

SELF-INDUCED SYSTEMS

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ABSTRACT. A minimal Cantor system is said to be self-induced whenever it is conjugate to one of its induced systems. Substitution subshifts and some odometers are classical examples, and we show that these are the only examples in the equicontinuous or expansive case. Nevertheless, we exhibit a zero entropy self-induced system that is neither equicontinuous nor expansive. We also provide non-uniquely ergodic self-induced systems with infinite entropy. Moreover, we give a characterization of self-induced minimal Cantor systems in terms of substitutions on finite or infinite alphabets.

1. INTRODUCTION

From the Poincaré's recurrence theorem one can define the notion of a first return map on a set of positive measure and, as a consequence, what is now named an induced dynamical system. In the 1940's S. Kakutani initiated, in the measurable framework, the study of such systems. Intensive studies of the induction structures for fifty years contributed to a better understanding of the orbital structures of dynamical systems, see for example [34, 27, 39, 8, 16]. However, in the topological context this notion has been less studied despite the powerful strategy it provides as shown in [22] (see also [25]).

Since, it has been observed in several well-known families of dynamical systems the phenomenon of self-induction, that is, systems being conjugate to one of its induced systems.

For interval exchange transformations (IETs), where induction plays a crucial role through the so-called Rauzy-Veech induction [40, 37], a stationary behaviour may appear in the scheme of successive inductions. The IETs whose interval lengths lie in a quadratic number field, as quadratic rotations, are conjugate to an induced system on a subinterval [4]. Let us point out this is not particular to the quadratic fields since there are IETs defined on cubic fields that are self-induced on canonical intervals [3, 2, 19, 5, 26] and even for some algebraic fields of arbitrary degree [1].

Let us also mention that there exist self-induced exchanges of domains like for the Rauzy fractal [38] where the conjugacy is given by a similarity. A last example is given by the Pascal adic transformation [31] that can be proven to be self-induced. In most of the former papers the self-induction property comes from a measure theoretical conjugacy to a substitution subshift. This observation was enlightened

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in [2] where it is shown that any system with a σ -structure, that is a sequence of finite partitions having specific properties, is conjugate to a substitution subshift.

The self-induction property is sensitive to the class of sets on which the induction is made. For instance, inducing on measurable sets instead of intervals, each irrational rotation on the torus is an induced system of any other irrational rotation [34], thus all rotations are self-induced (in the measure theoretical framework). A similar phenomena arises in the topological context. Indeed, we show in Section 6 that given two minimal systems (X, T) and (Y, S) , on Cantor sets, there always exists a closed set $C \subset X$ such that the induced map T_C is well-defined and (C, T_C) is topologically conjugate to (Y, S) (Theorem 35). This incidentally proves that all such systems are self-induced on some proper closed subsets.

In this paper we focus on minimal topological dynamical systems (X, T) with a self-induced behaviour: that is topological dynamical systems that are conjugate to one of their induced systems on proper open (and necessarily closed) sets. The minimality of the system ensures the induced systems are well defined. Since a self-induced system has closed and open sets, called clopen sets, it is natural to consider dynamics on Cantor sets, called Cantor systems. From the Abramov formula for the entropy of induced systems, the self-induced property implies that the entropy is 0 or $+\infty$ (see Proposition 5). Among zero entropy systems, it is a folklore result that p -odometers (*i.e.*, addition of 1 in the set \mathbb{Z}_p of p -adic integers) or minimal substitution subshifts [36] are self-induced (see propositions 11 and 13).

The goal of this paper is to characterize self-induced minimal Cantor systems. In Section 2, we give preliminary results on such systems. Among them, we show that the clopen set on which the system is self-induced can be taken inside any open set.

The next section is devoted to the characterization of self-induced systems in the equicontinuous and expansive cases. For the equicontinuous case (*i.e.*, odometers) we prove: an odometer is self-induced if and only if it factorizes onto a p -odometer (Proposition 11). Hence, the odometer built as the inverse limit of $(\mathbb{Z}/p_1 \cdots p_n \mathbb{Z})$, where $(p_i)_i$ is the sequence of primes, is not self-induced whereas the 2-odometer (thus, on \mathbb{Z}_2) is. For expansive minimal Cantor system (*i.e.*, minimal subshift) it is equivalent to be conjugate to a minimal substitution subshift (Proposition 13 and Theorem 14).

We provide in Section 4 two examples of self-induced minimal Cantor systems that are neither equicontinuous nor expansive. One example has zero entropy and the other $+\infty$. Moreover this last example is not uniquely ergodic.

In Section 5, we characterize self-induced minimal Cantor systems by means of *generalized substitution subshifts i.e.*, systems generated by substitution maps on a compact zero-dimensional (but not necessarily finite) alphabet (theorems 32 and 33).

In the last section, we define the notion of Poincaré sections and we prove Theorem 35.

2. PRELIMINARIES

2.1. Dynamical systems. For us a *topological dynamical system*, or just a dynamical system, is a couple (X, T) where X is a compact metric space and T is a homeomorphism $T : X \rightarrow X$. A dynamical system is *minimal* if every orbit is

dense in X , or equivalently if the only non empty closed invariant set is X . We call (X, T) a *Cantor system* if X is a Cantor set, that is, if X is a compact metric space with a countable basis of closed and open sets and no isolated points. We recall that any such space is homeomorphic to the standard Cantor ternary set. When X is a subset of $A^{\mathbb{Z}}$ where A is a finite set and S is the shift map $S(x)_n = x_{n+1}$, the Cantor system (X, S) is called a *subshift*. The system (X, T) is said to be *aperiodic* if T has no periodic points. Observe that minimal Cantor systems are always aperiodic.

Let (X_1, T_1) and (X_2, T_2) be two dynamical systems. We say (X_2, T_2) is a *factor* of (X_1, T_1) if there is a continuous and onto map $\varphi : X_1 \rightarrow X_2$ such that $\varphi \circ T_1 = T_2 \circ \varphi$. Then, φ is called *factor map*. If φ is one-to-one we say it is a *conjugacy*, and (X_1, T_1) and (X_2, T_2) are *conjugate*.

For a minimal system (X, T) and for any open set $U \subset X$, we can define the *return time function* $r_U : X \rightarrow \mathbb{N}$ by

$$r_U : x \mapsto \inf\{n > 0 : T^n(x) \in U\}.$$

It is well known that for a minimal Cantor system and $A \subset X$ a clopen set, the function r_A is locally constant hence continuous. The *induced map* $T_A : A \rightarrow A$ is then defined by

$$T_A : x \mapsto T^{r_A(x)}(x).$$

Such a map $T_A : A \rightarrow A$ is a homeomorphism and the Cantor system (A, T_A) is called the *induced system* on A . If the system (X, T) is minimal and $A \subset X$ is clopen then, (A, T_A) is also minimal.

Conversely, let (X, T) be a Cantor system and $r : X \rightarrow \mathbb{Z}_+$ a continuous map, the *exduced system* (\tilde{X}_r, \tilde{T}) is the dynamical system defined by the set $\tilde{X}_r = \{(x, i) : x \in X, 0 \leq i < r(x)\}$ and the map $\tilde{T} : \tilde{X}_r \rightarrow \tilde{X}_r$ defined by

$$(x, i) \mapsto \begin{cases} (x, i+1) & \text{if } 0 \leq i < r(x) - 1 \\ (Tx, 0) & \text{if } i = r(x). \end{cases}$$

It is plain to check that the exduced system of (X, T) is a minimal Cantor system if (X, T) is. Moreover, the induced system of (\tilde{X}_r, \tilde{T}) on the clopen set $X \times \{0\}$ is conjugate to the system (X, T) . More precisely, if (X, T) is a minimal Cantor system, U a clopen set of X and r_U the associated return function then, $(x, i) \mapsto T^i x$ is a conjugacy between the systems $(\tilde{U}_{r_U}, \tilde{T})$ and (X, T) .

Definition 1. A minimal Cantor system (X, T) is self-induced if there exists a non-empty clopen proper subset $U \subset X$ such that the induced system (U, T_U) is conjugate to the system (X, T) .

While we work within the topological category in our study of self-induced systems, it will be relevant to consider invariant Borel probability measures. By the Bogoliouboff-Krylov theorem, associated to any topological dynamical system (X, T) there is a nonempty compact, convex space $M(X, T)$ of T -invariant Borel probability measures [28], see also [35]. When $M(X, T)$ is a singleton set, as is the case for minimal substitution subshifts and odometer systems, we say that (X, T) is *uniquely ergodic*. The proposition below is well-known and describes the relation between $M(X, T)$ and the space of measures for an induced system of (X, T) .

Proposition 2. *Let (X, T) be a minimal Cantor system and suppose $U \subset X$ is clopen. Let $F : M(X, T) \rightarrow M(U, T_U)$ be the function where $F(\mu)$ is the measure ν on U defined by $\nu(B) = \frac{\mu(B)}{\mu(U)}$ where $B \subset U$ is a Borel set. Then, F is a bijection.*

Proof. First note that any such measure $\nu = \frac{1}{\mu(U)}\mu$ defines a Borel probability measure on U . Also, for any Borel set $B \subset U$,

$$\nu(T_U^{-1}B) = \frac{\mu(T_U^{-1}B)}{\mu(U)} = \frac{\mu(B)}{\mu(U)} = \nu(B).$$

Thus, the map F is well-defined.

We will prove that F is bijective. To this end, let $U_k = \{x \in U : r_U(x) = k\}$. Then, U_k is clopen for all k , and for some K , U is a disjoint union of U_1, U_2, \dots, U_K . It follows that the collection $\{T^i U_k : 0 \leq i < k, 1 \leq k \leq K\}$ forms a clopen partition of X .

Suppose there exist two measures μ, μ' in $M(X, T)$ such that $F(\mu)(B) = F(\mu')(B)$ for every measurable subset B of U . Set $r = \mu(U)/\mu'(U)$. Then, for every subset B of U , $\mu(B) = r\mu'(B)$ and

$$1 = \mu(X) = \sum_k k\mu(U_k) = \sum_k kr\mu'(U_k) = r \sum_k k\mu'(U_k) = r\mu'(X) = r.$$

Then, there is a clopen set B such that $\mu(B) \neq \mu'(B)$. But

$$\mu(B) = \sum_{k=1}^K \sum_{j=0}^{k-1} \mu(B \cap T^j U_k) = \sum_{k=1}^K \sum_{j=0}^{k-1} \mu(T^{-j} B \cap U_k).$$

Therefore, $\mu(T^{-j} B \cap U_k) \neq \mu'(T^{-j} B \cap U_k)$ for some j, k . So we get $F(\mu)(T^{-j} B \cap U_k) \neq F(\mu')(T^{-j} B \cap U_k)$, a contradiction. Hence F is injective.

Let ν be a T_U -invariant Borel probability measure on U . Let $\rho = \sum_{k=1}^K k\nu(U_k)$. For a Borel set $B \subset X$, define the T -invariant measure

$$\mu(B) = \frac{1}{\rho} \sum_{k=1}^K \sum_{j=0}^{k-1} \nu(T^{-j} B \cap U_k).$$

We wish to see that $F(\mu) = \nu$. Notice that for $B \subset U$ a Borel set

$$\mu(B) = \frac{1}{\rho} \sum_{k=1}^K \nu(B \cap U_k) = \frac{\nu(B)}{\rho}.$$

So, for $B = U$, we get $\mu(U) = 1/\rho$ and $\nu(B) = \frac{\mu(B)}{\mu(U)} = F(\mu)(B)$. Therefore, F is onto. \square

2.2. Observations about self-induced systems. We will make a series of observations about self-induced systems. We see that Abramov's Formula and the Variational Principle imply that the topological entropy, denoted $h_{\text{top}}(T)$, of any self-induced minimal Cantor system (X, T) is either 0 or ∞ . See [35] for an introduction to entropy and these results.

Suppose (X, T) is a self-induced minimal Cantor system. We note that the conjugacy to an induced system may be iterated.

Proposition 3. *Suppose (X, T) is a minimal Cantor system conjugate via φ to the induced system (U, T_U) where $U \subsetneq X$ is clopen. Then, $\varphi|_{\varphi^k(X)}$ is a conjugacy*

from the induced system $(\varphi^k(X), T_{\varphi^k(X)})$ to $(\varphi^{k+1}(X), T_{\varphi^{k+1}(X)})$ for any $k \geq 0$. In particular (X, T) and $(\varphi^k(X), T_{\varphi^k(X)})$ are conjugate by φ^k .

