Discrete weak-KAM methods for stationary uniquely ergodic setting

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Abstract

The Frenkel-Kontorova model describes how an infinite chain of atoms minimizes the total energy of the system when the energy takes into account the interaction of nearest neighbors as well as the interaction with an exterior environment. An almost-periodic environment leads to consider a family of interaction energies which is stationary with respect to a minimal topological dynamical system. We introduce, in this context, the notion of calibrated configuration (stronger than the standard minimizing condition) and, for continuous superlinear interaction energies, we prove its existence for some environment of the dynamical system. Furthermore, in one dimension, we give sufficient conditions on the family of interaction energies to ensure the existence of calibrated configurations for any environment when the underlying dynamics is uniquely ergodic. The main mathematical tools for this study are developed in the frameworks of discrete weak KAM theory, Aubry-Mather theory and spaces of Delone sets.

Keywords: almost-periodic environment, Aubry-Mather theory, calibrated configuration, Delone set, Frenkel-Kontorova model, Mañé potential, Mather set, minimizing holonomic probability, weak KAM theory

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1 Introduction

A *minimizing configuration* $\{x_k\}_{k\in\mathbb{Z}}$ for an *interaction energy* $E : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a chain of points in $\mathbb{R}^d$ arranged so that the energy of each finite segment $(x_m, x_{m+1}, \ldots, x_n)$ cannot be lowered by changing the configuration inside the segment while fixing the two boundary points. Define

$$E(x_m, x_{m+1}, \ldots, x_n) := \sum_{k=m}^{n-1} E(x_k, x_{k+1}).$$

Then $\{x_k\}_{k\in\mathbb{Z}}$ is said to be minimizing if, for all $m < n$, for all $y_m, y_{m+1}, \ldots, y_n \in \mathbb{R}^d$ satisfying $y_m = x_m$ and $y_n = x_n$, one has

$$E(x_m, x_{m+1}, \ldots, x_n) \leq E(y_m, y_{m+1}, \ldots, y_n). \tag{1}$$

If the interaction energy is C^0, *coercive* and *translation periodic*,

$$\lim_{R\to+\infty} \inf_{\|y-x\|\geq R} E(x,y) = +\infty, \tag{2}$$

$$\forall\, t \in \mathbb{Z}^d,\ \forall\, x, y \in \mathbb{R}^d, \quad E(x+t, y+t) = E(x,y), \tag{3}$$

it is easy to show (see [14]) that minimizing configurations do exist. If $d = 1$ and E is a smooth *strongly twist* translation periodic interaction energy,

$$\frac{\partial^2 E}{\partial x \partial y} \leq -\alpha < 0, \tag{4}$$

a minimizing configuration admits in addition a rotation number (see Aubry and Le Daeron [2]). The interaction energy E is supposed to model the interaction between two successive points as well as the interaction between the chain and the environment.

For environments which are aperiodic, namely, with trivial translation group, few results are known (see, for instance, [9, 13, 24]). If $d = 1$ and E is a twist interaction energy describing a quasicrystal environment, Gambaudo, Guiraud and Petite [13] showed that minimizing configurations do exist, they all have a rotation number and any prescribed real number is the rotation number of a minimizing configuration.

We shall make slightly more general assumptions on the properties of E. We say that E is *translation bounded* if

$$\forall R > 0, \quad \sup_{\|y-x\| \leq R} E(x,y) < +\infty, \tag{5}$$

translation uniformly continuous if

$$\forall R > 0, \quad E(x,y) \text{ is uniformly continuous in } \|y-x\| \leq R, \tag{6}$$

and *superlinear* if

$$\lim_{R \to +\infty} \inf_{\|y-x\| \geq R} \frac{E(x,y)}{\|y-x\|} = +\infty. \tag{7}$$

A modification of the arguments given by Zavidovique [25, Appendix] shows that semi-infinite minimizing configurations do exist for a superlinear, translation bounded and translation uniformly continuous E. We give a short proof of this result in Appendix A, proposition 60. It is not clear that there exist bi-infinite minimizing configurations in this general context.

We call *ground energy* the lowest energy per site for all configurations

$$\bar{E} := \lim_{n \to +\infty} \inf_{x_0, \ldots, x_n} \frac{1}{n} E(x_0, \ldots, x_n). \tag{8}$$

A configuration $\{x_n\}_{n \in \mathbb{Z}}$ is *calibrated* at the level $\bar{E}$ if, for every $k < l$,

$$\big[E(x_k, \ldots, x_l) - (l-k)\bar{E}\big] \leq \inf_{n \geq 1} \inf_{y_0 = x_k, \ldots, y_n = x_l} \big[E(y_0, \ldots, y_n) - n\bar{E}\big]. \tag{9}$$

Notice that the number of sites on the right hand side is arbitrary. A calibrated configuration is obviously minimizing; the converse is false in general, as discussed in Appendix A. More generally, a configuration which is calibrated at some level c (replace $\bar{E}$ by c in (9)) is also minimizing.

If $d \geq 1$ and E is C^0, coercive and translation periodic (conditions (2) and (3)), an argument using the notion of weak KAM solutions as in [15, 11, 14] shows that there exist calibrated configurations at the level $\bar{E}$. Conversely, if $d = 1$ and E is twist translation periodic, every minimizing configuration is calibrated for some modified energy $E_\lambda(x,y) = E(x,y) - \lambda(y-x)$, $\lambda \in \mathbb{R}$, with ground energy $\bar{E}(\lambda)$. If $d = 1$ and E is arbitrary (at least translation bounded, translation uniformly continuous and superlinear), it is not known in general that a calibrated configuration does exist.

In order to give a positive answer to the question of the existence of calibrated configurations, we will consider in this paper an interaction energy which has almost periodic behavior. This leads to look at a family of interaction energies parameterized by a minimal dynamical system.

Concretely, we will assume there exists a family of interaction energies $\{E_\omega\}_\omega$ depending on an environment ω. Let Ω denote the collection of all possible environments. We assume that a chain $\{x_k + t\}_{k\in\mathbb{Z}}$ translated in the direction $t \in \mathbb{R}^d$ and interacting with the environment ω has the same local energy that $\{x_k\}_{k\in\mathbb{Z}}$ interacting with the shifted environment $\tau_t(\omega)$, where $\{\tau_t : \Omega \to \Omega\}_{t\in\mathbb{R}^d}$ is supposed to be a group of bijective maps. More precisely, each environment ω defines an interaction $E_\omega(x, y)$ which is assumed to be *topologically stationary* in the following sense

$$\forall\,\omega \in \Omega,\ \forall\, t \in \mathbb{R}^d,\ \forall\, x, y \in \mathbb{R}^d, \quad E_\omega(x + t, y + t) = E_{\tau_t(\omega)}(x, y). \tag{10}$$

In order to ensure the topological stationarity, the interaction energy will be supposed to have a *Lagrangian form*. Formally, we will use the following notations.

Notation 1. *The space of environments* $\big(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}\big)$ *is said to be almost periodic if* Ω *is a compact metric space equipped with a minimal* $\mathbb{R}^d$*-action* $\{\tau_t\}_{t\in\mathbb{R}^d}$*, that is, a family of homeomorphisms* $\tau_t : \Omega \to \Omega$ *satisfying the cocycle property* $\tau_s \circ \tau_t = \tau_{s+t}$ *for all* $s, t \in \mathbb{R}^d$*, and*

– $\tau_t(\omega)$ *is jointly continuous with respect to* (t, ω)*,*

– $\forall\,\omega \in \Omega,\ \{\tau_t(\omega)\}_{t\in\mathbb{R}^d}$ *is dense in* Ω*.*

We say that the family of interaction energies $\{E_\omega\}_{\omega\in\Omega}$ *derive from a Lagrangian if there exists a continuous function* $L : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\forall\,\omega \in \Omega,\ \forall\, x, y \in \mathbb{R}^d, \quad E_\omega(x, y) := L(\tau_x(\omega), y - x). \tag{11}$$

We call the set of data $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ *an almost periodic interaction model.*

Notice that the expression "almost periodic" shall not be understood in the sense of H. Bohr. The almost periodicity in the Bohr sense is canonically relied to the uniform convergence. See [3] for a discussion on the different concepts of almost periodicity accordingly to the uniform topology or, for instance, the compact open topology.

Because of the particular form of $E_\omega(x, y)$, these energies are translation bounded and translation continuous uniformly in ω and in $\|y - x\| \le R$. We make precise the two notions of *coerciveness* and *superlinearity* in the Lagrangian form.

Definition 2. *Let* $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ *be an almost periodic interaction model. The Lagrangian* L *is said to be coercive if*

$$\lim_{R\to+\infty}\ \inf_{\omega\in\Omega}\ \inf_{\|t\|\ge R} L(\omega, t) = +\infty.$$

L *is said to be superlinear if*

$$\lim_{R\to+\infty}\ \inf_{\omega\in\Omega}\ \inf_{\|t\|\ge R} \frac{L(\omega, t)}{\|t\|} = +\infty.$$

L *is said to be ferromagnetic if, for every* $\omega \in \Omega$*,* E_ω *is of class* $C^1(\mathbb{R}^d \times \mathbb{R}^d)$ *and, for every* $\omega \in \Omega$ *and* $x, y \in \mathbb{R}^d$*,*

$$x \in \mathbb{R}^d \mapsto \frac{\partial E_\omega}{\partial y}(x, y) \in \mathbb{R}^d \quad \text{and} \quad y \in \mathbb{R}^d \mapsto \frac{\partial E_\omega}{\partial x}(x, y) \in \mathbb{R}^d$$

are homeomorphisms.

Note that if there is a constant $\alpha > 0$ such that $\sum_{i,j=1}^d \frac{\partial^2 E_\omega}{\partial x \partial y} v_i v_j \leq -\alpha \sum_{i=1}^d v_i^2$ for all $\omega \in \Omega, x, y \in \mathbb{R}^d$, then L is ferromagnetic and superlinear.

Let us illustrate our abstract notions by three typical examples.

Example 3. *The classical periodic one-dimensional Frenkel-Kontorova model takes into account the family of interaction energies $E_\omega(x,y) = W(y-x) + V_\omega(x)$, with $\omega \in \mathbb{S}^1$, written in Lagrangian form as*

$$L(\omega, t) = W(t) + V(\omega) = \frac{1}{2}|t-\lambda|^2 + \frac{K}{(2\pi)^2}\big(1-\cos 2\pi\omega\big), \tag{12}$$

where λ, K are constants. Here $\Omega = \mathbb{S}^1$ and $\tau_t : \mathbb{S}^1 \to \mathbb{S}^1$ is given by $\tau_t(\omega) = \omega + t$. We observe that $\{\tau_t\}_t$ is clearly minimal.

The following example comes from [13].

Example 4. *Consider, for an irrational $\alpha \in (0,1) \setminus \mathbb{Q}$, the set*

$$\omega(\alpha) := \{n \in \mathbb{Z} : \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor = 1\},$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Notice that the distance between two consecutive elements of $\omega(\alpha)$ is $\lfloor \frac{1}{\alpha} \rfloor$ or $\lfloor \frac{1}{\alpha} \rfloor + 1$. Now let U_0 and U_1 be two real valued smooth functions with supports respectively in $(0, \lfloor \frac{1}{\alpha} \rfloor)$ and $(0, \lfloor \frac{1}{\alpha} \rfloor + 1)$. Let $V_{\omega(\alpha)}$ be the function defined by $V_{\omega(\alpha)}(x) = U_{\omega_{n+1}-\omega_n - \lfloor \frac{1}{\alpha} \rfloor}(x - w_n)$, where $\omega_n < \omega_{n+1}$ are the two consecutive elements of the set $\omega(\alpha)$ such that $\omega_n \leq x < \omega_{n+1}$. The associated interaction energy is the function

$$E_{\omega(\alpha)}(x,y) = \frac{1}{2}|x-y-\lambda|^2 + V_{\omega(\alpha)}(x). \tag{13}$$

We can directly extend the definition of $V_{\omega'}$ to any relatively dense set ω' of the real line such that the distance between two consecutive points is in $\{\lfloor \frac{1}{\alpha} \rfloor, \lfloor \frac{1}{\alpha} \rfloor + 1\}$. Let Ω' be the collection of all such sets. Then, for any $x, t \in \mathbb{R}$, we have the relation $V_{\omega'}(x+t) = V_{\omega'-t}(x)$, where $\omega' - t$ denotes the set of elements of $\omega' \in \Omega'$ translated by $-t$. In section 2, we explain how to associate a compact metric space $\Omega \subset \Omega'$, where the group of translations acts minimally, as well as a Lagrangian from which the family $\{E_\omega\}_{\omega \in \Omega}$ derives.

As we shall see in section 2, the construction given in example 4 extends to any *quasiscrystal* ω of $\mathbb{R}^d$, namely, to any set $\omega \subset \mathbb{R}^d$ which is relatively dense and uniformly discrete such that the difference set $\omega - \omega$ is discrete and any finite pattern repeats with a positive frequency (see definition 22). We will later focus on the class of environments of quasicrystal type (see definition 17). An example of almost periodic interaction model on $\mathbb{R}$ which is not of quasicrystal type can be constructed in the following way.

Example 5. *The underlying minimal flow is the irrational flow $\tau_t(\omega) = \omega + t(1, \sqrt{2})$ acting on $\Omega = \mathbb{T}^2$. The family of interaction energies E_ω derives from the Lagrangian*

$$L(\omega, t) := \frac{1}{2}|t-\lambda|^2 + \frac{K_1}{(2\pi)^2}\big(1-\cos 2\pi\omega_1\big) + \frac{K_2}{(2\pi)^2}\big(1-\cos 2\pi\omega_2\big), \tag{14}$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{T}^2$.

For an almost periodic interaction model, the notion of *ground energy* is given by the following definition.

Definition 6. *We call ground energy of a family of interactions $\{E_\omega\}_{\omega\in\Omega}$ of Lagrangian form $L : \Omega \times \mathbb{R}^d \to \mathbb{R}$ the quantity*

$$\bar{E} := \lim_{n\to+\infty} \inf_{\omega\in\Omega} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n} E_\omega(x_0,\ldots,x_n).$$

The above limit is actually a supremum by superadditivity and is finite as soon as L is assumed to be coercive. Besides, we clearly have *a priori* bounds

$$\inf_{\omega\in\Omega} \inf_{x,y\in\mathbb{R}^d} E_\omega(x,y) \le \bar{E} \le \inf_{\omega\in\Omega} \inf_{x\in\mathbb{R}^d} E_\omega(x,x). \tag{15}$$

The constant $\bar{E}$ plays the role of a drift and $E_\omega(x,y) - \bar{E}$ acts like a "signed distance". It is natural to modify the previous notion of minimizing configurations by saying that $\{x_n\}_{n\in\mathbb{Z}}$ is *calibrated at the level* $\bar{E}$ if $\sum_{k=m}^{n-1}[E(x_k,x_{k+1})-\bar{E}]$ realizes the smallest signed distance between x_m and x_n for every $m < n$. Hence, we consider the following key notions borrowed from the weak KAM theory (see, for instance, [10]).

Definition 7. *We call Mañé potential in the environment ω the function on $\mathbb{R}^d\times\mathbb{R}^d$ given by*

$$S_\omega(x,y) := \inf_{n\ge1} \inf_{x=x_0,\ldots,x_n=y} \big[E_\omega(x_0,\ldots,x_n) - n\bar{E}\big].$$

We say that a configuration $\{x_k\}_{k\in\mathbb{Z}}$ is calibrated for E_ω (at the level $\bar{E}$) if

$$\forall\, m < n, \quad S_\omega(x_m,x_n) = E_\omega(x_m,x_{m+1},\ldots,x_n) - (n-m)\bar{E}.$$

As discussed in section 3, the Mañé potential for any almost periodic environment is always finite. More importantly, calibrated configurations always exist for some environments ω in the projection of a specific set called the *Mather set*. The Mather set, denoted Mather(L), will be introduced properly in definition 11 of this section. It is a nonempty compact set of $\Omega\times\mathbb{R}^d$ and its first projection (the projected Mather set) by $pr\colon \Omega\times\mathbb{R}^d \to \Omega$, describes the set of environments for which there exists a calibrated configuration passing through the origin of $\mathbb{R}^d$.

The next theorem extends Aubry-Mather theory of the classical periodic model. It is the first main result of this paper and will be proved in section 3.

Theorem 8. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model, with L a C^0 superlinear function. Then, for all $\omega \in pr(Mather(L))$, there exists a calibrated configuration $\{x_k\}_{k\in\mathbb{Z}}$ for E_ω such that $x_0 = 0$ and $\sup_{k\in\mathbb{Z}} \|x_{k+1}-x_k\| < +\infty$.*

This theorem states that, in the almost periodic case, there exist at least one environment and one calibrated configuration for that environment (and thus for any environment in its orbit). It may happen that the projected Mather set does not meet every orbit of the system. Indeed, in the almost periodic Frenkel-Kontorova model described in example 5 , for $\lambda = 0$, we have $\bar{E} = 0$ which is attained by taking $x_n = 0$ for every $n\in\mathbb{Z}$. In addition, it is easy to check that the Mather set is reduced to the point $(0_{\mathbb{T}^2}, 0_{\mathbb{R}})$ and in particular the projected Mather set $\{0_{\mathbb{T}^2}\}$ meets a unique orbit. We shall later show (theorem 19) that this pathology disappears for a restricted class of one-dimensional almost periodic interaction models, which generalizes example 4 and will be called *weakly twist almost periodic interaction model of quasicrystal type* (see definitions 17 and 18).

