal{N}$-orbits, and the $\mathcal{R}$-invariance of θ implies that h is almost surely constant along the $\mathcal{R}$-orbits. This defines then a measurable map on the quotient space by $\mathcal{R}$ which is G-invariant, the ergodicity of $\overline{\theta}$ implies this map is almost surely constant. □

Notice that an invariant measure on $X_{\mathcal{P}(w)}^G$ can be decomposed locally into the product of a measure on a Cantor set by a measure along the leaves. Since the map R does not preserve the transversal measure on Ω in $X_{\mathcal{P}}^{\mathcal{N}}$, the holonomy groupoid of $X_{\mathcal{P}(w)}^G$ does not preserve the transversal measure on the Cantor set.
The G-action is locally free and acts transitively on each leaf, so each orbits inherits a hyperbolic 2-manifolds structure. Actually, $X_{\mathcal{P}(w)}^G$ can be equipped with a continuous metric with a constant curvature -1 in restriction to the leaves. The invariant measures of $X_{\mathcal{P}(w)}^G$ have then also a geometric interpretation in terms of *harmonic* measures, a notion introduced by L. Garnett in [**?**].

Definition 4.5. *A probability measure μ on M is harmonic if*

$$\int_M \Delta f d\mu = 0$$

for any continuous function f with restriction to leaves of class $\mathcal{C}^2$, where Δ denotes the Laplace-Beltrami operator in restriction on each leaf.

Actually, it is shown in [**?**] , that on $X_{\mathcal{P}(w)}^G$ the notions of harmonic and invariant measures are the same and such measures can be described in terms of inverse limit of vectoriel cones.

5. Transformation groupoids

We gather this section with results on groupoids and their C^*-algebras. Good material on this topic can be found in [**?**]. Let us fix first some notations. Let $\mathcal{G}$ be a locally compact groupoid, with base space X, range and source maps respectively $r : \mathcal{G} \to X$ and $s : \mathcal{G} \to X$. Recall that X can be viewed as a closed subset of $\mathcal{G}$ (the set of units). For any element x of X, we set

$$\mathcal{G}^x = \{\gamma \in \mathcal{G} \text{ such that } r(\gamma) = x\}$$

and

$$\mathcal{G}_x = \{\gamma \in \mathcal{G} \text{ such that } s(\gamma) = x\}.$$

Let us denote for any γ in $\mathcal{G}$ by $L_\gamma : \mathcal{G}^{s(\gamma)} \to \mathcal{G}^{r(\gamma)}$ the left translation by γ. Thourought this section, all the groupoids will be assumed locally compact and second countable. Recall that a Haar system λ for $\mathcal{G}$ is a family $(\lambda^x)_{x\in X}$ of borelian measures on $\mathcal{G}$ such that

(1) the support of λ^x is $\mathcal{G}^x$;
(2) for any f in $C_c(\mathcal{G})$, the map $X \to \mathbb{C}$; $x \to \int_{\mathcal{G}^x} f d\lambda^x$ is continuous;
(3) $L_{\gamma_*}\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ for all γ in Γ.

Our prominent examples of groupoid will be *semi-direct product groupoid*: let H be a locally compact group acting on a locally compact space X. The semi-direct product groupoid $X \rtimes H$ of X by H is defined by

- $X \times H$ as a topological space ;
- the base space is X and the structure maps are $r : X \rtimes H \to X$; $(x,h) \mapsto x$ and $s : X \rtimes H \to X$; $(x,h) \mapsto h^{-1}x$;
- the product is $(x,h)\cdot(h^{-1}x,h') = (x,hh')$ for x in X and h and h' in H.

Let μ be a left Haar mesure on H. Then the groupoid $X \rtimes H$ is equipped with a Haar system $\lambda^\mu = (\lambda^\mu_x)_{x\in X}$ given for any f in $C_c(X\times H)$ and any x in X by $\lambda^\mu_x(f) = \int_H f(x,h)d\mu(h)$.

5.1. **Suspension of a groupoid.** Recall that any automorphism α of a groupoid $\mathcal{G}$ induces a homeomorphism of its base space X that we shall denote by α_X.

Definition 5.1. *Let $\mathcal{G}$ be a groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x\in X}$. A groupoid automorphism $\alpha : \mathcal{G} \to \mathcal{G}$ is said to preserve the Haar system λ if there exists a continuous function $\rho_\alpha : \mathcal{G} \to \mathbb{R}^+$ such that for any x in X the measures $\alpha_*\lambda^x$ and $\lambda^{\alpha(x)}$ on $\mathcal{G}^{\alpha(x)}$ are in the same class and ρ_α restricted to $\mathcal{G}^{\alpha(x)}$ is $\frac{d\alpha_*\lambda^x}{d\lambda^{\alpha(x)}}$. The map ρ_α is called the density of α.*

Remark 5.2. *Let $\mathcal{G}$ be a groupoid with base space X and Haar system $\lambda = (\lambda^x)_{x\in X}$ and let $\alpha : \mathcal{G} \to \mathcal{G}$ be an automorphism of groupoid preserving the Haar system λ.*

(1) *Since $L_\gamma \circ \alpha = \alpha \circ L_{\alpha^{-1}(\gamma)}$ for any γ in $\mathcal{G}$, we get that*

$$L_{\gamma,*}\alpha_*\lambda^{\alpha^{-1}(s(\gamma))} = \alpha_* L_{\alpha^{-1}(\gamma),*}\lambda^{s(\alpha^{-1}(\gamma))} = \alpha_*\lambda^{r(\alpha^{-1}(\gamma))}.$$

Since $L_{\gamma,*}\alpha_*\lambda^{\alpha^{-1}(s(\gamma))}$ is a measure on $\mathcal{G}^{r(\gamma)}$ absolutly continuous with respect to $L_{\gamma,*}\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ with density $\rho_\alpha \circ L_{\gamma^{-1}}$ we see that $\rho_\alpha \circ L_{\gamma^{-1}}$ and ρ_α coincide on $\mathcal{G}^{r(\gamma)}$. In particular ρ_α is constant on $\mathcal{G}_x$ for any x in X.

(2) *The automorphism of groupoid $\alpha^{-1} : \mathcal{G} \to \mathcal{G}$ also preserves the Haar system λ and $\rho_{\alpha^{-1}} = 1/\rho_\alpha \circ \alpha$.*

Definition 5.3. *Let $\mathcal{G}$ be a groupoid with base space X, range and source map r and s and let $\alpha : \mathcal{G} \to \mathcal{G}$ be a groupoid automorphism. Using the notations of section* **??** *the suspension of the groupoid $\mathcal{G}$ respectively to α is the groupoid $\mathcal{G}_\alpha \overset{def}{=\!=} (\mathcal{G}\times\mathbb{R})/\mathcal{A}_\alpha$ with base space $X_\alpha \overset{def}{=\!=} (X\times\mathbb{R})/\mathcal{A}_{\alpha_X}$. For any γ in $\mathcal{G}$ and t in $\mathbb{R}$, let us denote by $[\gamma,t]$ the class of (γ,t) in $\mathcal{G}_\alpha$.*

- *The range map r_α and the source map s_α are defined in the following way:*
  - *$r_\alpha([\gamma,t]) = [r(\gamma),t]$ for every γ in $\mathcal{G}$ and t in $\mathbb{R}$;*
  - *$s_\alpha([\gamma,t]) = [s(\gamma),t]$ for every γ in $\mathcal{G}$ and t in $\mathbb{R}$;*
- *Let γ and γ' be elements of $\mathcal{G}$ such that $s(\gamma) = r(\gamma')$ and let t be in $\mathbb{R}$, then $[\gamma,t]\circ[\gamma',t] = [\gamma\circ\gamma',t]$;*

- $[\gamma, t]^{-1} = [\gamma^{-1}, t]$.

There is an action of $\mathbb{R}$ *on* $\mathcal{G}_\alpha$ *by automorphisms given for* s *in* $\mathbb{R}$ *and* $[\gamma, t]$ *in* $\mathcal{G}_\alpha$ *by* $s \cdot [\gamma, t] = [\gamma, t+s]$.

Lemma 5.4. *Let* $\mathcal{G}$ *be a groupoid with base space* X *equipped with a Haar system* $\lambda = (\lambda^x)_{x \in X}$ *and let* $\alpha : \mathcal{G} \to \mathcal{G}$ *be an automorphism preserving the Haar System* λ. *Let us assume that* $\rho_\alpha = \rho_\alpha \circ \alpha$. *Then* $\mathcal{G}_\alpha$ *admits a Haar system* $\lambda_\alpha = \left(\lambda_\alpha^{[x,t]}\right)_{[x,t] \in X_\alpha}$ *given for any* $[x, t]$ *in* X_α *and any continuous fonction* f *in* $C_c\left(\mathcal{G}_\alpha^{[x,t]}\right)$ *by*

$$\lambda_\alpha^{[x,t]}(f) = \int_{\mathcal{G}^x} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^x(\gamma).$$

Proof.

- Let us prove first that the definition of $\lambda_\alpha^{[x,t]}(f)$ for $[x, t]$ in X_α and f in $C_c\left(\mathcal{G}_\alpha^{[x,t]}\right)$ makes sense.

$$\begin{aligned}
\int_{\mathcal{G}^x} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^x(\gamma) &= \int_{\mathcal{G}^x} \rho_\alpha(\alpha(\gamma))^{-t} f([\alpha^{-1}(\alpha(\gamma)), t]) d\lambda^x(\gamma) \\
&= \int_{\mathcal{G}^{\alpha(x)}} \rho_\alpha(\gamma)^{-t+1} f([\alpha^{-1}(\gamma), t]) d\lambda^{\alpha(x)}(\gamma) \\
&= \int_{\mathcal{G}^{\alpha(x)}} \rho_\alpha(\gamma)^{-t+1} f([\gamma, t-1]) d\lambda^{\alpha(x)}(\gamma).
\end{aligned}$$

- It is clear that the continuity condition is fullfilled. Let us show then that $(\lambda^{[x,t]})_{[x,t] \in X_\alpha}$ is a Haar system. Let γ' be an element of $\mathcal{G}$, let t be a real number and let f be a function in $C_c\left(\mathcal{G}_\alpha^{[r(\gamma'),t]}\right)$. Then

$$\begin{aligned}
\lambda_\alpha^{[r(\gamma'),t]}(f) &= \int_{\mathcal{G}^{r(\gamma')}} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^{r(\gamma')}(\gamma) \\
&= \int_{\mathcal{G}^{s(\gamma')}} \rho_\alpha(\gamma' \cdot \gamma)^{-t} f([\gamma' \cdot \gamma, t]) d\lambda^{s(\gamma')}(\gamma) \\
&= \int_{\mathcal{G}^{s(\gamma')}} \rho_\alpha(\gamma)^{-t} f([\gamma' \cdot \gamma, t]) d\lambda^{s(\gamma')}(\gamma) \\
&= \lambda_\alpha^{[s(\gamma'),t]}(f \circ L_{[\gamma',t]}),
\end{aligned}$$

where the third equality holds in view of remark **??**.

□

5.2. **C^*-algebra of a suspension groupoid.** Let us recall first the construction of the reduced C^*-algebra $C_r^*(\mathcal{G}, \lambda)$ associated to a groupoid $\mathcal{G}$ with base X and Haar system $\lambda = (\lambda^x)_{x \in X}$. Let $\mathcal{L}^2(\mathcal{G}, \lambda)$ be the $C_0(X)$-Hilbert completion of $C_c(\mathcal{G})$ equipped with the $C_0(X)$-valued scalar product

$$\langle \phi, \phi' \rangle(x) = \int_{\mathcal{G}^x} \bar{\phi}(\gamma^{-1}) \phi'(\gamma^{-1}) d\lambda^x(\gamma)$$

for ϕ and ϕ' in $C_c(\mathcal{G})$ and x in X, i.e the completion of $C_c(\mathcal{G})$ with respect to the norm $\|\phi\| = \sup_{x \in X} \langle \phi, \phi \rangle^{1/2}$. The $C_0(X)$-module structure on $C_c(\mathcal{G})$ extends to $\mathcal{L}^2(\mathcal{G}, \lambda)$ and $\langle \bullet, \bullet \rangle$ extends to a $C_0(X)$-valued scalar product on $\mathcal{L}^2(\mathcal{G}, \lambda)$. Recall that an operator $T : \mathcal{L}^2(\mathcal{G},\lambda) \to \mathcal{L}^2(\mathcal{G},\lambda)$ is called adjointable if there exists an operator $T^* : \mathcal{L}^2(\mathcal{G},\lambda) \to \mathcal{L}^2(\mathcal{G},\lambda)$, called the adjoint of T such that

$$\langle T^*\phi, \phi'\rangle = \langle \phi, T\phi'\rangle$$

for all ϕ and ϕ' in $\mathcal{L}^2(\mathcal{G},\lambda)$. Notice that the adjoint, when it exists is unique and that operator that admits an adjoint are automatically $C_0(X)$-linear and continuous. The set of adjointable operators on $\mathcal{L}^2(\mathcal{G},\lambda)$ is then a C^*-algebra with respect to the operator norm. Then any f in $C_c(\mathcal{G})$ acts as an adjointable operator on $\mathcal{L}^2(\mathcal{G},\lambda)$ by convolution

$$f \cdot \phi(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} f(\gamma')\phi(\gamma'^{-1}\gamma)d\lambda^{r(\gamma)}(\gamma')$$

where ϕ is in $C_c(\mathcal{G})$, the adjoint of this operator being given by the action of $f^* : \gamma \mapsto \bar{f}(\gamma^{-1})$. The convolution product provides a structure of involutive algebra on $C_c(\mathcal{G})$ and using the action defined above, this algebra can be viewed as a subalgebra of the C^*-algebra of adjointable operators of $\mathcal{L}^2(\mathcal{G},\lambda)$. The reduced C^*-algebra $C_r^*(\mathcal{G},\lambda)$ is then the closure of $C_c(\mathcal{G})$ in the C^*-algebra of adjointable operators of $\mathcal{L}^2(\mathcal{G},\lambda)$. Namely, if we define for x in X the measure on $\mathcal{G}_x$ by $\lambda_x(\phi) = \int_{\mathcal{G}^x} \phi(\gamma^{-1})d\lambda^x(\gamma)$ for any ϕ in $C_c(\mathcal{G}_x)$, then $\mathcal{L}^2(\mathcal{G},\lambda)$ is a continuous field of Hilbert spaces over X with fiber $\mathcal{L}^2(\mathcal{G}_x,\lambda_x)$ at x in X. The corresponding $C_0(X)$-structure on $C_r^*(\mathcal{G},\lambda)$ is then given for h in $C_0(X)$ by the multiplication by $h \circ s$.

Example 5.5. *Let H be a locally compact group acting on a locally compact space X, and consider the semi-direct product groupoid $X \rtimes H$ equipped with a Haar system arising from the Haar measure on H. Then $C_r^*(X \times H, \lambda^\mu)$ is the usual reduced crossed product $C_0(X) \rtimes_r H$.*

Let us denote for any x in X by ν_x the representation of $C_r^*(\mathcal{G},\lambda)$ on the fiber $\mathcal{L}^2(\mathcal{G}_x,\lambda_x)$. Then for any f in $C_r^*(\mathcal{G},\lambda)$, we get that $\|f\|_{C_r^*(\mathcal{G},\lambda)} = \sup_{x\in X} \|\nu_x(f)\|$.

Lemma 5.6. *Let $\mathcal{G}$ be a locally compact groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x\in X}$ and let $\alpha : \mathcal{G} \to \mathcal{G}$ be an automorphism preserving the Haar System λ. Let us define the continuous map $\rho'_\alpha : \mathcal{G} \to \mathbb{R}; \gamma \mapsto \rho_\alpha(\gamma^{-1})$. Then there exists a unique automorphism $\tilde{\alpha}$ of the C^*-algebra $C_r^*(\mathcal{G},\lambda)$ such that for every f in $C_c(\mathcal{G})$ we have $\tilde{\alpha}(f) = (\rho'_\alpha\rho_\alpha)^{1/2} f \circ \alpha^{-1}$.*

Proof. The map $C_c(\mathcal{G}) \to C_c(\mathcal{G}); \phi \mapsto \rho_\alpha'^{1/2}\phi\circ\alpha^{-1}$ extends uniquely to a continuous linear and invertible map $W : \mathcal{L}^2(\mathcal{G},\lambda) \to \mathcal{L}^2(\mathcal{G},\lambda)$ such that

$$\langle W \cdot \phi, W \cdot \phi\rangle(x) = \langle \phi, \phi\rangle(\alpha^{-1}(x)),$$

for all x in X. Its inverse W^{-1} is defined by $W^{-1}(\phi) = ({\rho'}_\alpha \circ \alpha)^{-1/2}\phi \circ \alpha$ for all ϕ in $C_c(\mathcal{G})$. Let us define

$$\tilde{\alpha} : C_r^*(\mathcal{G},\lambda) \to C_r^*(\mathcal{G},\lambda);\, x \mapsto W \cdot x \cdot W^{-1}.$$

Then $W \cdot f \cdot W^{-1} = (\rho'_\alpha\rho_\alpha)^{1/2} f \circ \alpha^{-1}$ for all f in $C_c(\mathcal{G})$. □

Recall that if A is a C^*-algebra and if β is an automorphism of A then the mapping torus of A is the C^*-algebra

$$A_\beta = \{f \in C([0,1], A) \text{ such that } \beta(f(1)) = f(0)\}.$$

Namely, the mapping torus A_β can be viewed as the algebra of continuous function $h : \mathbb{R} \to A$ such that $h(t) = \beta(h(t+1))$ for all t in $\mathbb{R}$. In this picture, there is an action of $\mathbb{R}$ on A_β by translations defined for t in $\mathbb{R}$ and f in A_β by

$$t \cdot f(s) = f(s-t)$$

for any s in $\mathbb{R}$. Translations then define a strongly continuous action by automorphisms $\widehat{\beta}$ of $\mathbb{R}$ on A_β. By the mapping torus isomorphism, we have a natural Morita equivalence between $A \rtimes_\beta \mathbb{Z}$ and $A \rtimes_{\widehat{\beta}} \mathbb{R}$.

Let α be an automorphism of a groupoid $\mathcal{G}$ preserving a Haar system λ and with density ρ_α. For a function f in $C_c(\mathcal{G}_\alpha)$, we define $\hat{f}$ in $C_c([0,1] \times \mathcal{G}) \subset C([0,1], C_r^*(\mathcal{G},\lambda))$ by $\hat{f}(t,\gamma) = \rho_\alpha^{-t/2}(\gamma){\rho'}_\alpha^{-t/2}(\gamma) f([\gamma,t])$. We can check easily that $\hat{f}$ belongs to the mapping torus $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$.

Proposition 5.7. *Let $\mathcal{G}$ be a locally compact groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x\in X}$ and let $\alpha : \mathcal{G} \to \mathcal{G}$ be an automorphism preserving the Haar System λ such that $\rho_\alpha \circ \alpha = \rho_\alpha$. Then there is an unique automorphism of C^*-algebras*

$$\Lambda_\alpha : C_r^*(\mathcal{G}_\alpha, \lambda_\alpha) \to C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$$

such that $\Lambda_\alpha(f) = \hat{f}$ for any f in $C_c(\mathcal{G}_\alpha)$.

Proof. Let f be a function of $C_c(\mathcal{G}_\alpha)$. Then

$$\begin{aligned} \|\hat{f}\|_{C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}} &= \sup_{t\in[0,1]} \|\hat{f}(t,\bullet))\|_{C_r^*(\mathcal{G},\lambda)} \\ &= \sup_{t\in[0,1],\, x\in X} \|\nu_x(\hat{f}(t,\bullet))\| \end{aligned}$$

On the other hand,

$$\|f\|_{C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)} = \sup_{t\in[0,1],\, x\in X} \|\nu_{[x,t]}(f)\|,$$

where $\nu_{[x,t]}$ is the representation of $C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)$ on the fiber $\mathcal{L}^2(\mathcal{G}_{\alpha,[x,t]}, \lambda_{\alpha,[x,t]})$ at $[x,t] \in (X\times\mathbb{R})/\mathcal{A}_{\alpha_X}$. If we define for t in $[0,1]$ the map $\pi_t : \mathcal{G} \to \mathcal{G}_\alpha : \gamma \mapsto [\gamma,t]$, then

$$C_c(\mathcal{G}_{[x,t]}) \to C_c(\mathcal{G}_x) : \phi \mapsto {\rho'}_\alpha^{-t/2} \phi \circ \pi_t$$

extends to an isometry $W_t : \mathcal{L}^2(\mathcal{G}_{\alpha,[x,t]}, \lambda_{\alpha,[x,t]}) \to \mathcal{L}^2(\mathcal{G}_x,\lambda_x)$ and W_t conjugate $\nu_{[x,t]}(f)$ and $\nu_x(\hat{f}(t,\bullet))$. Thus $\|\hat{f}\|_{C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}} = \|f\|_{C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)}$ and

$$C_c(\mathcal{G}_\alpha) \to C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}};\, f \mapsto \hat{f}$$

extends to a monomorphism $\Lambda_\alpha : C_r^*(\mathcal{G}_\alpha,\lambda_\alpha) \to C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$. The set

$$\mathcal{A}_\alpha = \{h \in C_c([0,1]\times\mathcal{G}) \text{ such that } h(1,\alpha(\gamma)) = {\rho'}_\alpha^{t/2} \rho_\alpha^{t/2} h(0,\gamma) \text{ for all } \gamma \in \mathcal{G}\}$$

is dense in $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$. Let us define for an element h of $\mathcal{A}_\alpha$ the map $\tilde{h} : \mathcal{G}\times\mathbb{R} \to \mathbb{C}$ as the unique map such that

- $\tilde{h}(\gamma,t) = {\rho'}_\alpha^{-t/2} \rho_\alpha^{-t/2} h(t,\gamma)$ for all γ in $\mathcal{G}$ and t in $[0,1]$;
- $h(\alpha(\gamma),t) = h(\gamma,t+1)$ for all γ in $\mathcal{G}$ and t in $\mathbb{R}$.