Proof. The induced system of (X, T) on a clopen set $\varphi^k(X)$, is conjugate by φ to the induced system of (U, T_U) on $\varphi^{k+1}(X)$. We have just to notice that this induced system is the induced system of (X, T) on $\varphi^{k+1}(X)$. \square

Using the above, we can show that (X, T) is conjugate to an induced map T_U on a clopen set U where $U \subset X$ is an arbitrarily small set, where the smallness is expressed in terms of the invariant measures.

Proposition 4. *Let (X, T) be a minimal Cantor system conjugate via φ to the induced system (U, T_U) where $U \subsetneq X$ is clopen, then, $\lim_n \sup_{\mu \in M(X, T)} \mu(\varphi^n(X)) = 0$. In particular, $\bigcap_{n \in \mathbb{N}} \varphi^n(X)$ has measure 0 with respect to any T -invariant Borel measure μ .*

Proof. Since U^c is clopen and nonempty, $\mu(U) < 1$ for all $\mu \in M(X, T)$, and by compactness of $M(X, T)$, $\sup_{\mu \in M(X, T)} \mu(U) < 1$. Let $r = \sup_{\mu \in M(X, T)} \mu(U)$.

Since $\varphi : X \rightarrow U$ is a conjugacy, every measure in $M(U, T_U)$ is of the form $\nu \circ \varphi^{-1}$ for some $\nu \in M(X, T)$. Therefore, by Proposition 2, $\frac{1}{\mu(U)}\mu = \nu \circ \varphi^{-1}$ for some $\nu \in M(X, T)$. In particular,

$$r \geq \nu(U) = \nu \circ \varphi^{-1}(\varphi(U)) = \frac{\mu(\varphi(U))}{\mu(U)}.$$

So $\mu(\varphi(U)) \leq r\mu(U) \leq r^2$. It follows by induction that $\mu(\varphi^k(U)) \leq r^{k+1}$ for all $k \geq 1$. This ends the proof. \square

Proposition 5. *The topological entropy of any self-induced minimal Cantor system is 0 or ∞ .*

Proof. Suppose (X, T) is a minimal Cantor system and $U \subsetneq X$ is a clopen set such that (U, T_U) is conjugate to (X, T) . Let $1 > \epsilon > 0$. From propositions 3 and 4, one can take U such that $\mu(U) \leq \epsilon$ for all $\mu \in M(X, T)$.

For $\mu \in M(X, T)$ we set $\mu_U = \frac{1}{\mu(U)}\mu$. It is a probability measure defined on the Borel sets of U . Abramov's formula relates the measure-theoretic entropy of the systems (X, T, μ) and (U, T_U, μ_U) as follows: $h_\mu(T) = \mu(U)h_{\mu_U}(T_U)$. From the variational principle one gets

$$\begin{aligned} h_{\text{top}}(T) &= \sup_{\mu \in M(X, T)} h_\mu(T) = \sup_{\mu \in M(X, T)} \mu(U)h_{\mu_U}(T_U) \\ &\leq \epsilon \sup_{\mu \in M(X, T)} h_{\mu_U}(T_U) = \epsilon h_{\text{top}}(T_U), \end{aligned}$$

where the last equality comes from Proposition 2. Since (U, T_U) is conjugate to (X, T) , we have $h_{\text{top}}(T_U) = h_{\text{top}}(T)$, which ends the proof. \square

2.3. Bratteli-Vershik representations. In this section, we discuss the Bratteli-Vershik representation for a minimal Cantor system; every minimal Cantor system admits such a representation. We give a brief outline of such constructions emphasizing the notations used in this paper. The reader can find for more details on this theory in [25] (see also [22] and [13]). Such representations will be very helpful in many proofs or remarks below thanks to Proposition 6.

2.3.1. Bratteli diagrams. A Bratteli diagram is given by $((V_k)_{k \geq 0}, (E_k)_{k \geq 1})$ where for each $k \geq 1$ V_{k-1} is a finite set of vertices and E_k is a finite set of oriented edges from V_{k-1} to V_k . The set V_0 is a singleton $\{v_0\}$, and for $k \geq 1$, $V_k = \{1, \dots, C(k)\}$. For the purposes of non-degeneracy, we require that every vertex in V_k is the “end-point” of some edge in E_k for $k \geq 1$ and an “initial-point” of some edge in E_{k+1} for $k \geq 0$.

The *level k* of the diagram is the subgraph consisting of the vertices in $V_k \cup V_{k+1}$ and the edges E_{k+1} between these vertices. We describe the edge set E_k using a $V_k \times V_{k-1}$ matrix $M(k)$ over \mathbb{Z}_+ which is the adjacency matrix of the level k subgraph. The (i, j) -entry of $M(k)$ is the number of distinct edges in E_k joining vertex $j \in V_{k-1}$ with vertex $i \in V_k$. For $1 \leq k \leq l$ one defines $P(l, k) = M(l) \cdots M(k+1)$ with $P(k, k) = I$; where I is the identity map.

If for every k , $P(k+1, k)$ is positive, we say that the diagram (V, E) is *simple*.

2.3.2. Ordered Bratteli diagrams. We call a *range located order* (RL-order for short) on a graph (V, E) a partial order on its set of edges E such that two edges e, e' are comparable if and only if they have the same end-point. An *ordered* Bratteli diagram is a triple $B = ((V_k), (E_k), (\preceq_k))$ where $((V_k), (E_k))$ is a Bratteli diagram and each \preceq_k is an RL-order on the subgraph of level k .

Let $1 \leq k \leq l$ and let $E_{k,l}$ be the set of all paths in the graph joining vertices of V_{k-1} with vertices of V_l . The orders $\preceq_k, \dots, \preceq_l$ induce an order on $E_{k,l}$ given by $(e_k, \dots, e_l) \preceq_{k,l} (f_k, \dots, f_l)$ if and only if there is $k \leq i \leq l$ such that $e_j = f_j$ for $i < j \leq l$ and $e_i \preceq_i f_i$. Notice it is an RL-order on the graph $(V_k \cup V_l, E_{k,l})$.

Given a strictly increasing sequence of integers $(m_k)_{k \geq 0}$ with $m_0 = 0$ one defines the *contraction* of $B = ((V_k), (E_k), (\preceq_k))$ (with respect to $(m_k)_{k \geq 0}$) as

$$\left((V_{m_k})_{k \geq 0}, (E_{m_{k+1}, m_{k+1}})_{k \geq 0}, (\preceq_{m_{k+1}, m_{k+1}})_{k \geq 0} \right).$$

The inverse operation is called *microscoping* (see [25] for more details).

Given an ordered Bratteli diagram $B = ((V_k), (E_k), (\preceq_k))$ one defines X_B as the set of infinite paths (x_1, x_2, \dots) starting in v_0 such that for all $k \geq 1$ the end-point of $x_k \in E_k$ is the initial-point of $x_{k+1} \in E_{k+1}$. We topologize X_B by postulating a basis of open sets, namely the family of *cylinder sets*

$$[e_1, e_2, \dots, e_l]_B = \{(x_1, x_2, \dots) \in X_B : x_i = e_i, \text{ for } 1 \leq i \leq l\}.$$

Each $[e_1, e_2, \dots, e_l]_B$ is also closed, as is easily seen, and so we observe that X_B is a compact, totally disconnected metrizable space.

When there is a unique $x = (x_1, x_2, \dots) \in X_B$ such that x_i is maximal for any $i \geq 1$ and a unique $y = (y_1, y_2, \dots) \in X_B$ such that y_i is minimal for all $i \geq 1$, one says that B is a *properly ordered* Bratteli diagram. Call these particular points x_{\max} and x_{\min} respectively. In this case one defines a homeomorphism $V_B : X_B \rightarrow X_B$ called the *Vershik map* as follows. Let $x = (x_1, x_2, \dots) \in X_B \setminus \{x_{\max}\}$ and let $k \geq 1$ be the smallest integer so that x_k is not a maximal edge. Let y_k be the successor of x_k and (y_1, \dots, y_{k-1}) be the unique minimal path in $E_{1,k-1}$ connecting v_0 with the initial vertex of y_k . One sets $V_B(x) = (y_1, \dots, y_{k-1}, y_k, x_{k+1}, \dots)$ and $V_B(x_{\max}) = x_{\min}$.

The dynamical system (X_B, V_B) is called the *Bratteli-Vershik system* generated by B . In [25] it is proved that any minimal Cantor system (X, T) is conjugate to a Bratteli-Vershik system (X_B, V_B) on a simple, properly ordered Bratteli diagram. One says that (X_B, V_B) is a *Bratteli-Vershik representation* of (X, T) . In what

follows we identify a minimal Cantor system (X, T) with any of its Bratteli-Vershik representations.

A minimal Cantor system is of (topological) *finite rank* if it admits a Bratteli-Vershik representation such that the number of vertices per level $\#V_n$ is uniformly bounded by some integer d . The minimal such d is called the *topological rank* of the system. We observe that topological and measure theoretical finite rank are completely different notions. For instance, systems of topological rank one correspond only to odometers whereas the Chacon subshift has measure theoretical rank one (see [17]).

2.3.3. Induction and Bratteli-Vershik representations. Let us show that the induction process can be easily “observed” through Bratteli diagrams, as enlightened in [22].

Let U be a clopen set of (X_B, V_B) . It is a finite union of cylinder sets. We can suppose they all have the same length *i.e.*, for some n , $U = \cup_{p \in P} [p]_B$ where P is a set of paths from V_0 to V_n . Since the diagram is simple, we may assume that every vertex in V_n is the end-point for a path in P . Thus, we may define a contraction \hat{B} of B where levels 1 through n are contracted to level 1. To obtain a Bratteli-Vershik representation of the induced system on U it suffices to take the properly ordered Bratteli diagram B' which on level 1 consists of all edges in \hat{B} corresponding to a path in P , and is otherwise the same as the diagram \hat{B} , where the new order \preceq'_1 is induced by the one of \hat{B} . It is not too much work to prove that the induced system on U is conjugate to $(X_{B'}, V_{B'})$. To summarize we get the following, where the proof can be found in that of Theorem 3.8 in [22].

Proposition 6. *Let (X_B, V_B) be the Bratteli-Vershik dynamical system associated with a simple properly ordered Bratteli diagram $B = ((V_n), (E_n), (\preceq_n))$. Let P be a set of paths from V_0 to V_{n_0} . Then, a Bratteli-Vershik representation of the induced system of (X_B, V_B) on the clopen set $U = \cup_{p \in P} [p]_B$ is defined by the following Bratteli diagram $((V'_n), (E'_n), (\preceq'_n))$, where:*

- *The set of end-points of paths in P is V'_1 ;*
- *The paths P are the edges from $V'_0 = V_0$ to V'_1 ;*
- *The set of edges in E_{n_0+1} with initial-points in V'_1 is E'_2 ;*
- *for $n \geq 2$, $V'_n = V_{n-1+n_0}$ and $E'_{n+1} = E_{n+n_0}$ with the induced order.*

We say a Bratteli diagram $((V_n), (E_n), (\preceq_n))$ is *stationary* whenever the sequences (V_n) , (E_n) and (\preceq_n) are constant. Minimal Cantor systems having such a representation are exactly substitution subshifts or odometers with constant base (see Section 3 or [15]). A straightforward consequence of Proposition 6 shows these systems are self-induced.

Below we make use of the Bratteli-Vershik representation to prove a technical lemma which will be useful later in the paper.

Proposition 7. *Let (X, T) be a self-induced minimal Cantor system and let $W \subset X$ be clopen, nonempty. Then, (X, T) is conjugate to an induced system (U, T_U) where $U \subset W$.*

Proof. Let (X, T) be a self-induced minimal Cantor system and let $W \subset X$ be a nonempty clopen set. By iterating the self-induction conjugacy φ sufficiently many times, Proposition 4 implies that (X, T) is conjugate to an induced map (U, T_U) where $\mu(U) < \mu(W)$ for all μ in $M(X, T)$.

For large enough n , there exist sets, P_U and P_W , of paths ending at level n satisfying $U = \cup_{p \in P_U} [p]_B$ and $W = \cup_{p \in P_W} [p]_B$. Since $\mu(U) < \mu(W)$ for all μ in $M(X, T)$, taking n sufficiently large, we may assume that for each vertex $v \in V_n$, the number of paths in P_U which end at v is less than the number of paths in P_W which end at v (This is not a trivial fact, but is rather standard by now. It follows from the simplicity of the diagram and the Ergodic Theorem. For example see proof of Lemma 2.5 in [24]).