We now present the definition of the Mather set. Let $\underline{\omega} \in \Omega$ be fixed. The ground energy (in the environment $\underline{\omega}$) measures the mean energy per site of a configuration $\{x_n\}_{n\geq 0}$ which distributes in $\mathbb{R}^d$ so that $\frac{1}{n}E_{\underline{\omega}}(x_0,\ldots,x_n) \to \bar{E}$. Notice that the previous mean can be understood as an expectation of $L(\omega,t)$ with respect to a probability measure $\mu_{n,\underline{\omega}} := \frac{1}{n}\sum_{k=0}^{n-1}\delta_{(\tau_{x_k}(\underline{\omega}),\, x_{k+1}-x_k)}$:

$$\frac{1}{n}E_{\underline{\omega}}(x_0,\ldots,x_n) = \int L(\omega,t)\,\mu_{n,\underline{\omega}}(d\omega,dt). \tag{16}$$

Notice also that $\mu_{n,\underline{\omega}}$ satisfies the following property of pseudoinvariance

$$\int f(\omega)\,\mu_{n,\underline{\omega}}(d\omega,dt) - \int f(\tau_t(\omega))\,\mu_{n,\underline{\omega}}(d\omega,dt) = \frac{1}{n}\Big(f\circ\tau_{x_n}(\underline{\omega}) - f\circ\tau_{x_0}(\underline{\omega})\Big). \tag{17}$$

This suggests to consider the set of all weak* limits of $\mu_{n,\underline{\omega}}$ as $n\to+\infty$. Following [20], we call these limit measures *holonomic probabilities.*

Definition 9. *A probability measure μ on $\Omega\times\mathbb{R}^d$ is said to be holonomic if*

$$\forall\, f\in C^0(\Omega),\qquad \int f(\omega)\,\mu(d\omega,dt) = \int f(\tau_t(\omega))\,\mu(d\omega,dt).$$

Let $\mathbb{M}_{hol}$ denote the set of all holonomic probability measures.

The set $\mathbb{M}_{hol}$ is certainly not empty since it contains any $\delta_{(\omega,0)}$, $\omega\in\Omega$. It is then natural to look for holonomic measures that minimize L. We show that minimizing holonomic measures do exist and that the lowest mean value of L is the ground energy.

Proposition 10. *If L is C^0 coercive, then $\bar{E} = \inf\{\int L\,d\mu : \mu\in\mathbb{M}_{hol}\}$ and the infimun is attained by some holonomic probability measure.*

A measure that attains the previous infimum is called *minimizing.*

Definition 11. *We denote by $\mathbb{M}_{min}$ the set of minimizing measures. We call Mather set of L the set*

$$\mathrm{Mather}(L) := \cup_{\mu\in\mathbb{M}_{min}}\mathrm{supp}(\mu) \subseteq \Omega\times\mathbb{R}^d.$$

The projected Mather set is just $pr(\mathrm{Mather}(L))$, where $pr:\Omega\times\mathbb{R}^d\to\Omega$ is the first projection.

Proposition 12.

1. *If L is C^0 coercive, then*

$$\exists\,\mu\in\mathbb{M}_{min} \quad \textit{with} \quad \mathrm{Mather}(L) = \mathrm{supp}(\mu).$$

In particular, $\mathrm{Mather}(L)$ *is closed.*

2. If L is C^0 superlinear, then Mather(L) *is compact.*

The set of holonomic measures may be seen as a dual object to the set of coboundaries $\{u - u\circ\tau_t : u \in C^0(\Omega),\ t \in \mathbb{R}^d\}$. Proposition 10 admits thus a dual version that will actually be proved first.

Proposition 13 (The sup-inf formula). *If L is C^0 coercive, then*

$$\bar{E} = \sup_{u\in C^0(\Omega)} \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d} \big[L(\omega,t) + u(\omega) - u\circ\tau_t(\omega)\big].$$

We do not know whether the above supremum is achieved in the aperiodic case (*i.e.* when any map τ_t with $t \neq 0$ has no fixed point). There is finally a third way to compute the ground energy, which says that the exact choice of the environment ω is irrelevant.

Proposition 14. *If L is C^0 coercive, then*

$$\forall\, \omega \in \Omega, \quad \bar{E} = \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}^d} \frac{1}{n} E_\omega(x_0,\ldots,x_n).$$

We present now the definition of a *weakly twist interaction model of quasicrystal type* (generalizing example 4). We decided to work in a slightly more general frame than the usual one for quasicrystals (see section 2). The definition is presented only for the one-dimensional case, nevertheless the description can be done in any dimension. We begin by introducing the notions of flow boxes, transverse section, and box decomposition.

Definition 15. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be an almost periodic environment.*
– An open set $U \subset \Omega$ is said to be a flow box of size $R > 0$ if there exists a compact subset $\Xi \subset \Omega$, called transverse section, such that:
- *the induced topology on Ξ admits a basis of closed and open subsets, called clopen subsets,*
- *$\tau(t,\omega) = \tau_t(\omega)$, $(t,\omega) \in \mathbb{R}\times\Xi$, is a homeomorphism from $B_R(0)\times\Xi$ onto U.*

We shall later write $B_R = B_R(0)$ and $\tau_{(i)}^{-1} = \tau_{|U_i}^{-1} : U_i \to B_R\times\Xi$ for a flow box U_i.
– Two flow boxes $U_i = \tau[B_{R_i}\times\Xi_i]$ and $U_j = \tau[B_{R_j}\times\Xi_j]$ are said to be admissible if, whenever $U_i\cap U_j \neq \emptyset$, there exists $a_{i,j}\in\mathbb{R}$ such that

$$\tau_{(j)}^{-1}\circ\tau(t,\omega) = (t - a_{i,j}, \tau_{a_{i,j}}(\omega)), \quad \forall\,(t,\omega) \in \tau_{(i)}^{-1}(U_i\cap U_j).$$

– A flow box decomposition $\{U_i\}_{i\in I}$ is a cover of Ω by admissible flow boxes.

Typical examples of these structures are given by the suspensions of minimal homeomorphisms on a Cantor set with a locally constant roof functions.

The notion of transversally constant Lagrangian has been introduced in [13]. In the periodic case, equation (3) shows that the interaction energy keeps a constant value by moving the whole configuration by a distance equal to a multiple of the period. In example 4, equation (13) and the minimality of the action by an irrational rotation on the circle show that, given any finite configuration, the interaction energy keeps the same value for infinitely many translated configurations. Moreover, this set of translations is a relatively dense set in $\mathbb{R}$ depending on the configuration. We formalize this idea in the following definition.

Definition 16. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be an almost periodic interaction model admitting a flow box decomposition.*
– A flow box $\tau[B_R \times \Xi]$ is said to be compatible with respect to a flow box decomposition $\{U_i\}_{i\in I}$, where $U_i = \tau[B_{R_i} \times \Xi_i]$, when, for every $|t| < R$, there exist $i \in I$, $|t_i| < R_i$ and a clopen subset $\tilde{\Xi}_i$ of Ξ_i such that $\tau_t(\Xi) = \tau_{t_i}(\tilde{\Xi}_i)$.
– L is said to be transversally constant with respect to a flow box decomposition $\{U_i\}_{i\in I}$ if, for every flow box $\tau[B_R \times \Xi]$ compatible with respect to $\{U_i\}_{i\in I}$,

$$\forall\, \omega, \omega' \in \Xi,\ \forall\, |x|, |y| < R, \quad E_{\omega'}(x, y) = E_\omega(x, y).$$

We extend the case treated in [13] for quasicrystals to the almost periodic interaction models. Similarly to studies for the Hamilton-Jacobi equation (see, for instance, [6, 7, 8, 19]), we will consider here a stationary ergodic setting.

Definition 17. *An almost periodic interaction model $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ is said to be of quasicrystal type if the action $\{\tau_t\}_{t\in\mathbb{R}}$ is uniquely ergodic (with unique invariant probability measure λ) and L is transversally constant with respect to some flow box decomposition.*

The strongly twist property (4) is the main assumption in Aubry-Mather theory ([2, 21]). We slightly extend this property.

Definition 18. *A one-dimensional almost periodic interaction model $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ satisfies the weakly twist property if there exists a C^0 function $U : \Omega \to \mathbb{R}$ such that, for every $\omega \in \Omega$, the function $\tilde{E}_\omega(x, y) := E_\omega(x, y) + U(\tau_x(\omega)) - U(\tau_y(\omega))$ is C^2, and*

$$\forall\, x, y \in \mathbb{R}, \omega \in \Omega \quad \frac{\partial^2 \tilde{E}_\omega}{\partial x \partial y}(x, \cdot) < 0 \quad \text{and} \quad \frac{\partial^2 \tilde{E}_\omega}{\partial x \partial y}(\cdot, y) < 0 \qquad \text{a.e.}$$

Now we state the second main result of this paper, which says that, in the quasicrystal case, for any environment, there always exists a calibrated configuration. Its proof is detailed in section 4.

Theorem 19. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be a one-dimensional weakly twist interaction model of quasicrystal type. Then the projected Mather set meets uniformly any orbit of the flow τ_t. More precisely, for every $\omega \in \Omega$, there exists a calibrated configuration for E_ω, say $\{x_{k,\omega}\}_{k\in\mathbb{Z}}$, with bounded jumps and at a bounded distance from the origin uniformly in ω:*

$$\forall\, m < n, \quad S_\omega(x_{m,\omega}, x_{n,\omega}) = \sum_{k=m}^{n-1} E_\omega(x_{k,\omega}, x_{k+1,\omega}) - (n-m)\bar{E},$$

$$\sup_{\omega\in\Omega} \sup_{k\in\mathbb{Z}} |x_{k+1,\omega} - x_{k,\omega}| < +\infty, \quad \sup_{\omega\in\Omega} |x_{0,\omega}| < +\infty.$$

As in examples 3 and 4 as well as in the general setting described in section 2, interaction models of quasicrystal type are easily built when the interaction energy has the form $E_\omega(x, y) = W(y - x) + V_1(\tau_x(\omega)) + V_2(\tau_y(\omega))$, where W is superlinear weakly convex and V_1 and V_2 are locally transversally constant and smooth along the flow.

Definition 20. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be an almost periodic interaction model. A function $V : \Omega \to \mathbb{R}$ is said to be locally transversally constant on a flow box decomposition $\{U_i\}_{i\in I}$, where $U_i = \tau(B_{R_i} \times \Xi_i)$, if*

$$\forall\, i \in I,\ \forall\, \omega, \omega' \in \Xi_i,\ \forall\, |x| < R_i, \quad V(\tau_x(\omega)) = V(\tau_x(\omega')).$$

Notice that, in example 5, the locally transversally constant property is not verified.

Corollary 21. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be a one-dimensional almost periodic interaction model. Assume that $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ is uniquely ergodic. Let $V_1, V_2 : \Omega \to \mathbb{R}$ be C^0 locally transversally constant functions on the same flow box decomposition that are C^2 along the flow (namely, for all ω, the function $t \in \mathbb{R} \mapsto V_i(\tau_t(\omega))$ is C^2, $i = 1, 2$). Let $W : \mathbb{R} \to \mathbb{R}$ be a C^2 superlinear weakly convex function (namely, $W''(t) > 0$ a.e. and $|W'(t)| \to +\infty$ as $|t| \to +\infty$). Define*

$$L(\omega, t) = W(t) + V_1(\omega) + V_2(\tau_t(\omega)).$$

Then L is C^0, superlinear and transversally constant, $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ is a one-dimensional weakly twist interaction model of quasicrystal type and all conclusions of theorem 19 apply.

2 Backgrounds on quasicrystals

In this section, we recall the basic definitions and properties concerning Delone sets and specially quasicrystals. More details on Delone sets can be found, for instance, in [4, 17, 18]. Associated to Delone sets, we will consider strongly equivariant functions. We recall their main properties here and we refer the reader to [13, 16] for the proofs.

Definition of quasicrystal. A Delone set ω is a discrete subset of the Euclidean space $\mathbb{R}^d$ for which there exist two positive real numbers r_ω and R_ω satisfying the following properties:

- *uniform discreteness*: each open ball of radius r_ω in $\mathbb{R}^d$ contains at most one point of ω.
- *relative density*: each closed ball of radius R_ω in $\mathbb{R}^d$ contains at least one point of ω.

If precision is required, we will say that ω is r_ω-uniformly discrete and R_ω-relatively dense.

For $R > R_\omega$, we say that a subset $\mathtt{P}$ of a Delone set ω is a *R-patch* (or a *pattern* for short) of ω if, for some $y \in \omega$, one has

$$\mathtt{P} = \omega \cap \overline{B_R(y)},$$

where $B_R(y)$ denotes the open ball of a radius R centered at y. We will say that the patch is centered at y (notice that the center may not be unique). The collection of return vectors associated to the patch $\mathtt{P}$ is the set

$$\mathcal{R}_{\mathtt{P}}(\omega) = \{v \in \mathbb{R}^d : \mathtt{P} + v \text{ is a patch of } \omega\},$$

where $\mathtt{P} + v$ denotes the translation of all the points of $\mathtt{P}$ by the vector v. The set of occurrences of $\mathtt{P}$ is defined as $\omega_{\mathtt{P}} := x_{\mathtt{P}} + \mathcal{R}_{\mathtt{P}}(\omega)$.

Definition 22. *A Delone set $\omega \subset \mathbb{R}^d$ is* repetitive *if it satisfies all the two following properties:*

– finite local complexity*: for any real $R > 0$, the Delone set ω has just a finite number of R-patches up to translations;*

– repetitivity*: for each $R > 0$, there is a real number $M(R) > 0$ such that any closed ball of radius $M(R)$ contains at least one occurrence of every R-patch of ω.*

A repetitive Delone set $\omega \subset \mathbb{R}^d$ is a quasicrystal *if in addition it satisfies*

– uniform pattern distribution*: for any pattern P of ω, uniformly in $x \in \mathbb{R}^d$, the following limit exists*

$$\lim_{N\to+\infty} \frac{\#\left(\{z \in \mathbb{R}^d : z \textit{ is an occurrence of } P\} \cap B_N(x)\right)}{Leb(B_N(x))} = \nu(P) > 0.$$

Notice that the finite local complexity is equivalent to the property that the intersection of the difference set $\omega - \omega$ with any bounded set is finite (see [18]).

Basic examples of quasicrystals are derived from Beatty sequences: for a real number $\alpha \in (0,1)$, the associated the set is $\omega(\alpha) := \{n \in \mathbb{Z} : \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor = 1\}$. For details, we refer to [18].

Observe that, when ω is a repetitive Delone set (respectively, a quasicrystal), then $\omega + v$, obtained by translating any point of ω by $v \in \mathbb{R}^d$, is also a repetitive Delone set (respectively, a quasicrystal). A Delone set is said to be *aperiodic* if $\omega + v = \omega$ implies $v = 0$, and *periodic* if its stabilizers contains a cocompact subgroup of $\mathbb{R}^d$. In the former example, it is simple to check that the quasicrystal $\omega(\alpha)$ is aperiodic if, and only if, α is irrational, as in example 4.

We introduce now a combinatorial background. For a Delone set ω and a real number $R > 0$, the *R-atlas* $\mathcal{A}_\omega(R)$ of ω is the collection of all the R-patches centered at a point of ω and translated to the origin. More precisely, we set

$$\mathcal{A}_\omega(R) := \{\omega \cap \overline{B_R(x)} - x : \ x \in \omega\}.$$

Notice that ω has finite local complexity if, and only if, $\mathcal{A}_\omega(R)$ is finite for every R. For a quasicrystal ω and a patch $\mathtt{P}$, it is plain to check that the collection of return vectors $\mathcal{R}_{\mathtt{P}}(\omega)$ is also a quasicrystal. Hence $\omega_{\mathtt{P}}$, the set of all the occurrences of $\mathtt{P}$, is also a quasicrystal.

In order to avoid an unnecessary dichotomy, we will mainly focus on aperiodic quasicrystals. The following lemma is well-known and its proof is plain by contradiction.

Lemma 23. *If ω is an aperiodic quasicrystal, then, given $S > 0$, there exists a constant $R_S > 0$ such that, for any $R \geq R_S$ and any R-patch $\mathtt{P}$ of ω, the quasicrystal $\omega_{\mathtt{P}}$ is S-uniformly discrete.*

Hull of a quasicrystal. As we already mentioned, a translation of a repetitive Delone set ω_* is also a repetitive Delone set. We will equipped the set $\omega_* + \mathbb{R}^d$ of all the translations of ω_* with a topology that reflects its combinatorial properties: the Gromov-Hausdorff topology. Roughly speaking, two Delone sets in this set will be close whenever they have the same pattern in a large neighborhood of the origin, up to a small translation.

