THE VALUATIVE CAPACITY OF SUBSHIFTS OF FINITE TYPE

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ABSTRACT. The characteristic sequence of a subset E of the integers enable to generalize the factorial. The asymptotic limit of this sequence, called the valuative capacity, is actually related to the transfinite diameter of this set embedded into the set of p-adic integers. We use tools of potential theory to obtain formulas to compute this capacity. As an example, we provide a family of sets, invariant by a family of graph-directed iterated function systems, whose valuative capacities are algebraic numbers.

1. INTRODUCTION

M. Bhargava [3], using the notion of integer-valued polynomial, gives a generalization of the notion of factorial. He associates, in an algebraic way, with each subset E of $\mathbb{Z}$, a factorial sequence $(n!_E)_{n\in\mathbb{N}}$ that preserves the properties of the classical one. This notion is linked to the notion of *p-ordering sequence* defined via the p-adic valuation v_p for a prime number p: A sequence $(a_n)_{n\geq 0} \subset E$ is a p-ordering if for any $n \geq 1$, a_n realizes the minimum $\min_{x\in E} v_p\big((a_{n-1}-x)\cdots(x-a_0)\big)$. An important property is that the value of this minima, possibly infinite, does not depend on the sequence $(a_n)_n$, and is denoted $w_E(n,p)$. The factorial of Bhargava is then the value $n!_E := \prod_p p^{-w_E(n,p)}$ where the product is taken over all the prime numbers. One can show that the sequence $(w_E(n,p))_n$ is superadditive, so the limit $\lim_{n\to\infty} \frac{w_E(n,p)}{n} =: L_p(E)$ exists when it is finite, it is called the *valuative capacity* of the set E [5].

An other way to study the set E is to consider its closure $\bar{E}$ into the p-adic integers $\mathbb{Z}_p$ equipped with the p-adic metric and use potential theory and harmonic analysis. One can check that the valuative capacity $L_p(E)$ is related to the *transfinite diameter* $d_\infty(E)_p$ of E via the formula $d_\infty(E)_p = p^{-L_p(E)}$. This implies, in particular, the following variational equality:

$$L_p(E) = \inf_{\mu\in\mathcal{P}(\bar{E})} \iint v_p(x-y)d\mu(x)d\mu(y),$$

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where $\mathcal{P}(\bar{E})$ denotes the set of probability measures with support in the set $\bar{E}$. Classical results assert that there exists a measure (called an *equilibrium measure*) realizing the infimum. When the infimum is finite, this measure is unique and is invariant by any isometry of $\bar{E}$.

These properties enable us to recover some existing results but proven in an algebraic way like in [9] and to obtain new ones, *e.g.*, any p-ordering sequence of E is distributed according to te equilibrium measure. Conversely the algebraic methods enable to perform explicit computation of the capacity.

We are mainly interested by the following problem: Given an infinite compact set E in $\mathbb{Z}_p$ (or in $\mathbb{C}_p$), invariant by some continuous map, can we explicitly compute the capacity of this set? L. DeMarco and R. Rumely give in [6] the answer to this question for a compact set invariant by a polynomial map. In this paper, by using technics elaborated in [7, 9], we treat the case where the set E in $\mathbb{Z}_p$ is invariant under a graph directed iterated function system coded by a subshift of finite type on an alphabet with at most p letters. We obtain that such capacities are always algebraic numbers. Surprisingly we get that the capacity associated with the Fibonacci subshift of finite type is $\sqrt{2}$ (see Section 5). Notice that the formula in [6] implies that the valuative capacity of a set invariant by a polynomial is always rational.

This paper is structured as follow: Section 2 and Section 3 are devoted to the necessary background on respectively p-ordering sequences and potential theory. We deduce in the next section an equidistribution result of p-ordering sequences. We also recover, using potential theory arguments, a formula to compute valuative capacities. Its original proof was given by K. Johnson in [9], but with combinatorial arguments. Finally, Section 5 concerns the capacity computation of compact sets invariant by graph-directed iterated function systems.

2. p-ordering sequences

2.1. **Backgrounds.** Our interest will focus on subsets of the ring of p-adic rational numbers $\mathbb{Q}_p$ where throughout the paper p denotes a prime number.

Recall that for a rational number $x \in \mathbb{Q}^*$, the *p-adic valuation* $v_p(x) = r \in \mathbb{Z}$ if x may be written as $p^r \frac{a}{b}$ with the integers a, b, p relatively prime, and we define $v_p(0) = +\infty$. The classical properties of this valuation are

$$\begin{aligned} v_p(xy) &= v_p(x) + v_p(y) && \text{for any } x, y \in \mathbb{Q}, \\ v_p(x+y) &\leq \inf(v_p(x), v_p(y)) && \text{with equality if } v_p(x) \neq v_p(y). \end{aligned}$$

The *p-adic norm* of x is then $|x|_p = p^{-v_p(x)}$ and is an ultra-metric. The completions of $\mathbb{Q}$ and $\mathbb{Z}$ with this norm are respectively the p-adic field $\mathbb{Q}_p$ and the p-adic ring $\mathbb{Z}_p$. The p-adic valutation, with its properties, extends to those sets. Note that any element $x \in \mathbb{Q}_p$ can be written as $x = \sum_{k=v_p(x)}^{+\infty} x_k p^k$ with $x_k \in \{0, 1, \dots, p-1\}$.

In order to study properties of a relatively compact subset E of $\mathbb{Q}_p$, we will use the notions of integer-valued polynomials on E and of p-ordering sequence.
We denote, for $N \in \mathbb{N} \cup \{\infty\}$, by $\mathrm{Int}_N(E, \mathbb{Z}_p)$ the ring of integer-valued polynomial on E with degree lesser than N, that is

$$\mathrm{Int}_N(E, \mathbb{Z}_p) = \{f \in \mathbb{Q}_p[X] \mid \forall x \in E, f(x) \in \mathbb{Z}_p, \deg f \leq N\}.$$

It is well known that $\mathrm{Int}_\infty(\mathbb{Z}, \mathbb{Z}_p)$ is a $\mathbb{Z}$-module generated by the binomial polynomials $\binom{X}{n} = \frac{\prod_0^{n-1}(X-k)}{n!}$. It is also known, see [4], that $\mathrm{Int}_\infty(E, \mathbb{Z}_p)$ admits a regular basis $(f_n)_{n\in\mathbb{N}}$, i.e., a basis such that $\deg f_n = n$, for all $n \in \mathbb{N}$. Such a basis can be given using a *p-ordering sequence* of E, defined as follows.