Then $\tilde{h}$ defines a continuous map of $C_c(\mathcal{G}_\alpha)$ whose image under Λ_α is h. Hence Λ_α has dense range in $C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}$ and thus is surjective. $\square$

Remark 5.8. *With the notations of above proposition, let us define for a real s the automorphism of groupoid $\theta_s : \mathcal{G}_\alpha \to \mathcal{G}_\alpha$; $[\gamma, t] \mapsto [\gamma, s + t]$. Then θ_s is preserving the Haar system $\lambda_\alpha = (\lambda^{[x,t]})_{[x,t]\in X_\alpha}$ with density*

$$\mathcal{G}_\alpha \to \mathbb{R};\ [\gamma, t] \mapsto \rho_\alpha(\gamma)^s.$$

We obtain from lemma **??** *an automorphism $\tilde{\theta}_s$ of $C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)$ which gives rise to a strongly continuous action of $\mathbb{R}$ on $C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)$ by automorphism. The isomorphism*

$$\Lambda_\alpha : C_r^*(\mathcal{G}_\alpha, \lambda_\alpha) \to C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$$

of proposition **??** *is then $\mathbb{R}$-equivariant, where the action of $\mathbb{R}$ on $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$ is the action $\widehat{\tilde{\alpha}}$ associated to a mapping torus.*

6. The dynamic of the uncolored Penrose tiling under translations

As we have seen before, the closure $X_{\mathcal{P}}^{\mathcal{N}}$ of $\mathcal{N} \cdot \mathcal{P}$ for the tiling topology is the suspension $(\Omega \times \mathbb{R})/\mathcal{A}_o$ of the odometer homeomorphism $o : \Omega \to \Omega$; $x \mapsto x + 1$, where Ω is the dyadic completion of the integers. The $\mathbb{R}$-algebra $C((\Omega \times \mathbb{R})/\mathcal{A}_o)$ is then the mapping torus algebra of $C(\Omega)$ with respect to automorphism induced by o. In consequence, the crossed product algebras $C(X_{\mathcal{P}}^{\mathcal{N}}) \rtimes \mathbb{R}$ and $C(\Omega) \rtimes \mathbb{Z}$ are Morita equivalent. The purpose of this section is to recall the explicit description of the isomorphism $C(\Omega) \rtimes \mathbb{Z} \xrightarrow{\cong} C(X_{\mathcal{P}}^{\mathcal{N}}) \rtimes \mathbb{R}$ arising from this Morita equivalence.

For this, let us define on $C_c(\Omega \times \mathbb{R})$ the $C(\Omega) \rtimes \mathbb{Z}$-valued inner product

$$\langle \xi, \xi' \rangle(\omega, k) = \int_{\mathbb{R}} \bar{\xi}(\omega, s)\xi'(\omega - k, s + k)ds$$

for ξ and ξ' in $C_c(\Omega \times \mathbb{R})$ and (ω, k) in $(\Omega \times \mathbb{R})$. This inner product is positive and gives rise to a right $C(\Omega) \rtimes \mathbb{Z}$-Hilbert module $\mathcal{E}$, the action of $C(\Omega) \rtimes \mathbb{Z}$ being given for h in $C_c(\Omega \times \mathbb{Z})$ and ξ in $C_c(\Omega \times \mathbb{R})$ by

$$\xi \cdot h(\omega, t) = \sum_{n\in\mathbb{Z}} \xi(n + \omega, t - n)h(n + \omega, n)$$

for (ω, k) in $\Omega \times \mathbb{R}$. The right $C(\Omega) \rtimes \mathbb{Z}$-Hilbert module $\mathcal{E}$ is also equipped with a left action of $C\left((\Omega \times \mathbb{R})/\mathcal{A}_o\right) \rtimes \mathbb{R}$ given for f in $C_c\left((\Omega \times \mathbb{R})/\mathcal{A}_o \times \mathbb{R}\right)$ and ξ in $C_c(\Omega \times \mathbb{R})$ by

$$f \cdot \xi(\omega, t) = \int_{\mathbb{R}} f([\omega, t], s)\xi(\omega, t - s)ds$$

for (ω, k) in $\Omega\times\mathbb{R}$. We get in this way a $C\left((\Omega \times \mathbb{R})/\mathcal{A}_o\right)\rtimes\mathbb{R}-C(\Omega)\rtimes\mathbb{Z}$ imprimitivity bimodule which implements the Morita equivalence we are looking for. Actually, there is an isomorphism of right $C(\Omega) \rtimes \mathbb{Z}$-Hilbert module

$$\Psi : \mathcal{E} \to L^2([0, 1]) \otimes C(\Omega) \rtimes \mathbb{Z}$$

defined in a unique way by $\Psi(g) = g \otimes u$ for g in $C_c(\mathbb{R})$ supported in $(0, 1)$, where u is the unitary of $C(\Omega) \rtimes \mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$. Using the right $C\left((\Omega \times \mathbb{R})/\mathcal{A}_o\right) \rtimes \mathbb{R}$-module structure of the $C\left((\Omega \times \mathbb{R})/\mathcal{A}_o\right) \rtimes \mathbb{R} - C(\Omega) \rtimes \mathbb{Z}$ imprimitivity bimodule $\mathcal{E}$ and the isomorphism Ψ, we get an isomorphism

$$C\left((\Omega \times \mathbb{R})/\mathcal{A}_o\right) \rtimes \mathbb{R} \xrightarrow{\cong} \mathcal{K}(L^2([0, 1])) \otimes C(\Omega) \rtimes \mathbb{Z}. \tag{6.1}$$

This isomorphism can be described as follows. Let us define for f and g in $L^2([0, 1])$ the rank one operator

$$\Theta_{f,g} : L^2([0, 1]) \to L^2([0, 1]);\ h \mapsto f\langle g, h\rangle.$$

We define for ξ and ξ' in $C_c(\Omega\times\mathbb{R})$ the continuous function of $C_c\left((\Omega\times\mathbb{R})/\mathcal{A}_o\times\mathbb{R}\right)$

$$\Theta^{\Omega}_{\xi,\xi'}([\omega,s],t)=\sum_{k\in\mathbb{Z}}\xi(\omega+k,s-k)\bar{\xi}'(\omega+k,s-t-k)$$

for all ω in Ω and s and t in $\mathbb{R}$. It is straightforward to check that $\Theta^{\Omega}_{\xi,\xi'}$ is well defined and that

$$\Theta^{\Omega}_{\xi,\xi'}\cdot\eta=\xi\langle\xi',\eta\rangle$$

for all η in $C_c(\Omega\times\mathbb{R})$. If we set for f and g in $C_c(\mathbb{R})$ with support in $(0,1)$ and for ϕ in $C(\Omega)$, $\xi=1\otimes f$, $\xi'=\phi\otimes g$ and $\xi'':\Omega\times\mathbb{R}\to\mathbb{R};(\omega,t)\mapsto g(t+1)$, then the image of $\Theta^{\Omega}_{\xi,\xi'}$ under the isomorphism of equation (**??**) is $\Theta_{f,g}\otimes\phi\in\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$ and moreover,

$$\Theta^{\Omega}_{\xi,\xi'}([\omega,s],t)=\sum_{k\in\mathbb{Z}}f(s-k)\bar{\phi}(\omega+k)\bar{g}(s-t-k). \tag{6.2}$$

The image of $\Theta^{\Omega}_{\xi,\xi''}$ under the isomorphism of equation (**??**) is $\Theta_{f,g}\otimes u\in\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$ and moreover,

$$\Theta^{\Omega}_{\xi,\xi''}([\omega,s],t)=\sum_{k\in\mathbb{Z}}f(s-k)\bar{g}(s+1-t-k). \tag{6.3}$$

Let us define the automorphism α of the groupoid $(\Omega\times\mathbb{R})/\mathcal{A}_o\rtimes\mathbb{R}$ in the following way

- $\alpha([\omega,s],t)=([\omega/2,s/2],t/2)$ if ω is even;
- $\alpha([\omega,s],t)=([(\omega+1)/2,(s+1)/2],t/2)$ if ω is odd.

Notice that $\alpha^{-1}([\omega,s],t)=([2\omega,2s],2t)$ for all ω in Ω and s and t in $\mathbb{R}$. Then α preserves the Haar system of $(\Omega\times\mathbb{R})/\mathcal{A}_o\rtimes\mathbb{R}$ arising from the Haar mesure on $\mathbb{R}$ and has constant density $\rho_\alpha=2$. Hence according to lemma **??**, the automorphism of groupoid α induces an automorphism $\tilde{\alpha}$ of C^*-algebra $C\left((\Omega\times\mathbb{R})/\mathcal{A}_o\right)\rtimes\mathbb{R}$ such that $\tilde{\alpha}(h)=2h\circ\alpha^{-1}$ for all h in $C\left((\Omega\times\mathbb{R})/\mathcal{A}_o\times\mathbb{R}\right)$. We are now in position to describe how $\tilde{\alpha}$ is transported under the isomorphism of equation (**??**) to an automorphism Υ of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$. With ξ, ξ' and ξ'' as defined above,

$$\begin{aligned}\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi'})([\omega,s],t)&=2\Theta^{\Omega}_{\xi,\xi'}([2\omega,2s],2t)\\&=2\sum_{k\in\mathbb{Z}}f(2s-k)\bar{\phi}(2\omega+k)\bar{g}(2s-2t-k)\\&=2\sum_{k\in\mathbb{Z}}f(2s-2k)\bar{\phi}(2\omega+2k)\bar{g}(2s-2t-2k)+\\&\quad 2\sum_{k\in\mathbb{Z}}f(2s-2k-1)\bar{\phi}(2\omega+2k-1)\bar{g}(2s-2t-2k-1)\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi''})([\omega,s],t) &= 2\Theta^{\Omega}_{\xi,\xi''}([2\omega,2s],2t)\\
&= 2\sum_{k\in\mathbb{Z}} f(2s-k)\bar{g}(2s+1-2t-k)\\
&= 2\sum_{k\in\mathbb{Z}} f(2s-2k)\bar{g}(2s+1-2t-2k)+\\
&\qquad 2\sum_{k\in\mathbb{Z}} f(2s-2k-1)\bar{g}(2s-2t-2k).
\end{aligned} \tag{6.5}$$

To complete the description of the automorphism Υ of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$ corresponding to $\tilde{\alpha}$, we need to introduce some further notations. We define the partial isometries U_0, U_1 and V of $L^2([0,1])$ by

- $U_0 f(t) = \sqrt{2} f(2t)$ if $t\in[0,1/2]$ and $U_0 f(t) = 0$ otherwise;
- $U_1 f(t) = \sqrt{2} f(2t-1)$ if $t\in[1/2,1]$ and $U_1 f(t) = 0$ otherwise;
- $V f(t) = f(t+1/2)$ if $t\in[0,1/2]$ and $V f(t) = 0$ otherwise,

for f in $C([0,1])$. Let use define also the endomorphisms W_0 and W_1 of the C^*-algebra $C(\Omega)$ by $W_0\phi(\omega)=\phi(2\omega)$ and $W_1\phi(\omega)=\phi(2\omega+1)$, for ϕ in $C(\Omega)$ and ω in Ω. Using this notations, equations (**??**) and (**??**) can be rewriten as

$$\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi'})([\omega,s],t) = \sum_{k\in\mathbb{Z}} U_0 f(s-k) W_0\bar{\phi}(\omega+k) U_0\bar{g}(s-t-k) + \sum_{k\in\mathbb{Z}} U_1 f(s-k) W_1\bar{\phi}(\omega+k) U_1\bar{g}(s-t-k)$$

and

$$\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi''})([\omega,s],t) = \sum_{k\in\mathbb{Z}} U_0 f(s-k) U_1\bar{g}(s-t-k+1) + \sum_{k\in\mathbb{Z}} U_1 f(s-k) U_0\bar{g}(s-t-k).$$

Thus, in view of equations (**??**) and (**??**), we get that

$$\Upsilon(\Theta_{f,g}\otimes\phi) = \Theta_{U_0 f, U_0 g}\otimes W_0\phi + \Theta_{U_1 f, U_1 g}\otimes W_1\phi$$

and

$$\Upsilon(\Theta_{f,g}\otimes u) = \Theta_{U_0 f, U_1 g}\otimes u + \Theta_{U_1 f, U_0 g}\otimes 1.$$

From this we deduce

$$\Upsilon(k\otimes\phi) = U_0\cdot k\cdot U_0^*\otimes W_0\phi + U_1\cdot k\cdot U_1^*\otimes W_1\phi$$

and

$$\begin{aligned}
\Upsilon(k\otimes u) &= U_0\cdot k\cdot U_1^*\otimes u + U_1\cdot k\cdot U_0^*\otimes 1\\
&= U_0\cdot k\cdot U_0^*\cdot V\otimes u + U_1\cdot k\cdot U_1^*\cdot V^*\otimes 1\\
&= (U_0\cdot k\cdot U_0^* + U_1\cdot k\cdot U_1^*)\cdot(V\otimes u + \cdot V^*\otimes 1)
\end{aligned}$$

where the second equality holds since $V^*\cdot U_0 = U_1$ and $V\cdot U_1 = U_0$ and the third holds since $V^*U_1 = VU_0 = 0$. In consequence, if we extends Υ to the multiplier algebra of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$, we finally obtain that the automorphism Υ is the unique homomorphism of C^*-algebra such that

$$\Upsilon(k\otimes\phi) = U_0^*\cdot k\cdot U_0\otimes W_0\phi + U_1^*\cdot k\cdot U_1\otimes W_1\phi$$

and

$$\Upsilon(1\otimes u) = V\otimes u + V^*\otimes 1, \tag{6.6}$$

where k is in $\mathcal{K}(L^2([0,1]))$, ϕ is in $C(\Omega)$ and $1\otimes u$ and $V\otimes u+V^*\otimes 1$ are viewed as multipliers of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$.

The following lemma will be helpful to compute the K-theory of the C^*-algebra of the Penrose hyperbolic tiling. For short, we will denote from now on $\mathcal{K}(L^2([0,1]))$ by $\mathcal{K}$.

Lemma 6.1. *Let A be the unitarisation of $\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$ and let f be a norm one function of $L^2([0,1])$. Then the unitaries*

$$(1-\Theta_{f,f}\otimes 1)+\Theta_{f,f}\otimes u$$

and

$$\Theta_{U_0 f,U_1 f}\otimes u+\Theta_{U_1 f,U_0 f}\otimes 1+1-\Theta_{U_0 f,U_0 f}\otimes 1-\Theta_{U_1 f,U_1 f}\otimes 1 \tag{6.7}$$

of A are homotopic.

Proof. If we set $f_0=f$ and complete to a Hilbertian base $f_0,\dots,f_n,\dots$ of $L^2([0,1])$, then $U_0f_0,\dots,U_0f_n,\dots;U_1f_0,\dots,U_1f_n,\dots$ is a Hilbertian basis of $L^2([0,1])$. In this base the unitary of equation (**??**) can be written down as

$$\left(\begin{array}{ccc|ccc} 0 & & & u & & \\ & 1 & & & 0 & \\ & & \ddots & & & \ddots \\ \hline 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & \ddots & & & \ddots \end{array}\right)$$

which is homotopic to

$$\left(\begin{array}{ccc|ccc} u & & & 0 & & \\ & 1 & & & 0 & \\ & & \ddots & & & \ddots \\ \hline 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & \ddots & & & \ddots \end{array}\right).$$

All unitaries that can be writen down in such way in some hilbertian basis of $L^2([0,1])$ are homotopic and since this is the case for $1-\Theta_{f,f}\otimes 1+\Theta_{f,f}\otimes u$, we get the result. □

7. The C^*-algebra of a Penrose hyperbolic tiling

Let us consider the semi-direct product groupoid $\mathcal{G}=(X_{\mathcal{P}}^{\mathcal{N}}\times Z_w)\rtimes\mathbb{R}$ corresponding to the diagonal action of $\mathbb{R}$ on $X_{\mathcal{P}}^{\mathcal{N}}\times Z_w$, by translations on $X_{\mathcal{P}}^{\mathcal{N}}$ and trivial on Z_ω. Let us denote by $\lambda=(\lambda_{(\mathcal{P}',\omega)})_{(\mathcal{P}',\omega)\in X_{\mathcal{P}}^{\mathcal{N}}\times Z_w}$ the Haar system provided by the left Haar mesure on $\mathbb{R}$. Let us define the groupoid automorphism $\alpha_w:\mathcal{G}\to\mathcal{G};\,(\mathcal{P}',w',t)\mapsto(R\cdot\mathcal{P}',\sigma(w'),2t)$. Then α_w preserves the Haar system λ with constant density $\rho_{\alpha_w}=1/2$ and thus according to lemma **??** the suspension groupoid $\mathcal{G}_{\alpha_w}$ admits a Haar system λ_{α_w}. The semi-direct product groupoid $X_{\mathcal{P}(w)}^G\rtimes\mathbb{R}$, where $\mathbb{R}$ acts on $X_{\mathcal{P}(w)}^G$ by translations, is equipped with an action of $\mathbb{R}$ by automorphisms $\beta_t:X_{\mathcal{P}(w)}^G\rtimes\mathbb{R}\to X_{\mathcal{P}(w)}^G\rtimes\mathbb{R};\,(\mathcal{T},s)\mapsto(2^t\cdot\mathcal{T},2^ts)$ for any t in $\mathbb{R}$. The automorphism β_t preserves the Haar system with constant density 2^{-t} and thus in view of proposition **??** induced a strongly continuous action of $\mathbb{R}$ on the crossed product C^*-algebra $C(X_{\mathcal{P}(w)}^G)\rtimes\mathbb{R}$.

Lemma 7.1. *Let w be an element of $\{1,\ldots,r\}^{\mathbb{Z}}$. Then there is a unique isomorphism of groupoids $\Phi_w : \mathcal{G}_{\alpha_w} \longrightarrow X^G_{\mathcal{P}(w)} \rtimes \mathbb{R}$ such that:*

(1) $\Phi_w([\mathcal{P}+x,w,y,0]) = (\mathcal{P}(w)+x,y)$ *for all x and y in $\mathbb{R}$;*
(2) Φ_w *is equivariant with respect to the actions of $\mathbb{R}$;*
(3) $\Phi_{w,*}\lambda_{\alpha_w}$ *is the Haar system on $X^G_{\mathcal{P}(w)} \rtimes \mathbb{R}$ provided by the Haar measure on $\mathbb{R}$.*

Proof. With notations of the proof of proposition **??**, let us define $\mathcal{T}(w') = \overline{\Phi}([T,w,1])$, were $\mathcal{T}$ is in $X^{\mathcal{N}}_{\mathcal{P}}$ and w' is in $\{1,...,r\}^{\mathbb{Z}}$. Then the map

$$X^{\mathcal{N}}_{\mathcal{P}} \times Z_w \to X^G_{\mathcal{P}(w)};\ (\mathcal{T},w') \mapsto \mathcal{T}(w')$$

is continuous and since $(R\cdot\mathcal{T})(\sigma(w')) = R\cdot\mathcal{T}(w')$, the continuous map

$$\mathcal{G}\times\mathbb{R} \to X^G_{\mathcal{P}(w)} \rtimes \mathbb{R};\ (\mathcal{T},w',x,y) \mapsto (R_{2^y}\mathcal{T}(w'),2^y x)$$

induces a continuous homomorphism of groupoids

$$\Phi_w : \mathcal{G}_{\alpha_w} \to X^G_{\mathcal{P}(w)} \rtimes \mathbb{R}.$$

This map is clearly one-to-one since the equality $R_{2^t}\mathcal{T}(w') = \mathcal{T}'(w'')$ for t in $\mathbb{R}$, $\mathcal{T}$ and $\mathcal{T}'$ in $X^{\mathcal{N}}_{\mathcal{P}}$ and w'' and w' in Z_w holds if and only if t is integer, $w'' = \sigma^t(w')$ and $R_{2^t}\mathcal{T} = \mathcal{T}'$. To prove that Φ_w is onto, let us remark that any element of $X^G_{\mathcal{P}(w)}$ can be written as $R_{2^a}\mathcal{T}(w')$, with a in $\mathbb{R}$, $\mathcal{T}$ in $X^{\mathcal{N}}_{\mathcal{P}}$ and w' in Z_ω. We get then

$$\Phi_w([\mathcal{T},w',2^{-a}t,a]) = (R_{2^a}\mathcal{T}(w'),t)$$

for all t in $\mathbb{R}$.