Such a topology is metrizable and an associated metric can be defined as follows (for details, see [4, 16]): given ω and $\underline{\omega}$ two translations of ω_*, their distance is

$$D(\omega, \underline{\omega}) := \inf \Big\{ \frac{1}{r+1} : \exists\, |v|, |\underline{v}| < \frac{1}{r} \ \text{ s.t. } \ (\omega + v) \cap B_r(0) = (\underline{\omega} + \underline{v}) \cap B_r(0) \Big\}.$$

The *continuous hull* $\Omega(\omega_*)$ of the Delone set ω_* is the completion of such a metric space. The finite local complexity hypothesis implies that $\Omega(\omega_*)$ is a compact metric space and that any element $\omega \in \Omega(\omega_*)$ is a Delone set which has, up to translations, the same patterns as ω_*, namely, $\mathcal{A}_\omega(R) = \mathcal{A}_{\omega_*}(R)$ for any $R > 0$ (see [17, 4]). Moreover, $\Omega(\omega_*)$ is equipped with a continuous $\mathbb{R}^d$-action given by the homeomorphisms

$$\tau_v \colon \omega \mapsto \omega - v \qquad \text{for } \omega \in \Omega(\omega_*).$$

Given $\omega \in \Omega(\omega_*)$ and $S > 0$ such that $\omega \cap B_S(0) \neq \emptyset$, the associated *cylinder set* is defined as

$$\Xi_{\omega,S} := \{ \underline{\omega} \in \Omega(\omega_*) : \omega \cap \overline{B_S(0)} = \underline{\omega} \cap \overline{B_S(0)} \}.$$

The translations of cylinder sets,

$$U_{\omega,S,\epsilon} := \{ \underline{\omega} + v : v \in B_\epsilon(0),\ \underline{\omega} \in \Xi_{\omega,S} \}, \qquad \text{for } \epsilon > 0,\ S > 0,\ \omega \in \Omega(\omega_*),$$

form a base for the topology of $\Omega(\omega_*)$.

The dynamical system $(\Omega(\omega_*), \mathbb{R}^d)$ has a dense orbit (namely, the orbit of ω_*). Actually, the repetitivity hypothesis is equivalent to the *minimality* of the action, and so any orbit is dense. The uniform pattern distribution is equivalent to the *unique ergodicity*: the $\mathbb{R}^d$-action has a unique invariant probability measure. For details on these properties, we refer the reader to [17, 4]. We summarize these facts in the following proposition.

Proposition 24 ([17, 4])**.** *Let ω_* be a quasicrystal of $\mathbb{R}^d$. Then the dynamical system $(\Omega(\omega_*), \mathbb{R}^d)$ is minimal and uniquely ergodic.*

The *canonical transversal* $\Xi_0(\omega_*)$ of the hull $\Omega(\omega_*)$ of a quasicrystal is the set of quasicrystals ω in $\Omega(\omega_*)$ such that the origin 0 belongs to ω. The designation of transversal comes from the obvious fact that the set $\Xi_0(\omega_*)$ is transverse to the action: for any vector v smaller than the uniform discreteness constant, clearly $\tau_v(\omega) \notin \Xi_0(\omega_*)$ for any $\omega \in \Xi_0(\omega_*)$. This gives a Poincaré section.

Proposition 25 ([17]). *The canonical transversal $\Xi_0(\omega_*)$ and the cylinder sets $\Xi_{\omega,S}$ of an aperiodic quasicrystal ω_* are Cantor sets. If ω_* is a periodic quasicrystal, these sets are finite.*

It follows, in one dimension, that the hull admits a flow box decomposition. This can be generalized straightforwardly in any dimension.

Lemma 26. *Let ω_* be an aperiodic repetitive Delone set of $\mathbb{R}$ with constant of relative denseness R_{ω_*}. Then, for any large enough $R > 0$, there exist elements $\omega_1, \dots, \omega_n \in \Xi_0(\omega_*)$ such that the collection of open sets $\{U_{\omega_i,R,R_{\omega_*}}\}_{i=1}^n$ is a flow box decomposition of the almost periodic environment $(\Omega(\omega_*), \{\tau_t\}_{t\in\mathbb{R}})$.*

Proof. By lemma 23, for all large enough R and for any R-patch $\mathtt{P}$ of ω_*, the discreteness constant $r_{\omega_{\mathtt{P}}}$ of the occurrence set $\omega_{\mathtt{P}}$, is greater than $4R_{\omega*}$. Notice that, by the definition of the constant R_{ω_*}, for all $S > 0$, the collection $\{U_{\omega,S,R_{\omega_*}}\}_{\omega\in\Xi_0(\omega_*)}$ is a cover of the hull $\Omega(\omega_*)$. Moreover, the choice of the constant R implies that, for any $\omega \in \Xi_0(\omega_*)$, the map $\tau\colon B_{R_{\omega_*}}(0) \times \Xi_{\omega,R+2R_{\omega_*}} \to U_{\omega,R+2R_{\omega_*},R_{\omega_*}}$ is an homeomorphism. This choice also implies that, for any $\omega_1, \omega_2 \in \Xi_0(\omega_*)$, there is at most one vector $a \in B_{2R_{\omega*}}(0)$ such that $\tau_a\Xi_{\omega_1,R+2R_{\omega_*}} \cap \Xi_{\omega_2,R+2R_{\omega_*}} \neq \emptyset$. Indeed, if there are $a, a' \in B_{2R_{\omega*}}(0)$ and $\omega, \omega' \in \Xi_{\omega_1,R+2R_{\omega_*}}$ such that

$$\tau_a\omega \cap B_{R+2R_{\omega_*}}(0) = \tau_{a'}\omega' \cap B_{R+2R_{\omega_*}}(0),$$

we have in particular

$$\omega \cap B_R(a) - a = \omega' \cap B_R(a') - a',$$

which means that $a-a'$ is an occurrence of an R-patch, and then $a = a'$ by the choice of R. Therefore, if two open sets $U_{\omega_1,R+2R_{\omega_*},R_{\omega_*}}$ and $U_{\omega_2,R+2R_{\omega_*},R_{\omega_*}}$ are intersecting, there are $t, t' \in B_{R_{\omega_*}}(0)$ such that $\tau_t(\Xi_{\omega_1,R+2R_{\omega_*}})$ intersects $\tau_{t'}(\Xi_{\omega_2,R+2R_{\omega_*}})$. It follows that the vector $t - t'$ is unique, and the two open sets $U_{\omega_1,R+2R_{\omega_*},R_{\omega_*}}$ and $U_{\omega_2,R+2R_{\omega_*},R_{\omega_*}}$ are admissible. Thus, any finite subcover of the collection $\{U_{\omega,R,R_{\omega_*}}\}_{\omega\in\Xi_0(\omega_*)}$ is a flow box decomposition. ☐

For a more dynamical description of the hull in one dimension, we consider the *return time* function $\varrho : \Xi_0(\omega_*) \to \mathbb{R}^+$ given by

$$\varrho(\omega) := \inf\{t > 0 : \tau_t(\omega) \in \Xi_0(\omega_*)\}, \quad \forall\, \omega \in \Xi_0(\omega_*).$$

The finite local complexity implies that this function is locally constant. The *first return map* $T\colon \Xi_0(\omega_*) \to \Xi_0(\omega_*)$ is then given by

$$T(\omega) := \tau_{\varrho(\omega)}(\omega), \quad \forall\, \omega \in \Xi_0(\omega_*).$$

It is straightforward to check that, for a repetitive Delone set ω_*, the dynamical system $(\Omega(\omega_*), \mathbb{R})$ is conjugate to the suspension of the map T on the set $\Xi_0(\omega_*)$ with the time map given by the function ϱ. Thus, when ω_* is periodic, the continuous hull $\Omega(\omega_*)$ is homeomorphic to a circle. Otherwise, $\Omega(\omega_*)$ has a laminated structure: it is locally the product of a Cantor set by an interval. For the quasicrystal case, the unique invariant probability measure on $\Omega(\omega_*)$ induces a finite measure on $\Xi_0(\omega_*)$ that is T-invariant (see [13]).

Associated to a repetitive Delone set ω of $\mathbb{R}^d$, we will mainly consider strongly ω-equivariant functions as introduced in [16].

Definition 27. *Let ω be a repetitive Delone set of $\mathbb{R}^d$. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is strongly ω-equivariant with range $R > 0$ if, for $x, y \in \mathbb{R}^d$,*

$$\big(B_R(x) \cap \omega\big) - x = \big(B_R(y) \cap \omega\big) - y \quad \Rightarrow \quad f(x) = f(y).$$

In example 4, the function $V_{\omega(\alpha)}$ is strongly $\omega(\alpha)$-equivariant with range $\lfloor \frac{1}{\alpha} \rfloor + 1$. Let us mention another example from [16], which holds for any repetitive Delone set ω_*. Let $\delta := \sum_{x \in \omega_*} \delta_x$ be the Dirac comb supported on the points of a quasicrystal ω_* and let $g\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support. Then, one may check that the convolution product $\delta * g$ is a smooth strongly ω_*-equivariant function. Actually, any strongly ω-equivariant function can be defined by a similar procedure [16].

The following lemma shows that strongly ω_*-equivariant functions arise from functions on the space $\Omega(\omega_*)$ that are constant on the cylinder sets.

Lemma 28 ([13, 16])**.** *Given a repetitive Delone set ω_* of $\mathbb{R}^d$, let $V_{\omega_*}\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous strongly ω_*-equivariant function with range R. Then, there exists a unique continuous function $V\colon \Omega(\omega_*) \to \mathbb{R}$ such that*

$$V_{\omega_*}(x) = V \circ \tau_x(\omega_*), \quad \forall\, x \in \mathbb{R}^d.$$

Moreover, there exists $S > R$ such that V is constant on any cylinder set $\Xi_{\omega,S}$, $\omega \in \Omega(\omega_*)$. In addition, if V_{ω_*} is C^2, then V is C^2 along the flow (that is, for all ω, the function $x \in \mathbb{R}^d \mapsto V(\tau_x(\omega))$ is C^2).

Thus, for $d = 1$ and with the notation of the former lemma, we get that, for any large enough constant $R' > S + R_{\omega_*}$, the function $V\colon \Omega(\omega_*) \to \mathbb{R}$ is transversally constant on a flow box decomposition $\{U_{\omega_i,R',R_{\omega_*}}\}_{i=1}^n$ given by lemma 26. This comes from the fact that $\tau_x(\omega') \in \Xi_{\tau_x(\omega),S}$ whenever $x \in B_{R_{\omega_*}}(0)$, $\omega, \omega' \in \Xi_{\omega_i,S+R_{\omega_*}}$, and V is constant on such cylinder sets.

3 Mather set

The Mather set describes the set of environments for which there exist calibrated configurations. The Mather set is defined in terms of holonomic minimizing measures. Before proving propositions 10, 13 and 14, we note temporarily

$$\bar{E}_\omega = \lim_{n \to +\infty} \inf_{x_0,\ldots,x_n \in \mathbb{R}^d} \frac{1}{n} E_\omega(x_0, \ldots, x_n), \quad \bar{L} := \inf \Big\{ \int L\, d\mu : \mu \in \mathbb{M}_{hol} \Big\},$$
$$\text{and} \quad \bar{K} := \sup_{u \in C^0(\Omega)} \inf_{\omega \in \Omega,\ t \in \mathbb{R}^d} \big[L(\omega, t) + u(\omega) - u \circ \tau_t(\omega) \big].$$

We first show that the infimum is attained in proposition 10.

Proof of proposition 10. We shall prove later that $\bar L = \bar E$. We prove now that the infimum is attained in $\bar L := \inf\{\int L\,d\mu : \mu \in \mathbb{M}_{hol}\}$. Let

$$C := \sup_{\omega\in\Omega} L(\omega,0) \geq \bar L \quad \text{and} \quad \mathbb{M}_{hol,C} := \Big\{\mu \in \mathbb{M}_{hol} : \int L\,d\mu \leq C\Big\}.$$

We equip the set of probability measures on $\Omega\times\mathbb{R}^d$ with the weak topology (convergence of sequence of measures by integration against compactly supported continuous test functions). By coerciveness, for every $\epsilon>0$ and $M>\inf L$ such that $\epsilon > (C-\inf L)/(M-\inf L)$, there exists $R(\epsilon)>0$ with $\inf_{\omega\in\Omega,\|t\|\geq R(\epsilon)} L(\omega,t) \geq M$. By integrating $L-\inf L$, we get

$$\forall \mu \in \mathbb{M}_{hol,C},\quad \mu\big(\Omega\times\{t : \|t\|\geq R(\epsilon)\}\big) \leq \int \frac{L-\inf L}{M-\inf L}\,d\mu \leq \frac{C-\inf L}{M-\inf L} < \epsilon.$$

We have just proved that the set $\mathbb{M}_{hol,C}$ is tight. Let $(\mu_n)_{n\geq0}\subset \mathbb{M}_{hol,C}$ be a sequence of holonomic measures such that $\int L\,d\mu_n \to \bar L$. By tightness, we may assume that $\mu_n\to\mu_\infty$ with respect to the strong topology (convergence of sequence of measures by integration against bounded continuous test functions). In particular, μ_∞ is holonomic. Moreover, for every $\phi \in C^0(\Omega,[0,1])$, with compact support,

$$0 \leq \int (L-\bar L)\phi\,d\mu_\infty = \lim_{n\to+\infty}\int (L-\bar L)\phi\,d\mu_n \leq \liminf_{n\to+\infty}\int (L-\bar L)\,d\mu_n = 0.$$

Therefore, μ_∞ is minimizing. □

We next show that there is no need to take the closure in the definition of the Mather set. We will show later that it is compact.

Proof of proposition 12 – Item 1. We show that $\mathrm{Mather}(L) = \mathrm{supp}(\mu)$ for some minimizing measure μ. Let $\{V_i\}_{i\in\mathbb{N}}$ be a countable basis of the topology of $\Omega\times\mathbb{R}^d$ and let

$$I := \{i \in \mathbb{N} : V_i \cap \mathrm{supp}(\nu) \neq \emptyset \text{ for some } \nu \in \mathbb{M}_{min}\}.$$

We reindex $I = \{i_1, i_2, \ldots\}$ and choose for every $k\geq1$ a minimizing measure μ_k so that $V_{i_k}\cap\mathrm{supp}(\mu_k)\neq\emptyset$ or equivalently $\mu_k(V_{i_k})>0$. Let $\mu := \sum_{k\geq1}\frac{1}{2^k}\mu_k$. Then μ is minimizing. Suppose some V_i is disjoint from the support of μ. Then $\mu(V_i)=0$ and, for every $k\geq1$, $\mu_k(V_i)=0$. Suppose by contradiction that $V_i\cap\mathrm{supp}(\nu)\neq\emptyset$ for some $\nu\in\mathbb{M}_{min}$, then $i=i_k$ for some $k\geq1$ and, by the choice of μ_k, $\mu_k(V_i)>0$, which is not possible. Therefore, V_i is disjoint from the Mather set and we have just proved $\mathrm{Mather}(L)\subseteq\mathrm{supp}(\mu)$ or $\mathrm{Mather}(L)=\mathrm{supp}(\mu)$. □

Item 2 of proposition 12 will be proved later. We shall need the fact $\Phi = L-\bar L$ on the Mather set, that will be proved in lemma 36.

The two formulas given in propositions 10 and 13 are two different ways to compute $\bar E$. It is not an easy task to show that the two values are equal. It is the purpose of lemma 29 to give a direct proof of this fact. We also give a second proof using the minimax formula (see remark 31).

Since we do not yet know that $\bar E_\omega = \bar L = \bar K = \bar E$, we first prove the following result.

Lemma 29. *If L is C^0 coercive, then $\bar{L} = \bar{K}$ and there exists $\mu \in \mathbb{M}_{hol}$ such that $\bar{L} = \int L\, d\mu$.*

Proof. Part 1. We show that $\bar{L} \geq \bar{K}$. Indeed, for any holonomic measure μ and any function $u \in C^0(\Omega)$,

$$\begin{aligned} \int L\, d\mu &= \int [L(\omega,t) + u(\omega) - u \circ \tau_t(\omega)]\, \mu(d\omega, dt) \\ &\geq \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d} \big[L(\omega,t) + u(\omega) - u\circ\tau_t(\omega)\big]. \end{aligned}$$

We conclude by taking the supremum on u and the infimum on μ.