Definition 2.1. [3] *Let $N \in \mathbb{N} \cup \{\infty\}$. A sequence $(a_n)_{0\leq n\leq N} \subset E$ is said to be a p-*ordering *of length N if for every $1 \leq n \leq N$, one has*

$$v_p\Big(\prod_{k=0}^{n-1}(a_n - a_k)\Big) \leq v_p\Big(\prod_{k=0}^{n-1}(x - a_k)\Big) \text{ for every } x \in E.$$

Notice, since E is bounded and the valuation is discrete, for any point $a \in E$, there always exists a p-orderings sequence $(a_n)_{0\leq n\leq N} \subset E$ with $a_0 = a$. More generally it is possible to complete a p-ordering sequence: if $(a_n)_{0\leq n\leq N}$ is a p-ordering sequence, there exists an element $a_{N+1} \in E$ such that $(a_n)_{0\leq n\leq N+1}$ is still a p-ordering sequence but of length $N+1$. The next proposition is due to M. Bhargava.

Proposition 2.2. [3] *The sequence $(a_n)_{0\leq n\leq N}$ is a p-ordering of E if and only if the polynomials $f_n(X) = \prod_{k=0}^{n-1} \frac{X-a_k}{a_n-a_k}$, $1 \leq n \leq N$, form a basis of the $\mathbb{Z}_p$-module $\mathrm{Int}_N(E, \mathbb{Z}_p)$.*

For a positive integer N, the N^{th}-*factorial ideal* is the ideal

$$\{y \in \mathbb{Z}_p \mid y\mathrm{Int}_N(E, \mathbb{Z}_p) \subseteq \mathbb{Z}_p[X]\}.$$

The Proposition 2.2 implies that this ideal is generated by the product $\prod_{k=0}^{N-1}(a_N - a_k)$ for any p-ordering sequence $(a_n)_{0\leq n\leq N}$ of E. We deduce the following characterization of p-ordering sequences.

Proposition 2.3. *Let $E \subset \mathbb{Q}_p$ be a bounded set. Then for any p-ordering $(a_n)_{0\leq n\leq N}$ of E, we have for any $0 < n \leq N$*

$$v_p\Big(\prod_{k=0}^{n-1}(a_n - a_k)\Big) = \sup_{b_0,\dots,b_{n-1}\in E} \inf_{x\in E} v_p\Big(\prod_{k=0}^{n-1}(x - b_k)\Big) =: w_E(n,p). \tag{1}$$

In particular the value $w_E(n)$ is independent of the choice of the p-ordering sequence $(a_n)_{0<n\leq N}$ of E.

Conversely, any sequence $(a_n)_{0\leq n\leq N}$ of E satisfying the equality (1) for any $0 < n \leq N$, is a p-ordering of E.

The integer sequence $(w_E(n,p))_n$ is called the *characteristic sequence* of the subset E.
Notice that the first statement was already in [5], we propose here a more direct proof.

Proof. Assume first that $(a_n)_{0<n\leq N}$ is a p-ordering of E. The inequality

$$v_p(\prod_{k=0}^{n-1}(a_n-a_k)) \leq \sup_{b_0,\dots,b_{n-1}\in E} \inf_{x\in E} v_p(\prod_{k=0}^{n-1}(x-b_k)). \tag{2}$$

follows from the very definition of p-ordering sequence. Let us show the converse inequality. By the previous proposition, $\prod_0^{n-1}(a_n-a_k)$ generates the n^{th}-factorial ideal.
Let $b_0,\dots,b_{n-1},b_n\in E$ realizing the extremal value, *i.e.*,

$$\sup_{b_0,\dots,b_{n-1}\in E} \inf_{x\in E} v_p(\prod_{k=0}^{n-1}(x-b_k)) = v_p(\prod_{k=0}^{n-1}(b_n-b_k)).$$

Hence the polynomial $\prod_{k=0}^{n-1}\frac{X-b_k}{b_n-b_k}$ is in $\mathrm{Int}_n(E,\mathbb{Z}_p)$, and $\prod_0^{n-1}(a_n-a_k)/\prod_0^{n-1}(b_n-b_k)$ is in $\mathbb{Z}_p$. Henceforth, $\prod_0^{n-1}(a_n-a_k)$ is divisible by $\prod_0^{n-1}(b_n-b_k)$ and this shows the converse of Inequality (2).

Conversely, let $(a_n)_{0\leq n\leq N}$ be a sequence of E satisfying Equation (1) for any $0<n\leq N$. Let us show by induction on n it is a p-ordering. For $n=1$, let $a_0,\tilde{a}_1$ be a p-ordering sequence, so by the first part of this proof, we have

$$\begin{aligned} v_p(a_1-a_0) = &\quad w_E(1,p) = v_p(\tilde{a}_1-a_0) \\ = &\quad \inf\nolimits_{x\in E} v_p(x-a_0). \end{aligned}$$

Hence, a_0,a_1 is also a p-ordering sequence. Let us assume now that $a_0,\dots,a_{n-1}$ is a p-ordering sequence. Let $\tilde{a}_n\in E$ such that $a_0,\dots,a_{n-1},\tilde{a}_n$ is a p-ordering sequence. Then the first part of this proof implies

$$\begin{aligned} v_p(\prod_{k=0}^{n-1}(a_n-a_k)) = &\quad w_E(n,p) = v_p(\textstyle\prod_{k=0}^{n-1}(\tilde{a}_n-a_k)) \\ = &\quad \inf\nolimits_{x\in E} v_p(\textstyle\prod_{k=0}^{n-1}(x-a_k)). \end{aligned}$$

This shows that $a_0,\dots,a_{n-1},a_n$ is a p-ordering sequence. $\square$

Let us consider basic examples in this framework.

Example 1. For every prime p, $\mathrm{Int}_\infty(\mathbb{N},\mathbb{Z}_p)$ is an $\mathbb{Z}_p$-module generated by the binomial polynomials $\binom{X}{n}=\frac{\prod_0^{n-1}(X-k)}{n!}$. Hence, the sequence $0,1,2,\dots,n,\dots$

is a p-ordering of $\mathbb{N}$. Using the Legendre Formula, we get

$$\begin{aligned} w_{\mathbb{N}}(n,p) &= v_p\left(\prod_0^{n-1}(n-k)\right) = v_p(n!) \\ &= \sum_{k\geq 1}\left[\frac{n}{p^k}\right], \end{aligned}$$

where $[\cdot]$ denote the integer part.

Example 2. A sequence $(u_n)_n$ of elements in $\mathbb{Z}_p$ is said to be *concordant* if

$$v_p(u_n - u_m) \leq v_p(u_{n+r} - u_{m+r})$$

for every $n, m, \in \mathbb{N}$. According to [10], if $(u_n)_n$ is a concordant sequence of elements of $\mathbb{Z}_p$, then the sequence $(u_n)_n$ is a p-ordering of $E = \{u_n \mid n \in \mathbb{N}\}$. Thus, for every polynomial $f \in \mathbb{Z}_p[X]$ and every $a \in \mathbb{Z}_p$, the sequence $(f^n(a))_{n\geq 0}$ (of iterated of a) is a p-ordering of its orbit $\{f^n(a) \mid n \in \mathbb{N}\}$.