It is then straightforward to check that condition (3) of the lemma is satisfied. The uniqueness of Φ_w is a consequence on one hand of its equivariance and on the other hand of the density of the $\mathbb{R}$-orbit of $\mathcal{P}$ in $X^{\mathcal{N}}_{\mathcal{P}}$. ◻

As a consequence of lemma **??**, we get

Corollary 7.2. *The map*

$$C_c(X^G_{\mathcal{P}(w)} \rtimes \mathbb{R}) \to C_c(\mathcal{G}_{\alpha_w});\ f \mapsto f\circ\Phi_w$$

induces an $\mathbb{R}$-equivariant isomorphism

$$\widetilde{\Phi}_w : C_0(X^G_{\mathcal{P}(w)}) \rtimes \mathbb{R} \to C^*_r(\mathcal{G}_{\alpha_w},\lambda_{\alpha_w}).$$

Proposition 7.3. *Using the notations of lemmas **??** and **??**, the C^*-algebras $C(X^G_{\mathcal{P}(w)}) \rtimes G$ and $C^*_r(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z}$ are Morita equivalent.*

Proof. Recall that $G = \mathbb{R} \rtimes \mathbb{R}^*_+$, where the group $(\mathbb{R}^*_+,\cdot)$ acts on $(\mathbb{R},+)$ by multiplication. Iterate crossed products leads to an isomorphism

$$C(X^G_{\mathcal{P}(w)}) \rtimes G \cong (C(X^G_{\mathcal{P}(w)}) \rtimes \mathbb{R}) \rtimes \mathbb{R}^*_+.$$

If we identify the groups $(\mathbb{R},+)$ and $(\mathbb{R}^*_+,\cdot)$ using the isomorphism

$$\mathbb{R} \to \mathbb{R}^*_+;\ t \mapsto 2^t,$$

this provides the action under consideration in lemma **??** of $\mathbb{R}$ on $C(X^G_{\mathcal{P}(w)}) \rtimes \mathbb{R}$ and hence, the algebras $C(X^G_{\mathcal{P}(w)}) \rtimes G$ and $C^*(\mathcal{G}_{\alpha_w},\lambda_{\alpha_w}) \rtimes \mathbb{R}$ are isomorphic. In view of lemma **??** and of remark **??**, the C^*-algebra $C(X^G_{\mathcal{P}(w)}) \rtimes G$ is isomorphic to $C^*_r(\mathcal{G},\lambda)_{\tilde{\alpha_w}} \rtimes \mathbb{R}$. But since $C^*_r(\mathcal{G},\lambda)_{\tilde{\alpha_w}}$ is the mapping torus algebra with respect to the automorphism $\tilde{\alpha_w} : C^*_r(\mathcal{G},\lambda) \to C^*_r(\mathcal{G},\lambda)$, the crossed product C^*-algebra

$C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha_w}} \rtimes \mathbb{R}$ is Morita equivalent to $C_r^*(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z}$ and hence we get the result. □

8. The K-theory of the C^*-algebra of a Penrose hyperbolic tiling

Let us consider the semi-direct groupoid $\mathcal{G} = (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rtimes \mathbb{R}$ corresponding to the diagonal action of $\mathbb{R}$ on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w$, by translations on $X_{\mathcal{P}}^{\mathcal{N}}$ and trivial on Z_w. According to proposition **??** we have an isomorphism

$$K_*(C(X^G_{\mathcal{P}(w)}) \rtimes G) \xrightarrow{\cong} K_*(C_r^*(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z})$$

induced by the Morita equivalence. In order to compute this K-theory group, we need to recall some basic facts concerning the K-theory group of a crossed product of a C^*-algebra A by an action of $\mathbb{Z}$ provided by an automorphism θ of A. This K-theory can be computed by using the Pimsner-Voiculescu exact sequence [**?**]

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\theta_*-Id} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \rtimes_\theta \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\theta \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\theta_*-Id} & K_1(A) \end{array},$$

where ι_* is the homomorphism induced in K-theory by the inclusion $\iota : A \hookrightarrow A \rtimes_\theta \mathbb{Z}$ and θ_* is the homomorphism in K-theory induced by θ. The vertical maps are given by the composition

$$K_*(A \rtimes_\theta \mathbb{Z}) \xrightarrow{\cong} K_*(A_\theta \rtimes_{\widehat{\theta}} \mathbb{R}) \xrightarrow{\cong} K_{*+1}(A_\theta) \xrightarrow{ev_*} K_{*+1}(A),$$

where

- A_θ is the mapping torus of A with respect to the action θ endowed, with its associated action $\widehat{\theta}$ of $\mathbb{R}$;
- the first map is induced by the Morita equivalence between $A \rtimes_\theta \mathbb{Z}$ and $A_\theta \rtimes_{\widehat{\theta}} \mathbb{R}$;
- the second map is the Thom-Connes isomorphism;
- the third map is induced in K-theory by the evaluation map
$$ev : A_\theta \to A;\ f \mapsto f(0).$$

For an automorphism Ψ of an abelian group M, let us define $\operatorname{Inv} M$ as the set of invariant elements of M and by $\operatorname{Coinv} M = M/\{x - \Psi(x), x \in M\}$ the set of coinvariant elements. We then get short exact sequences

$$0 \to \operatorname{Coinv} K_0(A) \to K_0(A \rtimes_\theta \mathbb{Z}) \to \operatorname{Inv} K_1(A) \to 0 \tag{8.1}$$

and

$$0 \to \operatorname{Coinv} K_1(A) \to K_1(A \rtimes_\theta \mathbb{Z}) \to \operatorname{Inv} K_0(A) \to 0. \tag{8.2}$$

Moreover the inclusions in these exact sequences are induced by ι_*. The first step in the computation of $K_*(C_r^*(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z})$ is provided by next lemma, which is straightforward to prove.

Lemma 8.1. *Let Z be a Cantor set and let us denote by $C(Z,\mathbb{Z})$ the algebra of continuous and integer valued functions on Z.*

1. *we have an isomorphism $C(Z,\mathbb{Z}) \to K_0(C(Z))$; $\chi_E \mapsto [\chi_E]$.*
2. $K_1(C(Z)) = \{0\}$,

where for a compact-open subset E of Z, then χ_E stands for the characteristic function of E.

Plugging $C_r^*(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z}$ into the short exact sequences (**??**) and (**??**), we get

$$0\to \operatorname{Coinv} K_0(C_r^*(\mathcal{G},\lambda))\to K_0(C_r^*(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z})\to \operatorname{Inv} K_1(C_r^*(\mathcal{G},\lambda))\to 0 \tag{8.3}$$

and

$$0\to \operatorname{Coinv} K_1(C_r^*(\mathcal{G},\lambda))\to K_1(C_r^*(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z})\to \operatorname{Inv} K_0(C_r^*(\mathcal{G},\lambda))\to 0. \tag{8.4}$$

According to equation (**??**), the C^*-algebra $C_r^*(\mathcal{G},\lambda)$ is isomorphic to $C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$. The K-theory of $C_r^*(\mathcal{G},\lambda)$ can be the computed by using the Künneth formula: in view of lemma **??**, $K_0(C(Z_w))\cong C(Z_w,\mathbb{Z})$ is torsion free and $K_1(C(Z_w))=\{0\}$ and by Morita equivalence, we get that

$$K_0(C_r^*(\mathcal{G},\lambda))\cong C(Z_w,\mathbb{Z})\otimes K_0(C(\Omega)\rtimes\mathbb{Z})$$

and

$$K_1(C_r^*(\mathcal{G},\lambda))\cong C(Z_w,\mathbb{Z})\otimes K_1(C(\Omega)\rtimes\mathbb{Z}).$$

This isomorphism, up to the Morita equivalence and to the isomorphism of equation (**??**) are implemented by the external product in K-theory and will be precisely described later on. Once again, $K_*(C(\Omega)\rtimes\mathbb{Z})$ can be computed from the short exact sequences (**??**) and (**??**), and we get, using lemma **??** that

$$K_0(C(\Omega)\rtimes\mathbb{Z})\cong \operatorname{Coinv} C(\Omega,\mathbb{Z}) \tag{8.5}$$

and

$$K_1(C(\Omega)\rtimes\mathbb{Z})\cong \operatorname{Inv} C(\Omega,\mathbb{Z})\cong\mathbb{Z}. \tag{8.6}$$

The isomorphism of equation (**??**) is induced by the composition

$$C(\Omega,Z)\stackrel{\cong}{\to}K_0(C(\Omega))\to K_0(C(\Omega)\rtimes\mathbb{Z}),$$

which factorizes through $\operatorname{Coinv} C(\Omega,Z)$, where the first map is described in lemma **??**, and the second map is induced on K-theory by the inclusion $C(\Omega)\hookrightarrow C(\Omega)\rtimes\mathbb{Z}$. In the first isomorphism of equation (**??**) the class of $[u]$ in $K_1(C(\Omega)\rtimes\mathbb{Z})$ of the unitary u of $C(\Omega)\rtimes\mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$ is mapped to the constant function 1 of $C(\Omega,\mathbb{Z})$.

Lemma 8.2. *Let ν be the Haar measure on Ω. Then*

(1) *$\int f d\nu$ is in $\mathbb{Z}[1/2]$ for all f in $C(\Omega,\mathbb{Z})$;*
(2) *$C(\Omega,\mathbb{Z})\to\mathbb{Z}[1/2]; f\mapsto\int f d\nu$ factorizes through an isomorphism*

$$\operatorname{Coinv} C(\Omega,\mathbb{Z})\stackrel{\cong}{\to}\mathbb{Z}[1/2].$$

Proof. It is enought to check the first point for characteristic function of compact-open subset of Ω. For an integer n and k in $\{0,\dots,2^{n-1}\}$, we set $F_{n,k}=2^n\Omega+k$. Then $(F_{n,k})_{n\in\mathbb{N},\,0\le k\le 2^{n-1}}$ is a basis of compact-open neighborhoods for Ω and thereby, every compact-open subset of Ω is a finite disjoint union of some $F_{n,k}$. Since $\nu(F_{n,k})=2^{-n}$, we get the first point.

The measure μ being invariant by translation, the map

$$C(\Omega,\mathbb{Z})\to\mathbb{Z}[1/2]; f\mapsto\int f d\nu$$

factorizes through a group homomorphism $\operatorname{Coinv} C(\Omega,\mathbb{Z}) \to \mathbb{Z}[1/2]$. This homomorphism admits a cross-section

$$\mathbb{Z}[1/2] \to \operatorname{Coinv} C(\Omega,\mathbb{Z});\ 2^{-n} \mapsto [\chi_{F_{n,0}}]. \tag{8.7}$$

This map is well defined since $F_{n,0} = F_{n+1,0} \coprod (2^n + F_{n+1,0})$ and thus

$$[\chi_{F_{n,0}}] = [\chi_{F_{n+1,0}}] + [\chi_{2^n+F_{n+1,0}}] = 2[\chi_{F_{n+1,0}}]$$

in $\operatorname{Coinv} C(\Omega,\mathbb{Z})$. Since the $(\chi_{F_{n,k}})_{n\in\mathbb{N},\,0\leq k\leq 2^{n-1}}$ generates $C(\Omega,\mathbb{Z})$ as an abelian group, it is enought to check that the cross-section of equation (**??**) is a left inverse on $\chi_{F_{n,k}}$, which is true since $[\chi_{F_{n,k}}] = [\chi_{k+F_{n,0}}] = [\chi_{F_{n,0}}]$ in $\operatorname{Coinv} C(\Omega,\mathbb{Z})$. ☐

Proposition 8.3. *Let $C(Z_w,\mathbb{Z}[1/2]) \cong C(Z_w,\mathbb{Z})\otimes\mathbb{Z}[1/2]$ be the algebra of continuous function on Z_w, valued in $\mathbb{Z}[1/2]$ (equipped with the discrete topology). Then with the notations of the proof of lemma **??**, we have isomorphisms*

(1)

$$\begin{array}{ccc} C(Z_w,\mathbb{Z}[1/2]) & \xrightarrow{\cong} & K_0(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}) \\ \dfrac{\chi_E}{2^n} & \mapsto & [\chi_E\otimes\chi_{F_{n,0}}], \end{array}$$

where E is a compact-open subset of Z_w and χ_E is its characteristic function.

(2)

$$\begin{array}{ccc} C(Z_w,\mathbb{Z}) & \xrightarrow{\cong} & K_1(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}) \\ \chi_E & \mapsto & [\chi_E\otimes u + (1-\chi_E)\otimes 1], \end{array}$$

where u is the unitary of $C(\Omega)\rtimes\mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$.

Proof. As we have already mentionned, $K_*(C(Z_w))$ is torsion free and the Künneth formula provides isomorphisms

$$\begin{array}{ccc} K_0(C(Z_w))\otimes K_0(C(\Omega)\rtimes\mathbb{Z}) & \stackrel{\cong}{\to} & K_0(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}) \\ [p]\otimes[q] & \mapsto & [p\otimes q], \end{array}$$

where p and q are some matrix projectors with coefficients respectively in $C(Z_w)$ and $C(\Omega)\rtimes\mathbb{Z}$, and

$$\begin{array}{ccc} K_0(C(Z_w))\otimes K_1(C(\Omega)\rtimes\mathbb{Z}) & \stackrel{\cong}{\to} & K_1(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}) \\ [p]\otimes[v] & \mapsto & [p\otimes v + (I_k - p)\otimes I_l], \end{array}$$

where p is a projector in $M_l(C(Z_w))$ and v is a unitary in $M_k(C(\Omega)\rtimes\mathbb{Z})$. The proposition is then a consequence of lemmas **??**, **??** and of the discussion related to equations (**??**) and (**??**). ☐

In order to compute the invariants and the coinvariants of

$$K_*(C_r^*(\mathcal{G},\lambda)) \cong K_*(C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}),$$

we will need a carefull description of the action induced in K-theory by the automorphism $\sigma^*\otimes\Upsilon$ of $C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$, where σ^* is the automorphism of $C(Z_\omega)$ induced by the shift σ and where Υ was defined in section **??**.

Lemma 8.4. *If we equip $C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$ with the $\mathbb{Z}$-action provided by $\sigma^*\otimes\Upsilon$ and under the $\mathbb{Z}$-equivariant isomorphism*

$$C^*_r(\mathcal{G},\lambda)\cong C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z},$$

the action induced by α_w on $K_0(C^*_r(\mathcal{G},\lambda)\cong C(Z_w,\mathbb{Z}[1/2])$ and on $K_1(C^*_r(\mathcal{G},\lambda)\cong C(Z_w,\mathbb{Z})$ are given by the automorphisms of abelian groups

$$\begin{array}{rcl}\Psi_0: C(Z_w,\mathbb{Z}[1/2]) & \to & C(Z_w,\mathbb{Z}[1/2])\\ f & \mapsto & 2f\circ\sigma^{-1}\end{array}$$

and

$$\begin{array}{rcl}\Psi_1: C(Z_w,\mathbb{Z}) & \to & C(Z_w,\mathbb{Z})\\ f & \mapsto & f\circ\sigma^{-1}.\end{array}$$

Proof. According to proposition **??** and using the Morita equivalence between $C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}$ and $C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$, in order to describe Ψ_0, we have to compute the image under $(\sigma^*\otimes\Upsilon)_*$ of

$$[\chi_E\otimes\Theta_{f,f}\otimes\chi_{F_{n,0}}]\in K_0(C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})$$

where,

- χ_E is the characteristic function of a compact-open subset E of Z_w;
- $\chi_{F_{n,0}}$ is the characteristic function of $F_{n,0}=2^n\Omega$ for $n\geq 1$;
- $\Theta_{f,f}$ is the rank one projector associated to a norm 1 function f of $L^2([0,1])$.

We have

$$\begin{array}{rcl}\sigma^*\otimes\Upsilon(\chi_E\otimes\Theta_{f,f}\otimes\chi_{F_{n,0}}) & = & \chi_{\sigma(E)}\otimes\Theta_{U_0f,U_0f}\otimes W_0\chi_{F_{n,0}}+\chi_{\sigma(E)}\otimes\Theta_{U_1f,U_1f}\otimes W_1\chi_{F_{n,0}}\\ & = & \chi_{\sigma(E)}\otimes\Theta_{U_0f,U_0f}\otimes\chi_{F_{n-1,0}},\end{array}$$

where the last equality holds since $W_0\chi_{F_{n,0}}=\chi_{F_{n-1,0}}$ and $W_1\chi_{F_{n,0}}=0$. Since Θ_{U_0f,U_0f} is again a rank one projector, then up to the Morita equivalence between $C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}$ and $C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$, the image of $[\chi_E\otimes\chi_{F_{n,0}}]\in K_0(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z})$ under $(\sigma^*\otimes\Upsilon)_*$ is $[\chi_{\sigma(E)}\otimes\chi_{F_{n-1,0}}]\in K_0(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z})$. Using proposition **??**, this completes the description of Ψ_0. For Ψ_1, notice first that up to the isomorphism

$$K_0(C(Z_w))\otimes K_0(\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})\stackrel{\cong}{\to}K_1(C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})$$

provided by the Künneth formula, the action of $(\sigma^*\otimes\Upsilon)_*$ is $\sigma^*_*\otimes\Upsilon_*$ and then the result is a consequence of lemma **??** and of proposition **??**. □

Let us equip $C(Z_w,\mathbb{Z}[1/2])$ and $C(Z_w,\mathbb{Z})$ with the $\mathbb{Z}$-actions respectively provided by Ψ_0 and Ψ_1. Then since $\|\Psi_0(h)\|=2\|h\|$ for any h in $C(Z_w,\mathbb{Z}[1/2])$, we get that $\operatorname{Inv}C(Z_w,\mathbb{Z}[1/2])=\{0\}$ We are now in position to get a complete description of the K-theory of $C(X^G_{\mathcal{P}(w)})\rtimes G$. In view of the short exact sequences of equations (**??**) and (**??**), the two following theorems are then consequences of lemma **??** and of proposition **??**.

Theorem 8.5. *We have a short exact sequence*

$$0\to\operatorname{Coinv}C(Z_w,\mathbb{Z}[1/2])\stackrel{\iota_0}{\to}K_0(C(X^G_{\mathcal{P}(w)})\rtimes G)\to\operatorname{Inv}C(Z_w,\mathbb{Z})\to 0,$$

where up to the Morita equivalence $C(X^G_{\mathcal{P}(w)})\rtimes G\cong C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z}$*, the element* $\iota_0[2^{-n}\chi_E]$ *is the image of* $[\chi_E\otimes\Theta_{f,f}\otimes\chi_{F_{n,0}}]\in K_0(C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})$ *under the homomorphism induced in* K*-theory by the inclusion*

$$C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}\cong C^*_r(\mathcal{G},\lambda)\hookrightarrow C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z},$$

where

- χ_E *is the characteristic function of a compact-open subset* E *of* Z_w*;*
- $\chi_{F_{n,0}}$ *is the characteristic function of* $F_{n,0}=2^n\Omega$*;*
- $\Theta_{f,f}$ *is the rank one projector associated to a norm* 1 *function* f *of* $L^2([0,1])$*.*

Theorem 8.6. *We have an isomorphism*

$$\operatorname{Coinv} C(Z_w,\mathbb{Z})\xrightarrow{\cong}K_1(C(X^G_{\mathcal{P}(w)})\rtimes G)$$

induced on the coinvariants by the composition

$$C(Z_w,\mathbb{Z})\cong K_0(C(Z_w))\xrightarrow{\otimes[u]}K_1(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z})\cong K_1(C^*_r(\mathcal{G},\lambda)\rightarrow K_1(C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z}),$$

where

- $\otimes[u]$ *is the external product in* K*-theory by the class in* $K_1(C(\Omega)\rtimes\mathbb{Z})$ *of the unitary* u *of* $C(\Omega)\rtimes\mathbb{Z}$ *corresponding to the positive generator of* $\mathbb{Z}$*;*
- *the last map in the composition is the homomorphism induced in* K*-theory by the inclusion* $C^*_r(\mathcal{G},\lambda)\hookrightarrow C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z}$*.*

The short exact sequence of theorem **??**, admits an explicit splitting which can be described in the following way: assume first that (Z_w,σ) is minimal. In particular, $\operatorname{Inv} C(Z_w,\mathbb{Z})\cong\mathbb{Z}$ is generated by $1\in C(Z_w,\mathbb{Z})$. Let us considerer the following diagram, whose left square is commutative

$$\begin{tikzcd} K_1(C^*(\mathbb{R})) \arrow[r] \arrow[d] & K_1(C^*(\mathcal{G},\lambda)) \arrow[d] & \\ \mathbb{Z}\cong K_0(\mathbb{C}) \arrow[r] & K_0(C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_w)) \arrow[r, "ev_*"] & K_0(C(\Omega\times Z_w)) \end{tikzcd},$$

where

- the horizontal maps of the left square are induced by the inclusion $\mathbb{C}\hookrightarrow C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_w)$.
- vertical maps are the Thom-Connes isomorphisms.
- The map $ev: C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_w))\longrightarrow C(\Omega\times Z_w)$ is induced by the continuous map $\Omega\rightarrow X^{\mathcal{N}}_{\mathcal{P}}\cong(\Omega\times\mathbb{R})/\mathcal{A}_o$; $x\mapsto[x,0]$;

Up to the Morita equivalence between $C^*(\mathcal{G},\lambda)$ and $C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}$, the right down staircase is the boundary of the Pimsner-Voiculescu six-term exact sequence that computes $K_*(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z})$. From this, we see that $K_1(C^*(\mathcal{G},\lambda)\cong\mathbb{Z}$ is generated by the image of the generator ζ of $K_1(C^*(\mathbb{R}))$ corresponding under the canonical identification $K_1(C^*(\mathbb{R}))\cong K_1(C_0(\mathbb{R}))\cong K_0(\mathbb{C})\cong\mathbb{Z}$ to the class of any rank one projector in some $M_n(\mathbb{C})$. On the other hand, we have a diagram with commutative squares

$$\begin{tikzcd}
K_0(C^*(\mathbb{R})\rtimes\mathbb{R}_+^*) \arrow[r] \arrow[d] & K_0(C(X^G_{\mathcal{P}(w)})\rtimes G) \arrow[r, "\cong"'] \arrow[d] & K_0(C^*(\mathcal{G}_{\alpha_w},\lambda_{\alpha_w})\rtimes\mathbb{R}) \arrow[r, "\cong"'] \arrow[d] & K_0(C^*(\mathcal{G},\lambda)_{\alpha_w}\rtimes\mathbb{R}) \arrow[d] \\
K_1(C^*(\mathbb{R})) \arrow[r] & K_1(C(X^G_{\mathcal{P}(w)})\rtimes\mathbb{R}) \arrow[r, "\cong"'] & K_1(C^*(\mathcal{G}_{\alpha_w},\lambda_{\alpha_w})) \arrow[r, "\cong"'] & K_1(C^*(\mathcal{G},\lambda)_{\alpha_w}) \arrow[d, "ev_*"] \\
 & & & K_0(C^*(\mathcal{G},\lambda))
\end{tikzcd}$$

where,

- the horizontal maps of the left square are induced by the inclusion $C^*(\mathbb{R})\hookrightarrow C(X^G_{\mathcal{P}(w)})\rtimes\mathbb{R}$;
- the horizontal maps of the middle square are induced by the isomorphism of lemma **??**
- the horizontal maps of the right square are induced by the isomorphism of proposition **??**
- the first row of vertical maps are Thom-Connes isomorphisms.

It is then straightforward to check that the down staircase of the diagram is indeed induced by the inclusion $C^*(\mathbb{R})\hookrightarrow C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_w)\rtimes\mathbb{R}=C^*(\mathcal{G},\lambda)$. Notice that $K_0(C^*(\mathbb{R})\rtimes\mathbb{R}_+^*)\cong\mathbb{Z}$ (by Thom-Connes isomorphism). Moreover, under the inclusion $C_0(\mathbb{R}_+^*)\rtimes\mathbb{R}_+^*\hookrightarrow C_0(\mathbb{R})\rtimes\mathbb{R}_+^*\cong C^*(\mathbb{R})\rtimes\mathbb{R}_+^*$, any rang one projector e of $\mathcal{K}(L^2(\mathbb{R}_+^*))\cong C_0(\mathbb{R}_+^*)\rtimes\mathbb{R}_+^*$ provides a generator for $K_0(C^*(\mathbb{R})\rtimes\mathbb{R}_+^*)$ whose image under the left vertical map is the generator ζ for $K_1(C^*(\mathbb{R}))\cong K_0(\mathbb{C})\cong\mathbb{Z}$. Using the description of the boundary map of Pimsner-Voiculescu six-term exact sequence, we see that e, viewed as an element of $C(X^G_{\mathcal{P}(w)})\rtimes G$ whose class in K-theory provides a lift for $1\in C(Z_w,\mathbb{Z})$ in the short exact sequence of theorem **??**.