Part 2. We show that $\bar{K} \geq \bar{L}$. Let $X := C^0_b(\Omega\times\mathbb{R}^d)$ be the vector space of bounded continuous functions equipped with the uniform norm. A coboundary is a function f of the form $f = u\circ\tau - u$ or $f(\omega,t) = u\circ\tau_t(\omega) - u(\omega)$ for some $u \in C^0(\Omega)$. Let

$$\begin{aligned} A &:= \{(f,s) \in X\times\mathbb{R} :\ f \text{ is a coboundary and } s \geq \bar{K}\} \quad \text{and} \\ B &:= \{(f,s) \in X\times\mathbb{R} :\ \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d} (L-f)(\omega,t) > s\}. \end{aligned}$$

Then A and B are nonempty convex subsets of $X\times\mathbb{R}$. They are disjoint by the definition of $\bar{K}$ and B is open because L is coercive. By Hahn-Banach theorem, there exists a nonzero continuous linear form Λ on $X\times\mathbb{R}$ which separates A and B. The linear form Λ is given by $\lambda\otimes\alpha$, where λ is a continuous linear form on X and $\alpha\in\mathbb{R}$. The linear form λ is, in particular, continuous on $C^0_0(\Omega\times\mathbb{R}^d)$ and, by Riesz-Markov theorem,

$$\forall\, f \in C^0_0(\Omega\times\mathbb{R}^d), \quad \lambda(f) = \int f\, d\mu,$$

for some signed measure μ. By separation, we have

$$\lambda(f) + \alpha s \leq \lambda(u - u\circ\tau) + \alpha s',$$

for $u \in C^0(\Omega)$, $f \in X$ and $s, s' \in \mathbb{R}$ such that $\inf_{\Omega\times\mathbb{R}^d}(L-f) > s$ and $s' \geq \bar{K}$. By multiplying u by an arbitrary constant, one obtains

$$\forall\, u \in C^0(\Omega), \quad \lambda(u - u\circ\tau) = 0.$$

The case $\alpha = 0$ is not admissible, since otherwise $\lambda(f) \leq 0$ for every $f \in X$ and λ would be the null form, which is not possible. The case $\alpha < 0$ is not admissible either, since otherwise one would obtain a contradiction by taking $f = 0$ and $s \to -\infty$. By dividing by $\alpha > 0$ and changing λ/α to λ (as well as μ/α to μ), one obtains

$$\forall\, f \in X, \quad \lambda(f) + \inf_{\Omega\times\mathbb{R}^d}(L-f) \leq \bar{K}.$$

By taking $f = c\mathbb{1}$, one obtains $c(\lambda(\mathbb{1}) - 1) \leq \bar{K} - \inf_{\Omega\times\mathbb{R}^d} L$ for every $c\in\mathbb{R}$, and thus $\lambda(\mathbb{1}) = 1$. By taking $-f$ instead of f, one obtains $\lambda(f) \geq \inf_{\Omega\times\mathbb{R}^d} L - \bar{K}$ for every $f \geq 0$, which (again arguing by contradiction) yields $\lambda(f) \geq 0$. In particular, μ is a probability measure. We claim that

$$\forall\, u \in C^0(\Omega), \quad \int (u - u \circ \tau)\, d\mu = 0.$$

Indeed, given $R > 0$, consider a continuous function $0 \leq \phi_R \leq 1$, with compact support on $\Omega \times B_{R+1}(0)$, such that $\phi_R \equiv 1$ on $\Omega \times B_R(0)$. Then

$$u - u \circ \tau \geq (u - u \circ \tau)\phi_R + \min_{\Omega \times \mathbb{R}^d} (u - u \circ \tau)(1 - \phi_R).$$

Since λ and μ coincide on $C^0_0(\Omega \times \mathbb{R}^d) + \mathbb{R}\mathbb{1}$, one obtains

$$0 = \lambda(u - u \circ \tau) \geq \int (u - u \circ \tau)\phi_R\, d\mu + \min_{\Omega \times \mathbb{R}^d} (u - u \circ \tau) \int (1 - \phi_R)\, d\mu.$$

By letting $R \to +\infty$, it follows that $\int (u - u \circ \tau)\, d\mu \leq 0$ and the claim is proved by changing u to $-u$. In particular, μ is holonomic. We claim that

$$\forall\, f \in X, \quad \int f\, d\mu + \inf_{\Omega \times \mathbb{R}^d} (L - f) \leq \bar{K}.$$

Indeed, we first notice that the left hand side does not change by adding a constant to f. Moreover, if $f \geq 0$ and $0 \leq f_R \leq f$ is any continuous function with compact support on $\Omega \times B_{R+1}(0)$ which is identical to f on $\Omega \times B_R(0)$, the claim follows by letting $R \to +\infty$ in

$$\int f_R\, d\mu + \inf_{\Omega \times \mathbb{R}^d} (L - f) \leq \lambda(f_R) + \inf_{\Omega \times \mathbb{R}^d} (L - f_R) \leq \bar{K}.$$

We finally prove the opposite inequality $\bar{L} \leq \bar{K}$. Given $R > 0$, denote $L_R = \min(L, R)$. Since L is coercive, $L_R \in X$. Then $L - L_R \geq 0$ and $\int L_R\, d\mu \leq \bar{K}$. By letting $R \to +\infty$, one obtains $\int L\, d\mu \leq \bar{K}$ for some holonomic measure μ. $\square$

Remark 30. *The existence of a minimizing holonomic probability may be also obtained from basic properties of Kantorovich-Rubinstein topology on the set of probabilities measures on a Polish space (X, d). Given a point $x_0 \in X$, let us consider the set of probability measures on the Borel sets of X that admit a finite first moment,* i.e.,

$$\mathcal{P}^1(X) = \big\{\mu : \int_X d(x_0, x)\, d\mu(x) < +\infty\big\}.$$

Notice that this set does not depend on the choice of the point x_0. The Kantorovitch-Rubinstein distance on $\mathcal{P}^1(X)$ is defined for $\mu, \nu \in \mathcal{P}^1(X)$ by

$$D(\mu, \nu) := \inf \big\{ \int_{X \times X} d(x, y)\, d\gamma(x, y) : \ \gamma \in \Gamma(\mu, \nu)\big\},$$

where $\Gamma(\mu, \nu)$ denotes the set of all the probability measures γ on $X \times X$ with marginals μ and ν on the first and second factors, respectively.

Recall that a continuous function $L\colon X \to \mathbb{R}$ is said to be superlinear on a Polish space X if the map defined by $x \in X \mapsto L(x)/\big(1+d(x,x_0)\big) \in \mathbb{R}$ is proper. Notice that this definition is also independent of the choice of x_0 and, by considering the distance $\hat{d} := \min(d,1)$ on X, any proper function is superlinear for $\hat{d}$. The following well known property gives us a sufficient condition for the relative compactness in $\mathcal{P}^1(X)$ (for a detailed discussion, we refer the reader to [1]).

Property. *If L is a superlinear continuous function on a Polish space X, then the map $\mu \mapsto \int L\,d\mu$ is lower semi-continuous and proper, namely, for all $c \in \mathbb{R}$, the set $\{\mu \in \mathcal{P}^1(X) : \int L\,d\mu \le c\}$ is compact for the Kantorovich-Rubinstein topology.*

Applying this result to $X = \Omega \times \mathbb{R}^d$, one may guarantee the existence of minimizing holonomic probabilities for C^0 superlinear Lagrangians, since it is plain to check that the set of holonomic measures is a closed subset of $\mathcal{P}^1(\Omega \times \mathbb{R}^d)$ for the Kantorovich-Rubinstein topology.

Remark 31 (A second proof for the sup-inf formula)**.** *Notice that*

$$\begin{aligned}
\min_{\omega\in\Omega} L(\omega,0) &= \min_{\omega\in\Omega} \int (L+u-u\circ\tau)\,d\delta_{(\omega,0)} \qquad \forall\, u \in C^0(\Omega)\\
&\ge \inf_{\omega\in\Omega,\ t\in\mathbb{R}^d} \int (L+u-u\circ\tau)\,d\delta_{(\omega,t)}\\
&\ge \inf_{\mu\in\mathcal{P}^1(\Omega\times\mathbb{R}^d)} \int (L+u-u\circ\tau)\,d\mu\\
&\ge \min_{\omega\in\Omega,\ t\in\mathbb{R}^d} (L+u-u\circ\tau)(\omega,t)
\end{aligned}$$

clearly implies

$$\bar{K} = \sup_{u\in C^0(\Omega)} \inf_{\mu\in\mathcal{P}^1(\Omega\times\mathbb{R}^d)} \int (L+u-u\circ\tau)\,d\mu.$$

Besides, for a positive integer ℓ, we have the equality

$$\bar{K}_\ell := \sup_{\substack{u\in C^0(\Omega)\\ \|u\|_\infty\le\ell}} \inf_{\mu\in\mathcal{P}^1(\Omega\times\mathbb{R}^d)} \int (L+u-u\circ\tau)\,d\mu = \sup_{\substack{u\in C^0(\Omega)\\ \|u\|_\infty\le\ell}} \inf_{\mu\in C_\ell} \int (L+u-u\circ\tau)\,d\mu, \tag{18}$$

where the nonempty convex subset

$$C_\ell := \big\{\mu \in \mathcal{P}^1(\Omega\times\mathbb{R}^d) : \int L\,d\mu \le \min_{\omega\in\Omega} L(\omega,0) + 2\ell\big\} \tag{19}$$

is closed thanks to the property highlighted in the previous remark. Obviously, it follows that $\bar{K}_\ell \uparrow \bar{K} \le \min_{\omega\in\Omega} L(\omega,0)$.

We will use now a topological minimax theorem which is a generalization of Sion's classical result [22]. For a recent review on such a subject, see [23]. We state a particular case of theorem 5.7 there.

Topological Minimax Theorem. *Let X, Y be Hausdorff topological spaces, and $C \subset X, D \subset Y$ be nonempty closed subsets. Let $F(x,y)$ be a real-valued function on $C \times D$ for which*

– *there exists a real number* $\alpha^* > \sup_{y\in D}\inf_{x\in C} F(x,y)$ *such that, for every* $\alpha \in (\sup_{y\in D}\inf_{x\in C} F(x,y), \alpha^*)$*,*

- for every finite set $\emptyset \neq H \subset D$*, the set* $\cap_{y\in H}\{x \in C : F(x,y) \leq \alpha\}$ *is either empty or connected,*

- for every set $K \subset C$*, the set* $\cap_{x\in K}\{y \in D : F(x,y) > \alpha\}$ *is either empty or connected;*

– *for any* $y \in D$ *and* $x \in C$*,* $F(x,y)$ *is lower semi-continuous in* x *and upper semi-continuous in* y*;*

– *there exists* $y_0 \in D$ *such that* $x \mapsto F(x,y_0)$ *is proper.*

Then,

$$\inf_{x\in C}\sup_{y\in D} F(x,y) = \sup_{y\in D}\inf_{x\in C} F(x,y).$$

In order to apply such a result, we take then into account here the function $F : (\mu,u) \in \mathcal{P}^1(\Omega\times\mathbb{R}^d)\times C^0(\Omega) \mapsto \int (L+u-u\circ\tau)\,d\mu$ *and we consider the closed sets* C_ℓ *given in (19) and* $D_\ell := \{u \in C^0(\Omega) : \|u\|_\infty \leq \ell\}$*. Since* F *is affine in both variables, it satisfies the first point of the above theorem. The property stated in the previous remark shows that* F *also verifies the second and the third points. Thus, from equation (18), we get by the topological minimax theorem*

$$\bar{K}_\ell = \inf_{\mu\in C_\ell}\sup_{u\in D_\ell}\int (L+u-u\circ\tau)\,d\mu. \tag{20}$$

If $\mu_0 \in C_{\ell_0}$ *is not a holonomic probability, there exists a function* $u_0 \in C^0(\Omega)$ *such that* $\int \big(u_0(\omega) - u_0(\tau_t(\omega))\big)\,d\mu_0(\omega,t) > 0$*. Moreover, up to a multiplication by a scalar, we can suppose that* $\int (u_0 - u_0\circ\tau)\,d\mu_0 > \min_{\omega\in\Omega} L(\omega,0) - \inf_{\Omega\times\mathbb{R}^d} L$*. Thus,* μ_0 *may be disregarded in the infimum in (20) whenever* $\ell \geq \ell_0 + \|u_0\|_\infty$*. Since* μ_0 *is any non-holonomic probability with respect to which* L *is integrable, we finally conclude that*

$$\bar{K} = \lim_{\ell\to\infty}\inf_{\mu\in C_\ell}\sup_{u\in D_\ell}\int (L+u-u\circ\tau)\,d\mu = \inf_{\mu\in\mathbb{M}_{hol}}\int L\,d\mu = \bar{L}.$$

The holonomic condition shall not be confused with invariance in the usual sense of dynamical systems. We may nevertheless introduce a larger space than $\Omega\times\mathbb{R}^d$ and a suitable dynamics on such a space. We will apply Birkhoff ergodic theorem with respect to that dynamical system to prove that $\bar{L} \geq \bar{E}$.

Notation 32. *Consider* $\hat{\Omega} := \Omega\times(\mathbb{R}^d)^{\mathbb{N}}$ *equipped with the product topology and the Borel sigma-algebra.* $\hat{\Omega}$ *becomes a complete separable metric space. Any probability measure* μ *on* $\Omega\times\mathbb{R}^d$ *admits a unique disintegration along the the first projection* $pr : \Omega\times\mathbb{R}^d \to \Omega$*,*

$$\mu(d\omega,dt) := pr_*(\mu)(d\omega)P(\omega,dt),$$

where $\{P(\omega,dt)\}_{\omega\in\Omega}$ *is a measurable family of probability measures on* $\mathbb{R}^d$*. Let* $\hat{\mu}$ *be the Markov measure with initial distribution* $pr_*(\mu)$ *and transition probabilities* $P(\omega,dt)$*. For Borel bounded functions of the form* $f(\omega,t_0,\ldots,t_n)$*, we have*

$$\hat{\mu}(d\omega,d\underline{t}) = pr_*(d\omega)P(\omega,dt_0)P(\tau_{t_0}(\omega),dt_1)\cdots P(\tau_{t_0+\cdots+t_{n-1}}(\omega),dt_n).$$

If μ is holonomic, then $\hat\mu$ is invariant with respect to the shift map

$$\hat\tau : (\omega, t_0, t_1, \ldots) \mapsto (\tau_{t_0}(\omega), t_1, t_2, \ldots).$$

We will call $\hat\mu$ the Markov extension of μ. Conversely, the projection of any $\hat\tau$-invariant probability measure $\tilde\mu$ on $\Omega \times \mathbb{R}^d$ is holonomic. This gives a fourth way to compute $\bar E$

$$\bar E = \inf \Big\{ \int \hat L \, d\tilde\mu : \tilde\mu \textit{ is a } \hat\tau\textit{-invariant probability measure on } \hat\Omega \Big\},$$

where $\hat L(\omega, t_0, t_1, \ldots) := L(\omega, t_0)$ is the natural extension of L on $\hat\Omega$.

Proof of propositions 10, 13 and 14.

– Part 1: We know that $\bar K = \bar L$ by lemma 29.

– Part 2: We claim that $\bar E_\omega = \bar E$ for all $\omega \in \Omega$. By the topological stationarity (10) of E_ω and by the minimality of τ_t, for any $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\inf_{x_0,\ldots,x_n \in \mathbb{R}^d} E_\omega(x_0, \ldots, x_n) &= \inf_{x_0,\ldots,x_n \in \mathbb{R}^d} \inf_{t \in \mathbb{R}^d} E_\omega(x_0 + t, \ldots, x_n + t) \\
&= \inf_{x_0,\ldots,x_n \in \mathbb{R}^d} \inf_{t \in \mathbb{R}^d} E_{\tau_t(\omega)}(x_0, \ldots, x_n) \\
&= \inf_{x_0,\ldots,x_n \in \mathbb{R}^d} \inf_{\omega \in \Omega} E_\omega(x_0, \ldots, x_n),
\end{aligned}$$

which clearly yields $\bar E_\omega = \bar E$ for every $\omega \in \Omega$.

– Part 3: We claim that $\bar E \geq \bar K$. Indeed, given $c < \bar K$, let $u \in C^0(\mathbb{R}^d)$ be such that, for every $\omega \in \Omega$ and any $t \in \mathbb{R}^d$, $u(\tau_t(\omega)) - u(\omega) \leq L(\omega, t) - c$. Define $u_\omega(x) = u(\tau_x(\omega))$. Then,

$$\forall\, x, y \in \mathbb{R}^d, \quad u_\omega(y) - u_\omega(x) \leq E_\omega(x, y) - c,$$

which implies $\bar E \geq c$ for every $c < \bar K$, and therefore $\bar E \geq \bar K$.

– Part 4: We claim that $\bar L \geq \bar E$. Let μ be a minimizing holonomic probability measure with Markov extension $\hat\mu$ (see notation 32). If $(\omega, \underline{t}) \in \hat\Omega$, then

$$\sum_{k=0}^{n-1} \hat L \circ \hat\tau^k(\omega, \underline{t}) = E_\omega(x_0, \ldots, x_n) \quad \text{with} \quad x_0 = 0 \text{ and } x_k = t_0 + \cdots + t_{k-1},$$

and, by Birkhoff ergodic theorem,

$$\bar E \leq \int \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat L \circ \hat\tau^k \, d\hat\mu = \int L \, d\mu = \bar L.$$

□

A calibrated sub-action u as given by the Lax-Oleinik operator (see section 5) is not available in general for an almost periodic interaction energy E. The purpose of such a sub-action is to calibrate the energy in the following way

$$E_{\omega,u}(x, y) := E_\omega(x, y) - \big[u \circ \tau_y(\omega) - u \circ \tau_x(\omega)\big] - \bar E. \tag{21}$$

Actually, $E_{\omega,u}(x,y)$ is nonnegative and, depending whether u is forward or backward calibrated, if one of the variables x or y is fixed, the other one can be chosen so that the interaction becomes null. Notice that $U(\omega,t) := u\circ\tau_t(\omega) - u(\omega)$ is a cocycle, namely, it satisfies

$$\forall\,\omega\in\Omega,\ \forall\,s,t\in\mathbb{R}^d,\quad U(\omega,s+t) = U(\omega,s) + U(\tau_s(\omega),t). \tag{22}$$

An important ingredient of the proof of theorem 8 is the notion of Mañé subadditive cocycle.