In particular, for $q \in \mathbb{Z}_p$ and E_q the set $\{q^k \mid k \in \mathbb{N}\}$, the sequence $(q^n)_{n\geq 0}$ is a p-ordering of E_q. Hence, $w_{E_q}(n) = v_p\left(\prod_0^{n-1}(q^n - q^k)\right)$ or explicitely, we have, when $v_p(q) \geq 1$,

$$w_{E_q}(n,p) = \frac{n(n-1)}{2}v_p(q).$$

2.2. **Basic properties of the characteristic sequence.** The following properties are obvious for bounded subsets E, F of $\mathbb{Q}_p$.

(1) If E is a finite subset of cardinality s, every p-ordering sequence of E of lenght $s-1$ is formed by the s elements of E and, for $n \geq s$, we have $w_E(n,p) = +\infty$.
(2) If $E \subset F$ then, for $n \geq 0$, $w_F(n,p) \leq w_E(n,p)$.
(3) If $E \subset F$ and $\overline{E} = \overline{F}$, then a sequence $(a_n)_n$ of elements of E is a p-ordering of E if and only if it is a p-ordering of F.
(4) If $F = \{as + b \mid s \in E\}$ with $a, b \in \mathbb{Z}_p$, then we have

$$w_F(n,p) = w_E(n,p) + nv_p(a).$$

In particular, if a is a unit of $\mathbb{Z}_p$, then E and F share the same characteristic sequence.

Less obvious properties are the following

Proposition 2.4. [8, Proposition 4] *For any bounded set $E \subset \mathbb{Q}_p$ and any p-ordering sequence $(a_i)_{0\leq i\leq n}$ of E, we have*

$$\begin{aligned} 2\sum_{k=1}^{n} w_E(k,p) &= v_p\left(\prod_{\substack{0\leq j,k\leq n-1\\ j\neq k}} (a_k - a_j)\right) \\ &= \min_{x_0,\ldots,x_n\in E} v_p\left(\prod_{\substack{0\leq j,k\leq n-1\\ j\neq k}} (x_k - x_j)\right). \end{aligned}$$

Proposition 2.5. *For any bounded subset E of $\mathbb{Q}_p$, the sequence $(w_E(n,p))_{n\geq 0}$ is superadditive, that is for any integers $n, m \geq 0$,*

$$w_E(n+m,p) \geq w_E(n,p) + w_E(m,p).$$

Proof. Let $m, n \in \mathbb{N}$. For every $f \in \mathrm{Int}_n(E,\mathbb{Z}_p)$ and $g \in \mathrm{Int}_m(E,\mathbb{Z}_p)$, the product fg is an element of $\mathrm{Int}_{n+m}(E,\mathbb{Z}_p)$. In particular, for any element y in the $(n+m)^{\text{th}}$ factorial ideal, we have $yfg \in \mathbb{Z}_p[X]$ and so $w_E(n+m,p) \geq w_E(n,p) + w_E(m,p)$ □

The sequence $(w_E(n))_n$ is then superadditive, so by Fekete's lemma, its asymptotic limit, possibly infinite, exists.

Definition 2.6. *The valuative capacity of a bounded subset E of $\mathbb{Q}_p$ is*

$$L_p(E) = \lim_{n\to+\infty} \frac{w_E(n,p)}{n} = \sup_{n\geq 0} \frac{w_E(n,p)}{n} \in \mathbb{R}_+^* \cup \{+\infty\}.$$

For instance, the explicit computations of the characteristic sequences in Subsection 2.1 provides:

Example 1: For $E = \mathbb{N}$, we have $L_p(\mathbb{N}) = L_p(\mathbb{Z}_p) = \frac{p}{p-1}$.

Example 2: For $q \in \mathbb{Z}_p$ with $v_p(q) \geq 1$ and the set $E_q = \{q^n \mid n \in \mathbb{N}\}$, we get $L_p(E_q) = +\infty$.

The next proposition follows directly from basic properties of characteristic sequences.

Proposition 2.7. *Let $a \in \mathbb{Q}_p$ and E be a bounded subset of $\mathbb{Q}_p$. Then*

(1) $E + a$ et E have the same valuative capacity: $L_p(a+E) = L_p(E)$;
(2) $L_p(aE) = v_p(a) + L_p(E)$. In particular, $L_p(p^r E) = r + L_p(E)$ for every $r \in \mathbb{Z}$;
(3) $L_p(E) \geq \frac{p}{p-1}$, when $E \subset \mathbb{Z}_p$.

Let us also recall this theorem useful in the following.

Theorem 2.8. [5, Theorem 4.2] *For any bounded set $E \subset \mathbb{Q}_p$, we have*

$$L_p(E) = \lim_{n} \frac{1}{n(n-1)} \min_{x_0,\ldots,x_{n-1}\in E} \sum_{\substack{0\leq k,j\leq n-1\\ k\neq j}} v_p(x_k - x_j).$$

2.3. **Density and p-Ordering Sequence.**

Proposition 2.9. *For a fixed integer $n \geq 1$, the map $E \mapsto w_E(n,p)$ is continuous for the Hausdorff topology on compact sets of $\mathbb{Q}_p$.*

Proof. Actually, we will show that the map $E \mapsto w_E(n,p)$ is locally constant. Let $a_0, a_1, \ldots, a_n$ be a p-ordering of a compact set E and let $N = 1 + \max\{v(a_i - a_j) \mid 0 \leq i < n\}$. Let F be a compact set $F \subset \mathbb{Q}_p$ such that $d_{\text{Haus}}(E,F) \leq p^{-N}$, that is for any element $x \in E$ there exists $y \in F$ such that $v(x-y) \geq N$.

In particular, there are $b_0, \ldots b_n \in F$ so that $v(a_k - b_k) \geq N$. Since $b_i - b_j = b_i - a_i + a_i - a_j - b_j$, we get $v(b_i - b_j) = v(a_i - a_j)$ and Proposition 2.3 implies

$$w_E(n,p) \leq w_F(n,p).$$

A symmetric argument gives us the converse inequality and this proves the proposition. □

Let us mention also the following topological properties of p-ordering sequences.