In general, $\operatorname{Inv} C(Z_w,\mathbb{Z})$ is generated by characteristic functions of $\mathbb{Z}$-invariant compact-open subsets of Z_w. According to proposition **??**, any $\mathbb{Z}$-invariant compact-open subset E of Z_w provides a $\mathbb{R}$-invariant compact subset $\widetilde{E}$ of $X^G_{\mathcal{P}}$. Hence, with above notations, if $\chi_{\widetilde{E}}$ is the characteristic function for $\widetilde{E}$, then $\chi_{\widetilde{E}}e$ can be viewed as an element of $C(X^G_{\mathcal{P}(w)})\rtimes G$. Let $\upsilon:\operatorname{Inv} C(Z_w,\mathbb{Z})\to K_0(C(X^G_{\mathcal{P}(w)})\rtimes G)$ be the group homomorphism uniquely defined by $\upsilon(\chi_E)=\chi_{\widetilde{E}}e$ for E a $\mathbb{Z}$-invariant compact-open subset of Z_w. Then υ is a section for the short exact sequence of theorem **??**.

9. Topological invariants for the continuous hull

It is known that for Euclidian tilings, topological invariants of the continuous hull are closely related to the K-theory of the C^*-algebra associated to the tiling. The K-theory of the latter turn out to be isomorphic to the K-theory of the hull which is using the Chern character rationally isomorphic to the integral Cěch cohomology. Moreover, in dimension less or equal to 3, the Chern character can be defined valued in integral cohomology and we eventually obtain an isomorphism between the integral Cěch cohomology of the hull and the K-theory of the C^*-algebra associated to the tiling. In consequence of this fact, a lot of interest has been generated in the computation of topological invariants of the hull.

For Penrose hyperbolic tilings, since the group of affine isometries of the hyperbolic half-plane is isomorphic to a semi-direct product $\mathbb{R} \rtimes \mathbb{R}$, we get using the Thom-Connes isomorphism that

$$K_*(C(X^G_{\mathcal{P}(w)}) \rtimes G) \cong K^*(X^G_{\mathcal{P}(w)}). \tag{9.1}$$

Moreover, since the cohomological dimension of $X^G_{\mathcal{P}(w)}$ is 2, the Chern character can also be defined with values in integral Cěch cohomology and hence we get as in the Euclidian case of low dimension an isomorphism

$$K_*(C(X^G_{\mathcal{P}(w)}) \rtimes G) \cong \check{H}(X^G_{\mathcal{P}(w)}, \mathbb{Z}).$$

These topological invariants can be indeed computed directly using technics very closed to those used in section **??**. Indeed for a C^*-algebra A provided with an automorphism β, there is natural isomorphisms

$$K_0(A_\beta) \cong K_1(A \rtimes_\beta \mathbb{Z}) \quad \text{and} \quad K_1(A_\beta) \cong K_0(A \rtimes_\beta \mathbb{Z}), \tag{9.2}$$

called the mapping torus isomorphisms, where A_β is the mapping torus algebra constructed at the end of section **??**. Recall from proposition **??** that $X^G_{\mathcal{P}(w)}$ can be viewed as a double suspension

$$(((\Omega \times \mathbb{R})/\mathcal{A}_o) \times Z_\omega \times \mathbb{R})\,/\mathcal{A}_f \tag{9.3}$$

with $f : (\Omega \times \mathbb{R})/\mathcal{A}_o \to (\Omega \times \mathbb{R})/\mathcal{A}_o; ([x,t],\omega') \mapsto ([2x,2t],\sigma(\omega'))$. In regard of the mapping torus isomorphism, the double crossed product by $\mathbb{Z}$ corresponds in K-theory to the double suspension structure on $X^G_{\mathcal{P}(w)}$. For people interested in topological invariants, we explain how a straight computation can be carried out.

For a C^*-algebra A provided with an automorphism β, the short exact sequence

$$0 \to C_0((0,1), A) \to A_\beta \xrightarrow{ev} A \to 0, \tag{9.4}$$

where ev is the evaluation at 0 of elements of $A_\beta \subset C([0,1], A)$, gives rise to short exact sequences

$$0 \to \operatorname{Coinv} K_1(A) \to K_0(A_\beta) \to \operatorname{Inv} K_0(A) \to 0$$

and

$$0 \to \operatorname{Coinv} K_0(A) \to K_1(A_\beta) \to \operatorname{Inv} K_1(A) \to 0$$

where invariants and coinvariants are taken with respect to the action induced by β on $K_*(A)$ (see section **??**). In particular, if X is a compact set and $f : X \to X$ is a homeomorphism, and with the notations of section **??** the mapping torus of $C(X)$ with respect to the automorphism induced by f is $C((X \times \mathbb{R})/\mathcal{A}_f)$. We deduce short exact sequences

$$0 \to \operatorname{Coinv} K^1(X) \to K^0((X \times \mathbb{R})/\mathcal{A}_f) \to \operatorname{Inv} K^0(X) \to 0$$

and

$$0 \to \operatorname{Coinv} K^0(X) \to K^1((X \times \mathbb{R})/\mathcal{A}_f) \to \operatorname{Inv} K^1(X) \to 0.$$

Similarly, we have short exact sequences in Cěch cohomology

$$0 \to \operatorname{Coinv} \check{H}^{n-1}(X,\mathbb{Z}) \to \check{H}^n((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{Z}) \to \operatorname{Inv} \check{H}^n(X,\mathbb{Z}) \to 0,$$

derived from the inclusion

$$(0,1) \times X \hookrightarrow (X \times \mathbb{R})/\mathcal{A}_f. \tag{9.5}$$

Since the space $X^G_{\mathcal{P}(w)}$ has a structure of double suspension, we see following the same route as in section **??** that $X^G_{\mathcal{P}(w)}$ only has cohomology in degree 0, 1 and 2 and we get isomorphisms

$$K^0(X^G_{\mathcal{P}(w)}) \cong \operatorname{Inv} C(Z_w,\mathbb{Z}) \oplus \operatorname{Coinv} C(Z_w,\mathbb{Z}[1/2]) \tag{9.6}$$

$$K^1(X^G_{\mathcal{P}(w)}) \cong \operatorname{Coinv} C(Z_w,\mathbb{Z}) \tag{9.7}$$

$$\check{H}^0(X^G_{\mathcal{P}(w)},\mathbb{Z}) \cong \operatorname{Inv} C(Z_w,\mathbb{Z}) \tag{9.8}$$

$$\check{H}^1(X^G_{\mathcal{P}(w)},\mathbb{Z}) \cong \operatorname{Coinv} C(Z_w,\mathbb{Z}) \tag{9.9}$$

$$\check{H}^2(X^G_{\mathcal{P}(w)},\mathbb{Z}) \cong \operatorname{Coinv} C(Z_w,\mathbb{Z}[1/2]). \tag{9.10}$$

Recall that invariants and coinvariants of $C(Z_w,\mathbb{Z})$ are taken with respect to the automorphism

$$C(Z_w,\mathbb{Z}) \to C(Z_w,\mathbb{Z});\ f \mapsto f\circ\sigma^{-1},$$

and that coinvariants of $C(Z_w,\mathbb{Z}[1/2])$ are taken with respect to the automorphism

$$C(Z_w,\mathbb{Z}[1/2]) \to C(Z_w,\mathbb{Z}[1/2]);\ f \mapsto 2f\circ\sigma^{-1}.$$

Let us describe explicitly these isomorphisms. The identification of equation (**??**) yields to a continuous map

$$X^G_{\mathcal{P}(w)} \to (Z_w\times\mathbb{R})/\mathcal{A}_\sigma, \tag{9.11}$$

induced by the equivariant projection $((\Omega\times\mathbb{R})/\mathcal{A}_o)\times Z_\omega\times\mathbb{R} \to Z_w\times\mathbb{R}$. Together with the inclusion $Z_w\times(0,1)\hookrightarrow (Z_w\times\mathbb{R})/\mathcal{A}_\sigma$, this gives rise to a homomorphism $K^1(Z_w\times(0,1)) \to K^1(X^G_{\mathcal{P}(w)})$ inducing under Bott peridodicity the isomorphism of equation (**??**) (recall from lemma **??** that $K^0(Z_w)=K_0(C(Z_w))\cong C(Z_\omega,\mathbb{Z})$). The identification of equation (**??**) is obtained in the same way by using the isomorphism

$$C(Z_\omega,\mathbb{Z}) \cong \check{H}^0(Z_\omega,\mathbb{Z}) \stackrel{\cong}{\to} \check{H}^1(Z_\omega\times(0,1),\mathbb{Z})$$

provided by the cup product by the fundamental class of $\check{H}^1((0,1),\mathbb{Z})$. Recall that $\operatorname{Inv} C(Z_w,\mathbb{Z})$ is generated by characteristic functions of invariant compact-open subsets of Z_ω. If E is such a subset, then $(E\times\mathbb{R})/\mathcal{A}_{\sigma_{/E}}$ is a compact-open subset of $(Z_w\times\mathbb{R})/\mathcal{A}_\sigma$ and is pulled-back under the map of equation (**??**) to a compact-open subset $\widetilde{E}$ of $X^G_{\mathcal{P}(w)}$. The isomorphism of equation (**??**) identifies $\chi_E\in\operatorname{Inv} C(\mathbb{Z}_\omega,\mathbb{Z})$ with the class of $\widetilde{E}$ in $\check{H}^0(X^G_{\mathcal{P}(w)},\mathbb{Z})$ and the isomorphism of equation (**??**) identifies χ_E with the class of $\chi_{\widetilde{E}}$ in $K^0(Z_w)=K_0(C(Z_w))$. Using twice the inclusion of equation (**??**) for the double suspension structure of $X^G_{\mathcal{P}(w)}$, we obtain an inclusion $\Omega\times Z_\omega\times(0,1)^2\hookrightarrow X^G_{\mathcal{P}(w)}$ and hence by Bott periodicity a map $C(\Omega\times Z_\omega,\mathbb{Z})\to K^0(X^G_{\mathcal{P}(w)})$. Then, if E is a compact-open subset of Z_ω and n is an integer, the image of $\chi_{E\times 2^n\Omega}$ under this map is up to identification of equation (**??**) the class of $\chi_E/2^n$ in $\operatorname{Coinv} C(Z_w,\mathbb{Z}[1/2])$. The description of the identification of equation (**??**) is obtained in the same way by using the isomorphism

$$C(\Omega\times Z_\omega,\mathbb{Z}) \cong \check{H}^0(\Omega\times Z_\omega,\mathbb{Z}) \stackrel{\cong}{\to} \check{H}^2(\Omega\times Z_\omega\times(0,1)^2,\mathbb{Z})$$

provided by the cup product by the fundamental class of $\check{H}^2((0,1)^2,\mathbb{Z})$. Moreover, since the Chern character is natural and intertwins Bott periodicity and the cup product by the fundamental class of $\check{H}^1_c((0,1),\mathbb{Z})$, we deduce that up to the identifications of equations (**??**) to (**??**), it is given by the identity maps of $\operatorname{Inv} C(Z_w,\mathbb{Z})\oplus\operatorname{Coinv} C(Z_w,\mathbb{Z}[1/2])$ and of $\operatorname{Coinv} C(Z_w,\mathbb{Z})$. It is easy to guess how the generators of $K^*(X^G_{\mathcal{P}(w)})$ described in equations (**??**) and (**??**) should be identified with those of $K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)$ described in section **??** under Thom-Connes isomorphism of equation (**??**). Recall first that for a unital C^*-algebra A provided with an automorphism β,

- the mapping torus A_β is provided with an action $\widehat{\beta}$ of $\mathbb{R}$ by automorphisms (see section **??**) and moreover $A\rtimes_\beta\mathbb{Z}$ and $A_\beta\rtimes_{\widehat{\beta}}\mathbb{R}$ are Morita equivalent;
- the mapping torus isomorphisms are the composition of the Thom-Connes isomorphisms $K_0(A_\beta)\xrightarrow{\cong}K_1(A\rtimes_\beta\mathbb{Z})$ and $K_1(A_\beta)\xrightarrow{\cong}K_0(A\rtimes_\beta\mathbb{Z})$ with the isomorphism $K_*(A_\beta\rtimes_{\widehat{\beta}}\mathbb{R})\cong K_*(A\rtimes_\beta\mathbb{Z})$ induced with the above Morita-equivalence;

It is then straightforward to check that viewing $C(X^G_{\mathcal{P}(w)})\rtimes G$ as a double crossed product by $\mathbb{Z}$ as we did in section **??**, the Thom-Connes isomorphism

$$K_*(C(X^G_{\mathcal{P}(w)}))\stackrel{\cong}{\to}K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)$$

is obtained by using twice the mapping torus isomorphim (up to stabilisation for the second one). In view of our purpose of idenfying the generators of $K_*(C(X^G_{\mathcal{P}(w)}))$ with those of $K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)$, we will need the following alternative description of the mapping torus isomorphism using the bivariant Kasparov K-theory groups $KK^{\mathbb{Z}}_*(\bullet,\bullet)$ [**?**]. Let A be a C^*-algebra and let β be an automorphism of A. Since the action of $\mathbb{Z}$ on $\mathbb{R}$ by translations is free and proper, we have a Morita equivalence between A_β and $C_0(\mathbb{R},A)\rtimes\mathbb{Z}$, where $C_0(\mathbb{R},A)\cong C_0(\mathbb{R})\otimes A$ is equipped with the diagonal action of $\mathbb{Z}$. Recall that the $\mathbb{Z}$-equivariant unbounded operator $\imath\frac{d}{dt}$ of $L^2(\mathbb{R})$ gives rise to a unbounded K-cycle and hence to an element y in $KK^{\mathbb{Z}}_1(C_0(\mathbb{R}),\mathbb{C})$. Then the mapping torus isomorphism of equation (**??**) is the composition

$$K_*(A_\beta)\xrightarrow{\cong}K_*(C_0(\mathbb{R},A)\rtimes\mathbb{Z})\stackrel{\otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}}J_{\mathbb{Z}}(\tau_A(y))}{\longrightarrow}K_{*+1}(A\rtimes_\beta\mathbb{Z})$$

where,

- the first map comes from the Morita equivalence;
- $J_{\mathbb{Z}}:KK^{\mathbb{Z}}_*(\bullet,\bullet)\to KK_*(\bullet\rtimes\mathbb{Z},\bullet\rtimes\mathbb{Z})$ is the Kasparov transformation in bivariant KK-theory;
- for any C^*-algebra B equipped with an action of Γ by automorphism τ_B : $KK^\Gamma_*(\bullet,\bullet)\to KK^\Gamma_*(\bullet\otimes B,\bullet\otimes B)$ is the tensorisation operation;
- $\otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}}J_{\mathbb{Z}}(\tau_A(y))$ stands for the right Kasparov product by $J_{\mathbb{Z}}(\tau_A(y))$.

Then the identification between the generators of $K^*(X^G_{\mathcal{P}(w)})$ and of $K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)$ can be achieved using the next two lemmas.

Lemma 9.1. *Let A be a unital C^*-algebra together with an automorphism β. Let e be invariant projector in A and let x_e be the class in $K_0(A_\beta)$ of the projector $[0,1]\to A;\ t\mapsto e$. Then the image of x_e under the mapping torus isomorphism is equal to the class of the unitary $1-e+e\cdot u$ of $A\rtimes_\beta\mathbb{Z}$ in $K_1(A\rtimes_\beta\mathbb{Z})$ (here u is the unitary of $A\rtimes_\beta\mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$);*

Proof. The invariant projector e gives rise to an equivariant map $\mathbb{C}\to A$; $z\to ze$ and hence to a homomorphism $C(\mathbb{T})\to A_\beta$. By naturality of the mapping torus, this amounts to prove the result for $A=\mathbb{C}$ which is done in [**?**, Example 6.1.6]. □

Lemma 9.2. *Let A be a unital C^*-algebra together with an automorphism β and let x be an element in $K_*(A)$. The two following elements then coincide:*

- *the image of x under the composition*

$$K_*(A) \stackrel{\cong}{\to} K_{*+1}(C((0,1),A)) \to K_{*+1}(A_\beta) \to K_{*+1}(A \rtimes_\beta \mathbb{Z}),$$

  where
  - *the first map is the Bott periodicity isomorphism;*
  - *the second map is induced by the inclusion $C((0,1),A) \hookrightarrow A_\beta$;*
  - *the third map is the mapping torus isomorphism.*
- *the image of x under the map $K_*(A) \to K_*(A \rtimes_\beta \mathbb{Z})$ induced by the inclusion $A \hookrightarrow A \rtimes_\beta \mathbb{Z}$.*

Proof. Let us first describe the imprimity bimodule implementing the Morita equivalence between A_β and $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$. Indeed, in a more general setting, if

- X is a locally compact space equipped with a proper action of $\mathbb{Z}$ by homeomorphisms,
- B is a C^*-algebra provided with an action of $\mathbb{Z}$ by automorphisms;
- $B_X^{\mathbb{Z}}$ stands for the algebra of equivariant continuous maps $f : X \to B$ such that $\mathbb{Z}.x \mapsto \|f(x)\|$ belongs to $C_0(X/\mathbb{Z})$.

then, if we equip $C_0(X, B) \cong C_0(X) \otimes B$ with the diagonal action of $\mathbb{Z}$, there is an imprimitivity $B_X^{\mathbb{Z}} - C_0(X, B) \rtimes \mathbb{Z}$-bimodule defined in the following way: let us consider on $C_c(X, B)$ the $C_0(X, B) \rtimes \mathbb{Z}$-valued inner product

$$\langle \xi, \xi' \rangle(n) = n(\xi^*)\xi',$$

for ξ and ξ' in $C_c(X, B)$ and n in $\mathbb{Z}$. This inner product is namely positive and gives rise to a right $C_0(X, B) \rtimes \mathbb{Z}$-Hilbert module $\mathcal{E}(B, X)$, the action of $C_0(X, B) \rtimes \mathbb{Z}$ on the right being given for ξ in $C_c(X, B)$ and h in $C_c(\mathbb{Z} \times X, B) \subset C_c(\mathbb{Z}, C_0(X, B))$ by

$$\xi \cdot h(t) = \sum_{n \in \mathbb{Z}} n(\mathcal{E}(n+t))n(h(n, n+t)).$$

The action by pointwise multiplication of $B_X^{\mathbb{Z}} \subset C_b(X, B)$ on $C_0(X, B)$ extends to a left $B_X^{\mathbb{Z}}$-module structure on $\mathcal{E}(B, X)$. Let us denote by $[\mathcal{E}(B, X)]$ the class of the $B_X^{\mathbb{Z}} - C_0(X, B) \rtimes \mathbb{Z}$-bimodule $\mathcal{E}(B, X)$ in $KK_*(B_X^{\mathbb{Z}}, C_0(X, B) \rtimes \mathbb{Z})$. It is straightforward to check that

- $[\mathcal{E}(B, X)]$ is natural in both variable, in particular, if Y is an open invariant subset of X and let us denote by $\iota_{Y,X,B} : C_0(Y, B) \to C_0(X, B)$ and $\iota_{Y,X,B}^{\mathbb{Z}} : B_Y^{\mathbb{Z}} \to B_X^{\mathbb{Z}}$ the homomorphisms induced by the inclusion $Y \hookrightarrow X$ and respectively by $[\iota_{Y,X,B}]$ and $[\iota_{Y,X,B}^{\mathbb{Z}}]$ the corresponding classes in $KK_*^{\mathbb{Z}}(C_0(Y, B), C_0(X, B))$ and $KK_*(B_Y^{\mathbb{Z}}, B_X^{\mathbb{Z}})$, then

$$[\iota_{Y,X,B}^{\mathbb{Z}}] \otimes_{B_X^{\mathbb{Z}}} [\mathcal{E}(B, X)] = [\mathcal{E}(B, Y)] \otimes_{C_0(Y,B)} J_{\mathbb{Z}}([\iota_{Y,X,B}]).$$

- up to the identification $B_{\mathbb{Z}}^{\mathbb{Z}} \cong B$, the class $[\mathcal{E}(B, \mathbb{Z})]$ is induced by the composition

$$B \longrightarrow C_0(\mathbb{Z}, B) \hookrightarrow C_0(\mathbb{Z}, B) \rtimes \mathbb{Z}$$

  where the first map is $b \mapsto \delta_0 \otimes b$.
- if V is any locally compact space, and if we consider $\mathbb{Z}$ acting trivially on it, then up to the identifications $B_{X \times V}^{\mathbb{Z}} \cong B_X^{\mathbb{Z}} \otimes C_0(V)$ and $C_0(X \times V, B) \rtimes \mathbb{Z} \cong C_0(X, B) \rtimes \mathbb{Z} \otimes C_0(V)$, we have $[\mathcal{E}(B, V \times X)] = \tau_{C_0(V)}([\mathcal{E}(B, X)])$.

Noticing that for a C^*-algebra A provided with an automorphism β, we have a natural identification $A_\beta \cong A^{\mathbb{Z}}_{\mathbb{R}}$, the mapping torus isomorphism of equation (**??**) is obtained by right Kasporov product with $[\mathcal{E}(A,\mathbb{R})] \otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}} J_{\mathbb{Z}}(\tau_A([y]))$. From this, we see that the composition in the statement of the lemma is given by right Kasparov product with

$$z = \tau_A([\partial]) \otimes_{C_0((0,1),A)} [\iota^{\mathbb{Z}}_{(0,1)\times\mathbb{Z},\mathbb{R},A}] \otimes_{A_\beta} [\mathcal{E}(A,\mathbb{R})] \otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}} J_{\mathbb{Z}}(\tau_A(y)) \tag{9.12}$$

where

- $[\partial]$ in $KK_1(\mathbb{C}, C_0(0,1))$ is the boundary of the evaluation at 0 extension
$$0 \to C_0(0,1) \to C_0[0,1) \to \mathbb{C} \to 0;$$
- we have used the identification $(0,1)\times\mathbb{Z} \cong \mathbb{R}\setminus\mathbb{Z}$ to see $(0,1)\times\mathbb{Z}$ as an invariant open subset of $\mathbb{R}$, in particular we have $A^{\mathbb{Z}}_{(0,1)\times\mathbb{Z}} \cong C_0((0,1),A)$.