Definition 33. *Let L be a coercive Lagrangian. We call Mañé subadditive cocycle associated to L the function defined on $\Omega\times\mathbb{R}^d$ by*

$$\Phi(\omega,t) := \inf_{n\geq 1}\ \inf_{0=x_0,x_1,\ldots,x_n=t}\ \sum_{k=0}^{n-1}\big[L(\tau_{x_k}(\omega),x_{k+1}-x_k) - \bar{E}\big].$$

We call Mañé potential in the environment ω the function on $\mathbb{R}^d\times\mathbb{R}^d$ given by

$$S_\omega(x,y) := \Phi(\tau_x(\omega),y-x) = \inf_{n\geq 1}\ \inf_{x=x_0,\ldots,x_n=y}\big[E_\omega(x_0,\ldots,x_n) - n\bar{E}\big].$$

The very definitions of Φ and $\bar{E}$ show that

$$\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad \Phi(\omega,0)\geq 0\ \text{ and }\ \Phi(\omega,t)\leq L(\omega,t)-\bar{E}. \tag{23}$$

(The sequence $\{\bar{E}_n(\omega,0) := \inf_{x_1,\ldots,x_{n-1}} E_\omega(0,x_1,\ldots,x_{n-1},0)\}_n$ is subadditive in n and $\bar{E}\leq\lim_{n\to\infty}\frac{1}{n}\bar{E}_n(\omega,0)$.) Moreover, Φ is upper semi-continuous (lemma 36) and a subadditive cocycle:

$$\forall\,\omega\in\Omega,\ \forall\,s,t\in\mathbb{R},\quad \Phi(\omega,s+t)\leq\Phi(\omega,s)+\Phi(\tau_s(\omega),t). \tag{24}$$

This shows in particular that $\Phi(\omega,t)\geq\bar{E}-L(\tau_t(\omega),-t)$ and thus $\Phi(\omega,t)$ takes always real values. The nontrivial part is to prove that Φ is Mather-calibrated.

Definition 34. *A measurable function $U:\Omega\times\mathbb{R}^d\to[-\infty,+\infty[$ is called a Mather-calibrated subadditive cocycle if the following properties are satisfied:*

- $\forall\,\omega\in\Omega,\ \forall\,s,t\in\mathbb{R}^d,\quad U(\omega,s+t)\leq U(\omega,s)+U(\tau_s(\omega),t),$
- $\forall\,\omega\in\Omega,\ \forall\,s,t\in\mathbb{R}^d,\quad U(\omega,t)\leq L(\omega,t)-\bar{L}$ *and* $U(\omega,0)\geq 0,$
- $\forall\,\mu\in\mathbb{M}_{hol}$ *with* $\int L\,d\mu<+\infty\ \Rightarrow\ \int U(\omega,\sum_{k=0}^{n-1}t_k)\,\hat{\mu}(d\omega,d\underline{t})\geq 0,\ \ \forall\,n\geq 1,$ *where $\hat{\mu}$ is the Markov extension of μ.*

Notice that, provided we know in advance that U is finite, $U(\omega,0)\geq 0$ by replacing $s=t=0$ in the subadditive cocycle inequality.

Lemma 35. *A Mather-calibrated subadditive cocycle U satisfies in addition*

- *$U(\omega,t)$ is finite everywhere,*
- $\sup_{\omega\in\Omega,t\in\mathbb{R}^d}\ |U(\omega,t)|/(1+\|t\|)<+\infty,$
- $\forall\,\mu\in\mathbb{M}_{min},\ \forall\,n\geq 1,\quad U(\omega,\sum_{k=0}^{n-1}t_k)=\sum_{k=0}^{n-1}[\hat{L}-\bar{L}]\circ\hat{\tau}^k(\omega,\underline{t})\quad \hat{\mu}$ *a.e.*

Proof. Part 1. We show that U is sublinear. Let $K := \sup_{\omega\in\Omega,\ \|t\|\le 1}[L(\omega,t)-\bar L]$. Fix $t\in\mathbb{R}^d$ and choose the unique integer n such that $n-1\le\|t\|<n$. Let $t_k=\frac{k}{n}t$ for $k=0,\dots,n-1$. Then the subadditive cocycle property implies, on the one hand,

$$\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad U(\omega,t)\le\sum_{k=0}^{n-1}U(\tau_{t_k}(\omega),t_{k+1}-t_k)\le nK\le(1+\|t\|)K.$$

On the other hand, thanks to the hypothesis $U(\omega,0)\ge 0$, we get the opposite inequality

$$\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad U(\omega,t)\ge U(\omega,0)-U(\tau_t(\omega),-t)\ge-(1+\|t\|)K.$$

We also have shown that U is finite everywhere.

Part 2. Suppose μ is minimizing. Since

$$\forall\,\omega\in\Omega,\ \forall\,t_0,\dots,t_{n-1}\in\mathbb{R}^d,\quad \sum_{k=0}^{n-1}\big[\hat L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t})\ge U\Big(\omega,\sum_{k=0}^{n-1}t_k\Big),$$

by integrating with respect to $\hat\mu$, the left hand side has a null integral whereas the right hand side has a nonnegative integral. The previous inequality is thus an equality that holds almost everywhere. □

Lemma 36. *If L is C^0 coercive, then the Mañé subadditive cocycle Φ is upper semi-continuous and Mather-calibrated. In particular, $\Phi=L-\bar L$ on* Mather(L), *or more precisely*

$$\forall\,\mu\in\mathbb{M}_{min},\ \forall\,(\omega,\underline{t})\in\operatorname{supp}(\hat\mu),\ \forall\,i<j,$$
$$\Phi\Big(\tau_{\sum_{k=0}^{i-1}t_k}(\omega),\sum_{k=i}^{j-1}t_k\Big)=\sum_{k=i}^{j-1}\big[L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t})$$

(or in an equivalent manner, if $x_0=0$ and $x_{k+1}=x_k+t_k$, $\forall\,k\ge0$, the semi-infinite configuration $\{x_k\}_{k\ge0}$ is calibrated for E_ω as in definition 7).

Proof. Part 1. We first show the existence of a measurable Mather-calibrated subadditive cocycle $U(\omega,t)$. From the sup-inf formula (proposition 13), for every $p\ge1$, there exists $u_p\in C^0(\Omega)$ such that

$$\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad u_p\circ\tau_t(\omega)-u_p(\omega)\le L(\omega,t)-\bar L+1/p.$$

Let $U_p(\omega,t):=u_p\circ\tau_t(\omega)-u_p(\omega)$ and $U:=\limsup_{p\to+\infty}U_p$. Then U is clearly a subadditive cocycle and satisfies $U(\omega,0)=0$. Besides, U is finite everywhere, since $0=U(\omega,0)\le U(\omega,t)+U(\tau_t(\omega),-t)$ and $U(\omega,t)\le L(\omega,t)-\bar L$. We just verify the last property in definition 34. Let $\mu\in\mathbb{M}_{hol}$ be such that $\int L\,d\mu<+\infty$. For $n\ge1$, let

$$\hat S_{n,p}(\omega,\underline{t}):=\sum_{k=0}^{n-1}\Big[\hat L-\bar L+\frac{1}{p}\Big]\circ\hat\tau^k(\omega,\underline{t})-U_p\Big(\omega,\sum_{k=0}^{n-1}t_k\Big)\ge0.$$

By integrating with respect to $\hat\mu$, we obtain

$$0 \le \int \inf_{p\ge q} \hat S_{n,p}\, d\hat\mu \le \inf_{p\ge q} \int \hat S_{n,p}(\omega,\underline{t})\, d\hat\mu \le n \int \Big[L-\bar L+\frac{1}{q}\Big]\, d\mu.$$

By Lebesgue's monotone convergence theorem, we obtain

$$\int \Big[n(\hat L-\bar L) - U\Big(\omega, \sum_{k=0}^{n-1} t_k\Big)\Big] d\hat\mu \le \int n[L-\bar L]\, d\mu \quad \text{and}$$

$$\int U\Big(\omega, \sum_{k=0}^{n-1} t_k\Big)\, \hat\mu(d\omega, d\underline{t}) \ge 0.$$

Part 2. We next show that Φ is Mather-calibrated. We have already noticed that Φ satisfies the subadditive cocycle property, besides $\Phi \le L-\bar L$ by definition. We point out that $\Phi(\omega,0)\ge 0$, since, for $x_0=0, x_1,\ldots,x_{n-1}, x_n=0$, denoting $y_{\ell n+i}=x_i$, $\forall\, \ell=0,\ldots,k$, $\forall\, i=0,\ldots,n-1$, we have

$$kE_\omega(x_0,\ldots,x_n) = E_\omega(y_0,\ldots,y_{kn}) \ge \inf_{0=y_0,\ldots,y_{kn}=0} E_\omega(y_0,\ldots,y_{kn}),$$

which, thanks to proposition 14, implies

$$\inf_{0=x_0,\ldots,x_n=0} \frac{1}{n} E_\omega(x_0,\ldots,x_n) \ge \inf_{0=y_0,\ldots,y_{kn}=0} \frac{1}{nk} E_\omega(y_0,\ldots,y_{kn}) \xrightarrow{k\to\infty} \bar E.$$

In particular, $\Phi(\omega,t)$ is finite everywhere. Moreover, $\Phi(\omega,t)\ge U(\omega,t)$ and the third property of definition 34 is thus automatic.

Part 3. We show that Φ is upper semi-continuous. For $n\ge 1$, let

$$S_n(\omega,t) := \inf\{E_\omega(x_0,\ldots,x_n) : x_0=0,\ x_n=t\}.$$

Then $\Phi=\inf_{n\ge1}(S_n-n\bar E)$ is upper semi-continuous if we prove that $S_n(\omega,t)$ is continuous whenever $\omega\in\Omega$ and $\|t\|\le D$. Let $c_0 := \inf_{\omega,x,y} E_\omega(x,y)$ and $K := \sup_{\omega\in\Omega,\ \|t\|\le D} E_\omega(0,\ldots,0,t)$. By coerciveness, there exists $R>0$ such that

$$\forall\, x,y\in\mathbb{R}^d, \quad \|y-x\|>R \Rightarrow\ \forall\,\omega\in\Omega,\ E_\omega(x,y) > K-(n-1)c_0.$$

Suppose $\omega, x_0,\ldots,x_n$ are such that $E_\omega(x_0,\ldots,x_n)\le K$. Suppose by contradiction that $\|x_{k+1}-x_k\|>R$. Thus

$$K \ge E_\omega(x_0,\ldots,x_n) \ge (n-1)c_0 + E_\omega(x_k,x_{k+1}) > K,$$

which is impossible. We have proved that the infimum in the definition of $S_n(\omega,t)$, for every $\omega\in\Omega$ and $\|t\|\le D$, can be realized by some points $\|x_k\|\le kR$. By the uniform continuity of $E_\omega(x_0,\ldots,x_n)$ on the product space $\Omega\times\Pi_k\{\|x_k\|\le kR\}$, we obtain that S_n is continuous on $\Omega\times\{\|t\|\le D\}$.

Part 4. Let μ be a minimizing measure with Markov extension $\hat\mu$. We show that every $(\omega,\underline{t})$ in the support of $\hat\mu$ is calibrated. Let

$$\hat\Sigma := \Big\{(\omega,\underline{t})\in\Omega\times(\mathbb{R}^d)^{\mathbb{N}} : \forall\, n\ge1,\ \Phi\Big(\omega, \sum_{k=0}^{n-1} t_k\Big) \ge \sum_{k=0}^{n-1} \big[L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t})\Big\}.$$

The set $\hat\Sigma$ is closed, since Φ is upper semi-continuous. By lemma 35, $\hat\Sigma$ has full $\hat\mu$-measure and therefore contains $\mathrm{supp}(\hat\mu)$. Thanks to the subadditive cocycle property of Φ and the $\hat\tau$-invariance of $\mathrm{supp}(\hat\mu)$, we obtain the calibration property

$$\forall\,(\omega,\underline{t})\in\hat\Sigma,\ \forall\,0\le i<j,\quad \Phi\Big(\tau_{x_i}(\omega),\sum_{k=i}^{j-1}t_k\Big)=\sum_{k=i}^{j-1}\big[L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t}). \qquad \square$$

Proof of proposition 12 – Item 2. We now assume that L is superlinear. From lemma 35, the Mañé subadditive cocycle is at most linear. There exists $R>0$ such that

$$\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad |\Phi(\omega,t)|\le R(1+\|t\|).$$

By superlinearity, there exists $B>0$ such that

$$\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad L(\omega,t)\ge 2R\|t\|-B.$$

Let μ be a minimizing measure. Since $\Phi=L-\bar L$ μ a.e. (lemma 35), we obtain

$$\|t\|\le(R+B+|\bar L|)/R,\quad \mu(d\omega,dt)\ \text{a.e.}$$

We have proved that the support of every minimizing measure is compact. In particular, the Mather set is compact. $\square$

Proof of theorem 8. We show that, for every environment ω in the projected Mather set, there exists a calibrated configuration for E_ω passing through the origin. Let μ be a minimizing measure such that $\mathrm{supp}(\mu)=\mathrm{Mather}(L)$. Let $\hat\mu$ denote its Markov extension. For $n\ge1$, consider

$$\hat\Omega_n:=\Big\{(\omega,\underline{t})\in\Omega\times(\mathbb{R}^d)^{\mathbb{N}}:\Phi\Big(\omega,\sum_{k=0}^{2n-1}t_k\Big)\ge\sum_{k=0}^{2n-1}\big[L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t})\Big\}.$$

From lemma 36, $\mathrm{supp}(\hat\mu)\subseteq\hat\Omega_n$. From the upper semi-continuity of Φ, $\hat\Omega_n$ is closed. To simplify the notations, for every $\underline{t}$, we define a configuration $(x_0,x_1,\ldots)$ by

$$x_0=0,\ x_{k+1}=x_k+t_k\quad\text{so that}\quad \hat\tau^k(\omega,\underline{t})=(\tau_{x_k}(\omega),(t_k,t_{k+1},\ldots)).$$

Notice that, if $(\omega,\underline{t})\in\hat\Omega_n$, thanks to the subadditive cocycle property of Φ and the fact that $\Phi\le L-\bar L$, the finite configuration $(x_0,\ldots,x_{2n})$ is calibrated in the environment ω, that is,

$$\forall\,0\le i<j\le 2n,\quad \Phi\Big(\tau_{x_i}(\omega),\sum_{k=i}^{j-1}t_k\Big)=\sum_{k=i}^{j-1}\big[L-\bar L\big]\circ\hat\tau^k(\omega,\underline{t}),$$

or written using the family of interaction energies E_ω,

$$\forall\,0\le i<j\le 2n,\quad S_\omega(x_i,x_j)=E_\omega(x_i,\ldots,x_j)-(j-i)\bar E.$$

Thanks to the sublinearity of S_ω, there exists a constant $R>0$ such that, uniformly in $\omega\in\Omega$ and $x,y\in\mathbb{R}^d$, we have $|S_\omega(x,y)|\le R(1+\|y-x\|)$. Besides, thanks to the superlinearity of E_ω, there exists a constant $B > 0$ such that $E_\omega(x, y) \geq 2R\|y - x\| - B$. Since $S_\omega(x_k, x_{k+1}) = E_\omega(x_k, x_{k+1}) - \bar{E}$, we thus obtain a uniform upper bound $D := (R + B + |\bar{E}|)/R$ on the jumps of calibrated configurations:

$$\forall\, (\omega, \underline{t}) \in \hat{\Omega}_n,\ \forall\, 0 \leq k < 2n, \quad \|x_{k+1} - x_k\| \leq D.$$

Let $\hat{\Omega}'_n = \hat{\tau}^n(\hat{\Omega}_n)$. Thanks to the uniform bounds on the jumps, $\hat{\Omega}'_n$ is again closed. Since $\hat{\mu}(\hat{\Omega}_n) = 1$, $\hat{\mu}(\hat{\Omega}'_n) = 1$ by invariance of $\hat{\tau}$. Let $\nu := pr_*(\mu)$ be the projected measure on Ω. Then $\mathrm{supp}(\nu) = pr(\mathrm{Mather}(L))$. By the definition of $\hat{\Omega}'_n$, we have

$$\begin{aligned} pr(\hat{\Omega}'_n) = \{\omega \in \Omega : \exists\, (x_{-n}, \ldots, x_n) \in \mathbb{R}^d \ \text{ s.t. } \ x_0 = 0 \ \text{ and } \\ S_\omega(x_{-n}, x_n) \geq E_\omega(x_{-n}, \ldots, x_n) - 2n\bar{E}\}. \end{aligned}$$

Again by compactness of the jumps, $pr(\hat{\Omega}'_n)$ is closed and has full ν-measure. Thus, $pr(\hat{\Omega}'_n) \supseteq pr(\mathrm{Mather}(L))$. By a diagonal extraction procedure, we obtain, for every $\omega \in \mathrm{Mather}(L)$, a bi-infinite calibrated configuration with uniformly bounded jumps passing through the origin. $\square$

4 Calibrated configurations for quasicrystals

This section is devoted to the proof of the second main result of this paper: theorem 19. We first collect elementary results on flow boxes in lemma 37. The notions of flow boxes and flow box decomposition have been introduced in definition 15. In general, a minimal flow does not possess a cover of flow boxes. Flow boxes are open sets obtained by taking the union of every orbits of size R starting from any point belonging to a closed transverse Poincaré section. The restricted topology on a transverse section must be special: it must admit a basis of clopen sets. We then explain in lemma 38 how to build a transversally constant Lagrangian from a locally transversally constant potential. It is indeed easy to built such a potential in the context of Delone sets as explained in section 2. We show in lemma 40 how to construct a suspension with locally constant return maps that we call Kakutani-Rohlin tower. We then assume the flow to be uniquely ergodic and recall in lemma 41 the construction of a unique transverse measure associated to each transverse section.