Proposition 2.10. [8]. *Let E be a bounded subset of $\mathbb{Q}_p$, then every p-ordering of E is dense in E.*

Corollary 2.11. *Let E and F two bounded subsets of $\mathbb{Q}_p$ such that $E \subseteq F$. Then E is dense in F if and only if for every $n \in \mathbb{N}$, $w_E(n,p) = w_F(n,p)$.*

Proof. Obviously if E is dense in F then for every $n \in \mathbb{N}$, $w_F(n,p) = w_E(n,p)$. Conversely, suppose that for every $n \in \mathbb{N}$, $w_F(n,p) = w_E(n,p)$. Let $(a_n)_n$ be a p-ordering sequence of E, then $(a_n)_n$ is also a p-ordering sequence of F (Proposition 2.3) and is dense in F. □

2.4. **Regular Subset.** For a given infinite subset E of $\mathbb{Z}_p$, it is not easy, in general, to give explicitly a p-ordering sequence of E or to determine its characteristic sequence. In this subsection we recall the results obtained in [7] to compute the characteristic sequence for specific compact subsets, introduced in [1] and called *regular*. We start with the definitions and properties related to such subsets.

For every $x \in E \subset \mathbb{Z}_p$ et $r \in \mathbb{N}^*$, we denote by:

$$E(x,r) = \{z \in E \mid v_p(x-z) \geq r\}$$

and $E(x,r)/p^{r+1}$ the sets of all cosets of $E(x,r)$ modulo p^{r+1} It is obvious that for every $x, x' \in E$, $E(x,r)/p^{r+1} = E(x',r)/p^{r+1}$ if and only if $v_p(x-x') \geq r+1$.

Proposition 2.12. *The following assertions are equivalents:*

(1) for every $r \in \mathbb{N}^*$ and every $x, x' \in E$, $card(E(x,r)/p^{r+1}) = card(E(x',r)/p^{r+1})$,

(2) for every $r \in \mathbb{N}^*$, there exists α_r such that for every x of E, the ball $E(x,r)$ is a disjoint union of α_r bals of the form $E(z,r+1), z \in E$.

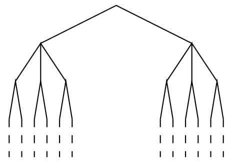
The straightforward proof is left to the reader.

Definition 2.13. *A subset E of $\mathbb{Z}_p$ is said to be a* regular subset *if E verifies one of the conditions of the previous proposition.*

When E is a regular subset, the sequence $(q_r(E))_{r\geq 0} = (card(E \bmod p^r))_{r\geq 0}$ is called the *structure sequence* of E. When there is no confusion, we will write q_r instead of $q_r(E)$. Obviously, the sequence $(q_r)_{r\geq 0}$ is non-decreasing. By definition we have the following equality for any integer $r \geq 1$ and $x \in E$,

$$q_{r+1} = q_r \mathrm{card}(E(x,r)/p^{r+1}).$$

We may represent a regular set E by a graph where any vertex is a class of E mod p^r for some positive integer r and two classes mod p^r and mod p^{r+1} are related if they are the same mod p^r. For the structure sequence $q_1 = 2$, $q_2 = 6$, $q_3 = 12$, $\ldots$, we get the graph:



For instance, the set $\mathbb{Z}_p$ is a regular set with structure sequence $q_r = p^r$ for any $r \geq 0$.

If E is a sub-group of $(\mathbb{Z}_p, +)$ or $(\mathbb{Z}_p^*, \times)$ then E is a regular set. In particular, if $q \in \mathbb{Z}_p$ et $v_p(q) = 0$ then $E_q = \{q^k \mid k \in \mathbb{N}\}$ is a regular set.

For a regular set, we have the following formula.

Proposition 2.14. [1, 7] *Let $(q_r)_r$ be a structure of a regular subset E. Then the characteristic sequence $(w_E(n,p))_n$ of E is given by:*

$$\forall n \in \mathbb{N},\ w_E(n,p) = \sum_{i=1}^{\infty} \left[\frac{n}{q_i}\right],$$

where $[\cdot]$ denotes the integer part.

In particular, if $\sum_{i\geq 1} \frac{1}{q_i}$ converges, then the valuative capacity $L_p(E)$ of E, is finite.

3. Background on potential theory

We recall here the classical definitions and properties of the *logarithmic capacity* of a relatively compact subset $E \subset \mathbb{Q}_p$, we refer to [2] for an introduction to potential Theory on Berkovich's spaces and the details of the proofs. The exposition is adapted to our context, and it can be extended to more general choices of potentials or compact spaces.

3.1. **Equilibrium measure.** Let $\mathcal{P}(E)$ be the set of probability measures μ with support contained in E, i.e., $\mu(F) = 1$ for any closed set F containing E. The *energy* associated with a measure $\mu \in \mathcal{P}(E)$ is the integral

$$I(\mu) = \iint -\log_p d_p(x,y) d\mu(x) d\mu(y) = \iint v_p(x-y) d\mu(x) d\mu(y),$$

where $d_p(x,y) = p^{-v_p(x-y)}$ denotes the p-adic distance.

For instance, for the Dirac measure δ_x based on a point $x \in E$, the energy is $I(\delta_x) = v_p(x-x) = \infty$. It follows that a measure $\mu \in \mathcal{P}(E)$ giving a positive weight to a point has an infinite energy.

Definition 3.1. *The* Robin constant *of the set E is the value*

$$V_p(E) = \inf_{\mu \in \mathcal{P}(E)} \iint v_p(x-y) d\mu(x) d\mu(y),$$

and the logarithmic capacity $C_p(E)$ *of the set E is the non negative scalar*

$$C_p(E) = p^{-V_p(E)},$$

with the convention $p^{-\infty} = 0$.

When E is a compact set, we can equip the set $\mathcal{P}(E)$ with the weak* topology. This is a metrizable topology such that a sequence $(\mu_n)_{n\geq 0}$ of $\mathcal{P}(E)$ converges to a measure $\mu \in \mathcal{P}(E)$ if and only if for any continuous function $f \colon E \to \mathbb{R}$, $\lim_n \int f d\mu_n = \int f d\mu$.
For this topology, $\mathcal{P}(E)$ is a compact metric space, and the energy $I \colon \mathcal{P}(E) \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous. Thus the infimum in Definition 3.1 is a minimum, and a measure realizing this minimum is called an *equilibrium measure.*

A fundamental result, linked to the Maximum Principle, ensures the unicity of the equilibrium measure.

Proposition 3.2. [2] *Let $E \subset \mathbb{Q}_p$ be a compact subset with positive logarithmic capacity. Then the equilibrium measure is unique.*

Hence, if E is invariant by some isometry $f \colon \mathbb{Q}_p \to \mathbb{Q}_p$: that is $f(E) = E$, then by the very definition, the image of any equilibrium measure μ of E by f, (i.e., the measure defined by $f_*\mu(A) = \mu(f^{-1}(A))$ for any Borel set A) is also an equilibrium measure. So when the equilibrium measure is unique, it is f-invariant, *i.e.*, $f_*\mu = \mu$.