According to the naturality properties of $[\mathcal{E}(\bullet, A)]$ listed above, we get that

$$\begin{aligned}
[\iota^{\mathbb{Z}}_{(0,1)\times\mathbb{Z},\mathbb{R},A}] \otimes_{A_\beta} [\mathcal{E}(\mathbb{R},A)] &= [\mathcal{E}((0,1)\times\mathbb{Z},A)] \otimes_{C_0(\mathbb{Z},A)\rtimes\mathbb{Z}\otimes C_0(0,1)} J_{\mathbb{Z}}([\iota_{(0,1)\times\mathbb{Z},\mathbb{R},A}]) \\
&= \tau_{C_0(0,1)}([\mathcal{E}(\mathbb{Z},A)]) \otimes_{C_0(\mathbb{Z},A)\rtimes\mathbb{Z}\otimes C_0(0,1)} J_{\mathbb{Z}}([\iota_{(0,1)\times\mathbb{Z},\mathbb{R},A}])
\end{aligned} \tag{9.13}$$

Using commutativity of exterior Kasparov product, we get from equation (**??**) that

$$z = [\mathcal{E}(\mathbb{Z},A)] \otimes_{C_0(\mathbb{Z},A)\rtimes\mathbb{Z}} J_{\mathbb{Z}}(\tau_{C_0(\mathbb{Z},A)}([\partial]) \otimes_{C_0((0,1)\times\mathbb{Z},A)} [\iota_{(0,1)\times\mathbb{Z},\mathbb{R},A}] \otimes \tau_A(y)). \tag{9.14}$$

Let y' be the element of $KK_1(C_0(0,1),\mathbb{C})$ corresponding in the unbounded picture to the unbounded operator $\imath\frac{d}{dt}$ on $L^2(0,1)$ and let $[\mathcal{F}]$ be the element of $KK^{\mathbb{Z}}(C_0(\mathbb{Z}),\mathbb{C})$ corresponding to the equivariant representation by compact operator of $C_0(\mathbb{Z})$ onto $\ell^2(\mathbb{Z})$ (equipped with the left regular representation) given by pointwise multiplication. Then it is straightforward to check that

$$[\iota_{(0,1)\times\mathbb{Z},\mathbb{R},A}] \otimes_{C_0(\mathbb{R},A)} \tau_A(y) = \tau_A(\tau_{C_0(\mathbb{Z})}(y') \otimes_{C_0(\mathbb{Z})} [\mathcal{F}]).$$

But it is a standard fact that $[\partial] \otimes_{C_0(0,1)} y' = 1$ in the ring $KK_0(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$ and hence we eventually get that

$$z = [\mathcal{E}(A,\mathbb{Z})] \otimes_{C_0(\mathbb{Z},A)\rtimes\mathbb{Z}} J_{\mathbb{Z}}(\tau_A([\mathcal{F}])).$$

A direct inspection of the right hand side of this equality shows that z is indeed the class of $KK_*(A, A\rtimes\mathbb{Z})$ induced by the inclusion $A \hookrightarrow A\rtimes\mathbb{Z}$. $\square$

Recall that in section **??**, we have established isomorphisms

$$K_0(C(X^G_{\mathcal{P}(w)})\rtimes G) \cong \operatorname{Coinv} C(Z_w, \mathbb{Z}[1/2]) \oplus \operatorname{Inv} C(Z_w,\mathbb{Z}) \tag{9.15}$$

and

$$K_1(C(X^G_{\mathcal{P}(w)})\rtimes G) \cong \operatorname{Coinv} C(Z_w,\mathbb{Z}). \tag{9.16}$$

Under this identification, and using lemma **??** and twice lemma **??**, we are now in position to describe the image of the generators of $K^*(X^G_{\mathcal{P}(w)})$ under the double Thom-Connes isomorphism.

Corollary 9.3. *Under the identification of equations (**??**), (**??**), (**??**) and (**??**), the double Thom-Connes isomorphism*

$$K^*(X^G_{\mathcal{P}(w)}) \xrightarrow{\cong} K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)$$

corresponds to the identity maps of $\operatorname{Coinv} C(Z_w,\mathbb{Z}[1/2]) \oplus \operatorname{Inv} C(Z_w,\mathbb{Z})$ *and* $\operatorname{Coinv} C(Z_w,\mathbb{Z})$.

Proof. The statement concerning the factor $\operatorname{Inv} C(Z_w,\mathbb{Z})$ is indeed a consequence of the discussion at the end of section **??**. Before proving the statements concerning the factors $\operatorname{Coinv} C(Z_w,\mathbb{Z})$ and $\operatorname{Coinv} C(Z_w,\mathbb{Z}[1/2])$, we sum up for convenience of the reader the main features, described in sections **??** and **??** of the dynamic of the continuous hull for the coloured and the uncoloured Penrose hyperbolic tilings.

- the closure $X_{\mathcal{P}}^{\mathcal{N}}$ of $\mathcal{N}_{\mathcal{P}}\cdot\mathcal{P}$ in $X_{\mathcal{P}}^{G}$ for the tiling topolology of the Penrose hyperbolic tiling $\mathcal{P}$ is homeomorphic to the suspension of the homeomorphism $o:\Omega\to\Omega;\ \omega\mapsto\omega+1$;
- the continuous hull $X_{\mathcal{P}(w)}^{G}$ of the coloured Penrose hyperbolic tiling $\mathcal{P}(w)$ is homeomorphic to the suspension of the homeomorphism $X_{\mathcal{P}}^{\mathcal{N}}\times Z_\omega\to X_{\mathcal{P}}^{\mathcal{N}}\times Z_\omega:\ (\mathcal{T},w')\mapsto(R\cdot\mathcal{T},\sigma(w'))$;
- If we provide $X_{\mathcal{P}}^{\mathcal{N}}\times Z_\omega$ with the diagonal action of $\mathbb{R}$, by translations on $X_{\mathcal{P}}^{\mathcal{N}}$ and trivial on Z_ω, and equip the groupoid $\mathcal{G}=(X_{\mathcal{P}}^{\mathcal{N}}\times Z_\omega)\rtimes\mathbb{R}$ with the Haar system arising from the Haar measure of $\mathbb{R}$, then $C(X_{\mathcal{P}(w)}^{G})\rtimes\mathbb{R}$ is the mapping torus of $C_r^*(\mathcal{G},\lambda)$ with respect to the automorphim $\tilde{\alpha_\omega}$ arising from the automorphism of groupoid $\mathcal{G}\to\mathcal{G};\ (\mathcal{T},w',t)\mapsto(R\cdot\mathcal{T},\sigma(\omega'),2t)$ (see section **??**);
- $C_r^*(\mathcal{G},\lambda)$ is Morita-equivalent to the crossed product $C(\Omega\times Z_\omega)\rtimes\mathbb{Z}$ for the action of $\mathbb{Z}$ on $C(\Omega\times Z_\omega)$ arising from $o\times Id_{Z_\omega}$ (see section **??**).

Let us consider the following diagram:
(9.17)

$$\begin{array}{ccccc}
 & & K_i(C_r^*(\mathcal{G},\lambda)) & \longrightarrow & K_i(C(X_{\mathcal{P}(w)}^{G})\rtimes G)\\
 & & {\scriptstyle =}\big\uparrow & & \big\uparrow{\scriptstyle TC}\\
K_i(C(\Omega\times Z_w)) & \longrightarrow & K_i(C_r^*(\mathcal{G},\lambda)) & \longrightarrow & K_{i+1}(C(X_{\mathcal{P}(w)}^{G})\rtimes\mathbb{R}),\\
{\scriptstyle =}\big\uparrow & & \big\uparrow{\scriptstyle TC} & & \big\uparrow{\scriptstyle TC}\\
K_i(C(\Omega\times Z_w)) & \longrightarrow & K_{i+1}(C(X_{\mathcal{P}}^{\mathcal{N}}\times Z_w)) & \longrightarrow & K_i(C(X_{\mathcal{P}(w)}^{G}))
\end{array}$$

where

- the bottom and the right middle horizontal arrows are the maps defined for any C^*-algebra A provided by an automorphism β as the composition
$$K_i(A)\to K_{i+1}(C_0((0,1),A)\to K_{i+1}(A_\beta)$$
of the Bott peridocity isomorphism with the homomorphism induced in K-theory by the inclusion $C_0((0,1),A)\hookrightarrow A_\beta$;
- the left middle horizontal arrow is up to the Morita equivalence between $C_r^*(\mathcal{G},\lambda)$ and $C(\Omega\times Z_\omega)\rtimes\mathbb{Z}$ induced by the inclusion $C(\Omega\times Z_\omega)\hookrightarrow C(\Omega\times Z_\omega)\rtimes\mathbb{Z}$;
- the top horizontal arrow is up to the Morita equivalence between $C_r^*(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_\omega}}\mathbb{Z}$ and $C(X_{\mathcal{P}(w)}^{G})\rtimes G$ is induced by the inclusion $C_r^*(\mathcal{G},\lambda)\hookrightarrow C_r^*(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_\omega}}\mathbb{Z}$.
- the vertical maps TC stand for the Thom-Connes isomorphisms.

Then the inclusion $\operatorname{Coinv} C(Z_w,\mathbb{Z}[1/2])\hookrightarrow K^0(X_{\mathcal{P}(w)}^{G})$ of equation (**??**) is induced by the composition of the bottom arrows and the inclusion $\operatorname{Coinv} C(Z_w,\mathbb{Z}[1/2])\hookrightarrow K_0(C(X_{\mathcal{P}(w)}^{G})\rtimes G)$ of equation (**??**) is induced by the upper staircase. According to lemma **??**, the left bottom and the right top squares are commutative. Hence, the proof of the statements regarding to the $\operatorname{Coinv} C(Z_w,\mathbb{Z}[1/2])$ summand amounts to show that the bottom right square is commutative. To see this, let us equip $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)$ with the action of $\mathbb{R}$ by homeomorphisms

$$X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1) \times \mathbb{R} \longrightarrow X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1);\ (\mathcal{T}, \omega', s, t) \mapsto (\mathcal{T} + 2^{-s}t, \omega', s).$$

If we restrict this action to $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)$, then the inclusion $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1) \hookrightarrow X_{\mathcal{P}(w)}^{G}$ is $\mathbb{R}$-equivariant and the Bott periodicity isomorphism is the boundary of the equivariant short exact sequence

$$0 \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)) \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)) \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \to 0 \tag{9.18}$$

provided by evaluation at 0. This equivariant short exact sequence gives rise to a short exact sequence for crossed products

$$0 \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)) \rtimes \mathbb{R} \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)) \rtimes \mathbb{R} \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rtimes \mathbb{R} \to 0. \tag{9.19}$$

and since the Thom-Connes isomorphism is natural, it intertwins the corresponding boundary maps and hence we get a commutative diagram

$$\begin{tikzcd}
K_{i+1}(C(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rtimes \mathbb{R}) \arrow[r] & K_i(C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1) \rtimes \mathbb{R}) \arrow[r] & K_i(C(X_{\mathcal{P}(w)}^{G}) \rtimes \mathbb{R}) \\
K_i(C(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w)) \arrow[r] \arrow[u, "TC"] & K_{i+1}(C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)) \arrow[r] \arrow[u, "TC"'] & K_{i+1}(C(X_{\mathcal{P}(w)}^{G})) \arrow[u, "TC"']
\end{tikzcd}, \tag{9.20}$$

where the left horizontal arrows are induced by the boundary maps corresponding to the exact sequences of equation (**??**) and (**??**) and the right horizontal arrows are induced by the equivariant inclusion $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1) \hookrightarrow X_{\mathcal{P}(w)}^{G}$. Let us consider the family groupoids $(0,1) \times \mathcal{G}$ and $[0,1) \times \mathcal{G}$. Notice that if $X_{\mathcal{P}(w)}^{G} \rtimes \mathbb{R}$ is viewed as the suspension of the groupoid $\mathcal{G}$, then $(0,1) \times \mathcal{G}$ is the restriction of $X_{\mathcal{P}(w)}^{G} \rtimes \mathbb{R}$ to a fundamental domain. The reduced C^*-algebras of these two groupoids are respectively $C_0((0,1), C_r^*(\mathcal{G}, \lambda)))$ and $C_0([0,1), C_r^*(\mathcal{G}, \lambda))$ and the automorphism of groupoids

$$[0,1) \times \mathcal{G} \to (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)) \rtimes \mathbb{R};\ (\mathcal{T}, \omega', s, t) \mapsto (\mathcal{T}, \omega', s, 2^s t)$$

gives rise to a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & C_0((0,1), C_r^*(\mathcal{G}, \lambda)) \arrow[r] \arrow[d] & C_0([0,1), C_r^*(\mathcal{G}, \lambda)) \arrow[r] \arrow[d] & C_r^*(\mathcal{G}, \lambda) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)) \rtimes \mathbb{R} \arrow[r] & C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)) \rtimes \mathbb{R} \arrow[r] & C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rtimes \mathbb{R} \arrow[r] & 0.
\end{tikzcd}$$

Using naturality of the boundary map, we see that the composition of the top horizontal arrows in diagram (**??**) is the composition

$$K_i(C_r^*(\mathcal{G}, \lambda)) \to K_{i+1}(C_0((0,1), C_r^*(\mathcal{G}, \lambda)) \to K_{i+1}(C(X_{\mathcal{P}(w)}^{G}) \rtimes \mathbb{R}))$$

of the Bott peridocity isomorphism with the homomorphism induced in K-theory by the inclusion $C_0((0,1), C_r^*(\mathcal{G}, \lambda)) \hookrightarrow C(X_{\mathcal{P}(w)}^{G}) \rtimes \mathbb{R}$. This concludes the proof for the statement concerning the summand $\operatorname{Coinv} C(Z_\omega, \mathbb{Z}[1/2])$. The statement concerning the summand $\operatorname{Coinv} C(Z_\omega, \mathbb{Z})$ is a consequence of the commutativity of the top square of diagram **??** and of lemma **??** applied to the middle bottom vertical arrow. $\square$

10. The cyclic cocycle associated to a harmonic probabilty

Recall that according to the discussion ending section **??**, a probability is harmonic if and only if it is G-invariant. In this section, we associate to a harmonic probability a 3-cyclic cocycle on the smooth crossed product algebra of $X^G_{\mathcal{P}(\omega)} \rtimes G$. This cyclic cocycle is indeed builded from a 1-cyclic cocycle on the algebra of smooth (along the leaves) functions on $X^G_{\mathcal{P}(\omega)}$ by using the analogue in cyclic cohomology of the Thom-Connes isomorphism (see [**?**]). We give a description of this cocycle and we discuss an odd version of the gap-labelling.

10.1. **Review on smooth crossed products.** We collect here results from [**?**] concerning smooth crossed products by an action of $\mathbb{R}$ that we will need later on.

Let $\mathcal{A}$ be a Frechet algebra with respect to an increasing family of semi-norms $(\|\bullet\|_k)_{k\in\mathbb{N}}$.

Definition 10.1. *A smooth action on $\mathcal{A}$ is a homomorphism $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{A}$ such that*

1. *For every t in $\mathbb{R}$ and a in $\mathcal{A}$, the function $t \mapsto \alpha_t(a)$ is smooth.*
2. *For every integers k and m, there exist integers j and n and a real C such that $\left\|\frac{d^k}{dt^k}\alpha_t(a)\right\|_m \leq C(1+t^2)^{j/2}\|a\|_n$ for every a in $\mathcal{A}$.*

If α is a smooth action on $\mathcal{A}$, then the smooth crossed product $\mathcal{A}\rtimes_\alpha\mathbb{R}$ is defined as the set of smooth functions $f : \mathbb{R} \to \mathcal{A}$ such that

$$\|f\|_{k,m,n} \stackrel{\text{def}}{=\!=} \sup_{t\in\mathbb{R}}(1+t^2)^{k/2}\left\|\frac{d^m}{dt^m}f(t)\right\|_n < +\infty$$

for all integers k, m and n. The smooth crossed product $\mathcal{A}\rtimes_\alpha\mathbb{R}$ provided with the family of semi-norm $\|\bullet\|_{k,m,n}$ for k, m and n integers together with the convolution product

$$f * g(t) = \int f(s)\alpha_s(g(t-s))dt$$

is then a Frechet algebra. Notice that a smooth action α on a Frechet algebra $\mathcal{A}$ gives rise to a bounded derivation Z_α of $\mathcal{A}\rtimes_\alpha\mathbb{R}$ defined by $Z_\alpha(f)(t) = tf(t)$ for all f in $\mathcal{A}\rtimes_\alpha\mathbb{R}$ and t in $\mathbb{R}$.

Let $\mathcal{A}^G_{\mathcal{P}(\omega)}$ be the algebra of continuous and smooth along the leaves functions on $X^G_{\mathcal{P}(\omega)}$, i.e functions whose restrictions to leaves admit at all order differential which are continuous as functions on $X^G_{\mathcal{P}(\omega)}$. Let β^0 and β^1 be the two actions of $\mathbb{R}$ on $\mathcal{A}^G_{\mathcal{P}(\omega)}$ respectively induced by

$$\mathbb{R} \times X^G_{\mathcal{P}(\omega)} \to X^G_{\mathcal{P}(\omega)};\ (t,\mathcal{T}) \mapsto \mathcal{T} + t$$

and

$$\mathbb{R} \times X^G_{\mathcal{P}(\omega)} \to X^G_{\mathcal{P}(\omega)};\ (t,\mathcal{T}) \mapsto R_{2^t}\cdot\mathcal{T}.$$

Let X and Y be respectively the vector fields associated to β^0 and β^1. Then $\mathcal{A}^G_{\mathcal{P}(\omega)}$ is a Frechet algebra with respect to the family of semi-norms

$$f \mapsto \sup_{X^G_{\mathcal{P}(\omega)}} |X^kY^l(f)|,$$

where k and l run through integers. It is clear that β^0 is a smooth action on $\mathcal{A}^G_{\mathcal{P}(\omega)}$. Moreover,

$$\mathbb{R} \times \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \to \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}; (t, f) \mapsto [s \mapsto \beta^1(f(2^{-t}s))]$$

is an action of $\mathbb{R}$ on $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ by automorphisms. This action is not smooth in the previous sense. Nevertheless, the action β^1 satisfies conditions (1),(2) and (3) of [?, Section 7.2] with respect to the family of functions $\rho_n : \mathbb{R} \to \mathbb{R}; t \mapsto 2^{2n|t|}$, where n runs through integers. In this situation, we can define the smooth crossed product $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}$ of $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ by β^1 to be the set of smooth functions $f : \mathbb{R} \to \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ such that

$$\|f\|_{k,l,m} \stackrel{\text{def}}{=\!=} \sup_{t \in \mathbb{R}} \rho_k(t) \left\| \frac{d^l}{dt^l} f(t) \right\|_m < +\infty$$

for all integers k, l and m (we have reindexed for convenience the family of semi-norms on $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ using integers). Then $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}$ provided with the family of semi-norms $\| \bullet \|_{k,l,m}$ for k, l and m integers together with the convolution product is a Frechet algebra. Moreover, this algebra can be viewed as a dense subalgebra of $C(X^G_{\mathcal{P}(\omega)}) \rtimes G$. As for smooth actions, the action β^1 gives rise to a derivation Z_{β^1} of $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}$ (defined by the same formula).

10.2. **The 3-cyclic cocycle.** Let η be a G-invariant probability on $X^G_{\mathcal{P}(\omega)}$. Define

$$\tau_{w,\eta} : \mathcal{A}^G_{\mathcal{P}(\omega)} \times \mathcal{A}^G_{\mathcal{P}(\omega)} \to \mathbb{C}; (f, g) \mapsto \int Y(f) g d\eta.$$

Using the Leibnitz rules and the invariance of G, it is straightforward to check that $\tau_{w,\eta}$ is 1-cyclic cocycle. In [?] was constructed for a smooth action α on a Frechet algebra $\mathcal{A}$ a homomorphism $H^n_\lambda(\mathcal{A}) \to H^{n+1}_\lambda(\mathcal{A} \rtimes_\alpha \mathbb{R})$, where $H^*_\lambda(\bullet)$ stands for the cyclic cohomology. This homomorphism is indeed induced by a homomorphism at the level of cyclic cocycles $\#_\alpha : Z^n_\lambda(\mathcal{A}) \to Z^{n+1}_\lambda(\mathcal{A} \rtimes_\alpha \mathbb{R})$ and commutes with the periodisation operator S. Hence it gives rise to a homomorphism in periodic cohomology $HP^*(\mathcal{A}) \to HP^{*+1}(\mathcal{A} \rtimes_\alpha \mathbb{R})$ which turns out to be an isomorphism. This isomorphism is for periodic cohomology the analogue of the Thom-Connes isomorphism in K-theory.

We give now the description of $\#_{\beta^0} \tau_{w,\eta}$. Let us define first

$$X_{\beta^0} : \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \to \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$$

and

$$Y_{\beta^0} : \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \to \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$$

respectively by $X_{\beta^0} f(t) = X(f)(t)$ and $Y_{\beta^0} f(t) = Y(f)(t)$, for all f in $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ and t in $\mathbb{R}$. Using the relation $Y \circ \beta^0_t = \beta^0_t \circ Y - t \ln 2\, \beta^0_t \circ X$ and applying the definition of $\#^0_\beta$ (see [?, section 3.3]), we get:

Proposition 10.2. *For any elements f, g and h in $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \rtimes \mathbb{R}$, we have*

(1)

$$\#_{\beta^0} \tau_{w,\eta}(f, g, h) = -2\pi i \eta(Y_{\beta^0} f * g * Z_{\beta^0} h(0) + Z_{\beta^0} f * g * Y_{\beta^0} h(0)$$
$$-2\pi i \ln 2(\eta(1/2 Z_{\beta^0}{}^2 f * g * X_{\beta^0} h(0) + Z_{\beta^0} f * Z_{\beta^0} g * X_{\beta^0}(h)(0) - 1/2 X_{\beta^0} f * g * Z_{\beta^0}{}^2(0))$$

(2)
$$\#_{\beta^0}\tau_{w,\eta}(\beta^1_t f, \beta^1_t g, \beta^1_t h) = \#_{\beta^0}\tau_{w,\eta}(f,g,h)$$
for all t in $\mathbb{R}$, i.e the cocycle $\#_{\beta^0}\tau_{w,\eta}$ is β^1-invariant.