For every vertex v in V_n , let $u(v)$ equal the number of P_U -paths ending at v and $w(v)$ the number of P_W -paths ending at v . By assumption $u(v) < w(v)$ for each v .

Define a set $W' \subset W$ which is the union over $v \in V_n$ of the union of cylinder sets of the first $u(v)$ P_W -paths ending at v . Then, we have Bratteli-Vershik representations of the induced map on U and the induced map on W' on diagrams which both have exactly $u(v)$ edges from v_0 to $v \in V_1$ and are identical below level n . Thus, (U, T_U) is conjugate to $(W', T_{W'})$ where $W' \subset W$. \square

3. EQUICONTINUOUS AND EXPANSIVE CASES

We recall that, from the previous section, to be self-induced it is necessary to have zero entropy or infinite entropy. Among zero entropy dynamical systems is the family of those having finite topological rank. A result of Downarowicz and Maass [10] asserts that in this family the Cantor systems are either equicontinuous, that is odometers, or expansive, that is subshifts. Here we treat these both cases characterizing the self-induced Cantor systems of finite topological rank.

3.1. Equicontinuous case: the odometers. Let $(q_n)_{n \geq 1}$ be a integer sequence such that $q_n \geq 2$. We set $p_n = q_n \cdots q_2 q_1$. The set of (q_n) -adic integers is the inverse limit

$$\mathbb{Z}_{(q_n)} = \left\{ (x_n) \in \prod_{n=1}^{\infty} \mathbb{Z}/p_n \mathbb{Z} : x_n \equiv x_{n+1} \pmod{p_n} \right\}.$$

We endow $\prod_{n=1}^{\infty} \mathbb{Z}/p_n \mathbb{Z}$ with the product topology of the discrete topologies. The set of (q_n) -adic integers $\mathbb{Z}_{(q_n)}$ endowed with the induced topology forms a Cantor set. It is a topological group (Exercise). The dual group of $\mathbb{Z}_{(q_n)}$ (i.e., the group of continuous characters) is conjugate to the group of $(q_n)_n$ -adic rationals

$$\mathbb{Q}_{(q_n)} = \left\{ \frac{j}{p_n} : j \in \mathbb{Z}, n \geq 1 \right\}.$$

A base of the topology of $\mathbb{Z}_{(q_n)}$ is given by the sets

$$[a_1, a_2, \dots, a_n] = \{(x_n) \in \mathbb{Z}_{(q_n)} : x_i = a_i, 1 \leq i \leq n\},$$

with $a_i \in \mathbb{Z}/p_i \mathbb{Z}$, $i = 1, \dots, n$.

When $p_n = p^n$ for all n , it defines the classical ring of p -adic integers \mathbb{Z}_p . Let $R : \mathbb{Z}_{(q_n)} \rightarrow \mathbb{Z}_{(q_n)}$ be the map $x \mapsto x + 1$. As it will always be clear from the context these maps will be denoted by R for any sequence (q_n) . The Cantor system $(\mathbb{Z}_{(q_n)}, R)$ is called *odometer in base (p_n)* or *with characteristic sequence (q_n)* . This system is minimal and aperiodic.

Following the notation of Section 2.3, a Bratteli-Vershik representation of the odometer $(\mathbb{Z}_{(q_n)}, R)$ is obtained taking a single vertex at each level, $\#V_n = 1$, $n \geq 0$, and setting q_{n+1} edges in E_{n+1} . As there is a unique vertex at each level, the ordering on the E_n 's is arbitrary. Consequently, odometers are of topological

rank 1 and it is easy to prove that rank one Cantor systems are odometers. See [13] for more details. We say it is a *one-vertex* Bratteli diagram.

Let us recall that a minimal Cantor system (X, T) which is *equicontinuous* (i.e., the family of maps $\{T^n\}_{n \in \mathbb{Z}}$ is equicontinuous) is conjugate to an odometer, and, the two following well known results (see [29]).

Lemma 8. *The following are equivalent :*

- (1) *The odometer $(\mathbb{Z}_{(q'_n)}, R)$ is a factor of the odometer $(\mathbb{Z}_{(q_n)}, R)$;*
- (2) *$\mathbb{Q}_{(q'_n)}$ is included in $\mathbb{Q}_{(q_n)}$;*
- (3) *For all n there exists k such that p'_n divides p_k .*

Lemma 9. *The following are equivalent.*

- (1) *$(\mathbb{Z}_{(q'_n)}, R)$ and $(\mathbb{Z}_{(q_n)}, R)$ are conjugate.*
- (2) *$\mathbb{Q}_{(q'_n)} = \mathbb{Q}_{(q_n)}$.*

This last condition is true if and only if for all prime number p we have

$$(1) \quad \lim_n \max \{k : p^k \text{ divides } p_n\} = \lim_n \max \{k : p^k \text{ divides } p'_n\}.$$

As a corollary, we obtain the following. The proof is left to the reader.

Corollary 10. *Let $(\mathbb{Z}_{(q_n)}, R)$ be an odometer.*

- (1) *Any permutation of the elements of the characteristic sequence will define the same odometer, up to conjugacy.*
- (2) *$(\mathbb{Z}_{(q_n)}, R)$ is conjugate to the odometer with characteristic sequence (s_n) where s_n are prime numbers such that $p_{i+1}/p_i = s_{n_i} s_{n_i+1} \cdots s_{n_{i+1}-1}$ for some strictly increasing sequence (n_i) with $n_0 = 1$.*

Thanks to these results we deduce a characterization of self induced odometers. The proof will use Bratteli diagrams. One can also prove this using the periodic structure of $\mathbb{Z}_{(q_n)}$. This would be, in some sense, easier if one is more familiar with (q_n) -adic groups than Bratteli diagrams. However, we want to emphasize how useful Bratteli diagrams can be when studying Poincaré recurrence problems.

Proposition 11. *Let $(\mathbb{Z}_{(q_n)}, R)$ be an odometer. The following are equivalent.*

- (1) *$(\mathbb{Z}_{(q_n)}, R)$ is self-induced.*
- (2) *There exists a prime number q dividing infinitely many q_n , with $q_n = p_{n+1}/p_n$, i.e., $\lim_n \max \{k : q^k \text{ divides } p_n\} = \infty$.*
- (3) *There is a homomorphism from $\mathbb{Z}_{(q_n)}$ to \mathbb{Z}_p for some prime number p .*
- (4) *(\mathbb{Z}_p, R) is a factor of $(\mathbb{Z}_{(q_n)}, R)$.*

Proof. We leave as an exercise to prove that (2), (3) and (4) are equivalent. We prove (2) and (1) are equivalent.

(2) \implies (1) Again using Corollary 10, we may assume $p_1 = q_1 = q$. Then, Lemma 9 together with Equality (1) imply that the odometer with characteristic sequence $(q_n)_{n \geq 1}$ is conjugate to the odometer with characteristic sequence $(q_{n+1})_{n \geq 1}$.

Let B be the one-vertex Bratteli diagram with q_n edges in E_n for $n \geq 1$. Select a single edge e from E_1 and let $U = [e]$ (i.e., the clopen set of infinite paths starting with e). Proposition 6 gives us that the induced system on U is conjugate to an odometer with characteristic sequence (q_{n+1}) . Thus, $(\mathbb{Z}_{(q_n)}, R)$ is self-induced.

(1) \implies (2) Suppose $(\mathbb{Z}_{(q_n)}, R)$ is self-induced. There exists a proper clopen set U such that (U, R_U) is conjugate to $(\mathbb{Z}_{(q_n)}, R)$. Since (U, R_U) is an induced system of $(\mathbb{Z}_{(q_n)}, R)$, Proposition 6 implies (U, R_U) is an odometer with characteristic sequence (q'_n) where, for all $n \geq 2$, $q'_n = q_{n+n_0-1}$, for some n_0 , and $q'_1 < q_1 q_2 \cdots q_{n_0}$.

By contradiction, let us assume that there is no prime number p such that $\lim_n \max \{k : p^k \text{ divides } q_1 q_2 \cdots q_n\} = \infty$. Then, from Lemma 9, for all prime numbers p , we should have

$$\begin{aligned} & \lim_n \max \{k : p^k \text{ divides } q_1 q_2 \cdots q_n\} \\ &= \lim_n \max \{k : p^k \text{ divides } q'_1 q_{n_0+1} \cdots q_{n+n_0-1}\} < \infty. \end{aligned}$$

This is in contradiction with the fact that q'_1 is strictly less than $q_1 q_2 \cdots q_{n_0}$. \square

3.2. Expansive case: the substitution subshifts. In this section we prove that a self-induced minimal Cantor system (X, T) is expansive if and only if it is conjugate to a substitution subshift on a finite alphabet. This result may be of interest on its own, it generalizes a similar result in the case where the induced map is of the form T^k for some fixed k [11].

Recall that a homeomorphism $T : X \rightarrow X$ on a compact metric space (X, d) is *expansive* if there exists a constant $\delta > 0$ such that given any $x, y \in X$, $x \neq y$, there exists an $n \in \mathbb{Z}$ such that $d(T^n x, T^n y) > \delta$. A classical theorem states that if X is a Cantor space and (X, T) is an expansive dynamical system then, (X, T) is conjugate to a subshift.

3.2.1. Subshifts and induction. Let \mathcal{A} be an *alphabet*, that is, a finite set. We endow $\mathcal{A}^{\mathbb{Z}}$ with the infinite product of the discrete topology and consider the shift map $S : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$. That is, for a sequence $x = (x_i) \in \mathcal{A}^{\mathbb{Z}}$, $S(x)$ is the sequence defined by $S(x)_i = x_{i+1}$. By a *subshift* on \mathcal{A} we shall mean a couple (X, S) where $X \subset \mathcal{A}^{\mathbb{Z}}$ is closed and shift-invariant ($S(X) = X$). See [30] for a thorough introduction to subshifts.

A *word* is an element of the free monoid \mathcal{A}^* generated by \mathcal{A} where the neutral element is denoted by ϵ and is called the *empty word*. For $w = w_1 \dots w_n \in \mathcal{A}^*$, $w_i \in \mathcal{A}$, we use $|w|$ to denote the *length* of w , that is $|w| = n$. We denote by \mathcal{A}^n the set of words of length n . We set $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\epsilon\}$.

For a point $x = (x_i) \in \mathcal{A}^{\mathbb{Z}}$, and $i, j \in \mathbb{Z}$ with $i \leq j$ let $x[i, j]$ denote the word $x[i, j] = x_i \dots x_j$. We say i is an occurrence of the word u in x if $u = x[i, i + |u| - 1]$. We keep the same notation for finite words.

For a subshift (X, S) , we call *language* of X the subset of \mathcal{A}^* defined by $\mathcal{L}(X) = \{x[i, j] : x \in X, i \leq j\}$. We set $\mathcal{L}_k(X) = \{x[i, i + k - 1] : x \in X, i \in \mathbb{Z}\}$. A base for the subspace topology on X consists of the family of the sets

$$[w.v]_X = \{x \in X : x[-|w|, |v| - 1] = wv\},$$

where $w, v \in \mathcal{A}^*$. These sets $[w.v]_X$ are called *cylinders*. When it will not create confusion we will write $[w.v]$ instead of $[w.v]_X$. We set $[v] = [\epsilon.v]$, where ϵ is the empty word of \mathcal{A}^* .

Every shift commuting continuous function $F : X \rightarrow Y$ between two subshifts is given by a *sliding block code* [30]. That is, a map f from the set of words of length $2r + 1$ in X to letters appearing in elements of Y such that for all $x \in X$ and $i \in \mathbb{Z}$, $F(x)_i = f(x[i - r, i + r])$.

Suppose $X \subset \mathcal{A}^{\mathbb{Z}}$ is a subshift. The k -block presentation of X is the subshift $X^{[k]}$ with alphabet \mathcal{A}^k defined as the set of the sequences $(w_i) \in (\mathcal{A}^k)^{\mathbb{Z}}$ such that

- (1) each $w_i = w_{i,1}w_{i,2} \dots w_{i,k}$ is in $\mathcal{L}_k(X)$,
- (2) $w_{i,2} \dots w_{i,k} = w_{i+1,1} \dots w_{i+1,k-1}$ for all $i \in \mathbb{Z}$,
- (3) the sequence $(w_{i,1})$ is in X .

It is well-known that the shift map on the k -block presentation of X is conjugate to the shift map on X [30].

Let us explain a way to see the induction process via words in minimal subshifts. Let (X, S) be a minimal subshift defined on the alphabet \mathcal{A} and U be a clopen subset of X . Then, there exist words $u_1^-, \dots, u_n^-, u_1^+, \dots, u_n^+$ such that $U = \cup_{i=1}^n [u_i^-.u_i^+]$. Observe that these words can be chosen of equal lengths.