Proposition 3.3. *Let $E \subset \mathbb{Q}_p$ be a compact subset with positive logarithmic capacity. Then the equilibrium measure is invariant by any isometry preserving E.*

For instance, for $E = \mathbb{Z}_p$, any translation is an isometry, and the only invariant probability measure by any isometry is the Haar probability measure.

Notice, the map $M_r\colon z \mapsto p^r z$ on $\mathbb{Q}_p$ is a contraction for $r > 0$. A direct computation provides that if μ is an equilibrium measure of a set E, $M_{r*}\mu$ is an equilibrium measure of the set $M_r(E)$.

3.2. **Capacitary functions.** In the classical theory, there are two important capacitary functions: the *transfinite diameter* and the *Chebyshev constant*.

To define the transfinite diameter, for any integer $n \geq 2$, let us define for a compact set $E \subset \mathbb{Q}_p$,

$$d_{(n)}(E)_p := \sup_{x_1,\dots,x_n \in E} \big(\prod_{i \neq j} d_p(x_i, x_j)\big)^{1/n(n-1)}.$$

It is straightforward to check the sequence $(d_{(n)}(E)_p)_{n\geq 2}$ is monotone decreasing. Hence it converges to a constant $d_\infty(E)_p$ called the *transfinite diameter* of E.

In the same way the, to define the *Chebyshev constant*, let

$$P_n(E,p) = \inf_{x_1,\dots,x_n \in E} (\sup_{x\in E} \prod_{i=1}^{n} d_p(x_i, x))^{1/n}.$$

Notice Property 2.3 gives us, for any positive integer n,

$$P_n(E,p) = p^{-\frac{w_E(n,p)}{n}}. \tag{3}$$

As n goes to infinity, the value $P_n(E,p)$ converges to a constant $CH(E,p)$ called the *algebraic Chebyshev constant*. It follows by the very definitions and Relation (3), we have an identification between the algebraic Chebyshev constant and the valuative capacity:

$$CH(E,p) = p^{-L_p(E)}.$$

Actually the notions of Chebyshev constant, transfinite diameter and Chebyshev algebraic constant coincide.

Theorem 3.4. [2] *For any compact set $E \subset \mathbb{Q}_p$, the logarithmic capacity, the transfinite diameter and the Chebyshev constant are equal:*

$$C_p(E) = d_\infty(E)_p = CH(E,p).$$

It follows from this theorem, we have the equality between the valuative capacity and the Robin constant:

$$V_p(E) = L_p(E).$$

In the next sections, we will use this relation and the next properties to deduce properties on the p-ordering sequences.

3.3. **Basic properties.** A first property concern the monotonicity of the logarithmic capacity: if $E_1 \subset E_2$, then

$$C_p(E_1) \leq C_p(E_2).$$

We can deduce the capacity depends continuously on the set E, in the following sense:

Proposition 3.5. [2] *For any compact set $E \subset \mathbb{Q}_p$,*

$$C_p(E) = \sup_{\substack{F \subset E \\ F \text{ compact}}} C_p(F), \text{ or equivalently, } L_p(E) = \inf_{\substack{F \subset E \\ F \text{ compact}}} L_p(F).$$

As a direct corollary from these formulas, the capacity does not change when we add a set of null capacity:

Proposition 3.6. [2] *Let $F \subset \mathbb{Q}_p$ be a set of null capacity, then for any compact set $E \subset \mathbb{Q}_p$, the logarithmic capacity of $F \cup E$ is*

$$C_p(F \cup E) = C_p(E).$$

4. Application of potential theory to p-ordering sequences

The following result gives the equi-distribution of a p-ordering sequence of a compact set in $\mathbb{Q}_p$ to the equilibrium measure on the set. It is a classical result and the proof comes form the analytical properties of an associated Green's function. We propose here a less elaborated proof, adapted to our context.

Theorem 4.1. *Let $(a_n)_{n\geq 0}$ be a p-ordering sequence of a compact set $E \subset \mathbb{Q}_p$. Then*

(1) Any accumulation point, for the weak- topology, of the set of measures $\{\frac{1}{n}\sum_{k=0}^{n-1}\delta_{a_k}, n \geq 1\}$ is an equilibrium measure of E.*
(2) The logarithmic capacity of the set $\{a_n, n \geq 0\}$ is $C_p(E)$.

Recall when the set E has a positive logarithmic capacity, the equilibrium measure μ is unique (Proposition 3.2). We deduce then any p-ordering sequence $(a_n)_n$ in E is distributed according to this measure μ, *i.e.*, the sequence of measures $(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{a_k})_{n\geq 1}$ converges to the measure μ for the weak-* topology.

Proof. Notice first, up to consider a set $M_r(E)$ for some map $M_r \colon z \mapsto p^r z$ $r \geq 0$, we can assume that E is a subset of $\mathbb{Z}_p$. By compacity of the set $\mathcal{P}(E)$, we can also assume, up to consider a subsequence, the probability measure $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{a_k}$ converges, as n goes to infinity, to a measure $\mu \in \mathcal{P}(E)$. We have to show that for any measure $\mu_1 \in \mathcal{P}(E)$, $I(\mu_1) \geq I(\mu)$.

If the function $(x, y) \mapsto v_p(x - y)$ is not $\mu_1 \otimes \mu_1$ integrable then the result is trivial. We suppose then this function integrable. We will need the following lemma.

Lemma 4.2. *If a sequence of points $(x_k)_{k\geq 0}$ of a compact metric space E is such that $(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{x_k})_n$ converges to a measure ν, then the sequence of measures on $E \times E$*

$$\left(\frac{1}{n(n-1)}\sum_{\substack{0\leq k,j\leq n-1\\ k\neq j}}\delta_{(x_k,x_j)}\right)_n$$

converges to the product measure $\nu\otimes\nu$ for the weak- topology.*

Proof. It is enough to notice that

$$\frac{1}{n(n-1)}\sum_{\substack{0\leq k,j\leq n-1\\ k\neq j}}\delta_{(x_k,x_j)} = \frac{n^2}{n(n-1)}\frac{1}{n^2}\sum_{k=0}^{n-1}\delta_{x_k}\sum_{j=0}^{n-1}\delta_{x_j} - \frac{1}{n(n-1)}\sum_{k=0}^{n-1}\delta_{(x_k,x_k)}.$$

Then, by integrating a continuous function of $E\times E$ and passing through the limit in n, we obtain the claim. □

To use the properties of weak-* topology, we need to consider only continuous functions but unfortunately, the function $(x,y)\mapsto v_p(x-y)$ is not continuous. So we consider a truncation of it via the continuous function $v_{p,N}(\cdot) = \inf(v_p(\cdot), N)$ for an integer $N \geq 1$. Of course we have $v_{p,N}\to v_p$ as N goes to infinity. Let $(X_i\colon \Omega\to E)_i$ be a sequence of independent random variables defined on the same probability space $(\Omega,\mathcal{A},\mathbb{P})$ and with the same law μ (i.e., $\mathbb{P}(X_i\in B) = \mu_1(B)$ for any Borel set B). By the Strong Law of Large Numbers, the sequence of measures $(\frac{1}{n}\sum_{k=1}^{n}\delta_{X_k(\omega)})_n$ converges to the measure μ_1 for $\mathbb{P}$ a.e. ω. Let us fix a sequence realizing this convergence.