Supposing $(\Omega, \{\tau_t\}_{t \in \mathbb{R}}, L)$ to be weakly twist (definition 18), the fundamental Aubry crossing property is explained in lemma 43. Examples of weakly twist Lagrangian are given in corollary 21. We collect in lemmas 44, 46 and 47 several intermediate results, that are consequences of the weakly twist property, about the order of the points composing a minimizing configuration. We assume moreover L to be transversally constant. Our first nontrivial result is stated in proposition 48: a finite configuration $(x_0^n, \ldots, x_n^n)$ which realizes the minimum of the energy among all configurations of the same length must be strictly monotone, and must have uniformly bounded jumps $|x_k^n - x_{k-1}^n| \leq R$. If $E_\omega(x, x) = \bar{E}$ for some $\omega \in \Omega$ and $x \in \mathbb{R}$, the proof of theorem 19 is obvious. We thus suppose $E_\omega(x, x) > \bar{E}$ for every ω and x. Our second key result shows then that $\liminf_{n \to +\infty} \frac{1}{n}|x_n^n - x_0^n| > 0$: the frequency of points x_k^n in a flow box of sufficiently large size is positive. We finally conclude this section with the proof of theorem 19.

Lemma 37. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be an almost periodic $\mathbb{R}$-action. Assume that the action is not periodic ($t \in \mathbb{R} \mapsto \tau_t(\omega) \in \Omega$ is injective for every $\omega \in \Omega$). Then*

1. *If $\tau[B_R \times \Xi]$ is a flow box, then there exists R' such that*
$$\Omega = \tau[B_{R'} \times \Xi] = \{\tau_t(\omega) : |t| < R' \text{ and } \omega \in \Xi\}.$$
2. *If $\tau[B_R \times \Xi]$ is a flow box, then $\tau : \mathbb{R} \times \Xi \to \Omega$ is open and $\tau[B_R \times \Xi']$ is again a flow box for every clopen subset $\Xi' \subset \Xi$.*
3. *If $\tau[B_R \times \Xi]$ is a flow box, then, for every $R' > 0$ and $\omega \in \Xi$, there exists a clopen set $\Xi' \subset \Xi$ containing ω such that $\tau[B_{R'} \times \Xi']$ is again a flow box.*
4. *If $U = \tau[B_R \times \Xi]$ and $U' = \tau[B_{R'} \times \Xi']$ are two admissible flow boxes, if $\tau[B_{2R+2R'} \times \Xi]$ and $\tau[B_{2R+2R'} \times \Xi']$ are also flow boxes, then*
$$U \cap U' = \tau(\tilde{B} \times \tilde{\Xi}) = \tau(\tilde{B}' \times \tilde{\Xi}')$$
for some clopen sets $\tilde{\Xi}$, $\tilde{\Xi}'$ and some open convex subsets $\tilde{B} \subset B_R$, $\tilde{B}' \subset B_{R'}$.
5. *If $\{U_i\}_{i\in I}$ is a flow box decomposition, then, for every $\omega \in \Omega$ and $R > 0$, there exits a flow box $\tau[B_R \times \Xi]$, with a transverse section Ξ containing ω, that is compatible with respect to $\{U_i\}_{i\in I}$.*

Proof. Let $\theta_s : \mathbb{R} \times \Xi \to \mathbb{R} \times \Xi$ be the translation $(t, \omega) \mapsto (t + s, \omega)$. We observe the trivial conjugacy $\tau_s \circ \tau = \tau \circ \theta_s$ and note that both $\tau_s : \Omega \to \Omega$ and $\theta_s : \mathbb{R} \to \mathbb{R}$ are homeomorphisms.

Item 1. Let $U = \tau[B_R \times \Xi]$. The set $\cup_{t\in\mathbb{R}}\tau_t(U)$ is invariant, open, and therefore equal to Ω. By compactness $\Omega = \tau_{t_1}(U) \cup \ldots \cup \tau_{t_r}(U) = \tau[B_{R'} \times \Xi]$, with $R' = R + \max_i |t_i|$.

Item 2. Let V be an open subset of $\mathbb{R} \times \Xi$. Given $(t, \omega) \in V$, there exist $0 < \epsilon < R$ and a clopen set $\Xi' \subset \Xi$ containing ω such that $B_\epsilon(t) \times \Xi' \subset V$. Then
$$\tau(B_\epsilon(t) \times \Xi') = \tau \circ \theta_t(B_\epsilon(0) \times \Xi') = \tau_t \circ \tau(B_\epsilon(0) \times \Xi') \text{ is open in } \Omega.$$
If $\Xi' \subset \Xi$ is a clopen set, then $B_R(0) \times \Xi'$ is open in $B_R(0) \times \Xi$ and $\tau[B_R \times \Xi']$ is open in Ω.

Item 3. We may clearly assume $R' \geq R$. For every $\frac{3}{4}R \leq |s| \leq 2R'$, by aperiodicity, there exists a clopen set $\Xi_s \subset \Xi$ containing ω such that τ is injective on $[B_{R/4}(0) \cup B_{R/4}(s)] \times \Xi_s$. Furthermore, for every $|s| \leq \frac{3}{4}R$, τ is injective on $[B_{R/4}(0) \cup B_{R/4}(s)] \times \Xi$ by the definition of a flow box. Let $\{B_{R/4}(s_i)\}_i$ be a finite cover of $\overline{B_{2R'}(0)}$ so that τ is injective on each $[B_{R/4}(0) \cup B_{R/4}(s_i)] \times \Xi'$, where $\Xi' = \cap_i \Xi_{s_i}$. Then there exists $\epsilon >$ such that τ is injective on $[B_\epsilon(0) \cup B_\epsilon(s)] \times \Xi'$, for every $|s| \leq 2R'$. By conjugacy, τ is injective on $[B_\epsilon(s) \cup B_\epsilon(s')] \times \Xi'$, for every $|s|, |s'| \leq R'$. We thus have obtained that $\tau : B_{R'}(0) \times \Xi' \to \Omega$ is injective. Moreover, τ is open on $B_{R'}(0) \times \Xi'$ by *item 2.*

Item 4. Assume $U \cap U' \neq \emptyset$. There exists $a \in \mathbb{R}$ such that, if $\omega \in \Xi$, $\omega' \in \Xi'$, $|t| < R$, $|t'| < R$, then $\tau_t(\omega) = \tau_{t'}(\omega')$ if, and only if, $t' = t - a$ and $\omega' = \tau_a(\omega)$. In particular, a belongs to $B_R - B_{R'}$ and is unique. Then $\tilde\Xi := \Xi \cap \tau_a^{-1}(\Xi')$ is a clopen subset of Ξ and $\tilde B := B_R \cap (a + B_{R'})$ is an open convex subset of B_R.

Item 5. Let $\{U_i = \tau(B_{R_i} \times \Xi_i)\}$ be a flow box decomposition. Consider $\omega \in \Omega$ and $R > 0$. For every $|x| \le R$, $\tau_x(\omega) \in U_i$ for some box U_i. Then $x \in B_{R_i}(t_i)$ for some t_i such that $\omega_i := \tau_{t_i}(\omega) \in \Xi_i$. By compactness, one can find a finite set of indices I such that $\cup_{i\in I} B_{R_i}(t_i)$ covers $B_R(0)$. Let $i_0 \in I$ be such that $0 \in B_{R_{i_0}}(t_{i_0})$ and $\omega_{i_0} = \tau_{t_{i_0}}(\omega) \in \Xi_{i_0}$. We claim that, for every $i \in I$, there exists a clopen subset $\Xi_{i_0}^i \subset \Xi_{i_0}$ containing ω_{i_0} such that $\tau_{t_i - t_{i_0}}(\Xi_{i_0}^i)$ is a clopen subset of Ξ_i.

Assuming the claim is true, we denote $\Xi := \tau_{-t_{i_0}}(\cap_{i\in I} \Xi_{i_0}^i)$ and, by taking $\Xi_{i_0}^i$'s smaller if necessary, we choose Ξ sufficiently small so that $\tau(B_R \times \Xi)$ is a flow box. If $|x| < R$, $x \in B_{R_i}(t_i)$ for some index $i \in I$. Then $\tilde\Xi_i := \tau_{t_i - t_0}(\cap_{j\in I} \Xi_{i_0}^j)$ is a clopen subset of Ξ_i and

$$\tau_x(\Xi) = \tau_{x-t_i}(\tau_{t_i}(\Xi)) = \tau_{x-t_i}(\tilde\Xi_i).$$

We now prove the claim. We may assume that every $B_{R_i}(t_i)$ has a nonempty intersection with $B_{R'}(0)$. Let $i \in I$ and $x \in B_{R_i}(t_i) \cap B_{R'}(0)$. The segment $[0, x]$ can be split into successive segments $[x_{k-1}, x_k]$, $k = 1, \ldots, n$, each one included in a ball $B_{R_{i_k}}(t_{i_k})$ for some index i_k. The last index satisfies $i_n = i$. We construct by induction clopen subsets $\Xi_{i_0}^{(k)}$ of Ξ_{i_0} containing ω_{i_0} such that $\tau_{t_{i_k} - t_{i_0}}(\Xi_{i_0}^{(k)})$ is a clopen subset of Ξ_{i_k} containing ω_{i_k}. Let $\Xi_{i_0}^{(0)} = \Xi_{i_0}$. Since x_k belongs to both $B_{R_{i_k}}(t_{i_k})$ and $B_{R_{i_{k+1}}}(t_{i_{k+1}})$, we have

$$\begin{gathered}\tau_{(i_k)}(x_k - t_{i_k}, \omega_{i_k}) = \tau_{(i_{k+1})}(x_k - t_{i_{k+1}}, \omega_{i_{k+1}}), \\ \omega_{i_k} \in \Xi_{i_k}, \quad \omega_{i_{k+1}} \in \Xi_{i_{k+1}}, \\ a_k := t_{i_{k+1}} - t_{i_k},\ \omega_{i_{k+1}} = \tau_{a_k}(\omega_{i_k}),\ x_k - t_{i_{k+1}} = x_k - t_{i_k} - a_k.\end{gathered}$$

By admissability of the two flow boxes U_{i_k} and $U_{i_{k+1}}$, there exists a clopen subset Ξ'_{i_k} of $\tau_{t_{i_k} - t_{i_0}}(\Xi_{i_0}^{(k)})$ containing ω_{i_k} such that $\tau_{a_k}(\Xi'_{i_k}) \subset \Xi_{i_{k+1}}$. We have proved that $\Xi_{i_0}^{(k+1)} := \tau_{t_{i_0} - t_{i_k}}(\Xi'_{i_k})$ is a clopen subset of Ξ_{i_0} containing ω_{i_0} and that $\tau_{t_{i_{k+1}} - t_{i_0}}(\Xi_{i_0}^{(k+1)})$ is a clopen subset of $\Xi_{i_{k+1}}$. $\square$

An interaction model does not possess a canonical notion of vertical section. Such a notion naturally exists whenever the model admits a flow box decomposition (definition 15). We prove in the next lemma that locally transversally constant functions $V_1, V_2 : \Omega \to \mathbb{R}$ (a set of conditions checked on boxes of size R) enable to construct a transversally constant Lagrangian $L(\omega, t) = W(t) + V_1(\omega) + V_2(\tau_t(\omega))$ (a set of conditions checked on every sufficiently thin flow box). Corollary 21 follows from this lemma.

Lemma 38. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be an almost periodic interaction model admitting a flow box decomposition. Let $V_1, V_2 : \Omega \to \mathbb{R}$ be two locally transversally constant functions on the same flow box decomposition (definition 20), and $W = \mathbb{R} \to \mathbb{R}$ be any function. Define $L(\omega, t) = W(t) + V_1(\omega) + V_2(\tau_t(\omega))$. Then L is transversally constant (definition 16).*

Proof. Assume V_1 and V_2 are locally transversally constant on a flow box decomposition $\{U_i\}_{i\in I}$. Let $\tau[B_R\times\Xi]$ be a flow box which is compatible with respect to $\{U_i\}_{i\in I}$. If $|x|,|y|<R$ and $\omega,\omega'\in\Xi$, then

$$E_\omega(x,y)=W(y-x)+V_{1,\omega}(x)+V_{2,\omega}(y).$$

There exist $i\in I$, $|t_i|<R_i$ and $\tilde{\Xi}_i$ a clopen subset of Ξ_i such that $\tau_x(\Xi)=\tau_{t_i}(\tilde{\Xi}_i)$. Then $\tau_x(\omega)=\tau_{t_i}(\omega_i)$ and $\tau_x(\omega')=\tau_{t_i}(\omega_i')$ for some $\omega_i,\omega_i'\in\tilde{\Xi}_i$. We have

$$V_{1,\omega}(x)=V_{1,\omega_i}(t_i)=V_{1,\omega_i'}(t_i)=V_{1,\omega'}(x).$$

Similarly $V_{2,\omega}(y)=V_{2,\omega'}(y)$. We have thus proved $E_{\omega'}(x,y)=E_\omega(x,y)$. $\square$

The existence of a flow box decomposition (definition 15) enables us to build a global transverse section of the flow with locally constant return times. We extend for an almost periodic interaction model what has been done for quasicrystals in [13]. We first define the notion of Kakutani-Rohlin tower and show that an interaction model possessing a flow box decomposition admits a Kakutani-Rohlin tower.

Definition 39. *Let $(\Omega,\{\tau_t\}_{t\in\mathbb{R}})$ be a one-dimensional almost periodic interaction model possessing a flow box decomposition $\{U_i\}_{i\in I}$. We call Kakutani-Rohlin tower a partition $\{F_\alpha\}_{\alpha\in A}$ of Ω of the form*

$$F_\alpha=\tau\big([0,H_\alpha)\times\Sigma_\alpha\big)=\cup_{0\le t<H_\alpha}\tau_t(\Sigma_\alpha)$$

for some some height $H_\alpha>0$ and some transverse section Σ_α (closed set admitting a basis of clopen subsets), where $\tau\big((0,H_\alpha)\times\Sigma_\alpha\big)$ is a flow box (open and homeomorphic to $(0,H_\alpha)\times\Sigma_\alpha$), and $\cup_{\alpha\in A}\tau(\{H_\alpha\}\times\Sigma_\alpha)=\cup_{\alpha\in A}\tau(\{0\}\times\Sigma_\alpha)=\cup_{\alpha\in A}\Sigma_\alpha$. Moreover, we say that a Kakutani-Rohlin tower is compatible with respect to $\{U_i\}_{i\in I}$ if, for every $\alpha\in A$, there exist $i\in I$, $t_i\in\mathbb{R}$ and a clopen subset $\tilde{\Xi}_i\subset\Xi_i$ such that $\Sigma_\alpha=\tau_{t_i}(\tilde{\Xi}_i)$ and $[t_i,t_i+H_\alpha)\subset[-R_i,R_i)$.

Lemma 40. *Let $(\Omega,\{\tau_t\}_{t\in\mathbb{R}})$ be a one-dimensional almost periodic $\mathbb{R}$-action possessing a flow box decomposition $\{U_i\}_{i\in I}$. Then there exists a Kakutani-Rohlin tower $\{F_\alpha\}_{\alpha\in A}$ which is compatible with respect to $\{U_i\}_{i\in I}$.*

Proof. Let $\{U_i\}_{i=1}^n$ be a flow box decomposition, where $U_i=\tau[B_{R_i}\times\Xi_i]$. By definition, U_i is an open set of Ω. We denote $V_i:=\tau\big([-R_i,R_i)\times\Xi_i\big)$. We shall build by induction on $i=1,\dots,n$ a collection of flow boxes $\{\tau\big((0,H_{i,j})\times\Sigma_{i,j}\big)\}_j$ such that

- the sets $F_{i,j}:=\tau\big([0,H_{i,j})\times\Sigma_{i,j}\big)$ are pairwise disjoint,
- $V_i\setminus\cup_{k<i}V_k=\cup_j\tau\big([0,H_{i,j})\times\Sigma_{i,j}\big)=\cup_jF_{i,j}$,
- $\tau(\{-R_i\}\times\Xi_i)\setminus\cup_{k<i}V_k\subset\cup_j\tau(\{0\}\times\Sigma_{i,j})$,
- $\cup_{k<i}\tau(\{R_k\}\times\Xi_k)\cap(V_i\setminus\cup_{k<i}V_k)\subset\cup_j\tau(\{0\}\times\Sigma_{i,j})$,
- $\tau(\{H_{i,j}\}\times\Sigma_{i,j})\cap\cup_{k<i}V_k\subset\cup_{k<i}\cup_j\tau(\{0\}\times\Sigma_{k,j})$,
- $\tau(\{H_{i,j}\}\times\Sigma_{i,j})\setminus\cup_{k<i}V_k\subset\tau(\{R_i\}\times\Xi_i)\setminus\cup_{k<i}V_k$.