Let $\mathcal{R}_{X,U} \subset \mathcal{L}(X)$ be the set of *return words to U* , that is, the set of words $x[i, j-1]$, with $x \in X$ and $i < j$, where

- (1) $S^i x$ belongs to U ;
- (2) $S^j x$ belongs to U ;
- (3) there is no integer k , with $i < k < j$, such that $S^k x$ belongs to U .

Since the subshift (X, S) is minimal then, the length of the return words to U is finite and the set of return words to U is also finite.

When U consists of a single word and u^- is the empty word, then we recover the classical notion of return words (see [11, 20, 12]). In this situation, $U = [\epsilon.u^+]$, the return words w are characterized by the fact that u^+ is a prefix and a suffix of wu^+ , and, wu^+ has no other occurrence of u^+ than these two (see [15]).

The general situation is a bit different. We should keep the information of which u_i^\pm precedes and follows a return word w , for each occurrences of w : $u_i^-wu_j^+$ and $u_k^-wu_l^+$ could belong to $\mathcal{L}(X)$ for different i, j, k, l . Thus, this return word w represents two different “returns” to U (because it returns to different cylinders describing U). To this end we set

$$\tilde{\mathcal{R}}_{X,U} = \{(u^-, w, u^+) : u^-wu^+ \in \mathcal{L}(X), w \in \mathcal{R}_{X,U}, u^- \in U^-, u^+ \in U^+\},$$

where $U^- = \{u_1^-, \dots, u_n^-\}$ and $U^+ = \{u_1^+, \dots, u_n^+\}$.

Let θ be the morphism from $\tilde{\mathcal{R}}_{X,U}^*$ to \mathcal{A}^* defined by $\theta(u^-, w, u^+) = w$. As usual, we extend by concatenation θ to a map, also denoted θ , from $\tilde{\mathcal{R}}_{X,U}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$. It is easy to check that for each $x \in U$ there is a unique $y \in \tilde{\mathcal{R}}_{X,U}^{\mathbb{Z}}$ such that $x = \theta(y)$. Let Y be the set of all these y . It is invariant under the shift and hence (Y, S) is a subshift on the alphabet $\tilde{\mathcal{R}}_{X,U}$.

We left as an exercise to prove that $\theta : (Y, S) \rightarrow (U, S_U)$ is a conjugacy map. For the ideas to use we refer to [15] where this is treated when $U = [u^-.u^+]$.

3.2.2. Substitutive and primitive substitution subshifts. We recall here a small amount of the background necessary for this section on minimal substitution subshifts. For a much more in depth introduction see [15, 21, 36].

Suppose we have an alphabet \mathcal{A} and a map $\sigma : \mathcal{A} \rightarrow \mathcal{A}^+$. As usual, a substitution σ on an alphabet space \mathcal{A} can be extended to functions $\sigma : \mathcal{A}^+ \rightarrow \mathcal{A}^+$ by concatenation, and therefore iterated.

Because we wish to focus on minimal systems, we will assume that all of our substitutions are *primitive i.e.*, the following two properties hold.

- there is an $n \in \mathbb{N}$ such that for any two letters $a, b \in \mathcal{A}$, the letter b appears in the word $\sigma^n(a)$,
- there is an $a \in \mathcal{A}$ such that $\lim_{k \rightarrow \infty} |\sigma^k(a)| = \infty$.

Given a primitive substitution σ , let X_σ denote the set of sequences x in $\mathcal{A}^{\mathbb{Z}}$ with the property that every word that appears in x is a subword of $\sigma^k(a)$ for some $a \in \mathcal{A}$ and k . This set is S -invariant. As it will not create confusion, we continue to denote by S the restriction of S to X_σ . The couple (X_σ, S) is called *primitive substitution subshift* (see [36] for more details on these subshifts).

In the literature one can find less restrictive definitions of substitution subshifts that are, up to conjugacy, irrelevant in the minimal context as shown by Proposition 12 below. For instance, let us present a general definition of *substitutive subshifts*. Let σ be an endomorphism of the free monoid \mathcal{A}^* and let $\phi : \mathcal{A}^* \rightarrow \mathcal{B}^*$, for some finite alphabet \mathcal{B} . Let X be the set of infinite sequences x in $\mathcal{B}^{\mathbb{Z}}$ with the property that every word that appears in x is a subword of $\phi \circ \sigma^k(a)$ for some $a \in \mathcal{A}$ and k . It could happen that X is empty but it is a closed S -invariant subset of $\mathcal{B}^{\mathbb{Z}}$. We say (X, S) is a *substitutive subshift*. Within the category of minimal subshifts, the following result shows that our primitiveness hypothesis on the substitution is not restrictive. In fact, it proves that if you are given a non-primitive substitution generating a minimal subshift (X, S) then, there is a primitive substitution whose subshift is conjugate to (X, S) .

Proposition 12. [15, 14] *Let (X, S) be a minimal subshift. The following are equivalent.*

- (1) (X, S) is conjugate to a substitutive subshift.
- (2) (X, S) is conjugate to a primitive substitution subshift.

For example the Chacon substitution $0 \mapsto 0010, 1 \mapsto 1$ is not primitive but its subshift (X, S) is minimal and it is conjugate to a primitive substitution subshift (see [11]).

The following is well-known, e.g. see [15, Cor 12]. It is a direct consequence of Mossé's result on recognizability (see [32, 33]).

Proposition 13. *Aperiodic primitive substitution subshifts are self-induced minimal Cantor systems.*

Our goal here is to prove the converse, that expansive self-induced minimal Cantor systems must be substitution subshifts. Observe that self-induced minimal Cantor systems are necessarily aperiodic.

Theorem 14. *Let (X, S) be a self-induced expansive minimal Cantor system. Then, it is conjugate to a substitution subshift.*

Proof. Assume that (X, S) is an expansive minimal Cantor system (hence a subshift on some alphabet \mathcal{A} , up to conjugacy) that is self-induced. This means there is a clopen set $U \subsetneq X$ such that (X, S) and (U, S_U) are conjugate. We set $U = \bigcup_{i=1}^N [u_i^-, u_i^+]$ for some words u_i^\pm . Without loss of generality we can suppose that $|u_i^+| = |u_j^-|$, for all i, j .

Taking the notations of Section 3.2.1 and the observation we made, (U, S_U) is isomorphic to some subshift (Y, S) defined on the alphabet $\mathcal{R} = \tilde{\mathcal{R}}_{X,U}$. Notice that the map θ defined in this section is such that the length $|\theta(a)|$ is greater or equal to 1 for all $a \in \mathcal{R}$.

Let $f : \mathcal{A}^{2r+1} \rightarrow \mathcal{R}$ be a sliding block code defining a conjugacy from (X, S) to (Y, S) . We claim there exists an integer l such that

$$(2) \quad |\theta(y_1 \dots y_l)| \geq r + l$$

for all words $y_1 \dots y_l$ of $\mathcal{L}(Y)$. Indeed, (Y, S) being minimal it suffices to show there exists a letter $a \in \mathcal{R}$ such that $|\theta(a)| \geq 2$. Suppose it is not the case. This means that $S^k x$ belongs to U for all k and $x \in X$. Thus, from the minimality of (X, S) , we obtain that $U = X$ which contradicts the choice of U .

For all n , let \mathcal{A}_n be the alphabet

$$\{(x_{-n} \dots x_{-1} x_0 x_1 \dots x_n) : x_{-n} \dots x_{-1} x_0 x_1 \dots x_n \text{ is a word of } X\}.$$

Recall that $X^{[2n+1]}$ denote the $(2n+1)$ -block presentation of X , that is

$$X^{[2n+1]} = \{((x_{k-n} \dots x_{k-1} x_k x_{k+1} \dots x_{k+n}))_k : (x_k)_k \in X\}.$$

Of course the subshift (X, S) is conjugate to $(X^{[2n+1]}, S)$, by the sliding block code $h_n : \mathcal{A}^{2n+1} \rightarrow \mathcal{A}_n$ defined as $h_n(x_{-n} \dots x_n) = (x_{-n} \dots x_n)$. In the same way we can define \mathcal{R}_n and $Y^{[2n+1]}$. The map f naturally defines a sliding block code $g : \mathcal{A}_{l+r} \rightarrow \mathcal{R}_l$ asserting

$$\begin{aligned} g((x_{-(l+r)} \dots x_{-1} x_0 x_1 \dots x_{l+r})) &= (f(x_{-(l+r)} \dots x_{-1} x_0 x_1 \dots x_{l+r})) \\ &= (y_{-l} \dots y_{-1} y_0 y_1 \dots y_l). \end{aligned}$$

This provides a conjugacy from $(X^{[2(r+l)+1]}, S)$ to $(Y^{[2l+1]}, S)$. To conclude, it suffices to prove that $(Y^{[2l+1]}, S)$ is a substitution subshift. For this, let us construct a morphism $\tau : \mathcal{R}_l^* \rightarrow \mathcal{A}_{l+r}^*$.

For each $b = (y_{-l} \dots y_{-1} y_0 y_1 \dots y_l)$ in \mathcal{R}_l , we set, for $-l \leq k \leq l$, $\theta(y_k) = x_0^{(k)} \dots x_{|\theta(y_k)|-1}^{(k)} \in \mathcal{A}^*$. So we have

$$\theta(y_{-l} \dots y_{-1} y_0 y_1 \dots y_l) = x_0^{(-l)} \dots x_{|\theta(y_{-l})|-1}^{(-l)} \dots x_0^{(l)} \dots x_{|\theta(y_l)|-1}^{(l)}.$$

Observe that by (2), the length of the word $\theta(y_{-l} \dots y_{-1} y_0 y_1 \dots y_l)$, $\sum_{k=-l}^l |\theta(y_k)|$ is greater than $2(r+l)+1$. Let p (resp. s) be the prefix (resp. the suffix) of $\theta(y_{-l} \dots y_{-1} y_0 y_1 \dots y_l)$ of respective lengths $\sum_{k=-l}^{-1} |\theta(y_k)| - l - r$ and $\sum_{k=1}^l |\theta(y_k)| - l - r$. Let w be the word of length $|\theta(y_0)| + 2(l+r)$ such that $\theta(y_{-l} \dots y_0 \dots y_l) = pws$. Finally we define

$$\tau(b) = h_{l+r}(w) \in \mathcal{A}_{l+r}^*,$$

where h_{l+r} denote the sliding block code $\mathcal{A}^{2(l+r)+1} \rightarrow \mathcal{A}_{l+r}$.

It is plain to check that for every word $y_{i-l} y_{i-l+1} \dots y_{j+l}$ in $\mathcal{L}(Y)$, with $i \leq j$, the word $\tau((y_{i-l} \dots y_i \dots y_{i+l})(y_{i-l+1} \dots y_{i+l+1}) \dots (y_{i-l} \dots y_j \dots y_{j+l}))$ is a factor of the word $h_{l+r}(\theta(y_{i-l} y_{i-l+1} \dots y_{j+l}))$.

Consequently τ sends words of $\mathcal{L}(Y^{[2l+1]})$ to words of $\mathcal{L}(X^{[2(r+l)+1]})$ and $g \circ \tau$ defines an endomorphism of \mathcal{A}_l^* preserving the language $\mathcal{L}(Y^{[2l+1]})$.

We claim that for some $a \in \mathcal{R}_l$ the sequence of lengths $(|(g \circ \tau)^n(a)|)_n$ goes to infinity. This claim enables us to conclude because it implies that $g \circ \tau$ defines a non-empty substitution subshift included in $Y^{[2l+1]}$. From the minimality of $(Y^{[2l+1]}, S)$, both subshifts coincide and we prove the theorem using Proposition 12.