Recall that the function v_p is proper, so the map $\mathcal{P}(E\times E)\ni\nu\mapsto \iint v_p(x-y)d\nu(x,y)$ is lower semicontinuous. It follows from Lemma 4.2 that

$$I(\mu) \leq \liminf_n \frac{1}{n(n-1)}\sum_{\substack{0\leq k,j\leq n-1\\ k\neq j}} v_p(a_k-a_j), \tag{4}$$

$$= \liminf_n \frac{1}{n(n-1)} v_p(\prod_{\substack{0\leq k,j\leq n-1\\ k\neq j}}(a_k-a_j)). \tag{5}$$

Notice, for any $x,y\in\mathbb{Z}_p$, we have

$$v_{p,N}(xy)\leq v_{p,N}(x)+v_{p,N}(y).$$

Hence, from Property 2.4, we get

$$\frac{1}{n(n-1)}v_{p,N}(\prod_{\substack{0\leq k,j\leq n-1\\ k\neq j}}(a_k-a_j)) \leq \frac{1}{n(n-1)}\sum_{\substack{0\leq k,j\leq n-1\\ k\neq j}} v_{p,N}(X_k-X_j).$$

This inequality with Inequality (4) and again with Lemma 4.2, together imply

$$\inf(I(\mu), N) \leq \liminf_n \frac{1}{n(n-1)} v_{p,N}(\prod_{\substack{0\leq k,j\leq n-1\\ k\neq j}} (a_k - a_j)) \leq \iint v_{p,N}(x-y) d\mu_1 d\mu_1.$$

As N goes to infinity, the dominated Lebesgue convergence theorem gives us that that $\iint v_{p,N}(x-y)d\mu_1 d\mu_1$ tends to $I(\mu_1)$, so we get

$$I(\mu) \leq \liminf_n \frac{1}{n(n-1)} \sum_{\substack{0\leq k,j\leq n-1\\ k\neq j}} v_p(a_k - a_j) \leq I(\mu_1).$$

To obtain the point (2) of the statement, it is enough to consider the measure $\mu^1 = \mu$ in the last inequality, and to conclude by Proposition 2.4 and Theorem 2.8. $\square$

The following result, of main importance for our work, is due to K. Johnson [9]. We give here a proof by using tools of potential theory and, at the difference of [9], we consider the case with infinite valuative capacity (or a null logarithmic capacity).

Theorem 4.3. [9] *Let $E_1, \ldots, E_n$ be n compact sets in $\mathbb{Q}_p$ such that*

$$\forall\ 1 \leq i < j \leq n \qquad v(x_i - x_j) = 0 \qquad \forall x_i \in E_i, x_j \in E_j.$$

Then the logarithmic capacity of the compact set $E := \bigcup_{i=1}^n E_i$ satisfies

$$\frac{1}{\log(C_p(E))} = \sum_{i=1}^n \frac{1}{\log(C_p(E_i))},$$

with the convention $1/\infty = 0$.

Notice this formula does not ensure if a set have a null logarithmic capacity (or an infinite valuative capacity).

Proof. Let us show this result first for $n = 2$.
If the capacity of E_1 or E_2 is null then by Proposition 3.6, the equality is true. Otherwise, by the monotonicity of the capacity (Proposition 3.5), the set E has a measure μ of finite energy. Let μ_{E_i} be the induced measure of μ on E_i, i.e., the null measure if $\mu(E_i) = 0$ and otherwise

$$\mu_{E_i}(A) := \frac{\mu(A \cap E_i)}{\mu(E_i)}, \qquad \text{for any Borel set } A \subset \mathbb{Q}_p, i = 1, 2.$$

We have then $\mu = \mu(E_1)\mu_{E_1} + \mu(E_2)\mu_{E_2}$. Thus,

$$\begin{aligned}
I(\mu) &= \iint_{E_1\cup E_2} v(x-y)d\mu d\mu \\
&= \int_{E_1}\int_{E_1} v(x-y)d\mu d\mu + \int_{E_1}\int_{E_2} + \int_{E_2}\int_{E_1} + \int_{E_2}\int_{E_2} \\
&= \int_{E_1}\int_{E_1} v(x-y)d\mu d\mu + \int_{E_1} 0\mu(E_2)d\mu \\
&\qquad\qquad + \int_{E_2} 0\mu(E_1)d\mu + \int_{E_2}\int_{E_2} v(x-y)d\mu d\mu \\
&= \mu(E_1)^2 I(\mu_{E_1}) + (1-\mu(E_1))^2 I(\mu_{E_2}).
\end{aligned}$$

In an equivalent way, if $\mu_1 \in \mathcal{P}(E_1), \mu_2 \in \mathcal{P}(E_2)$ have finite energy, then for any $\lambda \in [0,1]$, the measure $\lambda\mu_1 + (1-\lambda)\mu_2 \in \mathcal{P}(E)$ and

$$I(\lambda\mu_1 + (1-\lambda)\mu_2) = \lambda^2 I(\mu_1) + (1-\lambda)^2 I(\mu_2).$$

Since for any non negative real x, y, $x+y \neq 0$, we have $\min_{\lambda\in[0,1]} \lambda^2 x + (1-\lambda)^2 y = \frac{xy}{x+y}$, we deduce

$$I(\lambda\mu_1 + (1-\lambda)\mu_2) \geq \frac{I(\mu_1)I(\mu_2)}{I(\mu_1)+I(\mu_2)}.$$

Hence

$$\log C_p(E_1 \cup E_2) = \frac{\log C_p(E_1))\log(C_p(E_2))}{\log(C_p(E_1)) + \log(C_p(E_2))}.$$

This shows the result for $n = 2$. We conclude the generalized case by a straightforward induction on n. $\square$

Corollary 4.4. *Let $r \geq 1$ and $E_1, \ldots, E_n$ be n compact sets of $\mathbb{Q}_p$ such that*

(1) $v_p(x_i - x_j) = r$ for any $x_i \in E_i$, $x_j \in E_j$, $i \neq j$
(2) $v_p(x-y) \geq r$ for any $x, y \in E_i$.