According to [?, Section 7.2], the action β^1 on $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \rtimes \mathbb{R}$ also gives rise to a homomorphism $\#_{\beta^1} : Z^n_\lambda(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}) \to Z^{n+1}_\lambda(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R})$ which induces an isomorphim $HP^*(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}) \xrightarrow{\cong} HP^{*+1}(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R})$. A direct application of the definition of $\#_{\beta^1}$ leads to

Lemma 10.3. *Let ϕ be a β_1-invariant 3-cyclic cocycle for $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$. Let us define for any f, g and h in $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}$.*
$$\widetilde{\phi}(f,g,h) = 2\pi\imath \int_{t_0+t_1+t_2=0} f(t_0)\beta^1_{t_0} g(t_1)\beta^1_{-t_2}(t_2).$$
Then
$$\begin{aligned}\#_{\beta^1}\phi(f_0,f_1,f_2,f_3) = -\widetilde{\phi}(f_0,f_1,f_2 * Z_{\beta^1} f_3) + \widetilde{\phi}(Z_{\beta^1} f_0 * f_1, f_2, f_3) \\ - \widetilde{\phi}(f_0, Z_{\beta^1} f_1 * f_2, f_3) - \widetilde{\phi}(Z_{\beta^1} f_0, f_1 * f_2, f_3)\end{aligned}$$

Definition 10.4. *With above notations, the 3-cyclic cocycle on $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}$ associated to the Penrose hyperbolic tiling coloured by w and to a G-invariant probability η on $X^G_{\mathcal{P}(\omega)}$ is*
$$\phi_{w,\eta} = \#_{\beta^1}\#_{\beta^0}\tau_{w,\eta}.$$

Notice that if we carry out this construction for a tiling $\mathcal{T}$ of the Euclidian space with continuous hull $X^{\mathbb{R}^2}_{\mathcal{T}}$ with respect to the $\mathbb{R}^2$-action by translations, we get taking twice the crossed product by $\mathbb{R}$ a 3-cyclic cocycle which is indeed equivalent (via the periodisation operator) to the 1-cycle cocycle on $C(X^{\mathbb{R}^2}_{\mathcal{T}}) \rtimes \mathbb{R}^2 \cong (C(X^{\mathbb{R}^2}_{\mathcal{T}}) \rtimes \mathbb{R}) \rtimes \mathbb{R}$ arising from the trace on $C(X^{\mathbb{R}^2}_{\mathcal{T}}) \rtimes \mathbb{R}$ associated to an $\mathbb{R}$-invariant probability on $X^{\mathbb{R}^2}_{\mathcal{T}}$.

The class $[\phi_{w,\eta}]$ of $\phi_{w,\eta}$ in $HP^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R})$ is the image of the class of $\tau_{w,\eta}$ under the composition of isomorphism
$$HP^1(\mathcal{A}^G_{\mathcal{P}(\omega)}) \xrightarrow{\cong} HP^0(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}) \xrightarrow{\cong} HP^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}).$$
Since pairing with periodic cohomology provides linear forms for K-theory groups, the 3-cyclic cocycle $\phi_{w,\eta}$ provides a linear map
$$\phi_{w,\eta,*} : K^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}) \to \mathbb{C};\ x \mapsto \langle [\phi_{w,\eta}], x\rangle.$$
The main issue in computing $\phi_{w,\eta,*}(K^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}))$ is that the Thom-Connes isomorphism a priori may not hold for $K^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R})$. If it were the case, the inclusion $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R} \hookrightarrow C(X^G_{\mathcal{P}(\omega)}) \rtimes G$ would induces an isomorphism $K_1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}) \xrightarrow{\cong} K_1(C(X^G_{\mathcal{P}(\omega)}) \rtimes G)$ and from this we could get that
$$\phi_{w,\eta,*}(K_1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R})) = \mathbb{Z}[\hat{\eta}] \stackrel{\text{def}}{=\!=} \{\hat{\eta}(E),\ E \text{ compact-open subset of } Z_w\},$$
where $\hat{\eta}$ is the probability on Z_w of proposition **??** in one-to-one correspondance with η.

Since $\mathbb{Z}[\hat{\eta}]$ is indeed the one dimension gap-labelling for the subshift corresponding to w, this would be viewed as an odd version of the gap labelling. Nevertheless, the right setting to state this generalisation of the gap-labelling seems to be the Frechet algebra and a natural question is whether we have

$$\{\langle[\phi_{w,\eta}], x\rangle;\, x \in K_1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^{\rho}_{\beta^1} \mathbb{R})\} = \mathbb{Z}[\hat{\eta}]$$

or if the pairing bring in new invariants.

Aknowledgments

The first author is indebted to J. Renault for some very helpful discussions on Haar systems for groupoids.

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On invariant measures of finite affine type tilings

Samuel Petite *

Abstract

In this paper, we consider tilings of the hyperbolic 2-space $\mathbb{H}^2$, built with a finite number of polygonal tiles, up to affine transformation. To such a tiling T, we associate a space of tilings: the continuous hull $\Omega(T)$ on which the affine group acts. This space $\Omega(T)$ inherits a solenoid structure whose leaves correspond to the orbits of the affine group. First we prove that the finite harmonic measures of this laminated space correspond to finite invariant measures for the affine group action. Then we give a complete combinatorial description of these finite invariant measures. Finally we give examples with an arbitrary number of ergodic invariant probability measures.

1 Introduction

Let N be either the hyperbolic 2-space $\mathbb{H}^2$, identified with the upper half complex plane: $\{z \in \mathbb{C} \,|\mathrm{Im}(z) > 0\}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, or the Euclidean plane $\mathbb{R}^2$.

A tiling $T = \{t_1, \ldots, t_n, \ldots\}$ of N, is a collection of convex compact polygons t_i with geodesic borders, called *tiles*, such that their union is the whole space N, their interiors are pairwise disjoint and they meet full edge to full edge. Let $\mathbf{G}$ denote a Lie group of isometries of N preserving the orientation. A tiling is said of $\mathbf{G}$-*finite type* if there exists a finite number of polygons $\{p_1, \ldots, p_n\}$ called *prototiles* such that each t_i is the image of one of these polygons by an element of $\mathbf{G}$. For instance, when F is a fundamental domain of a discrete co-compact group G of isometries of N, then $\{\gamma(F),\ \gamma \in G\}$ is a tiling of N. However the set of finite type tilings is much richer than the one given by discrete co-compact groups. When $N = \mathbb{R}^2$, R. Penrose [15] gave an example whose set of prototiles is made with teen rhombi: the Penrose's tiling. When $N = \mathbb{H}^2$, Penrose also constructed a finite type tiling made with a single prototile which is not stable for any Fuchsian group. This example is the typical example of tilings studied in this paper. The construction goes as follows.
Let P be the convex polygon with vertices A_p with affix $(p-1)/2 + i$ for $1 \leq p \leq 3$ and $A_4 : 2i + 1$ and $A_5 : 2i$ (see figure 1): P is a polygon with 5 geodesic edges. Consider the two maps:

$$R : z \mapsto 2z \text{ and } S : z \mapsto z + 1.$$

The hyperbolic Penrose's tiling is defined by $\mathcal{T} = \{R^k \circ S^n P | n, k \in \mathbb{Z}\}$ (see figure 2). This tiling is an example of $\mathcal{P}$-finite type tiling where $\mathcal{P}$ denote the group of *affine maps i.e.* isometries of $\mathbb{H}^2$ of the kind $z \mapsto az + b$ with a, b reals and $a > 0$.
The argument of Penrose is a homological one: he associates with the edge A_4A_5 a positive charge and two negative charges with edges A_1A_2, A_2A_3. If $\mathcal{T}$ was stable for a Fuchsian

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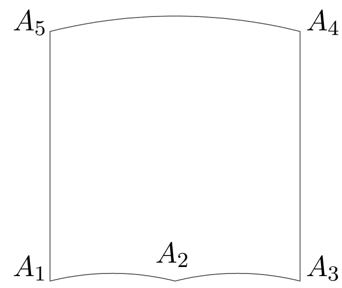

Figure 1: The prototile P

group, then P would tile a compact surface. Since the edge A_4A_5 can meet only the edges A_1A_2 or A_2A_3, the surface has a neutral charge. This is in contradiction with the fact P is negatively charged.
G. Margulis and S. Mozes [12] have generalized this construction to build a family of prototiles which cannot be used to tile a compact surface. Notice the group of isometries which preserves $\mathcal{T}$ is generated by the transformation R. In order to break this symmetry, it is possible to decorate prototiles to get a new finite type tiling which is not stable for any non trivial isometry (we say in this case that the tiling is *aperiodic*). Using the same procedure, C. Goodmann-Strauss [10] construct a set of polygons which can tile $\mathbb{H}^2$ only in an aperiodic way.

To understand the combinatorial properties of a tiling, it is useful to associate with this tiling, a set of tilings that we can study both from a geometric and dynamical point of view. The image of a $\mathbf{G}$ finite type tiling T by an element of $\mathbf{G}$ is again a $\mathbf{G}$ finite type tiling. We consider a compact metric space $\Omega(T)$, which is the completion of the set of tilings image of T by elements of $\mathbf{G}$, for a natural metrizable topology defined in section 2. The space $\Omega(T)$ is called the *continuous hull* of T. The group $\mathbf{G}$ acts continuously on this space. In this paper we are mainly interested in the situation when the $\mathbf{G}$-action on the hull is free (without fixed point). This is the case for the $\mathcal{P}$-action on the hulls of examples in [10] as well as for the translation group action on the hull of the Euclidean Penrose's tiling. In this case, the $\mathbf{G}$-action induces a specific laminated structure on the hull: a $\mathbf{G}$-solenoid structure, where leaves are orbits for the group $\mathbf{G}$-action (see section 2). The combinatorial properties of the tiling T are related to geometrical properties of $\Omega(T)$ and dynamical properties of $(\Omega(T), \mathbf{G})$. In particular, the distribution of tiles of the tiling, which is our main interest for this paper, can be described by the statistical properties of the leaves of the solenoid.
On the one hand, these properties can be grasped from a dynamical point of view. When the group $\mathbf{G}$ is amenable, the $\mathbf{G}$-action possesses finite invariant measures. R. Benedetti, J.-M. Gambaudo [2] show that a $\mathbf{G}$-solenoid can be seen as a projective limit $\lim_{\leftarrow}(\mathcal{B}_n, \pi_n)$ of branched manifold $\mathcal{B}_n$. Furthermore, when the group $\mathbf{G}$ is unimodular (for example when $N = \mathbb{R}^2$ and $\mathbf{G}$ is the translation group), authors of [2] prove that the notions of transverse invariant measure, foliated cycle and finite $\mathbf{G}$ invariant measure, are equivalent. Thanks to this, they characterize the finite $\mathbf{G}$-invariant measures as the elements of a projective limit of cones in the dim $\mathbf{G}$-homology groups of the branched manifolds $\mathcal{B}_n$. When the group $\mathbf{G}$ is amenable and not unimodular (this is the case when $\mathbf{G}$ is the affine group $\mathcal{P}$), their results do not apply. Actually, we prove that on $\mathcal{P}$-solenoid there is no transverse invariant measure (Proposition 3.1).

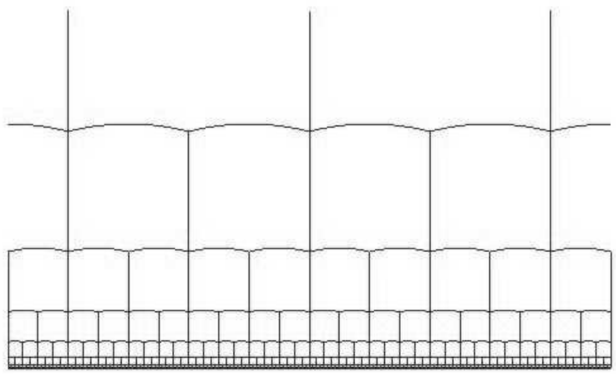

Figure 2: The hyperbolic Penrose's tiling

On the other hand, statistical properties of the leaves can be studied through a geometric point of view. Following the work of L. Garnett [7] on foliations, we can consider harmonic currents on the hull (such currents always exist on laminations). A riemannian metric on the leaves yields a correspondence between harmonic currents and finite harmonic measures and these measures give statistical properties of random path in a leaf of Brownian motions. More particulary, harmonic measures enable to define the average time of a generic path crossing an open subset of the hull. We prove that, for a $\mathcal{P}$-solenoid, both geometrical and dynamical approaches are related:

Theorem 1.1 *A finite measure on a $\mathcal{P}$-solenoid is harmonic if and only if it is invariant for the affine group action.*

Remark 1 *It is important to note that the proof of this theorem is totally independent from the structure of space of tilings.*

By using the structure of projective limit $\lim_{\leftarrow}(\mathcal{B}_n, \pi_n)$ of a $\mathcal{P}$-solenoid, we give a characterization of harmonic measures of a $\mathcal{P}$-solenoid:

Theorem 1.2 *There exists a sequence of linear morphisms A_n such that the set of harmonic measures is isomorphic to the projective limit of cones in 2 chains spaces of branched manifold $\mathcal{B}_n$, $\lim_{\leftarrow}(\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$.*

The linear morphisms A_n will be defined in section 4. We deduce from Theorem 1.2 that the number of ergodic invariant probability measures on the solenoid is bounded from above by the maximal number of faces of the branched manifolds. Finally we prove, by giving explicit examples:

Proposition 1.3 *For any integer $r \geq 1$, there exists a $\mathcal{P}$-finite type tiling T such that the $\mathcal{P}$-action on $\Omega(T)$ is free and minimal (all orbits are dense) and has exactly r invariant ergodic probability measures.*

This paper is organized as follows. In section 2, we recall some standard background on the tiling space, their solenoid structures and their description as projective limits of branched manifolds. Section 3 is devoted to harmonic currents and foliated cycles. We prove here that there exists no foliated cycle for a $\mathcal{P}$-solenoid. In Section 4, we prove Theorem 1.1 and Theorem 1.2. The last section, is devoted to the construction of examples which prove Proposition 1.3.

2 Background on tiling spaces

We recall here different useful notions defined in [11] and [2]

2.1 Action on the hull

Let $\mathbf{G}$ be the subgroup of isometries acting transitively, freely and preserving the orientation of the surface N, thus $\mathbf{G}$ is a Lie group homeomorphic to N. The metric on N gives a left multiplicative invariant metric on $\mathbf{G}$. We fix a point O in N that we call *origin*.
For a tiling T of $\mathbf{G}$ finite type and an isometry p in $\mathbf{G}$, the image of T by p^{-1} is again a tiling of N of $\mathbf{G}$ finite type. We denote by $T.\mathbf{G}$ the set of tilings which are image of T by isometries in $\mathbf{G}$. The group $\mathbf{G}$ acts on this set by the right action:

$$\begin{array}{ccc} \mathbf{G} \times T.\mathbf{G} & \longrightarrow & T.\mathbf{G} \\ (p, T') & \longrightarrow & T'.p = p^{-1}(T') \end{array}$$

We equip $T.\mathbf{G}$ with a metrizable topology, finer as one induced by the metric on N. A base of neighborhoods is defined as follows: two tilings are close one of the other if they agree, on a big ball of N centered at the origin, up to an isometry in $\mathbf{G}$ close to the identity. This topology can be generated by the metric δ on $T.\mathbf{G}$ defined by :

for T and T' be two tilings of $T.\mathbf{G}$, let

$$A = \{\epsilon \in [0,1] | \exists\ g \in B_\epsilon(Id) \subset \mathbf{G} \text{ s.t. } (T.g) \cap B_{1/\epsilon}(O) = T' \cap B_{1/\epsilon}(O)\}$$

where $B_{1/\epsilon}(O)$ is the set of points $x \in N$ such that $d(x, O) < 1/\epsilon$.
we define :

$$\delta(T, T') = \inf A \text{ if } A \neq \emptyset$$

$$\delta(T, T') = 1 \text{ else.}$$

The *continuous hull* of the tiling T, is the metric completion of $T.\mathbf{G}$ for the metric δ. We denote it by $\Omega(T)$. Actually this space is a set of tilings of N of $\mathbf{G}$-finite type. A *patch* of a tiling T is a finite set of tiles of T. It is straightforward to check that patches of tilings in $\Omega(T)$ are copies of patches of T. The set $\Omega(T)$ is then a compact metric set and the action of $\mathbf{G}$ can be extended to a continuous right action on this space. The dynamical system $(\Omega(T), \mathbf{G})$ has a dense orbit (the orbit of T).

We fix in each prototile *prot* of T, a marked point x_{prot} in its interior. Consequently, each tile t of a tiling $T' \in \Omega(T)$ admits a distinguished point x_t. Let $\Omega_0(T)$ denote the set of tilings of $\Omega(T)$ such that one x_t coincides with the origin O. With the induced topology, $\Omega_0(T)$ is compact and completely disconnected.

Definition 2.1 *A tiling T satisfies the repetitivity condition if for each patch P, there exists a real $R(P)$ such that every ball of N with radius $R(P)$ intersected with the tiling T contains a copy of the patch P.*

This definition can be interpreted from a dynamical point of view (see for instance [11]).

Proposition 2.2 *The dynamical system* $(\Omega(T), \mathbf{G})$ *is minimal (all orbits are dense) if and only if the tiling* T *satisfies the repetitivity condition.*

We call a tiling *non-periodic* if the action of $\mathbf{G}$ on $\Omega(T)$ is free: for all $p \neq Id$ of $\mathbf{G}$ and all tilings T' of $\Omega(T)$ we have $T'.p \neq T'$. In this case the space $\Omega_0(T)$ is a Cantor set. It is straightforward to show, for $N = \mathbb{R}^2$ and $\mathbf{G}$ is the translation group that when the stabilizer of T is reduced to the identity (T is aperiodic) and T is repetitive then T is non periodic. For example the Euclidean Penrose's tiling is a non-periodic repetitive tiling of $\mathbb{R}^2$ finite type. When $N = \mathbb{H}^2$ and $\mathbf{G}$ is the affine group $\mathcal{P}$, we saw that the hyperbolic Penrose's tiling is not aperiodic, however, using this example, we shall construct in the last section examples of repetitive and non-periodic affine finite type tilings (with specific ergodic properties).

2.2 Structure of G-solenoid

2.2.1 Solenoids

Let M be a compact metric space, suppose there exists a covering of M by open set U_i, called *boxes*, and homeomorphisms called *charts* $h_i : U_i \to V_i \times C_i$ where V_i is an open set of $\mathbf{G}$, considered as a Lie group, and C_i is a totally disconnected compact metric space. The collection of open set and homeomorphisms (U_i, h_i) is called an *atlas* of a $\mathbf{G}$-solenoid if the *transition map* $h_{i,j} = h_i \circ h_j^{-1}$, on their domains of definitions, read:

$$h_{i,j}(x, c) = (f_{i,j}.x, g_{i,j}(c))$$

where $f_{i,j}.x$ means the multiplication of $x \in V_j$ with an element $f_{i,j}$ of $\mathbf{G}$, independent of x and $c \in C_j$; and $g_{i,j}$ is a continuous map from C_j to C_i independent of x.

Two atlases are *equivalent* if their union is again an atlas. We will call a compact metric space M with an equivalence class of atlas, a $\mathbf{G}$-*solenoid*.

The transition maps structure provides the following important notions:

1. *slices* and leaves: a slice is a set of the kind $h_i^{-1}(V_i \times \{c\})$. The *leaves* are the union of the slices which intersect. The global space M is laminated by these leaves. Leaves are differentiable manifolds of dimension 2. A $\mathbf{G}$-solenoid M is called *minimal* if every leaf of M is dense in M.

2. *Vertical germs*: it is a set of the kind $h_i^{-1}(\{x\} \times C_i)$. Transition maps map vertical germs onto vertical germs, and thus this notion is well defined (independently of the charts).

These transition maps enable to define right multiplication by an element of $\mathbf{G}$ close to the identity. We suppose furthermore that each leaf is diffeomorphic to N and that this local $\mathbf{G}$ right action on a leaf extends to a free $\mathbf{G}$ right action on M. Leaves correspond to orbits of the action of $\mathbf{G}$ by right multiplication. This action is minimal if and only if the $\mathbf{G}$-solenoid is minimal.

Furthermore this action has locally constant return times: if an orbit (or a leaf) intersects two verticals V and V' at points v and $v.g$ where $g \in \mathbf{G}$, then for any point w of V close enough to v, $w.g$ belongs to V'.

It turns out that the hull of a tiling has a laminated structure (see for instance É. Ghys [8]). More precisely, in [2] authors prove that the hull $\Omega(T)$ of a non periodic $\mathbf{G}$ finite type tiling T, has a $\mathbf{G}$-solenoid structure. The boxes of $\Omega(T)$ are homeomorphic to spaces $V_i \times C_i$ where V_i is an open subset of $\mathbf{G} \simeq N$ and C_i is a closed and open subset of $\Omega_0(T)$. The charts are the inverse of the maps $f_i : V_i \times C_i \to U_i \subset \Omega(T)$ with $f_i(z, T') = z^{-1}(T')$.

The action of the group $\mathbf{G}$ on the solenoid coincides with the action of this group on the hull. This $\mathbf{G}$-action is *expansive*: there exists a positive real ϵ such that for every points T_1 and T_2 in the same vertical in $\Omega(T)$, if $\delta(T_1.g, T_2.g) < \epsilon$ for every $g \in \mathbf{G}$, then $T_1 = T_2$.

If furthermore T verifies the repetitivity condition, the hull $\Omega(T)$ is minimal, and the transversal in any point in any box is homeomorphic to a Cantor set.

2.2.2 Branched manifolds and projective limits

A *box decomposition* of a solenoid M is a finite collection of charts $B_1, \ldots, B_n$ such that: any two boxes are disjoint and the closure of the union of all boxes is the whole space M; furthermore each B_i is homeomorphic to a space $V_i \times C_i$, with C_i a totally disconnected set and V_i an open convex geodesic polyhedron in N. The *vertical boundary* of B_i is the set homeomorphic to $\partial V_i \times C_i$.