For $i=1$, we choose $H_{1,1}=2R_1$ and $\Sigma_{1,1}=\tau_{-R_1}(\Xi_1)$. Assume that we have built the sets $\tau\big([0,H_{k,j})\times\Sigma_{k,j}\big)$ for every $k<i$ and j. Thanks to the admissibility of

the flow boxes $\{U_i\}_{i\in I}$, the set $V_i \cap V_k$, if nonempty, is of the form $\tau(J_{i,k} \times \Xi_{i,k})$, where $J_{i,k} = [a_{i,k}, b_{i,k})$, with $-R_i \le a_{i,k} < b_{i,k} \le R_i$, and $\Xi_{i,k}$ is a clopen set of Ξ_i. The complement $V_i \setminus V_k$ is the union of sets of the form

$$\tau\big([-R_i, a_{i,k}) \times \Xi_{i,k}\big), \quad \tau\big([b_{i,k}, R_i) \times \Xi_{i,k}\big) \quad \text{or} \quad \tau\big([-R_i, R_i) \times (\Xi_i \setminus \Xi_{i,k})\big).$$

Hence, $V_i \setminus \cup_{k<i} V_k$ is obtained as a disjoint union of sets $\tau\big([c_\alpha, d_\alpha) \times \tilde\Sigma_\alpha\big)$, where $\tilde\Sigma_\alpha$ is any clopen set of the form $\cap_{k<i} S_k$, with either $S_k = \Xi_{i,k}$ or $S_k = \Xi_i \setminus \Xi_{i,k}$, and $[c_\alpha, d_\alpha)$ corresponds to any connected component of $[-R_i, R_i) \setminus \cup_{k<i} J_{i,k}$. We next rewrite $\tau\big([c_\alpha, d_\alpha) \times \tilde\Sigma_\alpha\big)$ as $\tau\big([0, H_{i,j}) \times \Sigma_{i,j}\big)$, with $j = j(\alpha)$, where $\Sigma_{i,j} = \tau_{c_\alpha}(\tilde\Sigma_\alpha)$ and $H_{i,j} = d_\alpha - c_\alpha$. By construction, for all $k < i$ with $V_i \cap \overline{V_k} \neq \emptyset$, $\tau(\{R_k\} \times \Xi_k) \cap V_i = \tau(\{b_{i,k}\} \times \Xi_{i,k})$ and its part which is not in $\cup_{l<i} V_l$ is included into $\cup_j \tau(\{0\} \times \Sigma_{i,j})$. Furthermore, $\tau(\{H_{i,j}\} \times \Sigma_{i,j})$ either is included into $\tau(\{R_i\} \times \Xi_i)$ or intersects V_k for some $k < i$ and therefore is included into $\cup_{k<i} \cup_j \tau(\{0\} \times \Sigma_{k,j})$. $\square$

When a Kakutani-Rohlin tower is built, we obtain a global transverse section $\cup_{\alpha\in A} \Sigma_\alpha$ with a return time constant on each Σ_α and equal to H_α. We can induce on a particular section Σ_{α_0} and build a second Kakutani-Rohlin tower with larger heights. We explain in the next paragraph the notations that will be used for these successive towers.

If $\{F^0_\alpha\}_{\alpha\in A^0}$ is a Kakutani-Rohlin tower of order 0, denote $F^0_\alpha := \tau\big([0, H^0_\alpha) \times \Sigma^0_\alpha\big)$. We say that $\Sigma^0 := \cup_\alpha \Sigma^0_\alpha$ is the basis of the tower. Let ω_* be a reference point of the base Σ^0. Consider α_0 such that $\omega_* \in \Sigma^0_{\alpha_0}$. The construction of the tower of order 1 is done by inducing the flow on $\Sigma^1 := \Sigma^0_{\alpha_0}$. We obtain a partition of Σ^1 given by $\{\Sigma^1_\beta\}_{\beta\in A^1}$, where $\beta = (\alpha_0, \ldots, \alpha_p)$, $p \ge 1$, $\alpha_p = \alpha_0$, $\alpha_i \neq \alpha_0$ for $i = 1, \ldots, p-1$,

$$\Sigma^1_\beta = \Sigma^0_{\alpha_0} \cap \tau^{-1}_{H^0_{\alpha_0}}(\Sigma^0_{\alpha_1}) \cap \ldots \cap \tau^{-1}_{H^0_{\alpha_0} + \ldots + H^0_{\alpha_{p-1}}}(\Sigma^0_{\alpha_p}).$$

By minimality, there is a finite collection of such nonempty sets Σ^1_β. Define then

$$H^1_\beta := H^0_{\alpha_0} + \ldots + H^0_{\alpha_{p-1}},$$
$$F^1_\beta := \tau\big([0, H^1_\beta) \times \Sigma^1_\beta\big) = \bigcup_{i=0}^{p-1} \tau\big([t_i, t_i + H^0_{\alpha_i}) \times \Sigma^0_{\alpha_i}\big), \text{ with } t_i = \sum_{j=0}^{i-1} H^0_{\alpha_j}. \tag{25}$$

We have just obtained a new Kakutani-Rohlin tower $\{F^1_\beta\}_{\beta\in A^1}$ of basis $\Sigma^0_{\alpha_0}$. We induced again on the section $\Sigma^1_{\beta_0}$ that contains ω_* and build the tower of order 2. We shall write $\{F^l_\alpha\}_{\alpha\in A^l}$ for the successive towers that are built using this procedure and F^l_* for the tower of height H^l_* whose basis Σ^l_* contains ω_* . The preceding construction gives $\min_{\alpha\in A^{l+1}} H^{l+1}_\alpha \ge H^l_*$ and in particular $H^{l+1}_* \ge H^l_*$. It may happen that $H^l_* = H^{l+1}_* = H^{l+2}_* = \ldots$ In that case, the flow is a suspension over Σ^l_* of constant return time H^l_* (and Ω is isomorphic to $\Sigma^l_* \times S^1$). In order to exclude this situation, we split the basis $\Sigma^l_{\alpha_0}$ which contains ω_* into two disjoint clopen sets $\Sigma^l_{\alpha_0} = \Sigma^l_{\alpha'_0} \cup \Sigma^l_{\alpha''_0}$. We obtain again a Kakutani-Rohlin tower and we induce as before on the subset which contains ω_*. If $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ is not periodic, we may choose the splitting so that $H^{l+1}_* > H^l_*$ at each step of the construction.

We now assume the flow $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ to be uniquely ergodic. Let λ be the unique ergodic invariant probability measure. The average frequency of return vectors to a transverse section of a flow box measures the thickness of the section. The next lemma gives a precise definition of a family of transverse measures $\{\nu_\Xi\}_\Xi$ parameterized by every transverse section Ξ.

Lemma 41. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ be an almost periodic and uniquely ergodic $\mathbb{R}$-action. Given Ξ a transverse section, let $\mathcal{R}_\Xi(\omega)$ be the set of return times to Ξ,*

$$\mathcal{R}_\Xi(\omega) := \{t \in \mathbb{R} : \tau_t(\omega) \in \Xi\}, \quad \forall\, \omega \in \Omega.$$

Then, for every nonempty clopen set $\Xi' \subset \Xi$, the following limit exists uniformly with respect to $\omega \in \Omega$ and is positive:

$$\nu_\Xi(\Xi') := \lim_{T\to+\infty} \frac{\#(\mathcal{R}_{\Xi'}(\omega) \cap B_T(0))}{\mathrm{Leb}(B_T(0))} > 0.$$

Moreover, ν_Ξ extends to a finite and nonnegative measure on Ξ, called transverse measure to Ξ, and, for every flow box $U = \tau[B_R \times \Xi]$,

$$\lambda(\tau(B' \times \Xi')) = \mathrm{Leb}(B')\nu_\Xi(\Xi'), \quad \forall\, B' \subset B_R(0),\ \forall\, \Xi' \subset \Xi \quad (\textit{Borel sets}).$$

Proof. Let $U = \tau[B_R \times \Xi]$ be a flow box. Let $t_1 \neq t_2$ be two return times of $\mathcal{R}_\Xi(\omega)$. Since τ is injective on $B_R(0) \times \Xi$, it is straightforward that $B_R(t_1) \cap B_R(t_2) = \emptyset$. For $\omega \in \Omega$ and $T > 0$, consider

$$\mu_{T,\omega}(U') = \frac{1}{\mathrm{Leb}(B_T(0))} \int_{B_T(0)} \mathbf{1}_{U'}(\tau_s(\omega))\, ds, \quad \forall\, U' \subset \Omega \quad \text{(Borel set)}.$$

The unique ergodicity of the action implies that, for all $\phi \in C^0(\Omega)$, $\mu_{T,\omega}(\phi)$ converges uniformly in ω to $\lambda(\phi)$ as $T \to +\infty$. Let $B' \subset B_R(0)$ be a Borel set and $\Xi' \subset \Xi$ be a nonempty clopen set. For $U' = \tau(B' \times \Xi')$, notice then that

$$\{s \in \mathbb{R}^d : \tau_s(\omega) \in U'\} = \bigcup_{t\in\mathcal{R}_{\Xi'}(\omega)} t + B', \;\; \mu_{T,\omega}(U') = \sum_{t\in\mathcal{R}_{\Xi'}(\omega)} \frac{\mathrm{Leb}(B_T(0) \cap (t + B'))}{\mathrm{Leb}(B_T(0))},$$

and, whenever $T > 2R$,

$$\mathrm{Leb}(B')\frac{\#(B_{T-R}(0) \cap \mathcal{R}_{\Xi'}(\omega))}{\mathrm{Leb}(B_T(0))} \le \mu_{T,\omega}(U') \le \mathrm{Leb}(B')\frac{\#(B_{T+R}(0) \cap \mathcal{R}_{\Xi'}(\omega))}{\mathrm{Leb}(B_T(0))}.$$

Moreover, clearly $\#(B_T(0) \cap \mathcal{R}_{\Xi'}(\omega)) \le \frac{\mathrm{Leb}(B_{T+R}(0))}{\mathrm{Leb}(B_R(0))}$ and $\lim_{T\to+\infty} \frac{\mathrm{Leb}(B_{T+R}(0))}{\mathrm{Leb}(B_T(0))} = 1$. Thus, if B' is open in $B_R(0)$, then U' is open in Ω and

$$\lambda(U') \le \liminf_{T\to+\infty} \mu_{T,\omega}(U') \le \frac{\mathrm{Leb}(B')}{\mathrm{Leb}(B_{2R}(0))}.$$

In particular, if B' is negligible, thanks to the regularity of Leb, $\lambda(U') = 0$. If B' is open, $\overline{B'} \subset B_R(0)$ and $\partial B'$ is negligible, then, for every $\epsilon > 0$, there exist nonnegative continuous functions $\phi \le \psi$ such that

$$\phi \le \mathbf{1}_{\tau(B'\times\Xi)} \le \mathbf{1}_{\tau(\overline{B'}\times\Xi)} \le \psi \quad \text{and} \quad \lambda(\psi - \phi) < \epsilon.$$

Therefore, $\mu_{T,\omega}(\tau(B'\times\Xi'))$ converges uniformly in ω to $\lambda(\tau(B'\times\Xi))$ as $T\to+\infty$.

On the one hand, for all clopen set $\Xi'\subset\Xi$, $\tau(B_R(0)\times\Xi')$ is a flow box and

$$\lim_{T\to+\infty}\frac{\#(B_T(0)\cap\mathcal{R}_{\Xi'}(\omega))}{\mathrm{Leb}(B_T(0))}:=\nu_\Xi(\Xi')\quad\text{(exists uniformly in }\omega).$$

On the other hand, for every $B'=B_{R'}(s')$, $s'\in B_R(0)$, $\|s'\|+R'<R$,

$$\lambda(\tau(B'\times\Xi'))=\lim_{T\to+\infty}\mu_{T,\omega}(\tau(B'\times\Xi'))=\mathrm{Leb}(B')\nu_\Xi(\Xi').$$

Hence, ν_Ξ extends to a measure on the Borel sets of Ξ and by the monotone class theorem $\lambda(\tau(B'\times\Xi'))=\mathrm{Leb}(B')\nu_\Xi(\Xi')$ for every Borel sets $B'\subset B_R(0)$ and $\Xi'\subset\Xi$.

We finally remark that $\nu_\Xi(\Xi')>0$ for every nonempty clopen set $\Xi'\subset\Xi$, since otherwise there would exist an open set of Ω of λ-measure zero. □

We come back to Kakutani-Rohlin towers of flows. Let $\{F^l_\alpha\}_{\alpha\in A^l}$ be such a tower of order l and $\{F^{l+1}_\beta\}_{\beta\in A^{l+1}}$ be the subsequent tower as introduced in (25). We recall the definition of the homology matrix as explained in lemma 2.7 of [13]. For every $\alpha\in A^l$ and $\beta\in A^{l+1}$, $\beta=(\alpha_0,\dots,\alpha_p)$, $\alpha_0=\alpha_p$, $\alpha_i\neq\alpha_0$ for $i=1,\dots,p-1$, we denote

$$M^l_{\alpha,\beta}:=\#\{0\le k\le p-1:\alpha_k=\alpha\}.$$

A flow box of order $l+1$, $\tau\big([0,H^{l+1}_\beta)\times\Sigma^{l+1}_\beta\big)$, is obtained as a disjoint union of flow boxes of order l of the type $\tau\big([t_i,t_i+H^l_{\alpha_i})\times\Sigma^l_{\alpha_i}\big)$. The integer $M^l_{\alpha,\beta}$ counts the number of times a flow box of order $l+1$ indexed by β cuts a flow box of order l indexed by α. The main result that we shall need is given by the following lemma.

Lemma 42. *Let $(\Omega,\{\tau_t\}_{t\in\mathbb{R}})$ be a one-dimensional almost periodic and uniquely ergodic $\mathbb{R}$-action. Let $\{F^l_\alpha\}_{\alpha\in A^l}$ be a sequence of Kakutani-Rohlin towers built as in (25). Let ν^l be the transverse measure associated to the transverse section $\cup_{\alpha\in A^l}\Sigma^l_\alpha$. If $\nu^l_\alpha:=\nu^l(\Sigma^l_\alpha)$, then*

$$\nu^l_\alpha=\sum_{\beta\in A^{l+1}}M^l_{\alpha,\beta}\nu^{l+1}_\beta.$$

Proof. Let $\Xi=\cup_{\beta\in A^{l+1}}\Sigma^{l+1}_\beta$. For $\omega\in\Xi$, let $0=t_0,t_1,t_2,\dots$ be its successive return times to Ξ. We introduce as in lemma 41 the set of return times to the transverse section Σ^l_α, say, $\mathcal{R}^l_\alpha(\omega):=\{t\in\mathbb{R}:\tau_t(\omega)\in\Sigma^l_\alpha\}$. The set $\mathcal{R}^{l+1}_\beta(\omega)$ is defined similarly. Since

$$\#\big(\mathcal{R}^l_\alpha(\omega)\cap[0,t_n)\big)=\sum_{\beta\in A^{l+1}}M^l_{\alpha,\beta}\ \#\big(\mathcal{R}^{l+1}_\beta(\omega)\cap[0,t_n)\big),$$

we divide by t_n and apply lemma 41 to conclude. □

The main property used in one-dimensional Aubry theory [2] is the twist property. It will not be used in the infinitesimal form. The following lemma is an easy consequence of definition 18. It shows that the energy of a configuration can be lower by exchanging the positions.

Lemma 43 (Aubry crossing lemma). *If L satisfies the weakly twist property, then, for every $\omega \in \Omega$, for every $x_0, x_1, y_0, y_1 \in \mathbb{R}$ verifying $(y_0 - x_0)(y_1 - x_1) < 0$,*

$$\big[E_\omega(x_0, x_1) + E_\omega(y_0, y_1)\big] - \big[E_\omega(x_0, y_1) + E_\omega(y_0, x_1)\big] = \alpha(y_0 - x_0)(y_1 - x_1) > 0,$$

with $\alpha = \frac{1}{(y_0-x_0)(y_1-x_1)} \int_{x_0}^{y_0} \int_{x_1}^{y_1} \frac{\partial^2 \tilde{E}_\omega}{\partial x \partial y}(x, y)\, dydx < 0$ and $\tilde{E}_\omega$ as in definition 18.

Proof. The inequality is obtained by integrating the function $\frac{\partial^2}{\partial x \partial y} \tilde{E}_\omega$ on the domain $[\min(x_0, y_0), \max(x_0, y_0)] \times [\min(x_1, y_1), \max(x_1, y_1)]$. □

The first consequence of Aubry crossing lemma is that minimizing configurations shall be strictly ordered. We begin by an intermediate lemma.

Lemma 44. *Let L be a weakly twist Lagrangian, $\omega \in \Omega$, $n \geq 2$, and $x_0, \ldots, x_n \in \mathbb{R}$ be a nonmonotone sequence (that is, a sequence which does not satisfy $x_0 \leq \ldots \leq x_n$ nor $x_0 \geq \ldots \geq x_n$).*

– If $x_0 = x_n$, then $E_\omega(x_0, \ldots, x_n) > \sum_{i=0}^{n-1} E_\omega(x_i, x_i)$.