Let us prove the claim by contradiction. Thus, for all words $w \in \mathcal{R}_l^*$ the sequence $(|(g \circ \tau)^n(w)|)_n$ is bounded. Let a be a letter of \mathcal{R} satisfying $|\theta(a)| \geq 2$ and

$b = (y_{-l} \dots y_{-1} y_0 y_1 \dots y_l) \in \mathcal{R}_l$ where $y_0 = a$. The system $(Y^{[2l+1]}, S)$ being minimal, there exists a constant K such that any word $w \in \mathcal{R}_l^*$ of length K has an occurrence of b . Let w be such a word. It has an occurrence of b and any other letter c appearing in w satisfies $|\theta(c)| \geq 1$. Thus, $|g \circ \tau(w)| \geq |w| + 1 > K$ and consequently for all $n \geq 1$, $|(g \circ \tau)^n(w)| \geq |w| + n$. We get a contradiction and this shows the claim. \square

3.3. The finite topological rank case. We recall that the topological rank of a minimal Cantor system (X, T) is the minimum over all its Bratteli-Vershik representations $B = ((V_n), (E_n), (\preceq_n))$ of the quantity $\sup_n \#V_n$. For example odometers have topological rank 1, Sturmian subshifts have topological rank 2 (see [6]), minimal substitution subshifts all have a finite topological rank (see [15]), as the Cantor version of interval exchange transformations (see [23]). It is well-known minimal Cantor systems with finite topological rank always have zero entropy. Let us mention that Toeplitz subshifts can have finite or infinite topological rank as some are substitution subshifts and some have positive topological entropy.

In [10] the authors proved the following nice result.

Theorem 15. [10] *A minimal Cantor systems with finite topological rank is necessarily expansive or equicontinuous.*

Thus, from the previous sections we deduce the following result.

Corollary 16. *A minimal Cantor systems with finite topological rank is self-induced if and only if it is a minimal substitution subshift or an odometer fulfilling Property (2) of Proposition 11.*

4. SELF-INDUCED EXAMPLES THAT ARE NEITHER EQUICONTINUOUS NOR EXPANSIVE

In this section, we provide examples of self-induced minimal Cantor systems that are non-expansive and non-equicontinuous. The first one has zero topological entropy whereas the second has an infinite topological entropy. To present them, we first recall the notion of *Toeplitz subshift*.

4.1. Toeplitz subshift. A *Toeplitz sequence* on a finite alphabet \mathcal{A} is a sequence $(x_n)_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ such that

$$\forall n \in \mathbb{Z}, \exists p \in \mathbb{Z} \text{ such that } x_{n+kp} = x_n, \forall k \in \mathbb{Z}.$$

A subshift (X, S) is Toeplitz when $X = \overline{\{S^i(x) : i \in \mathbb{Z}\}}$ for a Toeplitz sequence x . Let us recall some properties we need. We refer to [9] for a survey on Toeplitz subshifts. Any Toeplitz subshift is minimal and any aperiodic Toeplitz subshift (X, S) is an almost one-to-one extension of an odometer: *i.e.*, there exists a point with a unique preimage. Hence this odometer is the *maximal equicontinuous* factor of (X, S) , meaning that any equicontinuous factor of (X, S) is a factor of this odometer.

4.2. Self induced example with zero entropy. Our example is a product of a substitution subshift and an odometer. Let σ be the substitution defined on the alphabet $\{0, 1\}$ by

$$\sigma(0) = 01 \text{ and } \sigma(1) = 00.$$

Let (X_σ, S) be the subshift it generates. From a well-known result of Dekking [7], this is a minimal Cantor system and its maximal equicontinuous factor is the odometer defined on \mathbb{Z}_2 .

Let consider the Cantor system $(X_\sigma \times \mathbb{Z}_3, T)$ defined by

$$\begin{aligned} T: X_\sigma \times \mathbb{Z}_3 &\rightarrow X_\sigma \times \mathbb{Z}_3 \\ (x, z) &\mapsto (Sx, z + 1). \end{aligned}$$

Property 17. *The system $(X_\sigma \times \mathbb{Z}_3, T)$ is minimal.*

Even if it can be checked by using the very regular structure of the systems, we use a general result on Toeplitz systems in [9] to prove it.

Proof. From Lemma 12.1 in [9] and since (X_σ, S) does not have a topological periodic factor of period 3, we deduce that the (X_σ, S) and $(\mathbb{Z}_3, z \mapsto z + 1)$ are measurably disjoint. Hence the product system has the product measure as unique invariant probability measure, implying the minimality of the system. \square

Property 18. *The system $(X_\sigma \times \mathbb{Z}_3, T)$ is neither expansive nor equicontinuous.*

Proof. Let d_1 and d_2 be two distances defining the topologies of X_σ and \mathbb{Z}_3 , respectively. The action of T is not equicontinuous as it has an expansive factor, namely (X_σ, S) .

Let us show it is not expansive. Let $\epsilon > 0$ be arbitrary, it is enough to construct two points $(x, z) \neq (x', z') \in X_\sigma \times \mathbb{Z}_3$ such that $d_1(S^n x, S^n x') < \epsilon$ and $d_2(z + n, z' + n) < \epsilon$ for all integer n . Because the odometer \mathbb{Z}_3 is an isometry, we simply select any $x = x'$ and any $z \neq z'$ with $d_2(z, z') < \epsilon$. \square

Notice that the two previous claims also hold for the product of an odometer $(\mathbb{Z}_q, z \mapsto z + 1)$ with a Toeplitz subshift having $(\mathbb{Z}_p, z \mapsto z + 1)$ as maximal equicontinuous factor, when p and q are coprime.

Property 19. *The minimal Cantor system $(X_\sigma \times \mathbb{Z}_3, T)$ is self-induced.*

Proof. Define $\varphi : X_\sigma \times \mathbb{Z}_3 \rightarrow \sigma(X_\sigma) \times \mathbb{Z}_3$ by

$$\varphi(x, z) = (\sigma(x), 2z).$$

The map σ from X_σ onto its image satisfies $S^2 \circ \sigma = \sigma \circ S$. Moreover, since the substitution is injective on the letters and of constant length, the map $\sigma : X_\sigma \rightarrow \sigma(X_\sigma)$ is one-to-one. Thus, the return time for the map S to the clopen set $\sigma(X_\sigma)$ is constant and equal to 2. Let U be the clopen set $U = \sigma(X_\sigma) \times \mathbb{Z}_3$. Then, the induced map is

$$T_U : (x, z) \mapsto (S^2 x, z + 2).$$

Furthermore, we have

$$T_U \circ \varphi(x) = T_U(\sigma(x), 2z) = (S^2(\sigma(x)), 2z + 2)$$

and

$$\varphi \circ T(x) = \varphi(S(x), z + 1) = (\sigma(S(x)), 2(z + 1)) = (S^2(\sigma(x)), 2z + 2).$$

Hence, $T_U \circ \varphi = \varphi \circ T$, and φ is a conjugacy and $(X_\sigma \times \mathbb{Z}_3, T)$ is self-induced. \square

4.3. Self-induced example of infinite entropy. In this second example, we show that self-induced systems need not be uniquely ergodic, and need not be zero entropy.

The key ingredients will be that there exist Toeplitz subshifts that have a binary odometer as a factor, and that such systems can have positive entropy and can have multiple ergodic measures (see [41] or [9]).

Let (K_0, S_0) be any Toeplitz subshift with the odometer \mathbb{Z}_2 as a factor. The system (K_0, S_0) is the orbit closure of a Toeplitz sequence $z \in K_0$ that is *regularly recurrent* i.e., for any open neighborhood U of z , there is a $m \geq 0$ such that $S_0^{mn}(z) \in U$ for all $n \in \mathbb{Z}$. Because (K_0, S_0) has \mathbb{Z}_2 as a factor, there is a sequence of factor maps $\pi_n : (K_0, S_0) \rightarrow (\mathbb{Z}/2^n\mathbb{Z}, x \mapsto x+1)$ such that $\pi_n(z) = 0$.

For each integer $n \geq 0$, let $K_n := \pi_n^{-1}(0)$. We get that $K_0 \supset K_1 \supset K_2 \supset \dots$ and for each n , $\{K_n, S_0(K_n), \dots, S_0^{2^n-1}(K_n)\}$ forms a clopen partition of K_0 .

For $n \geq 1$, let S_n denote the induced map of S_0 on K_n , $S_n : K_n \rightarrow K_n$. So S_n is equal to $S_0^{2^n} = S_{n-1}^2$ on K_n for all $n \geq 1$.

For $n < 0$, we recursively define a sequence of spaces $K_0 \hookrightarrow K_{-1} \hookrightarrow K_{-2} \hookrightarrow \dots$ and maps $S_n : K_n \rightarrow K_n$ so that (K_{n-1}, S_{n-1}) is an exduced system of (K_n, S_n) for the constant function 2. In other words, $K_{n-1} = K_n \times \{0, 1\}$, $S_{n-1} : (x, 0) \mapsto (x, 1)$, and $S_{n-1} : (x, 1) \mapsto (S_n x, 0)$. We also observe $S_{n-1}^2(K_n \times \{0\}) = K_n \times \{0\}$ and $S_{n-1}^2 : (x, 0) \mapsto (S_n x, 0)$.

Consider $\mathbf{K} = \prod_{n \in \mathbb{Z}} K_n$, and let \mathbf{z} denote the sequence $(z_n)_n$ where $z_n = z$ for all $n \geq 0$ and $z_{n-1} = (z_n, 0)$ for $n \leq 0$. Note that for all n , the point z_n is regularly recurrent for the system (K_n, S_n) .

Define $\mathbf{T} : \mathbf{K} \rightarrow \mathbf{K}$, the map which takes the sequence $(x_n)_n$ to $(S_n(x_n))_n$. The map \mathbf{T} is a homeomorphism since each S_n is a homeomorphism. Consider $X \subset \mathbf{K}$, the orbit closure of \mathbf{z} under \mathbf{T} .

Property 20. *The system (X, \mathbf{T}) is a minimal Cantor system.*

Proof. It suffices to show that \mathbf{z} is regularly recurrent for \mathbf{T} . Consider an open set $U \subset \mathbf{K}$ containing \mathbf{z} . Without loss of generality, we may assume U is a basic set of the form $U = \prod U_n$ where $U_n = K_n$ for all $|n| \geq N$.

For each $|n| < N$, because $z_n \in K_n$ is regularly recurrent for S_n , there is a p_n such that $S_n^{kp_n}(z_n) \in U_n$ for all $k \in \mathbb{Z}$. Let $p = \prod_{n=-N}^N p_n$. Then, for all $k \in \mathbb{Z}$, $\mathbf{T}^{kp}(\mathbf{z}) \in U$. Therefore, \mathbf{z} is regularly recurrent for \mathbf{T} . \square

Property 21. *The system (X, \mathbf{T}) is self-induced.*

Proof. Notice that there is a natural embedding $\varphi_n : K_{n+1} \hookrightarrow K_n$ for all $n \in \mathbb{Z}$: For $n \geq 0$, φ_n is just inclusion, for $n \leq 0$, $\varphi_n(x_{n+1}) = (x_{n+1}, 0)$, $x_{n+1} \in K_{n+1}$. For all $n \in \mathbb{Z}$ we have the relation

$$\varphi_n \circ S_{n+1} = S_n^2 \circ \varphi_n.$$

Consider the homeomorphism φ on the product space for which $\varphi(x)_n = \varphi_n(x_{n+1})$ for all integer n where $x = (x_n)_n$. Then, we have

$$(3) \quad \varphi \circ \mathbf{T} = \mathbf{T}^2 \circ \varphi.$$

We claim that X is the disjoint union of $\varphi(X)$ and $\mathbf{T}(\varphi(X))$. To prove this, observe that $\varphi(X) \cup \mathbf{T}(\varphi(X))$ is a closed and \mathbf{T}^2 -invariant set by Equation (3). So, by minimality, we get $\varphi(X) \cup \mathbf{T}(\varphi(X)) = X$. Then, notice that $\varphi(X) \subset \{(x_n)_n \in$

$X : x_0 \in K_1\}$ and $\mathbf{T}(\varphi(X)) \subset \{(x_n)_n \in X : x_0 \in S_0(K_1)\}$. Hence these two sets are disjoint.

It follows that $\varphi(X)$ is a clopen proper subset of X and the induced map on $\varphi(X)$ is the map \mathbf{T}^2 . Finally Equation (3) enables to conclude the system (X, \mathbf{T}) is self-induced. \square

From here we observe that the projection map from (X, \mathbf{T}) onto the zero-th coordinate yields a factor map from (X, \mathbf{T}) onto (K_0, S_0) . By selecting (K_0, S_0) in a particular way, we obtain self-induced examples with infinite entropy and which are not uniquely ergodic.