The hull of a finite affine type tiling has a natural box decomposition, where boxes are homeomorphic to the product of a prototile of the tiling times a disconnected set. Boxes are sets of tilings having the same tile on the origin. We say that this box decomposition is *associated to* tiles of the tiling.

Let us consider a box decomposition on M. We consider now the equivalence relation $\sim$ generated by the relation $\approx$:

$$x \approx y \Leftrightarrow x \text{ and } y \text{ belong to the closure of the same box and are in the same vertical.}$$

Let B be the quotient space $M/\sim$ and let p be the projection of M onto B. Authors of [2] prove that the set B with the quotient topology, has a differentiable structure and is a branched manifold, a structure by R. Williams (see [22]). Actually, in the proof of Theorem 1.2 we will only use the simplex structure of B.

Example: consider a non-periodic tiling T which is a decorated hyperbolic Penrose's tiling (see section 5). The set of prototiles is a finite union of different copies of P. Let us consider now the box decomposition of $\Omega(T)$ associated to its prototiles. The quotient space $\Omega(T)/\sim$ is then homeomorphic to the collapsing of prototiles along edges. Points on prototiles are identified if somewhere, on T, their copies meet (see [1]). For the Penrose's tiling $\mathcal{T}$, this identification leads to a branched manifold $\mathcal{N}$ homeomorphic to P with edges identified as follows: edges A_1A_2, A_2A_3 and A_4A_5 are identified themselves and edge A_4A_1 is identified with A_5A_3. This space is homeomorphic to the mapping torus of the application $x \mapsto 2x \bmod 1$ on the circle $S^1 \simeq \mathbb{R}_{/\mathbb{Z}}$. There is a natural projection of $\Omega(T)/\sim$ onto $\mathcal{N}$.

We say that the box decomposition $\mathbf{B}_2$ is *zoomed out* of the box decomposition $\mathbf{B}_1$ if:

1. for each point x in a box B_1 in $\mathbf{B}_1$ and in a box B_2 in $\mathbf{B}_2$, the vertical of x in B_2 is contained in the vertical of x in B_1.

2. the vertical boundaries of the boxes of $\mathbf{B}_2$ are contained in the vertical boundaries of the boxes of $\mathbf{B}_1$.

3. for each box B_2 in $\mathbf{B}_2$, there exists a box B_1 in $\mathbf{B}_1$ such that $B_1 \cap B_2 \neq \emptyset$ and the vertical boundary of B_1 doesn't intersect the vertical boundary of B_2.

4. if a vertical in the vertical boundary of a box in $\mathbf{B}_1$ contains a point in a vertical boundary of a box in $\mathbf{B}_2$, then it contains the whole vertical.

A *tower system* of a solenoid M is a sequence of box decompositions $(\mathbf{B}_n)_{n\geq 1}$, such that for any $n \geq 1$, $\mathbf{B}_{n+1}$ is zoomed out of $\mathbf{B}_n$ and the diameters of the verticals in $\mathbf{B}_n$ go to zero when n goes to infinity. In [2] it is proved that any $\mathcal{P}$-solenoid admits a tower system$(\mathbf{B}_n)_n$.

From above, for every n, there exists a branched manifold $\mathcal{B}_n$ associated to the box decomposition $\mathbf{B}_n$ and a projection $p_n : M \to \mathcal{B}_n$. By definition, the set of verticals of boxes of $\mathbf{B}_{n+1}$ is included in the set of verticals of $\mathbf{B}_n$, this induces a natural map $\pi_n : \mathcal{B}_{n+1} \to \mathcal{B}_n$ such that $p_n = \pi_n \circ p_{n+1}$.

Theorem 2.3 (R. Benedetti, J.M. Gambaudo) *A* $\mathbf{G}$*-solenoid* M*, always posses a tower system* $(\mathbf{B}_n)_{n\geq 1}$*, and* M *is homeomorphic to the projective limit* $\lim_{\leftarrow}(\mathcal{B}_n, \pi_n)$*.*

We recall that $\lim_{\leftarrow}(\mathcal{B}_n, \pi_n)$ is a subspace of $\Pi\mathcal{B}_n$ defined by $\{(x_n) \in \Pi\mathcal{B}_n \mid x_n = \pi_n(x_{n+1})\}$ and equipped with the topology induced by the product topology.

3 Foliated cycles and harmonic currents

3.1 Foliated cycles

Let us fix an atlas. The leaves of a $\mathbf{G}$ solenoid M carry a 2-manifold structure. Following [8], we call *k-differential form* the data, in any box, of a family of real k-differential forms ($\mathcal{C}^\infty$) on slices $V_i \times \{c\}$ which depends continuously of the parameter c (in the $\mathcal{C}^\infty$-topology) and such that each family is mapped onto each other by the transition maps. We denote by $A^k(M)$ the set of k-differential forms on M. The differentiation along leaves gives an operator $d : A^k(M) \to A^{k+1}(M)$.

Foliated cycles, introduced by D. Sullivan [20], are a continuous linear forms $A^2(M) \to \mathbb{R}$ which are positive on positive forms and vanish on exact forms.

Proposition 3.1 *A* $\mathcal{P}$*-solenoid does not admit a foliated cycle.*

In order to prove this result, let us introduce the following definition.

Definition 3.2 *A finite transverse invariant measure on* M *is the data of a finite positive measure* μ_i *on each set* C_i *such that for any Borelian set* B *in some* C_i *which is contained in the definition set of the transition map* g_{ij} *then*

$$\mu_i(B) = \mu_j(g_{ij}(B))$$

The data of a transverse invariant measure for a given atlas provides another invariant transverse measure for any equivalent atlas and thus gives an invariant measure on each verticals. Thus it makes sense to consider a transverse invariant measure μ^t of a $\mathcal{P}$-solenoid. It turns out that finite transverse invariant measures are in one-to-one correspondence with foliated cycles (also called *Ruelle-Sullivan current*) and that conversely any foliated cycle implied the existence of a transverse invariant measure.

Proof of Proposition 3.1: if μ^t is a finite invariant transverse measure of a $\mathcal{P}$-solenoid Ω and λ is a left invariant Haar measure on Borelian sets of $\mathcal{P}$ (for example the measure induced by the standard metric on $\mathbb{H}^2$). We can define a global finite measure μ on Ω as follows. On a box $U_i \times C_i$ we consider the product measure $\lambda \otimes \mu^t$, which is well defined thanks the invariance properties of considered measures. Up to multiplication by a scalar, we can suppose the measure μ is a probability measure on Ω. As $\mathcal{P}$ acts on Ω, any element g of $\mathcal{P}$ defines an homeomorphism of Ω denoted τ_g.
Let f be a continuous function on Ω with value in $\mathbb{R}$ with support included in a box $B \simeq U \times C$. Thanks the locally constant return times property, we can decompose B into a finite disjoint union of boxes $b_i \simeq U \times C_i$ where C_i is a closed and open subset of C, such that b_i and $\tau(b_i)$ are included in the same box D_i. We consider now the probability measure $\tau_g * \mu$ obtained by the transport of μ by τ_g. We have

$$\int f d\tau_g * \mu = \sum_i \int_{b_i} f d\tau_g * \mu.$$

In each box D_i, $\int_{b_i} f d\tau_g * \mu = \int_{D_i} f(\tau_g^{-1}(x))\lambda \otimes \mu^t$. For a point $(z,c) \in U \times C_i$, we have $\tau_g^{-1}((z,c)) = (z.g^{-1}, c)$ where for $z = (x,y)$ in $\mathbb{H}^2$ and g^{-1} is the transformation $z \mapsto az + b$, the point $z.g^{-1} = (x + by, ay)$. Therefore we obtain $\int_{b_i} f d\tau_g * \mu = a \int_{b_i} f d\mu$ and

$$\int f d\tau_g * \mu = a \int f d\mu. \tag{$\sharp$}$$

By taking a partition of the unity associated with open sets of an atlas, it is possible to prove the equality ($\sharp$) holds true for any continuous function $f : \Omega \to \mathbb{R}$. Thus the measure $\tau_g * \mu$ is the measure $a\mu$. This is a contradiction with the fact that μ is a probability measure. $\square$

Remark 2 *When the Lie group* $\mathbf{G}$ *is unimodular, a* $\mathbf{G}$*-solenoid admits foliated cycles, which are characterized in [2].*

Remark 3 *The existence of a foliated cycle is a very strong hypothesis. The non existence of foliated cycle gives information on geometric behavior of leaves. Following J. Plante [16], it implies the exponential growth for every leaf of a* $\mathcal{P}$*-solenoid.*

3.2 Harmonic currents

Harmonic currents were introduced by L. Garnett in [7]. The Laplacian Δ in the leaf direction induces an operator $\Delta : A^0(M) \to A^2(M)$ and its image ($Im\Delta$) is contained in the space of exact forms. A *harmonic current* is a continuous operator $A^2(M) \to \mathbb{R}$ strictly positive on strictly positive form and null on $Im\Delta$. Foliated cycles are then specific harmonic current. Any lamination and in particular any $\mathbf{G}$-solenoid admits a harmonic current ([7]).
As for foliated cycles it is possible to associate to a harmonic current I a finite positive measure on M. We choose a metric on the tangent bundle of M. This defined a 2 differential form along the leaves, which enables us to identify $A^2(M)$ with the space of ($\mathcal{C}^\infty$) functions on M. Thanks to the positivity of I, it can be extended to a linear form on space of functions on M and it defines then a finite positive measure μ on M. These measures μ are called *harmonic measures* and are characterized by the following property. For any bounded

measurable function f on M, smooth in the leaf direction, the integral $\int \Delta f d\mu$ is null, where Δ denotes the the Laplacian in the leaf direction.

L. Garnett [7] gives the local structure of such measures. In a box $U_i \simeq V_i \times C_i$ a harmonic measure μ disintegrates into a probability measure ν_i on C_i times the measure $f_i(z,c)dz$ where dz denotes the Riemannian leaf measure and $f_i : V_i \times C_i \to \mathbb{R}^+$ denotes a function defined for almost all c of C_i and harmonic on all the slices $V_i \times \{c\}$. Thus for any Borelian B included in U_i:

$$\mu(B) = \iint_B f_i(z,c)dzd\nu_i(c)$$

This local decomposition is not unique. If two decompositions μ_i, f_i and μ'_i, f'_i define the same measure, then it exists a measurable application $\delta_i : C_i \to \mathbb{R}^+_*$ such that $\mu_i = \delta_i^{-1}(c)\mu'_i$ and $f_i(z,c) = \delta_i(c)f'_i(z,c)$.

Thus if we fix an atlas of M, harmonic functions $f_i(z,c)$ defined on slices are equal on intersecting slices up to a positive constant. Since in our case, leaves have no topology, it is possible to extend the harmonic function $f_i(z,c)$ defined on a slice, into a positive harmonic function on all the leaf.

Remark 4 *For a $\mathbb{R}^2$-solenoid, the leaves are homeomorphic to the plane. The harmonic function obtained is positive and defined on all the plane then it is a constant function. The harmonic measure associated with is locally disintegrated into a measure μ_i on C_i times the Riemannian measure, and μ_i is thus a transverse invariant measure.*

3.3 Harmonic measures and ergodic theorem

Let $x \in M$ be a point of the solenoid and let Γ_x be the set $\{\gamma : \mathbb{R}^+ \to L_x \text{ continuous } | \gamma(0) = x,\ \gamma(\mathbb{R}^+) \subset L_x\}$ where L_x is the leaf passing trough x. The set Γ_x is the set of continuous path beginning at x and strictly include in L_x. We equip this set with the topology of uniform convergence on compact sets. On the space of Borel sets, there exists a natural finite measure w_x called the *Wiener* measure. This measure is defined so that the motion $\Gamma_x \times \mathbb{R}^+ : (\gamma, t) \mapsto \gamma(t) \in L_x$ is a Brownian motion.
Let $\Gamma = \bigsqcup_{x \in M} \Gamma_x$ be the set of continuous paths of M strictly included in leaves, we equip again this set with the topology of uniform convergence on compact sets. If μ is a finite measure on M, then $\overline{\mu} = w_x \otimes \mu(x)$ is a finite measure on Γ.
The semi-group $\mathbb{R}^+$ acts on the space Γ by time translations: for $\tau > 0$ and $\gamma \in \Gamma$ we define the semi-group of transformations S_τ with $S_\tau(\gamma)(s) = \gamma(s+\tau)$. It is straightforward to check transformations S_τ preserve $\overline{\mu}$ if and only if μ is a harmonic measure. This is due to the fact that the Wiener measure is built with the heat kernel. For a harmonic measure μ, we can apply the Birkhoff ergodic theorem.

Theorem 3.3 (L. Garnett) *For any bounded continuous function f from M to $\mathbb{R}$ the limit $l(x,\gamma) = \lim_{n\to\infty} 1/n\Sigma_{i=0}^{n-1} f(\gamma(i))$ exists for μ almost all points x and w_x almost all paths γ of Γ_x.*
This limit is constant on leaves of M and $l(x,\gamma)$ is constant for w_x almost path γ.
Furthermore $\int l(x)d\mu(x) = \int f(x)d\mu(x)$.

Thanks to this theorem, we can define the average time of a generic path γ crossing a Borelian set B of M [8]. It is the limit $\lim_{T\to\infty} 1/T \int_0^T \chi_B(\gamma(t))dt$ where dt denotes the Lebesgue measure and χ_B the indicative function of B.

4 Invariant measures for the action

In this section we characterize invariant measures for the $\mathcal{P}$-action on a $\mathcal{P}$-solenoid M.

4.1 Proof of Theorem 1.1

These measures we are studying are defined on the Borel σ-algebra of the solenoid M. A measure m is *invariant* if for any $g \in \mathcal{P}$ and any measurable set $B \subset M$, $m(B.g) = m(B)$. Since the group $\mathcal{P}$ is the extension of two Abelian groups, $\mathcal{P}$ is amenable, and the set of invariant measures is a closed non-empty set for the weak topology. Actually, for a $\mathcal{P}$-solenoid invariant measures and harmonic measures are the same (Theorem 1.1).

First let us prove that a harmonic measure of M is an invariant finite measure for the $\mathcal{P}$-action. We will use the lemma:

Lemma 4.1 *Let $H : \mathbb{H}^2 \to \mathbb{R}$ be a positive harmonic map. If the quotient $\frac{H(x,y)}{y}$ is uniformly bounded, then $H(x, y) = \alpha y$ for some real α.*

Proof : It is a consequence of the Pick's formula (see [4] for example). Any positive harmonic map H reads $H(x, y) = \alpha y + \int_{-\infty}^{\infty} \frac{y}{(s-x)^2+y^2} d\lambda(s)$ where λ is a positive measure on $\mathbb{R}$ defined for any real $a < b$ by:

$$\lambda(]a, b]) = \lim_{y \to 0} \frac{1}{b-a} \int_{x=a}^{b} H(x, y)dx$$

and dx denotes here the standard Lebesgue measure on the real line. The fact the quotient $\frac{H(x,y)}{y}$ is uniformly bounded implies the measure λ is null. $\square$

Let μ be a harmonic measure of M and let ϕ be a continuous positive function with support included in a box $B \simeq U \times C$ of M. We identify the Lie group $\mathcal{P}$ with $\mathbb{H}^2$ and consider the function $F : \mathcal{P} \to \mathbb{R}$ defined by $F(\tau) = \int \phi d(\tau * \mu)$ where $\tau * \mu$ denotes the measure transported via the action of τ. Fix an element τ of $\mathcal{P}$ and a small positive real ϵ. Thanks the locally constant return times property, we can decompose B into a finite disjoint union of boxes $b_i \simeq U \times C_i$ with C_i a closed and open subset of C with a diameter smaller than ϵ; such that for each i, b_i and $b_i.\tau^{-1}$ are included in a same box D_i. By taking ϵ small enough, for every element g of a neighborhood of τ, we have also that b_i and $b_i.g^{-1}$ are included in D_i.
Therefore, we get

$$F(\tau) = \sum_i \int_{b_i} \phi d\tau * \mu.$$

In each box D_i, the measure μ reads $f_i(z, t)dzd\nu_i(t)$ with f_i a harmonic map in z. Then

$$\int_{b_i} \phi dg * \mu = \int_{D_i} \phi(z.g^{-1}, t) f_i(z, t)dzd\nu_i$$

$$= \int_{D_i} \phi(z, t) f_i(z.g, t) \frac{dz}{a} d\nu_i$$

where g is the map $z \mapsto az + b$. We recall here for $z = (x, y)$ in $\mathbb{H}^2$, $z.g = (x + by, ay)$.

As shown in section 3.2, the map $f_i(., t)$ for a fixed t, can be extended to a harmonic map on the whole half plane $\mathbb{H}^2$. The map $g \mapsto f_i(z.g, t)$ is defined on $\mathcal{P}$ and it is straightforward

to check it is a harmonic map. Thus the bounded map $g \in \mathbb{H}^2 \to \int_{b_i} \phi dg * \mu \in \mathbb{R}$ reads $(x, y) \mapsto \frac{H(x,y)}{y}$ with H a positive harmonic map. The lemma 4.1 enables us to conclude the function F is constant.
For a continuous function ϕ, by taking a partition of the unity associated with a cover of M by the open set of an atlas , we can prove the value $\int \phi d(\tau * \mu)$ is independent of τ, this concludes the first part of the proof.

Conversely let us prove that finite invariant measures are harmonic measures. This can be seen in the local expression of an invariant measure.

Lemma 4.2 *If a measure m on M is an invariant measure for the right $\mathcal{P}$-action then in each box, the measure m disintegrates into a transversal sum of leaf measures, where almost every leaf measure is a right invariant Haar measure of $\mathcal{P}$.*

Proof : Fix a box $V \times C$, we decompose m in this box into a transversal measure ν on C and a system of leaf measure σ_c on $V \times \{c\}$ for each c of C. Hence we have for any measurable function f with support included in the box,

$$\int f dm = \int_C \int_V f(z, c) d\sigma_c(z) d\nu(c).$$

We fix a point x of the box and a closed neighborhood K included in the box. Let A be the set of bounded measurable functions with support in K. If m is $\mathcal{P}$-invariant for any $f \in A$ and for any $g \in \mathcal{P}$ s.t. $K.g$ is included in the box, $\int f(x) - f(x.g) dm(x) = 0$.
We can decompose $f = f_1 + f_2$ where f_1 is the restriction of f to slices for which $\int_V f(x) - f(x.g) d\sigma_c > 0$; and f_2 is the restriction of f to slices for which the integral is negative. If m is invariant, then $\int f_i(x) - f_i(x.g) dm(x) = 0$ and thus

$$\nu\{c \in C | \int_V f_i(x) - f_i(x.g) d\sigma_c \neq 0\} = 0 \quad \text{for } i = 1, 2.$$

It follows that when m is invariant, for ν almost all c in C, $\int f(x) - f(x.g) d\sigma_c = 0$. Therefore, by identifying the leaf with the Lie group $\mathcal{P}$, for ν almost all c, σ_c is a right invariant Haar measure. □

When identifying the Lie group $\mathcal{P}$ with $\mathbb{H}^2$, a right invariant measure reads $\frac{\lambda}{y} dx dy$ for some constant $\lambda > 0$. Therefore an invariant measure m on M can be written in a box $\lambda_c \frac{dxdy}{y} d\nu(c)$, where $c \in C \mapsto \lambda_c \in \mathbb{R}^+$ is a measurable map. Then the measure m is harmonic. This ends the proof of Theorem 1.1.

As we know, the local decomposition of an invariant measure m is not unique. If $\frac{\lambda_c}{y} dx dy d\nu(c)$ and $\frac{\lambda'_c}{y} dx dy d\nu'(c)$ are two decompositions of the same measure m, the measures ν and ν' are in the same class, and thus there exists a positive measurable map defined almost everywhere $\delta : C \to \mathbb{R}^+_*$ such that $\nu = \frac{1}{\delta(.)} \nu'$ and $\lambda_c = \delta(c) \lambda'_c$. An important consequence is that the value $\int_C \lambda_c d\sigma(c)$ is well defined. Consider f the positive function $\mathbb{H}^2 \to \mathbb{R}$ defined by $f(x, y) = \int_C \lambda_c d\sigma(c).y$, then the measure of a *cylinder* $A \times C$ (where A is a measurable set of V) of the box is $m(A \times C) = \int_A f(x, y) \frac{dxdy}{y^2}$. We will use this function to characterize invariant measures.

4.2 Combinatorics of the invariant measures

For a branched manifold $\mathcal{B}$, let us denote by $\mathcal{C}_2(\mathcal{B},\mathbb{R})$ the finite dimensional $\mathbb{R}$ module space with basis the 2 faces of $\mathcal{B}$.Its elements are called 2 *chains*. For all the branched manifolds that we consider, the 2-faces are equipped with a natural orientation. Let $\mathcal{C}_2(\mathcal{B},\mathbb{R})^+$ be the cone of vectors of $\mathcal{C}_2(\mathcal{B},\mathbb{R})$ with positive coefficients, and let $\mathbf{P}(\mathcal{B},\mathbb{R})$ be the intersection of $\mathcal{C}_2(\mathcal{B},\mathbb{R})^+$ and the closed unit sphere centered in the origin for the norm $|(b_1,\ldots,b_q)|_1 = \Sigma_i|b_i|$. We denote by $\mathcal{M}(M)$ the set of finite positive measure of M invariant for the $\mathcal{P}$-action.

We consider first a box decomposition of the $\mathcal{P}$-solenoid M. With each box B and for an invariant measure m, we have seen that we can associate a non negative number $b = \int_C \lambda_c d\sigma(c) > 0$. The identification of elements belonging to the same vertical of the box decomposition leads to a fibration p of M over a branched manifold $\mathcal{B}$. We associate to the interior F_i of a 2-face of $\mathcal{B}$ a box $B_i = p^{-1}(F_i)$ with the fibration and then we consider the 2-chain $\Sigma_i b_i \overline{F_i} \in \mathcal{C}_2(\mathcal{B},\mathbb{R})^+$. Therefore the fibration $p : M \to \mathcal{B}$ induces a linear map $p_* : \mathcal{M}(M) \to \mathcal{C}_2(\mathcal{B},\mathbb{R})^+$.