Then the capacities of $\cup_i E_i$ and of the E_i satisfy

$$\frac{1}{\big(\log C_p(\cup_{i=1}^n E_i)\big) - r} = \sum_{i=1}^n \frac{1}{\big(\log C_p(E_i)\big) - r}.$$

Proof. Let E denote $\cup_{i=1}^n E_i$. We have

$$\frac{1}{p^r}E = \bigcup_{i=1}^n \frac{1}{p^r}E_i.$$

Then by Theorem 4.3 and the basic formulas in Proposition 2.7, we obtain the conclusion. $\square$

5. Computation of the capacity of some sets

We give in this section explicit computations of the capacity of some compact sets of $\mathbb{Z}_p$. These sets will be defined by combinatorics.

For this, we need first to recall some basic notions and introduce some notation. Let $A = \{0, 1, \dots, d-1\}$ be a finite alphabet. A *word* of *length* $n \in \mathbb{N}^*$ is a finite sequence $x_0 \cdots x_{n-1} \in A^n$, and each $x_0, \dots, x_{n-1} \in A$ is called a *letter*. The *concatenation* of the two words $u = x_0 \cdots x_n$, $v = y_0 \cdots y_p$, is the word $uv = x_0 \cdots x_n y_0 \cdots y_p$. The word u is then called a *factor* of the word uv. The collection of infinite sequences $x = (x_n)_{n \geq 0}$ with values in A, is denoted $A^{\mathbb{N}}$. Any finite word of the kind $x_i \cdots x_{i+n-1}$, for some integers i, n, is also called a *factor* of the sequence x. For the product topology, the set $A^{\mathbb{N}}$ is topologically a Cantor set.

Let $p \geq d$ be a prime number. There exists a canonical embedding of $A^{\mathbb{N}}$ into $\mathbb{Z}_p$ through the continuous map

$$\begin{array}{rcl} \varphi \colon A^{\mathbb{N}} & \to & \mathbb{Z}_p \\ (x_n)_{n \geq 0} & \mapsto & \sum_{k=0}^{\infty} x_k p^k. \end{array}$$

A first simple example of compact set is the following.

Lemma 5.1. *Let $w_1, w_2, \dots, w_s$ be $s \geq 2$ words with the same length ℓ such that all the first letters are distinct.*
Let $X \subset A^{\mathbb{N}}$ be the set of sequences such that any factor is a factor of a concatenation of the words $w_1, w_2, \dots, w_s$.
Then the set $E := \varphi(X) \subset \mathbb{Z}_p$ verifies

$$E = \bigcup_{i=1}^{s} x_i + p^\ell E, \qquad \text{with } x_i = \varphi(w_i 0^\infty). \tag{6}$$

It is a regular compact set and its valuative capacity is

$$L_p(\varphi(X)) = \frac{\ell}{s-1}.$$

Notice this provides also examples of sets with empty interior but with positive capacities.

Proof. It is plain to check that the set $\varphi(X)$ satisfies Relation (6). It is also straightforward to check ,with the definition of the set E, that for any $x \in E$, $r \geq 1$, $card(E(x,r)/p^{r+1}) = s^{[\frac{r}{\ell}] - [\frac{r-1}{\ell}]}$. So it is a regular set with a structure sequence $q_r = s^{[\frac{r}{\ell}]}$.

Let L denote the valuative capacity of the set $\varphi(X)$. It is finite by Propsition 2.14. From the corollary of Jonhson's theorem (Corollary 4.4) applied to the sets $E_i = x_i + p^\ell E$ of Relation (6), and basic formulas on valuative capacity (Proposition 2.7), we get

$$\frac{1}{L} = \sum_{i=1}^{s} \frac{1}{L+\ell} = \frac{s}{L+\ell}.$$

This shows the lemma. $\square$

Let T denotes the shift map onto $A^{\mathbb{N}}$ defined by $T((x_n)_{n\geq 0}) = (x_{n+1})_{n\geq 0}$. It is a continuous onto map for the product topology. A subshift $Y \subset A^{\mathbb{N}}$ is a closed T invariant set ($T(Y) = Y$).

A typical example of subshift is a *subshift of finite type* (SFT fort short), that is, given a finite family of words $\mathcal{F}$, we associate a subshift $X_{\mathcal{F}}$ consisting in all the sequences $(x_n)_{n\geq 0} \in A^{\mathbb{N}}$ such that any factor does not belong to $\mathcal{F}$. Equivalently, we define

$$X_{\mathcal{F}} = \{(x_n)_n;\ \forall n, i \geq 0, x_i \cdots x_{i+n} \notin \mathcal{F}\}.$$

Notice that, up to add forbiden words in $\mathcal{F}$, we can assume that all the words in $\mathcal{F}$ have the same length $N \geq 2$ (so $\mathcal{F} \subset A^N$). We can associate to the subshift $X_{\mathcal{F}}$ a directed graph $G = (V_G, E_G)$ where the set of vertices V_G is the collection of all words in A of length $N-1$. Two words $x_0 \cdots x_{N-2}$, $y_0 \cdots y_{N-2}$ of V_G are related by a directed edge in E_G if $x_1 \cdots x_{N-2} = y_0 \cdots y_{N-3}$ and $x_0 y_0 \cdots y_{N-2} \notin \mathcal{F}$. The adjancy matrix $M_{\mathcal{F}}$ associated with this graph, *i.e.*, $M_{\mathcal{F}x,y} = 1$ if $(x, y) \in E_G$ and 0 otherwise, is called the *associated matrix* to the subshift $X_{\mathcal{F}}$. Any sequence $(x_n)_n$ in $X_{\mathcal{F}}$ corresponds then to a non constant path in the graph G.

Let F be the map

$$\begin{array}{rcl} F\colon \mathbb{Z}_p & \to & \mathbb{Z}_p \\ x = \sum_{k=0}^{\infty} x_k p^k & \mapsto & \frac{x - x_0}{p} = \sum_{k=0}^{\infty} x_{k+1} p^k. \end{array}$$

This is a continuous, locally affine map, that extends the conjugation of the shift map by φ: $\varphi \circ T \circ \varphi^{-1}$. It follows that for any subshift $Y \subset A^{\mathbb{N}}$, the set $\varphi(Y)$ is F invariant.

Lemma 5.2. *For a set of forbidden words $\mathcal{F} \subset A^N$, let $X_{\mathcal{F}} \subset A^{\mathbb{N}}$ and $G = (V_G, E_G)$ be respectively the associated SFT and the associated graph.*
For a word $u = u_0 \cdots u_{N-2} \in A^{N-1}$, let E_u denote the set $(\varphi(u0^\infty) + p^{N-1}\mathbb{Z}_p) \cap \varphi(X_{\mathcal{F}})$ and $f_{u,v}$ denotes the affine map $z \mapsto \varphi(u_0 0^\infty) + pz$.