Proposition 22. *There exists a self-induced Cantor system with infinite entropy.*

Proof. There exists a Toeplitz subshift (K_0, S_0) with a binary odometer factor which have positive entropy [9, 41]. We may use the above construction to produce a self-induced Cantor system (X, \mathbf{T}) with (K_0, S_0) as a factor. Since self-induced systems can only have entropy 0 or ∞ , and the entropy of (X, \mathbf{T}) is greater than that of (K_0, S_0) , it must be infinite. \square

Proposition 23. *There exists a self-induced Cantor system which is not uniquely ergodic.*

Proof. There exist Toeplitz subshifts (K_0, S_0) with a binary odometer factor which have several ergodic measures [9, 41]. We may use the above construction to produce a self-induced Cantor system (X, \mathbf{T}) with (K_0, S_0) as a factor. Such a map must induce a surjection from the set of ergodic measures for (X, \mathbf{T}) onto the ergodic measures for (K_0, S_0) . \square

5. SELF-INDUCED SYSTEMS AS GENERALIZED SUBSTITUTIONS

We wish to give a characterization of all self-induced minimal Cantor systems as *generalized substitution subshifts*. The term generalized here refers to the fact that our substitutions may be defined on an infinite (compact zero dimensional) alphabet. In this section we define generalized substitutions and show that a minimal Cantor system is self-induced if and only if it is conjugate to a generalized substitution subshift.

5.1. Generalized Subshifts. We say that a topological space K is an *alphabet space* if it is a compact zero-dimensional metric space with at least two points. We will refer to the elements of an alphabet space as *letters*. A helpful example to keep in mind will be one where $K = \overline{\mathbb{N}_0} = \{0, 1, \dots, \infty\}$, the one-point compactification of \mathbb{N}_0 . We will continue to develop this example throughout this section.

Let K^n denote the set of words of length n on K , which we will write as $u_1 u_2 \dots u_n$ as opposed to the ordered n -tuple (u_1, u_2, \dots, u_n) , but the topology is the same. Let $K^+ = \bigcup_{n \geq 1} K^n$ denote the set of all words on K . If $u \in K^+$, let $|u|$ denote the length of u , that is, $|u| = n$ if u belongs to K^n . If d_K is a distance for K , then d_{K^+} will be the distance on K^+ defined by $d_{K^+}(u, v) = \max\{d_K(u_i, v_i) : i \leq \min(|u|, |v|)\}$.

Elements of the product space $K^{\mathbb{Z}}$, endowed with the product topology, are bi-infinite sequences in the alphabet space K . For a sequence $\mathbf{x} = (x_i)$ in $K^{\mathbb{Z}}$, we will use the notation $\mathbf{x}[i, j]$ to denote the word $x_i x_{i+1} \dots x_j$ belonging to K^{j-i+1} . We

may consider the action of the shift map S on the space of all sequences in $K^{\mathbb{Z}}$, $S(\mathbf{x})_i = x_{i+1}$ for every $\mathbf{x} = (x_i) \in K^{\mathbb{Z}}$.

A *generalized subshift* is a couple (Ω, S) where Ω is a closed S -invariant subset of $K^{\mathbb{Z}}$.

Note that if X is a Cantor set then, $X^{\mathbb{Z}}$ is also a Cantor set. Moreover, every continuous action of a Cantor set $T : X \rightarrow X$ is topologically conjugate to a generalized subshift via the conjugacy $h : X \rightarrow X^{\mathbb{Z}}$ where

$$h : x \mapsto \dots T^{-2}xT^{-1}x.xTxT^2x\dots$$

5.2. Generalized Substitutions. In order to reasonably define a substitution on an alphabet space K , we need to deal with several topological considerations that are trivial in the case where K is finite (and discrete). For a word $w \in K^+$ and $1 \leq j \leq |w|$, let $\pi_j(w)$ denote the j th letter of w . We say that $\sigma : K \rightarrow K^+$ is a *generalized substitution on K* if $a \mapsto |\sigma(a)|$ is continuous and the projection map $\pi_j \circ \sigma$ is continuous on the set $\{a \in K : |\sigma(a)| \geq j\}$. The words $\sigma(z)$, $z \in K$, are called σ -words.

For instance, we may define a generalized substitution ξ on $\overline{\mathbb{N}_0}$ by

$$\xi : j \mapsto 0(j+1), \quad \xi : \infty \mapsto 0\infty.$$

S. Ferenczi [18] observed that the system it generates is minimal, uniquely ergodic and measure theoretically conjugate to the odometer on \mathbb{Z}_2 . More generally, in this paper substitutions on countable alphabets are investigated.

As in the case of a finite alphabet, a generalized substitution σ on an alphabet space K can be extended to a function $\sigma : K^+ \rightarrow K^+$ by concatenation, and therefore iterated. Observe that σ^k is again a generalized substitution for any k . The generalized substitution map σ also extends to a function $\sigma : K^{\mathbb{Z}} \rightarrow K^{\mathbb{Z}}$ in a similar manner. For $\mathbf{x} \in K^{\mathbb{Z}}$, the image $\sigma(\mathbf{x})$ is the sequence formed by the concatenation of $\sigma(x_i)$ for all i , with $\sigma(x_0)$ starting at the origin

$$\sigma : \dots x_{-2}x_{-1}.x_0x_1x_2\dots \mapsto \dots \sigma(x_{-2})\sigma(x_{-1}).\sigma(x_0)\sigma(x_1)\sigma(x_2)\dots$$

Lemma 24. *Let K be an alphabet space and $\sigma : K \rightarrow K^+$ a generalized substitution on K . Then, $\sigma : K^{\mathbb{Z}} \rightarrow K^{\mathbb{Z}}$ is continuous.*

Proof. Fix $\mathbf{x} = (x_i) \in K^{\mathbb{Z}}$ and let $\epsilon > 0$ be given.

Since the topology on $K^{\mathbb{Z}}$ is given by the product topology, there is an $\epsilon_1 > 0$ and an $N > 0$ such that given $\mathbf{y} = (y_i), \mathbf{y}' = (y'_i) \in K^{\mathbb{Z}}$, $d(y_i, y'_i) < \epsilon_1$ for $|i| < N$ implies $d(\mathbf{y}, \mathbf{y}') < \epsilon$, where d is a distance defining the topology of $K^{\mathbb{Z}}$.

Since $z \mapsto |\sigma(z)|$ is a continuous function from K to \mathbb{N} , it is uniformly continuous. Similarly, each $\pi_j \circ \sigma$ is uniformly continuous. As such there is a $\delta_1 > 0$ such that for any $z, z' \in K$ with $d_K(z, z') < \delta_1$ we have $|\sigma(z)| = |\sigma(z')|$ and $d_{K^+}(\pi_j \circ \sigma(z), \pi_j \circ \sigma(z')) < \epsilon_1$ for all j with $1 \leq j \leq |\sigma(z)|$.

Again by the nature of the product topology, there is a $\delta > 0$ so that $d((x_i), (x'_i)) < \delta$ implies $d_K(x_i, x'_i) < \delta_1$ for $|i| < N$. Therefore, as $d((x_i), (x'_i)) < \delta$ we deduce $d_K(x_i, x'_i) < \delta_1$ for all i with $|i| < N$, and, thus, $d_K(\sigma(x)_i, \sigma(x')_i) < \epsilon_1$ for all i with $|i| < N$. Therefore, we finally obtain $d(\sigma((x_i)), \sigma((x'_i))) < \epsilon$. \square

Given a generalized substitution σ on an alphabet space K , we shall consider $\mathcal{L}(\sigma)$ the *language generated by σ* , again a trickier notion to define than in the classical case. Fix a letter a in the alphabet space K . By the *language generated by a* , denoted $\mathcal{L}(\sigma, a)$, we mean the set of words $w \in K^+$ such that w is a subword

of $\sigma^j(a)$ for some $j \in \mathbb{N}$, or w is the limit in K^n of such words. We set $\mathcal{L}(\sigma) = \bigcup_a \mathcal{L}(\sigma, a)$.

Returning to our example ξ , one can check that the word $\infty 0$ is not a subword of $\xi^k(j)$ for any $j \in \overline{\mathbb{N}_0}$. However, it is a limit of such words since $(n-1)0$ occurs as a subword of $\xi^n(0)$ for every $n \geq 2$. Thus, $\infty 0$ is in $\mathcal{L}(\xi, 0)$.

As in the classical case we wish to assume that our generalized substitution is *primitive* in the following sense.

Definition 25. Let $\sigma : K \rightarrow K^+$ be a generalized substitution on an alphabet space K . We say σ is *primitive* if given any non-empty open set $V \subset K$, there is an $j \in \mathbb{N}$ such that for any letter $a \in K$ and any $k \geq j$, one of the letters of $\sigma^k(a)$ is in the set V .

With the assumption of primitivity and that $\#K > 1$, we see that $|\sigma^n(a)| \rightarrow \infty$ for any $a \in K$.

One can check that our example ξ is primitive (recall that in this context open sets are complementary of finite sets in $\overline{\mathbb{N}_0}$).

Proposition 26. Let K be an alphabet space and let $\sigma : K \rightarrow K^+$ be a primitive generalized substitution. Given any $a \in K$, $\lim_{n \rightarrow \infty} |\sigma^n(a)| = \infty$.

Proof. By our axioms, K contains at least two nonequal letters a, b . Let U, V be two non-intersecting open sets containing a and b , respectively. By primitivity, for any $x \in K$, there is a k such that $\sigma^k(x)$ contains a point in U and a point in V , so in particular $|\sigma^k(x)| \geq 2$. It follows from the continuity of the functions $\pi_j \circ \sigma$ and $x \mapsto |\sigma(x)|$, and, the compactness of K that there is a j such that $|\sigma^j(x)| \geq 2$ for all $x \in K$.

Inductively, we see that $|\sigma^{jn}(x)| \geq 2^n$ for all $x \in K$, completing the proof. \square

The assumption of primitivity also simplifies the definition of the language of a generalized substitution to be the language generated by a for any $a \in K$: $\mathcal{L}(\sigma) = \mathcal{L}(\sigma, a)$. These are all the same as the following proposition shows.

Proposition 27. Suppose $\sigma : K \rightarrow K^+$ is a primitive generalized substitution on an alphabet space K . Then, for any two letters $a, b \in K$, $\mathcal{L}(\sigma, a) = \mathcal{L}(\sigma, b) = \mathcal{L}(\sigma)$.

Proof. Let $w \in \mathcal{L}(\sigma, b)$ and suppose $\epsilon > 0$ is given. Then, there are integers $k, i, j \geq 0$ such that the distance between $\sigma^k(b)[i, j]$ and w is less than $\epsilon/2$ in the $K^{|w|}$ -metric. By the continuity condition on σ^k , there is a $\delta > 0$ such that $d_K(b, b') < \delta$ implies the distance from $\sigma^k(b)[i, j]$ and $\sigma^k(b')[i, j]$ is less than $\epsilon/2$.

Let U be the ball of radius δ around b in K . By the primitivity of σ , there is an integer n and a $b' \in U$ such that b' occurs in $\sigma^n(a)$. It follows that there are integers $i', j' \geq 0$ such that the distance from $\sigma^{k+n}(a)[i', j']$ to w is less than ϵ . Thus, w belongs to $\mathcal{L}(\sigma, a)$ and $\mathcal{L}(\sigma, b)$ is included in $\mathcal{L}(\sigma, a)$.

Similarly, $\mathcal{L}(\sigma, a) \subset \mathcal{L}(\sigma, b)$. \square

For a primitive generalized substitution $\sigma : K \rightarrow K^+$, define $X_\sigma \subset K^{\mathbb{Z}}$ to be set of sequences $\mathbf{x} \in K^{\mathbb{Z}}$ such that $\mathbf{x}[-n, n] \in \mathcal{L}(\sigma)$ for all $n \geq 0$. It follows that X_σ is a generalized subshift, i.e., a closed, S -invariant subset of $K^{\mathbb{Z}}$.

When σ is a primitive substitution, we need to explicitly rule out the possibility that X_σ contains periodic points, an assumption that again is necessary even in the classical case. In such a situation we will say that σ is *aperiodic*.

Our example ξ being primitive, its language $\mathcal{L}(\xi)$ contains infinitely many letters and thus is aperiodic.

Because we wish to focus on self-induced minimal Cantor systems we shall require additional properties for our generalized substitutions, that they be primitive and aperiodic. Below, we establish an equivalent description of X_σ under these assumptions. This equivalent description is more or less the analogous technique to finding a fixed point for σ in the classical substitution case and taking its orbit closure.

For $\mathbf{x} \in K^\mathbb{Z}$ we denote by $\omega_\sigma(\mathbf{x})$ the *omega-limit set* of \mathbf{x} under the map σ : $\omega_\sigma(\mathbf{x}) = \bigcap_{n \in \mathbb{N}} \overline{\{\sigma^m(\mathbf{x}) : m \geq n\}}$.