If we consider now a tower decomposition $(\mathbf{B}_n)_n$, we obtain a sequence of fibration p_n over branched manifolds $\mathcal{B}_n$ and a sequence of map $\pi_n : \mathcal{B}_{n+1} \to \mathcal{B}_n$ such that $p_n = \pi_n \circ p_{n+1}$ and $M \simeq \lim_{\leftarrow}(\mathcal{B}_n, \pi_n)$. These maps induce linear maps $(p_n)_* : \mathcal{M}(M) \to \mathcal{C}_2(\mathcal{B}_n,\mathbb{R})^+$.

The relation between $(p_n)_*(m)$ and $(p_{n+1})_*(m)$ can be described as follows. We denote by $B_i^n \simeq F_i^n \times C_i^n$ the boxes of $\mathbf{B}_n$, where the index i is an enumeration of these boxes. Let $f_i(x,y)$ be the function $(x,y) \mapsto \int_{C_i^n} \lambda_{ic}^n d\sigma_i^n(c).y = b_i^n y$ for a local decomposition of the measure m. The intersection of B_i^n and B_j^{n+1} is either empty or a disjoint union of boxes $\bigsqcup_l D_{ij}^l$. In the non trivial case, there exists transition maps $h_{ij}^l : D_{ij}^l \cap B_i^n \to B_j^{n+1}$, with $h_{ij}^l(z,c) = (g_{ij}^l.z, \gamma_{ij}(c))$ and $g_{ij}^l \in \mathcal{P}$.

Thus for any cylinder $A \times C_i^n$ of B_i^n we have

$$m(A \times C_i^n) = \sum_j \sum_l m(h_{ij}^l((A \times C_i^n) \cap D_{ij}^l)$$

$$= \sum_j \sum_l \int_{g_{ij}^l(A)} f_j(x,y) \frac{dxdy}{y^2}$$

$$= \sum_j \sum_l \int_A \alpha(g_{ij}^l) b_j^{n+1} \frac{dxdy}{y}$$

where α is the morphism $\alpha(z \mapsto az+b) = a$

$$= \sum_j \sum_l \alpha(g_{ij}^l) \int_A f_j(x,y) \frac{dxdy}{y^2}.$$

Since this is true for any $A \subset V_i^n$, we have the relation:

$$b_i^n = \sum_j \sum_l \alpha(g_{ij}^l) b_j^{n+1} = \sum_j b_j^{n+1} \sum_l \alpha(g_{ij}^l).$$

Let us denote $p(n)$ the dimension of $\mathcal{C}_2(\mathcal{B}_n,\mathbb{R})$ and A_n the $p(n) \times p(n+1)$ matrix with positive coefficients $a_{i,j}^n = \sum_l \alpha(g_{ij}^l)$ when B_i^n and B_j^{n+1} intersect and 0 otherwise. We have the relation $(p_n)_*(m) = A_n((p_{n+1})_*(m))$, and thus the sequence $((p_n)_*(m))_n$ is an element of $Lim_{\leftarrow}(\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$. This enables us to extend maps $(p_n)_*$ to a map

$$p_* : \ \mathcal{M}(M) \to \lim_{\leftarrow}(\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n).$$

It is obvious that p_* maps the set of probability invariant measures to the set $\lim_{\leftarrow}(\mathbf{P}(\mathcal{B}_n, \mathbb{R}), A_n)$.

Actually this linear map is an isomorphism whose inverse can be constructed as follows. Let $(v_n)_n$ be an element of $\lim_{\leftarrow}(\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$. We consider the family of cylinder A such that there exists a box $B_i^n \simeq V_i^n \times C_i^n$ where $A \subset B_i^n$ and $A \simeq A_i^n \times C_i^n$ for some measurable subset A_i^n of V_i^n. Let $m(A)$ be the value $\int_{A_i^n} b_i^n \frac{dxdy}{y}$ where $v_n = (b_1^n, \dots, b_i^n, \dots, b_{p(n)}^n)$. Thanks to the relations between v_n and v_{n+1}, the value $m(A)$ is well defined and can be extended by additivity to the σ-algebra generated by cylinders A. This set is big enough so that its σ-algebra is actually the Borel σ-algebra. It is then straightforward to check that $p_*(m) = (v_n)_n$. Furthermore, since m disintegrates locally into a transverse measure times a measure of the kind $by\frac{dxdy}{y^2}$ on the slices, m is a harmonic measure, then from Theorem 1.1 m is also an invariant measure.

The above result can be summarized in the following theorem which is an explicit reformulation of Theorem 1.2 :

Theorem 4.3 *If M is a $\mathcal{P}$ solenoid and M is homeomorphic to a projective limit of branched manifolds $\mathcal{B}_n$, $\lim_{\leftarrow}(\mathcal{B}_n, p_n)$, constructed with a tower system.*
Then: $\mathcal{M}(M)$ is homeomorphic to
$\lim_{\leftarrow}(\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$, where A_n is a matrix with positive coefficients $A_n : \ \mathbb{R}^{p(n+1)} \to \mathbb{R}^{p(n)}$ with dim $\mathcal{C}_2(B_n, \mathbb{R}) = p(n)$.
The restriction to the set of invariant probability measure is then homeomorphic to $Lim_{\leftarrow}(\mathbf{P}(\mathcal{B}_n, \mathbb{R})^+, A_n)$.

This last theorem allows us to exhibit some criteria to bound the number of invariant probabilities.

Proposition 4.4 *With the same conditions as in Theorem 4.3 and M is minimal.*

1. *If the number of faces of $\mathcal{B}_n$ are uniformly bounded by N, then there is at most N ergodic invariant probability measures.*

2. *If furthermore M is minimal and the linear map A_n are uniformly bounded, then there is a unique invariant probability measure.*

Proof : Without loose of generality, we may assume that for all $n \geq 1$, $dim_{\mathbb{R}}\mathcal{C}_2(\mathcal{B}_n, \mathbb{R}) = N$. Let us consider N sequences $(w_j^n)_n \in \Pi_n \mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+$ for $j \in \{1, \dots, N\}$ where $w_j^n = (w_{j,1}^n, \dots, w_{ji}^n, \dots, w_{j,N}^n)$ and $w_{j,i}^n = 0$ if $j \neq i$ and 1 otherwise.

Fix an integer n, for any j in $\{1, \dots, N\}$ and $m > n$ let $w_j^{nm} = A_n \circ \dots \circ A_{m-1}(w_j^m)$. Up to a choice of a subsequence, we can suppose that the sequences $(w_j^{nm})_{m>n}$ converge to $w_j \in \mathbf{P}(\mathcal{B}_1, \mathbb{R})$. Let us denote $proj_n$ the projection of the product $\Pi_n \mathcal{C}_2(\mathcal{B}_n, \mathbb{R})$ onto $\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})$, and $Prob_n = proj_n(Lim_{\leftarrow}(\mathbf{P}(\mathcal{B}_n, \mathbb{R}), A_n)$. The set $Prob_n$ is a convex set and if H_m is the convex hull of $\{w_j^{nm} | j = 1, \dots, N\}$, we have $Prob_n = \bigcap_{m>n} A_n \circ \dots \circ A_{m-1}(H_m)$. Therefore

$Prob_n$ is the convex hull of $\{w_j | j = 1, \dots, N\}$. Suppose now there is more than N ergodic invariant probabilities then for n big enough, there would be more than N extremal points in $Prob_n$, a contradiction.

In order to prove the second statement, we show that for any n, $Prob_n$ is reduced to a point. For this we define the hyperbolic distance between two points x, y in $\mathbf{P}(\mathcal{B}_n, \mathbb{R})$.

$$d_h(x, y) = -ln \frac{(m + l).(m + r)}{l.r}$$

where m is the Euclidean length of the segment $[x, y]$ and l, r are the length of connected components of $S \backslash [x, y]$ where S is the largest line segment containing $[x, y]$ in $\mathbf{P}(\mathcal{B}_n, \mathbb{R})$. It is straightforward to check positive matrices contract this distance and the minimality of the action implies the positivity of matrices. Since linear maps A_n are uniformly bounded and defined on space with bounded dimension, the contraction is uniform. Therefore $Prob_n = \bigcap_{m>n} A_n \circ \ldots \circ A_{m-1}(\mathbf{P}(\mathcal{B}_m, \mathbb{R}))$ is reduced to a point. □

5 Examples and proof of Proposition 1.3

We give an example of a non periodic repetitive $\mathcal{P}$ finite type tiling with exactly r ergodic invariant probability measures, for any integer $r > 0$.

The idea is to decorate the Penrose's tiling with a non periodic bi-infinite sequence. We choose a sequence such that the action of the shift on the closure X of the orbit for the action, is minimal and has r ergodic invariant probability measures.

First, consider the case $r \geq 2$. Let Σ be the set $\{1, \dots, r\}$. We associate to each symbol in Σ a different color. Let P be the polygon defined in the introduction to build the Penrose's tiling. Let R and S be the affine maps defined in the introduction. For an element i of Σ, let P_i be the prototile P painted in the color i. To a sequence $w = (w_k)_{k \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$, we associate the decorated tiling $\mathcal{T}(w)$ of finite affine type, with prototiles P_i for i in Σ, defined by

$$\mathcal{T}(w) = \{R^q \circ S^n(P_{w_q}) | n, q \in \mathbb{Z}\}.$$

Its tiles are isometric to P and its stabilizer is included in $< R >$. To a sequence $(w_n)_{n \in \mathbb{Z}}$ the shift σ associates the sequence $(w'_n)_{n \in \mathbb{Z}}$ where $w'_n = w_{n+1}$. Thus we have $\mathcal{T}(w).R = \mathcal{T}(\sigma(w))$. Therefore if the sequence w is not periodic for the action of the shift, then $\mathcal{T}(w)$ is not stable for any element of $\mathcal{P}$.
The product space $\Sigma^{\mathbb{Z}}$ is equipped with the product topology and is a Cantor set. Let X denote the closure of the orbit of w by the action of the shift σ: $X = \overline{\{\sigma^n(w),\ n \in \mathbb{Z}\}}$. The set X is a compact metric space stable under the action of σ. When the dynamical system (X, σ) is minimal then $\Omega(\mathcal{T}(w))$ is minimal.

In [23], S. Williams generalizes an example of J. C. Oxtoby ([14]) and defines a Toeplitz sequence $w \in \Sigma^{\mathbb{Z}}$ for which the action of the shift is minimal and has r ergodic probability measures. We recall here the definition of this sequence.
Consider the sequence of natural numbers $(p_i)_{i \in \mathbb{N}}$ with $p_0 = 3$ and $p_{i+1} = 3^i.p_i$ and the sequence $s_i \equiv i \bmod r \in \Sigma$ for $i \in \mathbb{N}$.
Define then the sequence $w = (w_q)_{q \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ by inductive steps. The first step (step 1) is to set $w_q = s_1$ for all $q \equiv 0$ or $-1 \bmod p_1$. In general for $i \in \mathbb{N}$, k in $\mathbb{Z}$, let $J(i, k)$ denote the set

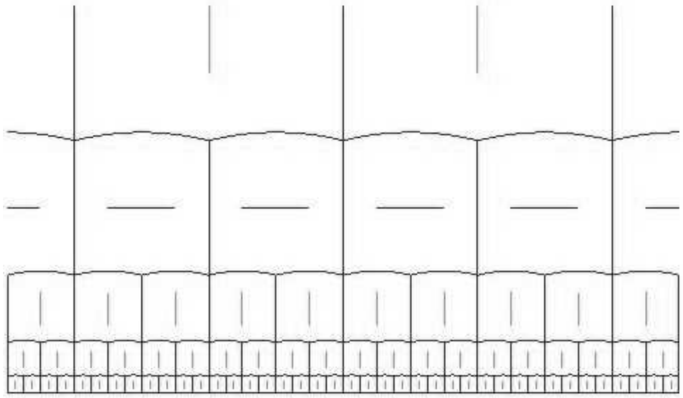

Figure 3: Decorated Penrose's tiling associated to an Oxtoby's sequence

of integers $q \in [kp_i, (k+1)p_i)$ for which w_q has been not yet defined at the end of the step i. The step $(i+1)$ is to set $w_q = s_{i+1}$ for $q \in J(i,k)$ with $k \equiv -1$ or 0 mod 3^i.
The dynamical system (X, σ) is minimal and X is a Cantor set.

Let us define now a sequence of atlas of words for the sequence w. Let $\mathcal{A}_0$ be the set of words $\{s_i, \ i = 1 \dots r\}$. Let $\mathcal{A}_1$ be the set of words $\{s_1 s_i^{p_1 - 2} s_1, \ i = 1, \dots, r\}$, where for two words a and b, ab denotes the concatenation of the two words and a^q denotes the concatenation of q times the word a. In the general case for any integer $q \geq 1$, we denote by $p_{q,i}$ $i \in \{1, \dots, r\}$ the word of $\mathcal{A}_q$ indexed by i and for $q > 1$, $\mathcal{A}_q$ is the set of words $\{p_{q-1,s_q}(p_{q-1,i})^{3^{q-1}-2} p_{q-1,s_q}, \ i = 1, \dots, r\}$. For any $q \in \mathbb{N}$ the sequence w is a bi-infinite sequence of words of $\mathcal{A}_q$.

The *suspension* of the action of σ on X, is the quotient space $\mathcal{X} = \mathbb{R} \times X/\sigma$ where points (t, x) and (s, x') are identified if $s - t \in \mathbb{Z}$ and $x = \sigma^{s-t}(x')$. The natural $\mathbb{R}$-action by time translation on the space $\mathbb{R} \times X$ induces a $\mathbb{R}$-action on the suspension. It turns out that the suspension $\mathbb{R} \times X/\sigma$ is a $\mathbb{R}$-solenoid ([2]) which has exactly r invariant ergodic probability measures ([23]). For any $q \geq 0$, $\mathcal{A}_q$ defines a box decomposition of the suspension $\mathcal{X}$. Each box is identified with a unique word of $\mathcal{A}_q$.

We will construct a tower system for $\Omega(\mathcal{T}(w))$ associated to the former box decompositions of the suspension, thanks to a collection of patches for the tiling $\mathcal{T}(w)$. For a word $b = w_{i_0} \dots w_{i_0+l}$ of w, let $\mathcal{P}a(b)$ be the patch $\bigcup_{j=0}^{l} \{R^{-j} \circ S^k(P_{w_{i_0+j}})$ for $k = 0, \dots, j\}$ of $\mathcal{T}(w)$. Now let us consider for $q \geq 0$ the collection of patches $\mathcal{P}a_q = \{\mathcal{P}a(p_{q,i}), \text{ for } i = 1, \dots, r\}$. For any q, the tiling $\mathcal{T}(w)$ is an union of elements of $\mathcal{P}a_q$, copies of patches meeting only on their borders. Remark that all the patches of $\mathcal{P}a_q$ have the same size and actually, the box decompositions of $\Omega(\mathcal{T}(w))$ associated to $\mathcal{P}a_q$ define a tower system of the hull.
If we denote by $\sim_q$ the relation generated by the identification of borders of patches of $\mathcal{P}a_q$ which meet somewhere in the tiling $\mathcal{T}(w)$ and $\mathcal{B}_q = \bigsqcup_{i=1}^{r} \mathcal{P}a_{p_{q,i}} / \sim_q$, we have applications π_q such that:

$$\Omega(\mathcal{T}(w)) \simeq \lim_{\leftarrow} (\mathcal{B}_q, \pi_q).$$

Now we construct a natural continuous map h from $\Omega(\mathcal{T}(w))$ onto $\mathcal{X}$. For an element

$g : z \mapsto az + b$ of the group $\mathcal{P}$, we define $h(\mathcal{T}(w).g) = [(\log_2(a), w)] \in \mathcal{X}$ where $[(t, x)]$ denotes the class of the element (t, x) in $\mathbb{R} \times X$ for the relation defined by σ. The map h is then a continuous map from $\mathcal{T}(w).\mathcal{P}$ to $\mathcal{X}$. Remark that if the origin O lies in a copy of a patch $\mathcal{P}a(p_{q,i})$ for some $q \geq 1$ and $i \in \Sigma$ in the tiling $\mathcal{T}(w).g$, then O lies also in a copy of the patch $\mathcal{P}a(p_{q,i})$ in the tiling $\mathcal{T}(\sigma^n(w))$, where n denotes the integer part of $\log_2(a)$. Thus the origin of the sequence $\sigma^n(w)$ lies in the word $p_{q,i}$. As $h(\mathcal{T}(w).g) = [(\log_2(a) - n, \sigma^n(w))]$, we get that $h(\mathcal{T}(w).g)$ is in the box of the suspension defined by the word $p_{q,i}$. It follows that for any $q \geq 1$, the map h sends the restriction to the orbit of $\mathcal{T}(w)$ of the box associated to the patch $\mathcal{P}a(p_{q,i})$ to the box of the suspension associated to the word $p_{q,i}$. Thus the map h is uniformly continuous.
It follows that h can be extended to a map from $\Omega(\mathcal{T}(w))$ onto $\mathcal{X}$ also denoted h. It is straightforward to check that each fiber of the map h is stable under the action of the group $\mathcal{N} = \{z \mapsto z + t,\ t \in \mathbb{R}\}$. Furthermore, as $\mathcal{P}$ is an extension over $\mathcal{N}$ and the group $\{z \mapsto az,\ a > 0\}$, the action of the group $\mathcal{P}$ preserves the set of fibers. Then the $\mathcal{P}$-action on the hull $\Omega(\mathcal{T}(w))$ defines through the application h, a $\mathcal{P}$-action on the suspension $\mathcal{X}$ and h is a semi-conjugacy from the hull $\Omega(\mathcal{T}(w))$ to $\mathcal{X}$. The group $\mathcal{N}$ acts trivially on $\mathcal{X}$. The invariant measures for the $\mathcal{P}$-action on $\mathcal{X}$ are the invariant measures for the $\mathbb{R}$-action. We claim that the map h sends the invariant measures of the hull onto the invariant measures of the suspension.
To prove this, we use a Følner's base of $\mathcal{P}$ that we denote $(A_n)_n$ and a right multiplicative invariant Haar measure on $\mathcal{P}$ that we denote λ. Let μ be a ergodic invariant probability measure for the $\mathcal{P}$-action on $\mathcal{X}$. By the ergodic theorem, there exists a point x in the suspension such that the sequence of probability measures $\mu_n = \frac{1}{\lambda(A_n)} \int_{A_n} \delta_{g.x} d\lambda(g)$ converges, when n grows to infinity, to the measure μ. Let y be a point in $\Omega(\mathcal{T}(w))$ such that $h(y) = x$. Then, up to the choice of a subsequence, the sequence of probability measures on $\Omega(\mathcal{T}(w))$ $\nu_n = \frac{1}{\lambda(A_n)} \int_{A_n} \delta_{g.y} d\lambda(g)$ converges to a probability measure ν invariant for the $\mathcal{P}$-action. As $h * \nu_n = \mu_n$, we get $h * \nu = \mu$. It follows that the map h sends the set of invariant measures of $\Omega(\mathcal{T}(w))$ onto the set of invariant measures of $\mathcal{X}$. Furthermore the map h sends ergodic measures on ergodic measures. Then $\Omega(\mathcal{T}(w))$ has at least r independent ergodic probability measures. From Proposition 4.4, we also know that the hull $\Omega(\mathcal{T}(w))$ admits at most r invariant ergodic probability measures. Thus there are exactly r probability measures.

To obtain an example of a minimal $\mathcal{P}$-solenoid with a single $\mathcal{P}$-invariant probability measure, we use the same strategy as before. We keep the same notations as the case $r = 2$ but we define an other Toeplitz sequence w on which the shift action is free, minimal and uniquely ergodic ([9]). We consider the *substitution* $\mathfrak{S}$ over the alphabet $\Sigma = \{1, 2\}$ defined by $\mathfrak{S}(1) = 112$, $\mathfrak{S}(2) = 122$. Using the extension of the substitution over the words by the concatenation, we can iterate the substitution. The sequence w is then the bi-infinite sequence defined by:

$$w = \lim_n \overleftarrow{\mathfrak{S}^n(2)}.\lim_n \overrightarrow{\mathfrak{S}^n(1)},$$

where the dot . is placed between the 0 and -1 coordinate.

Let $\mathcal{A}_0$ be the set $\{1, 2\}$, and for any integer $q \geq 1$, let $\mathcal{A}_q$ be the atlas of words $\{\mathfrak{S}^{q-1}(1)\mathfrak{S}^{q-1}(i)\mathfrak{S}^{q-1}(2),\ i = 1, 2\}$ for the sequence w. The sequence w is a bi-infinite sequence of words of $\mathcal{A}_q$. Now let us consider the collection of patches $\mathcal{P}a_q = \{\mathcal{P}a(wo),\ wo \in \mathcal{A}_q\}$. For any $q \geq 0$, the tiling $\mathcal{T}(w)$ is an union of elements of $\mathcal{P}a_q$ and the box decompositions

of $\Omega(\mathcal{T}(w))$ associated to $\mathcal{P}a_q$ define a tower system of the hull. The hull $\Omega(\mathcal{T}(w))$ is then homeomorphic to $\lim_{\leftarrow}(\mathcal{B}_q, \pi_q)$ where $\mathcal{B}_q = \bigsqcup_{wo\in\mathcal{A}_q} \mathcal{P}a(wo)/\sim_q$.
By Theorem 4.3, the space of invariant measures $\mathcal{M}(\Omega(\mathcal{T}(w)))$ is isomorphic to $\lim_{\leftarrow}(\mathcal{C}_2(\mathcal{B}_n, \mathbb{R})^+, A_n)$. A simple calculation shows that the linear applications A_n are defined by the matrices:

$$A_n = \begin{pmatrix} 1 + 2^{-3^n+1} & 1 \\ 2^{-3^n2+2} & 2^{-3^n+1} + 2^{-3^n2+2} \end{pmatrix}.$$

Proposition 4.4 enables us to conclude that the hull $\Omega(\mathcal{T}(w))$ admits only one $\mathcal{P}$-invariant probability measure.
Acknowledgments. The author would thank B. Deroin for helpful comments. He also thanks the the C.M.M. and the D.I.M. of the University of Chile, where part of this work has been done, for their warm hospitality. This work has been done thanks to the support from ECOS-Conicyt grant C03-E03.

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