Then, for any word $u \in A^{N-1}$ we have

$$E_u = \bigcup_{v \in A^{N-1}, (u,v) \in E_G} f_{u,v}(E_v).$$

Notice this lemma shows the compact set $\varphi(X_{\mathcal{F}})$ is the attractor of a graph-directed iterated functions system with affine maps.

Proof. For any $x = \sum_{k=0}^{\infty} x_k p^k \in E_u$, we have $u = x_0 \cdots x_{N-2}$, $x = \varphi(x_0 0^\infty) + pF(x)$ and $F(x) \in E_v$ for some word $v \in A^{N-1}$. Thus we have $(u, v) \in E_G$ and finally $E_u \subset \bigcup_{v,(u,v)\in E_G} f_{u,v}(E_v)$.

Conversely if $x = \sum_{k=0}^{\infty} x_k p^k \in f_{u,v}(E_v)$ with $(u, v) \in E_G$, then we have $u = x_0 \cdots x_{N-2}$ and $v = x_1 \cdots x_{N-1}$ and $x_n \cdots x_{n+N-2} x_{n+N-1} \notin \mathcal{F}$ for all $n \geq 1$.

Hence $x \in \varphi(X_{\mathcal{F}}) \cap (\varphi(u0^\infty) + p^N\mathbb{Z}_p) = E_u$ and we get $\bigcup_{v,(u,v)\in E_G} f_{u,v}(E_v) \subset E_u$. □

Let us see first the example given by the *Fibonacci subshift of finite type* defined as

$$X_{\mathrm{Fibo}} = \{(x_n)_{n\geq 0};\ x_n \in \{0,1\},\ x_n x_{n+1} \neq 11 \text{ for any } n\}.$$

The associated matrix M equals

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

For any prime number p, by Lemma 5.2, the set $E = \varphi(X_{\mathrm{Fibo}})$ satisfies

$$\begin{aligned} E_0 =& pE_1 \cup pE_0, \\ E_1 =& 1 + pE_{0,} \\ E_i =& \emptyset,\ \forall i \geq 2. \end{aligned}$$

Since $E = E_0 \cup E_1$, we get $E = pE \cup (1 + p^2E)$. Jonhson's theorem (Corollary 4.4) and basic calculus in Proposition 2.7 give us

$$\frac{1}{L_p(E)} = \frac{1}{L_p(E) - 1} + \frac{1}{L_p(E) - 2}. \tag{7}$$

It is important to notice that the subshift X_{Fibo} contains all the concatenations of the words 00 and 10. Hence Lemma 5.1 ensures the set $\varphi(X_{\mathrm{Fibo}})$ has a finite valuative capacity. Finally, with Equation (7), we obtain $L_p(E) = \sqrt{2}$, because $L_p(E) \geq 0$.
We deduce there is no simple relation between classical dynamical invariant as the entropy of the Fibonacci SFT -which equals the logarithm of the golden mean- and the valuative capacity.

We can generalize the former computation to the general case.

Theorem 5.3. *Let $\mathcal{F} \subset A^N$ be a set of forbidden words. Let $X_{\mathcal{F}} \subset A^{\mathbb{N}}$ and M denote the associated SFT and associated matrix. Let $E = \varphi(X_{\mathcal{F}}) \subset \mathbb{Z}_p$ for a prime number $p \geq d$. For any word u of length ℓ, E_u denotes the set $(\varphi(u0^\infty) + p^\ell\mathbb{Z}_p) \cap E$.*

Then the valuative capacities of the sets E, E_u, for words $u \in A^\ell$ with $1 \leq \ell \leq N-1$, verify

$$\left(\frac{1}{L_p(E_u) - N}\right)^t_{u\in A^{N-1}} = M\left(\tfrac{1}{L_p(E_u)}\right)^t_{u\in A^{N-1}}, \text{ and} \tag{8}$$

$$\frac{1}{L_p(E_v) - \ell - 1} = \textstyle\sum_{a\in A} \frac{1}{L_p(E_{va})-\ell-1}, \forall v \in A^\ell, \tag{9}$$

for some fixed order in A^{N-1} and with the convention $1/\infty = 0$.

Moreover, if for some power $n \geq 1$, the matrix $M^n = (M^{(n)}_{i,j})_{i,j\in A^{N-1}}$ has a diagonal entry $M^{(n)}_{u,u} \geq 2$, then the valuative capacity of E_u is finite.

This theorem provides a way to compute the valuative capacity: start computing the valuative capacities of the sets E_u for all the words $u \in A^{N-1}$ (with N given by the SFT) by solving Equation (8). Then, use these (supposed known) capacities to compute capacities of sets E_v for any words of length smaller via Equation (9). Iterate this second step until the computation of the capacities of the sets E_a for all the letters $a \in A$. Finally we obtain the capacity of the set $E = \bigcup_{a\in A} E_a$ via Jonhson's formula, explicitly:

$$\frac{1}{L_p(E)-1} = \sum_{a\in A} \frac{1}{L_p(E_a)-1}.$$

Since the equations (8) and (9) are algebraic, the valuative capacity of the set E, when finite, is an algebraic number.

When a set E_u has a finite valuative capacity, the monocity of the capacity implies that the set E has also a finite capacity. Notice also, when the matrix M is primitive, i.e., the matrix M^n has positive entries for some integer n, any set E_u has a finite capacity.

Proof. Lemma 5.2 provides us the relation

$$E_u = \bigcup_{v\in A^{N-1},(u,v)\in E_G} f_{u,v}(E_v).$$

If u is the word $u_0\cdots u_{N-1}$, any word v satisfying $(u,v) \in E_G$ is of the kind $v = u_1\cdots u_{N-1}v_{N-1}$ for some letter $v_{N-1} \in A$. It follows that any element of $f_{u,v}(E_v)$ is of the form $\varphi(uv_{N-1}\cdots)$. Thus we can apply Jonhson's formula (Corollary 4.4) and we obtain Equation (8).
Equation (9) also comes from Jonhson's Formula applied to the obvious equality $E_v = \bigcup_{a\in A} E_{va}$.

Let us assume now, that the matrix $M^n = (M^{(n)}_{i,j})_{i,j\in A^{N-1}}$ admits a word i such that $M^{(n)}_{i,i} \geq 2$. Standard graph theory tells us there exist at least two path of length n in the graph (V_G, E_G) associated with the SFT $X_{\mathcal{F}}$ starting and ending at the vertex i. Thus, there are two words u and $v \notin \mathcal{F}$, such that any concatenation of these two words is an allowed word in $X_{\mathcal{F}}$. Lemma 5.1 provides that the set E_i contains a regular compact subset with finite valuative capacity. By Proposition 3.5, we conclude the valuative capacity of E_i is finite. $\square$

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