Proposition 28. *Suppose K is an alphabet space and $\sigma : K \rightarrow K^+$ is a primitive generalized substitution, and let $a \in K$. Suppose bc appears in $\sigma^j(a)$ for some $j > 0$ and let \mathbf{x} be any point in $K^\mathbb{Z}$ with $x_{-1} = b \in K$ and $x_0 = c \in K$. Then, for $\mathbf{z} \in \omega_\sigma(\mathbf{x})$, X_σ is equal to the closure of the S -orbit of \mathbf{z} .*

Proof. Suppose $\mathbf{z} \in \omega_\sigma(\mathbf{x})$. Then, $\mathbf{z} = \lim_{i \rightarrow \infty} \sigma^{k_i}(\mathbf{x})$. Fix $m \in \mathbb{N}$ and let $\epsilon > 0$ be given. Then, there is an $i \geq 1$ such that the word $\mathbf{z}[-m, m]$ is within ϵ of a subword of $\sigma^{k_i}(bc)$, which implies $\mathbf{z}[-m, m]$ is within ϵ of a subword of $\sigma^{k_i+j}(a)$. Therefore, $\mathbf{z}[-m, m] \in \mathcal{L}(\sigma, a) = \mathcal{L}(\sigma)$. This shows that \mathbf{z} belongs to X_σ . But because X_σ is closed and shift-invariant, we see that $\overline{\{S^n(\mathbf{z}) : n \in \mathbb{Z}\}}$ is included in X_σ .

Conversely, fix an $\mathbf{y} \in X_\sigma$ and an $\epsilon' > 0$. Let us fix an integer $n \geq 0$, such that $d(\mathbf{y}, \mathbf{y}') < \epsilon'$ when $d_{K^+}(\mathbf{y}[-n, n], \mathbf{y}'[-n, n])$ is less than $\epsilon'/2$. Since $\mathbf{y}[-n, n] \in \mathcal{L}(\sigma) = \mathcal{L}(\sigma, c)$, there are integers k, i such that $d_{K^+}(\sigma^k(c)[i-n, i+n], \mathbf{y}[-n, n])$ is less than $\epsilon'/4$.

Note that there is a $\delta > 0$ such that if $d_K(c, c') < \delta$ then, the words $\sigma^k(c)$ and $\sigma^k(c')$ are within $\epsilon'/4$. Set U equal to the d_K -ball of radius δ around c . By the primitivity of σ , we can find a point in U in the word $\sigma^l(c)$ for all l large enough. Thus, there is an integer $i(l)$ such that $d_{K^+}(\sigma^{l+k}(c)[i(l)-n, i(l)+n], \mathbf{y}[-n, n])$ is less than $\epsilon'/2$. Since ϵ' and l are arbitrary, it follows that $\mathbf{y} \in \overline{\{S^n(\mathbf{z}) : n \in \mathbb{Z}\}}$. \square

Once again returning to our example ξ , letting \mathbf{x} be any sequence in $\overline{\mathbb{N}}_0$ with $x_{-1} = 0$ and $x_0 = 1$, we see that the omega-limit set of \mathbf{x} is simply the point

$$\mathbf{z} = \dots 0102010\infty.0102010301\dots$$

Note \mathbf{z} is a fixed point for $\widehat{\xi}$. The set X_ξ is the S -orbit closure of this point.

Proposition 29. *Suppose K is an alphabet space and $\sigma : K \rightarrow K^+$ is a primitive generalized substitution. For $\mathbf{z} \in X_\sigma$ and any $k \in \mathbb{N}$, the sequence \mathbf{z} is a concatenation of σ^k -words and $\mathbf{z} = S^i \sigma^k(\mathbf{y})$ for some $i \geq 0$ and $\mathbf{y} \in X_\sigma$.*

Proof. First let $\mathbf{z} \in \omega_\sigma(\mathbf{x})$ for some $\mathbf{x} \in K^\mathbb{Z}$. Then, $\mathbf{z} = \lim_{k \rightarrow \infty} \sigma^{n_k}(\mathbf{x})$. From Proposition 28 one can suppose \mathbf{x} belongs to X_σ . Consider the sequence $(\sigma^{n_k-1}(\mathbf{x}))$. By compactness a subsequence converges, say to a point \mathbf{y} belonging to X_σ . By continuity of σ , it follows that $\sigma(\mathbf{y}) = \mathbf{z}$. A similar argument shows that for any $k \in \mathbb{N}$, $\mathbf{z} = \sigma^k(\mathbf{y}')$, for some $\mathbf{y}' \in X_\sigma$.

By continuity and compactness, the maximal length of σ^k -words is bounded. Furthermore observe that $\bigcup_{0 \leq i < \max_{a \in K} |\sigma^k(a)|} S^i \sigma^k(X_\sigma)$ is a closed S -invariant set. By Proposition 28 and the first part of the proof, this set also contains X_σ . This proves the proposition. \square

This leaves us with the issue of *recognizability* in X_σ , namely the uniqueness of the decomposition of a sequence in X_σ into σ^k -words. It was proven by Mossé that a condition equivalent to our definition of recognizability below follows from primitivity and aperiodicity in the case of a classical substitution [32, 33]. We do not yet know if this theorem holds in the case of generalized substitutions, so we will assume recognizability as a separate axiom.

Definition 30. Let $\sigma : K \rightarrow K^+$ be a generalized substitution on an alphabet space K . We say σ is *recognizable* if for every $\mathbf{z} \in X_\sigma$, there is a unique set of integers $\{n_k : k \in \mathbb{Z}\}$ and unique $\mathbf{x} \in X_\sigma$ such that $\sigma(x_k) = \mathbf{z}[n_k, n_{k+1} - 1]$ for all $k \in \mathbb{Z}$.

The generalized substitution ξ in our example is recognizable. For $\mathbf{z} \in X_\xi$, since 0 appears in the image of each letter exclusively at the initial position, so either we have $(n_k) = (2k)$ or $(n_k) = (2k + 1)$. Moreover, ξ being injective on the letters, it follows immediately that ξ is recognizable.

Proposition 31. Suppose σ is recognizable and suppose $\mathbf{z} = \sigma(\mathbf{x})$ with $\mathbf{x} \in X_\sigma$. Then, the first return time to $\sigma(X_\sigma)$ of \mathbf{z} with respect to the shift map is $|\sigma(x_0)|$.

Proof. Let $r = |\sigma(x_0)|$. Then, $S^r(\mathbf{z}) = \sigma(S(\mathbf{x})) \in \sigma(X_\sigma)$. However, if $S^j(\mathbf{z}) = \sigma(\mathbf{y})$ for $0 < j < r$ where $\mathbf{y} \in X_\sigma$ then, this violates the uniqueness conditions in recognizability. \square

5.3. Self-Induced Systems and Generalized Substitutions. With all of these notions established for generalized substitutions, we are prepared to proceed in showing that a minimal Cantor system is self-induced if and only if it is conjugate to a recognizable, primitive, aperiodic generalized substitution subshift.

Theorem 32. Let (X_σ, S) be generated by a primitive generalized substitution $\sigma : K \rightarrow K^+$ then:

- (1) (X_σ, S) is minimal;
- (2) If σ is recognizable and aperiodic then, (X_σ, S) is a self-induced minimal Cantor system, where σ is a conjugacy from (X_σ, S) to the induced system $(\sigma(X_\sigma), S_{\sigma(X_\sigma)})$.

Proof. (1) **(X_σ, S) is a minimal Cantor system.** Let $\mathbf{x} \in K^\mathbb{Z}$ where $x_{-1}x_0$ is a subword of $\sigma^l(a)$ for a fixed $a \in K$ and $l > 0$. Let $\mathbf{z} \in \omega_\sigma(\mathbf{x})$. Thus, from Proposition 28, the S -orbit closure of \mathbf{z} is X_σ . To show the minimality it is enough to prove that for every open set U in X_σ containing \mathbf{z} , there is an $R > 0$ such that for any $j \in \mathbb{Z}$, there is a $0 \leq i < R$ such that $S^{j+i}(\mathbf{z}) \in U$. Notice that there are an $n \in \mathbb{N}$ and an $\epsilon > 0$ such that $S^i(\mathbf{z})$ in U whenever the distance from $\mathbf{z}[-n, n]$ to $\mathbf{z}[i - n, i + n]$ is less than ϵ .

Moreover, from the continuity of σ and the very definition of \mathbf{z} , there are a $\delta > 0$ and a $k > 0$ such that $d_K(a, a') < \delta$ implies that a word appears in $\sigma^k(a')$ which is within ϵ of $\mathbf{z}[-n, n]$.

By the primitivity of σ , there is an m such that for any $b \in K$, $\sigma^m(b)$ contains a letter within δ of a . Accordingly, $\sigma^{m+k}(b)$ contains a word within ϵ of $\mathbf{z}[-n, n]$.

Let R be equal to $\sup_{b \in K} |\sigma^{k+m}(b)|$. It is finite from the compactness and the continuity assumption. The sequence \mathbf{z} being a concatenation of σ^{m+k} -words (Proposition 29), for any $j \in \mathbb{Z}$, there is a $0 \leq i < R$ such that $S^{j+i}(\mathbf{z}) \in U$.

(2) **(X_σ, S) is self-induced.** Because the system is aperiodic, it is a minimal Cantor system. The map $\sigma : X_\sigma \rightarrow X_\sigma$ is continuous by assumption, thus, the

image set $U = \sigma(X_\sigma)$ is compact (closed) in X_σ . Furthermore, it is injective because σ is recognizable. Consequently, every point in X_σ can be decomposed into σ -words and every point in X_σ is in $S^i(U)$ for some $i \in \mathbb{Z}$.

Set $X_i = \{\mathbf{x} \in X_\sigma : |\sigma(x_0)| = i\}$ and $U_i = \sigma(X_i)$. Since the X_i 's are disjoint and σ is injective, so are the U_i 's. Proposition 31 ensures the return time to U on U_i is i . So we have that $\{S^k(U_i) : 0 \leq k < i, 1 \leq i \leq \max_{b \in K} |\sigma(b)|\}$ forms a disjoint collection of sets which covers X_σ . Thus, each $S^k(U_i)$ is clopen and U is clopen.

Let $\mathbf{x} \in X_\sigma$, and set $r = |\sigma(x_0)|$. Then, again from Proposition 31, r is the return time of $\sigma(\mathbf{x})$ to the set U and $S^r(\sigma(\mathbf{x})) = \sigma(S(\mathbf{x})) \in U$. \square

We now establish the reverse direction, that is to show that any self-induced minimal Cantor system is conjugate to a system generated by a primitive, recognizable, aperiodic generalized substitution.

Theorem 33. *Suppose (X, T) is a self-induced minimal Cantor system. Then, (X, T) is conjugate to a recognizable, primitive, aperiodic, generalized substitution subshift (X_σ, S) .*

Proof. Suppose $U \subset X$ is clopen and (X, T) is self-induced via a conjugacy $\varphi : (X, T) \rightarrow (U, T_U)$. Recall that in Proposition 7 we showed that without loss of generality we may assume $U \cap TU = \emptyset$.

Set $K = X$, define $\sigma : K \rightarrow K^+$ by

$$\sigma(x) = \varphi(x)T(\varphi(x))T^2(\varphi(x)) \dots T^{r_U(\varphi(x))-1}(\varphi(x)),$$

where $r_U(\varphi(x))$ is the return time of $\varphi(x)$ to $U = \varphi(X)$.

σ is a generalized substitution. To see that $m : x \mapsto |\sigma(x)|$ is continuous, we simply note that $|\sigma(x)| = r_U(\varphi(x))$ and that $r_U \circ \varphi$ is continuous. That $\pi_j \circ \sigma$ is continuous follows from observing $\pi_j \circ \sigma(x) = T^{j-1} \circ \varphi(x)$ if $1 \leq j \leq r_U(\varphi(x))$.

σ is primitive. We will show by induction on $n \geq 1$ that, for each $x \in X$,

$$(4) \quad \sigma^n(x) = \varphi^n(x)T(\varphi^n(x))T^2(\varphi^n(x)) \dots T^{r_{\varphi^n(X)}(\varphi^n(x))-1}(\varphi^n(x)).$$

Suppose it is true for n . For $0 \leq i \leq r_{\varphi^n(X)}(\varphi^n(x)) - 1$ we have

$$\sigma(T^i \varphi^n(x)) = (\varphi T^i \varphi^n(x))(T \varphi(T^i \varphi^n(x))) \dots T^{r_U(\varphi(T^i \varphi^n(x)))-1} \varphi(T^i \varphi^n(x)).$$

Since φ is a conjugacy map between (X, T) and (U, T_U) , we get $\varphi(T^i \varphi^n(x)) = T_U^i \varphi^{n+1}(x)$, so that for $0 \leq j \leq r_U(\varphi T^i \varphi^n(x)) - 1 = r_U(T_U^i \varphi^{n+1}(x)) - 1$, we have

$$\begin{aligned} T^j \varphi(T^i \varphi^n(x)) &= T^j T_U^i \varphi^{n+1}(x) \text{ and} \\ T^{r_U(T_U^i \varphi^{n+1}(x))-1} \varphi(T^i \varphi^n(x)) &= T^{-1} T_U^{i+1} \varphi^{n+1}(x). \end{aligned}$$

Then, to finish the proof of the induction, it remains to show that the last letter of $\sigma(T^{r_{\varphi^n(X)}(\varphi^n(x))-1} \varphi^n(x))$ is the correct one, that is

$$T^{r_{\varphi^{n+1}(X)}(\varphi^{n+1}(x))-1} \varphi^{n+1}(x) = T^{-1} T_U^{r_{\varphi^n(X)}(\varphi^n(x))} \varphi^{n+1}(x).$$

This last equality actually comes from

$$\begin{aligned} T_U^{r_{\varphi^n(X)}(\varphi^n(x))} \varphi^{n+1}(x) &= \varphi(T^{r_{\varphi^n(X)}(\varphi^n(x))} \varphi^n(x)) \\ &= \varphi(T_{\varphi^n(X)} \varphi^n(x)) \\ &= T_{\varphi^{n+1}(X)} \varphi^{n+1}(x), \end{aligned}$$

where the last equality is provided by Proposition 3.

Since (X, T) is minimal, for any open set $V \subset X$ there is an $R > 0$ such that for any point $z \in X$, $\{z, T(z), \dots, T^R(z)\} \cap V \neq \emptyset$. Thus, primitivity will follow from Formula (4) by showing that the return time to $\varphi^n(U)$ must go to infinity with n .

Because $U \cap TU = \emptyset$, for all $x \in X$ the T -return time for x to U is at least 2. Similarly, the T_U -return time for a point $x \in \varphi(U)$ to $\varphi(U)$ is at least 2, which implies the T -return time for a point $x \in \varphi(U)$ to $\varphi(U)$ is at least 4. Accordingly, the T -return time to $\varphi^n(U)$ is at least 2^n , and primitivity follows.

σ is recognizable. Recall that any word w in the language $\mathcal{L}(\sigma)$ is a limit of subwords of words $\sigma^n(a_n)$, that is, by Formula (4), w is a limit of words $x_n T x_n \dots T^{|w|-1} x_n$ for some points $x_n \in X$. The compactness of the space together with the continuity of the map T ensure the word w is of the form $x T x \dots T^{|w|-1} x$ for a point $x \in X$. Let \mathbf{z} be any sequence in X_σ . So any word $\mathbf{z}[-n, n]$, $n \geq 0$ is in $\mathcal{L}(\sigma)$, thus by the former remark, and applying a classical diagonal extraction, we obtain that \mathbf{z} is of the form $\mathbf{z} = \dots T^{-2} x T^{-1} x . x T x T^2 x \dots$ for some point $x \in X$. Moreover notice that this x is unique.

Let $n_0 \geq 0$ be the smallest non negative index j such that $z_j = T^j x \in U$. By the surjectivity of φ , there is a $y \in X$ such that $z_{n_0} = \varphi(y)$. It follows, from the definitions of σ and φ , that $S^{n_0} \mathbf{z} = \sigma(\dots T^{-2} y T^{-1} y . y T y T^2 y \dots)$.

Inductively define $n_0 < n_1 < n_2 < \dots$ and $n_0 > n_{-1} > n_{-2} > \dots$ so that $z_j \in U$ if and only if $j = n_k$ for some k . Then, for all k , $\mathbf{z}[n_k, n_{k+1}] = \sigma(T^k y)$.

To prove the recognizability, observe that a σ -word has a letter in U if and only if it is its first letter. Consequently, the sequence (n_k) is uniquely defined. Since the substitution σ is injective on the letters, we get σ is recognizable.

(X_σ, S) is conjugate to (X, T) .

We wish to show that there is a homeomorphism $h : X \rightarrow X_\sigma$ for which $h \circ T = S \circ h$. Note that the map $h : X \rightarrow X_\sigma$ given by

$$h : x \mapsto \dots T^{-2}(x) T^{-1}(x) . x T(x) T^2(x) \dots$$

gives a conjugacy from (X, T) onto its image with the shift map. It is left to show that $h(X) = X_\sigma$. The substitution σ being recognizable (see above), the system (X_σ, S) is minimal from Theorem 32. Consequently, it suffices to show that $h(X)$ is a subset of X_σ .

To that end, let $x \in X$ and consider the sequence $h(x)$. Let I be the infinite set $\{i \in \mathbb{Z} : T^i(x) \in U\}$. Since φ is bijective, for each $i \in I$, there is a unique $y_i \in X$ such that $T^i(x) = \varphi(y_i)$. Notice that

$$\begin{aligned} T^i(x) T^{i+1}(x) \dots T^{i+r_U(T^i(x))-1}(x) &= \varphi(y_i) T(\varphi(y_i)) \dots T^{r_U(\varphi(y_i))-1}(\varphi(y_i)) \\ &= \sigma(y_i). \end{aligned}$$

Thus, the sequence $h(x)$ is a concatenation of σ -words $\sigma(y_i)$.

By applying the same arguments to the generalized substitution σ^k for each integer $k \geq 1$, associated with the set $\varphi^{k-1}(U)$ and the return times to that set (see Formula (4)), we obtain that $h(x)$ is a concatenation of σ^k -words for all k . Therefore, $h(x) \in X_\sigma$. \square

6. POINCARÉ SECTIONS

We end with a note about the situation where we expand our notion of induced maps to return maps on closed, but not necessarily clopen, sets. Let (X, \mathcal{B}, μ, T)

be a measure theoretical dynamical system. Due to Poincaré Recurrence Theorem (see [35] for example), once we have $U \in \mathcal{B}$ with $\mu(U) > 0$ then, one can induce on U to define the induced dynamical system $(U, \mathcal{B} \cap U, \mu_U, T_U)$, where $T_U(x)$ is the first return of x in U . But it can happen that the induced map on a Borel set U is well defined even if $\mu(U) = 0$. For example, consider the full-shift $(\{0, 1\}^{\mathbb{Z}}, S)$ and U the set of sequences $\dots w_{-1}.w_0.w_1\dots$ where the $w_{2i}w_{2i+1}$ belongs to $\{00, 11\}$ for all i . Observe the return time to U of each element of U is well-defined and equals 2. Then, the induced map $S_U : U \rightarrow U$ is an homeomorphism and (X, S) is conjugate to (U, S_U) . This suggests the following definition.

Definition 34. *Let (X, T) be a minimal Cantor system. We say that a nonempty closed set C is a Poincaré section if the induced map $T_C : C \rightarrow C$ is a well-defined homeomorphism. We say that (X, T) is weakly self-induced if there exists a Poincaré section C such that (X, T) is conjugate to (C, T_C) .*

The main result in this section is the following.

Theorem 35. *Let (X, T) and (Y, R) be two minimal Cantor systems. There exists a Poincaré section C in (Y, R) such that (C, R_C) is conjugate to (X, T) .*

Before proving it, we need a technical lemma on embedding of ordered graphs. If $G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$ are two graphs with respective orders \preceq_1, \preceq_2 on the edges, we say that (G_1, \preceq_1) and (G_2, \preceq_2) are *isomorphic* if there is a graph isomorphism $i : G_1 \rightarrow G_2$ that is order preserving *i.e.*, for all $e, f \in E_1$

$$e \preceq_1 f \text{ if and only if } i(e) \preceq_2 i(f).$$

We refer to Section 2.3 for the definitions involve in the notion of Bratteli diagrams.

Lemma 36. *Let $B = ((V_n), (E_n), (\preceq_n))$ be a simple ordered Bratteli diagram such that the sequence $(\#V_n)$ is increasing. Let V be a subset of V_{n_0} for some level n_0 and let $K = (V, V', E)$ be a bipartite graph with a RL-order \preceq_K on E . Then there exists an integer k such that $(V_{n_0}, V_{n_0+k}, E_{n_0+1, n_0+k})$ contains a subgraph (V, V'', E') isomorphic to (K, \preceq_K) for the order \preceq_{n_0+1, n_0+k} restricted to E' .*

Proof. For a fixed set $V \subset V_{n_0}$, we show by induction on $n \geq 1$ that for every ordered bipartite graph $K = (V, V', E)$ with n edges and with order \preceq_K , there exist an integer k and a subgraph (V, V'', E') of $(V_{n_0}, V_{n_0+k}, E_{n_0+1, n_0+k})$ such that $((V, V'', E'), \preceq_{n_0+1, n_0+k})$ is isomorphic to (K, \preceq_K) .

The result is obvious if $n = 1$. Assume that $n > 1$ and that the result is true for $n - 1$. Let $((V, V', E), \preceq_K)$ be an ordered bipartite graph with n edges and let $e = (x_e, y_e) \in E$ be a maximal element for the order \preceq_K . Without loss of generality, we may assume thanks the induction hypothesis, that $(V, V', E \setminus \{e\})$ is a subgraph of $(V_{n_0}, V_{n_0+k}, E_{n_0+1, n_0+k})$ for some integer k , and the order \preceq_K coincides with \preceq_{n_0+1, n_0+k} on $E \setminus \{e\}$. Since the diagram B is simple, there is an injection $i : V' \rightarrow V_{n_0+k+2}$. By concatenating any edge (x, y) of $E \setminus \{e\}$ with the minimal path from y to $i(y)$, we obtain a subgraph of $(V_{n_0}, V_{n_0+k+2}, E_{n_0+1, n_0+k+2})$ isomorphic to $((V, V', E \setminus \{e\}), \preceq_K)$ for the restriction of \preceq_{n_0+1, n_0+k+2} . Then we identify the edge e with a maximal edge from $x_e \in V$ to $i(y_e) \in V_{n_0+k+2}$ in E_{n_0+1, n_0+k+2} . Since they are at least two paths from each vertex of V_{n_0+k} to each vertex of V_{n_0+k+2} , the maximal paths differ from the minimal ones. Hence we obtain a subgraph of $(V_{n_0}, V_{n_0+k+2}, E_{n_0+1, n_0+k+2})$ isomorphic to $((V, V', E), \preceq_K)$. \square

The strategy to prove Theorem 35 is the following: we start fixing a Bratteli diagram B representing (X, T) . Then by microscoping and contracting a Bratteli diagram associated to (Y, R) , we obtain one that contains a copy \tilde{B} of B . The Bratteli-Vershik system associated to \tilde{B} is then conjugated to (X, T) and an induced system of (Y, R) .

Proof of Theorem 35. Take a Bratteli-Vershik representation (X_B, V_B) of (X, T) given by the ordered Bratteli diagram $B = ((V_n), (E_n), (\preceq_n))$. Contracting and microscoping sufficiently a Bratteli diagram representing (Y, R) , we obtain an ordered Bratteli diagram $B' = ((V'_n), (E'_n), (\preceq'_n))$ where $(\#V'_n)$ is increasing and for each level n , $\#V'_n \geq \#V_n$. We inductively use Lemma 36 to get a contraction $B'' = ((V''_n), (E''_n), (\preceq''_n))$ of B' such that, for each level n , the bipartite ordered graph $((V_n, V_{n+1}, E_{n+1}), \preceq_{n+1})$ is isomorphic to $((\tilde{V}_n, \tilde{V}_{n+1}, \tilde{E}_{n+1}), \preceq''_{n+1})$ a subgraph of $(V''_n, V''_{n+1}, E''_{n+1})$. It follows that $\tilde{B} = ((\tilde{V}_n), (\tilde{E}_n), (\preceq''_n))$, is a subdiagram of B'' and the associated Bratteli-Vershik system $(X_{\tilde{B}}, V_{\tilde{B}})$, with $X_{\tilde{B}} \subset X_{B''}$, is a new representation of (X, T) . Observe $X_{\tilde{B}}$ is a non empty closed set being a decreasing intersection of closed sets. Since the induced orders coincide, the induced system of $(X_{B''}, V_{B''})$ on $X_{\tilde{B}}$ is conjugate to (X, T) . \square

Corollary 37. *Every minimal Cantor system is weakly self-induced.*

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