– If $x_0 \neq x_n$, then there exists a subset $\{i_0, i_1, \ldots, i_r\}$ of $\{0, \ldots, n\}$, with $i_0 = 0$ and $i_r = n$, such that $(x_{i_0}, x_{i_1}, \ldots, x_{i_r})$ is strictly monotone and

$$E_\omega(x_0, \ldots, x_n) > E_\omega(x_{i_0}, \ldots, x_{i_r}) + \sum_{i \notin \{i_0, \ldots, i_r\}} E_\omega(x_i, x_i).$$

Proof. We prove the lemma by induction.

Let $x_0, x_1, x_2 \in \mathbb{R}$ be a nonmonotone sequence. Then x_0, x_1, x_2 are three distinct points. Thus, $x_0 < x_1$ implies $x_2 < x_1$ and $x_1 < x_0$ implies $x_1 < x_2$. In both cases, lemma 43 tells us that

$$E_\omega(x_0, x_1) + E_\omega(x_1, x_2) > E_\omega(x_0, x_2) + E_\omega(x_1, x_1).$$

Let $(x_0, \ldots, x_{n+1})$ be a nonmonotone sequence. We have two cases: either $x_0 \leq x_n$ or $x_0 \geq x_n$. We shall only give the proof for the case $x_0 \leq x_n$.

Case $x_0 = x_n$. Then $(x_0, \ldots, x_n)$ is nonmonotone and by induction

$$\begin{aligned} E_\omega(x_0, \ldots, x_{n+1}) &> E_\omega(x_n, x_{n+1}) + \sum_{i=0}^{n-1} E_\omega(x_i, x_i) \\ &= E_\omega(x_0, x_{n+1}) + \sum_{i=1}^{n} E_\omega(x_i, x_i). \end{aligned}$$

Case $x_0 < x_n$. Whether $(x_0, \ldots, x_n)$ is monotone or not, we may choose a subset of indices $\{i_0, \ldots, i_r\}$ such that $i_0 = 0$, $i_r = n$, $x_{i_0} < x_{i_1} < \ldots < x_{i_r}$ and

$$E_\omega(x_0, \ldots, x_{n+1}) \geq \Big(E_\omega(x_{i_0}, \ldots, x_{i_r}) + \sum_{i \notin \{i_0, \ldots, i_r\}} E_\omega(x_i, x_i)\Big) + E_\omega(x_n, x_{n+1}).$$

If $x_n \leq x_{n+1}$, then $(x_0, \ldots, x_n)$ is necessarily nonmonotone and the previous inequality is strict. If $x_n = x_{n+1}$, the lemma is proved by modifying $i_r = n + 1$. If $x_n < x_{n+1}$, the lemma is proved by choosing $r + 1$ indices and $i_{r+1} = n + 1$.

If $x_{n+1} < x_n = x_{i_r}$, by applying lemma 43, one obtains

$$E_\omega(x_{i_{r-1}}, x_{i_r}) + E_\omega(x_n, x_{n+1}) > E_\omega(x_n, x_{i_r}) + E_\omega(x_{i_{r-1}}, x_{n+1}),$$

$$E_\omega(x_0, \ldots, x_{n+1}) > E_\omega(x_{i_0}, \ldots, x_{i_{r-1}}, x_{n+1}) + \sum_{i \notin \{i_0, \ldots, i_r\}} E_\omega(x_i, x_i) + E_\omega(x_n, x_n).$$

If $x_{i_{r-1}} < x_{n+1}$, the lemma is proved by choosing $i_r = n+1$. If $x_{i_{r-1}} = x_{n+1}$, the lemma is proved by choosing $r-1$ indices and $i_{r-1} = n+1$. If $x_{n+1} < x_{i_{r-1}}$, we apply again lemma 43 until there exists a largest $s \in \{0, \ldots, r\}$ such that $x_s < x_{n+1}$ or $x_{n+1} \leq x_0$. In the former case, the lemma is proved by choosing $s+1$ indices and by modifying $i_{s+1} = n+1$. In the latter case, namely, when $x_{n+1} \leq x_0 < x_n$, we have

$$E_\omega(x_0, \ldots, x_{n+1}) > E_\omega(x_0, x_{n+1}) + \sum_{i=1}^{n} E_\omega(x_i, x_i)$$

and the lemma is proved whether $x_{n+1} = x_0$ or $x_{n+1} < x_0$. □

The Mañé subadditive cocycle $\Phi(\omega, t)$ (definition 33) is obtained by minimizing a normalized energy $E_\omega(x_0, \ldots, x_n) - n\bar{E}$ on all the configurations satisfying $x_0 = 0$ and $x_n = t$. The following lemma shows that it is enough to minimize on strictly monotone configurations (unless $t = 0$).

Corollary 45. *If L satisfies the weakly twist property, then, for every $\omega \in \Omega$, the Mañé subadditive cocycle $\Phi(\omega, t)$ satisfies:*
– *if $t = 0$, $\Phi(\omega, 0) = E_\omega(0, 0) - \bar{E}$,*
– *if $t > 0$, $\Phi(\omega, t) = \inf_{n \geq 1} \inf_{0 = x_0 < x_1 < \ldots < x_n = t} [E_\omega(x_0, \ldots, x_n) - n\bar{E}]$,*
– *if $t < 0$, $\Phi(\omega, t) = \inf_{n \geq 1} \inf_{0 = x_0 > x_1 > \ldots > x_n = t} [E_\omega(x_0, \ldots, x_n) - n\bar{E}]$.*

Proof. Lemma 44 tells us that we can minimize the energy of $E_\omega(x_0, \ldots, x_n) - n\bar{E}$ by the sum of two terms:
– either $x_n = x_0$, then

$$E_\omega(x_0, \ldots, x_n) - n\bar{E} \geq \big[E_\omega(x_0, x_0) - \bar{E}\big] + \sum_{i \notin \{0, n\}} \big[E_\omega(x_i, x_i) - \bar{E}\big];$$

– or $x_n \neq x_0$, then for some $(x_{i_0}, \ldots, x_{i_r})$ strictly monotone, with $i_0 = 0$ and $i_r = n$,

$$E_\omega(x_0, \ldots, x_n) - n\bar{E} \geq \big[E_\omega(x_{i_0}, \ldots, x_{i_r}) - r\bar{E}\big] + \sum_{i \notin \{i_0, \ldots, i_r\}} \big[E_\omega(x_i, x_i) - \bar{E}\big].$$

We conclude the proof by noticing that $\bar{E} \leq \inf_{x \in \mathbb{R}} E_\omega(x, x)$. □

We recall that a finite configuration $(x_0, x_1, \ldots, x_n)$ is said to be minimizing in the environment ω if $E_\omega(x_0, x_1, \ldots, x_n) \leq E_\omega(y_0, y_1, \ldots, y_n)$ whenever $x_0 = y_0$ and $x_n = y_n$. The following lemmas show that, under certain conditions, a minimizing configuration is strictly monotone.

Lemma 46. *Suppose that L satisfies the weakly twist property. For every $\omega \in \Omega$, if $(x_0, \ldots, x_n)$ is a minimizing configuration, with $x_0 \neq x_n$, such that x_i is strictly between x_0 and x_n for every $0 < i < n-1$, then $(x_0, \ldots, x_n)$ is strictly monotone.*

Proof. Let $(x_0,\ldots,x_n)$ be such a minimizing sequence. We show, in part 1, it is monotone, and, in part 2, it is strictly monotone.

Part 1. Assume by contradiction that $(x_0,\ldots,x_n)$ is not monotone. According to lemma 44, one can find a subset of indices $\{i_0,\ldots,i_r\}$ of $\{0,\ldots,n\}$, with $i_0=0$ and $i_r=n$, such that $(x_{i_0},\ldots,x_{i_r})$ is strictly monotone and

$$E_\omega(x_0,\ldots,x_n) > E_\omega(x_{i_0},\ldots,x_{i_r}) + \sum_{i\notin\{i_0,\ldots,i_r\}} E_\omega(x_i,x_i).$$

We choose the largest integer r with the above property. Since $(x_0,\ldots,x_n)$ is not monotone, we have necessarily $r<n$. Since $(x_0,\ldots,x_n)$ is minimizing, one can find $i\notin\{i_0,\ldots,i_r\}$ such that $x_i\notin\{x_{i_0},\ldots,x_{i_r}\}$. Let s be one of the indices of $\{0,\ldots,r\}$ such that x_i is between x_{i_s} and $x_{i_{s+1}}$. Then, by lemma 43,

$$E_\omega(x_{i_s},x_{i_{s+1}}) + E_\omega(x_i,x_i) > E_\omega(x_{i_s},x_i) + E_\omega(x_i,x_{i_{s+1}}).$$

We have just contradicted the maximality of r. Therefore, $(x_0,\ldots,x_n)$ must be monotone.

Part 2. Assume by contradiction that $(x_0,\ldots,x_n)$ is not strictly monotone. Then $(x_0,\ldots,x_n)$ contains a subsequence of the form $(x_{i-1},x_i,\ldots,x_{i+r},x_{i+r+1})$ with $r\geq 1$ and $x_{i-1}\neq x_i=\ldots=x_{i+r}\neq x_{i+r+1}$. To simplify the proof, we assume $x_{i-1}<x_{i+r+1}$. We want built a configuration $(x'_{i-1},x'_i,\ldots,x'_{i+r},x'_{i+r+1})$ so that $x'_{i-1}=x_{i-1}$, $x'_{i+r+1}=x_{i+r+1}$ and

$$E_\omega(x_{i-1},x_i,\ldots,x_{i+r},x_{i+r+1}) > E_\omega(x'_{i-1},x'_i,\ldots,x'_{i+r},x'_{i+r+1}).$$

By changing by a coboundary as in definition 18, we may assume that $E_\omega(x,y)$ is C^2 in x and y. Indeed, since $(x_{i-1},\ldots,x_{i+r+1})$ is minimizing, we have

$$E_\omega(x_{i-1},\ldots,x_{i+r+1}) = E_\omega(x_{i-1},x_i+\epsilon,x_{i+1}-\epsilon,\ldots,x_{i+r}-\epsilon,x_{i+r+1}) + o(\epsilon^2).$$

Let

$$\alpha = \frac{1}{x_i-x_{i-1}}\int_{x_{i-1}}^{x_i}\frac{\partial^2 E_\omega}{\partial x\partial y}(x,x_i)\,dx < 0,$$

$$\beta = \frac{1}{x_{i+r+1}-x_{i+r}}\int_{x_{i+r}}^{x_{i+r+1}}\frac{\partial^2 E_\omega}{\partial x\partial y}(x_{i+r},y)\,dy < 0.$$

By Aubry crossing lemma,

$$\begin{aligned}&E_\omega(x_{i-1},x_i+\epsilon)+E_\omega(x_i+\epsilon,x_{i+1}-\epsilon)\\&\qquad = E_\omega(x_{i-1},x_{i+1}-\epsilon)+E_\omega(x_i+\epsilon,x_i+\epsilon)-2\epsilon(x_i-x_{i-1})\alpha+o(\epsilon).\end{aligned}$$

Since $x_i=x_{i+r}$, obviously $E_\omega(x_i+\epsilon,x_i+\epsilon)=E_\omega(x_{i+r}+\epsilon,x_{i+r}+\epsilon)$. Again by Aubry crossing lemma,

$$\begin{aligned}&E_\omega(x_{i+r}+\epsilon,x_{i+r}+\epsilon)+E_\omega(x_{i+r}-\epsilon,x_{i+r+1})\\&\qquad = E_\omega(x_{i+r}-\epsilon,x_{i+r}+\epsilon)+E_\omega(x_{i+r}+\epsilon,x_{i+r+1})-2\epsilon(x_{i+r+1}-x_{i+r})\beta+o(\epsilon).\end{aligned}$$

Then, for ϵ small enough, we have

$$E_\omega(x_{i-1},\dots,x_{i+r+1}) > E_\omega(x_{i-1},x_i-\epsilon,\dots,x_{i-r-1}-\epsilon,x_{i+r}+\epsilon,x_{i+r+1}),$$

which contradicts that $(x_{i-1},\dots,x_{i+r+1})$ is minimizing. We have thus proved that $(x_0,\dots,x_n)$ is strictly monotone. □

Lemma 47. *Let L be a weakly twist transversally constant Lagrangian. Then, there exists $R>0$ such that the fact $(x_0,\dots,x_n)\in\mathbb{R}$ is a minimizing configuration for an arbitrary environment $\omega\in\Omega$ and verifies $|x_n-x_0|\ge R$ implies that $(x_0,\dots,x_n)$ is strictly monotone.*

Proof. Let $\{U_i=\tau[B_{R_i}\times\Xi_i]\}_{i\in I}$ be a flow box decomposition with respect to which L is transversally constant. Since $\{U_i\}_{i\in I}$ is a finite cover, we may choose R large enough so that every orbit of size R meets every box entirely: for every ω, for every $|y-x|\ge R$, for every $i\in I$, there exists $t_i\in\mathbb{R}$ such that $(t_i-R_i,t_i+R_i)\subset[x,y]$ and $\tau_{t_i}(\omega)\in\Xi_i$.

We first show that there cannot exist $r\ge 0$ and $0<k<n-r$ such that

$$x_k<x_{k-1},\quad x_k=\ldots=x_{k+r}\quad\text{and}\quad x_k<x_{k+r+1}.$$

Otherwise, Aubry crossing lemma implies that

$$E_\omega(x_{k-1},x_k)+E_\omega(x_k,x_{k+r+1})>E_\omega(x_{k-1},x_{k+r+1})+E_\omega(x_k,x_k).$$

We rewrite the configuration $(x_0,\dots,x_{k-1},x_{k+r+1},\dots,x_n)$ as $(y_0,\dots,y_{n-r-1})$. Let U_i be a flow box containing $\tau_{x_k}(\omega)$. There exists $|s|<R_i$ and $\omega'\in\Xi_i$ such that $\tau_{x_k}(\omega)=\tau_s(\omega')$. By the choice of R, there exists t such that $(t-R_i,t+R_i)\subset[x_0,x_n]$ and $\tau_t(\omega)\in\Xi_i$. Let $z_0=\ldots=z_r:=t+s$ and $1\le l\le n-r-1$ be such that $y_{l-1}<z_0\le y_l$. Using the fact that L is transversally constant on U_i, we have

$$E_\omega(x_k,x_k)=E_{\omega'}(s,s)=E_{\tau_t(\omega)}(s,s)=E_\omega(z_0,z_0).$$

By applying again Aubry crossing lemma, we obtain

$$E_\omega(y_{l-1},y_l)+E_\omega(z_0,z_0)\ge E_\omega(y_{l-1},z_0)+E_\omega(z_0,y_l),$$

with a strict inequality if $z_0<y_l$. We have just obtained a new configuration $(y_0,\dots,y_{l-1},z_0,\dots,z_r,y_l,\dots,y_{n-r-1})$ of n points with a strictly lower energy, which contradicts the fact that $(x_0,\dots,x_n)$ is minimizing.

There cannot exist similarly $r\ge 0$ and $0<k<n-r$ such that

$$x_k>x_{k-1},\quad x_k=\ldots=x_{k+r}\quad\text{and}\quad x_k>x_{k+r+1}.$$

There cannot exists either a sub-configuration $(x_{k-1},x_k,\dots,x_{k+r},x_{k+r+1})$, $r\ge 1$, of the form $x_{k-1}\ne x_{k+r+1}$ and $x_k=\ldots=x_{k+r}$ strictly between x_{k-1} and x_{k+r+1} thanks to lemma 46. We are thus left to a configuration of the form

$$x_0=\ldots=x_r<\ldots<x_{n-r'}=\ldots=x_n\quad\text{or}\quad x_0=\ldots=x_r>\ldots>x_{n-r'}=\ldots=x_n$$

for some $r, r' \geq 0$. Assume by contradiction that $x_0 = x_1$ (the case $x_{n-1} = x_n$ is done similarly). As before, there exist U_i containing $\tau_{x_0}(\omega)$, $|s| < R_i$ and $\omega' \in \Xi_i$ such that $\tau_{x_0}(\omega) = \tau_s(\omega')$, as well as there exists $t \in \mathbb{R}$ such that $(t - R_i, t + R_i) \subset [\min\{x_0, x_n\}, \max\{x_0, x_n\}]$ and $\tau_t(\omega) \in \Xi_i$. One can show in an analogous way that, whenever $z := t + s$ belongs to $(\min\{x_{l-1}, x_l\}, \max\{x_{l-1}, x_l\}]$ for $2 \leq l \leq n$, $E(x_0, x_1, \dots, x_n) \geq E(x_1, \dots, x_{l-1}, z, x_l, \dots, x_n)$, with strict inequality if $z < \max\{x_{l-1}, x_l\}$. Since $(x_0, x_1, \dots, x_n)$ is as minimizing configuration, this implies that $z = \max\{x_{l-1}, x_l\}$ and thus $(x_1, \dots, x_{l-1}, z, x_l, \dots, x_n)$ is a minimizing configuration. The first part of this proof shows that this cannot happen.

The proof that $(x_0, \dots, x_n)$ is strictly monotone is complete. $\square$

Proposition 48. *Let L be a weakly twist transversally constant Lagrangian. Then, there exists $R > 0$ such that, for $\omega \in \Omega$, $n \geq 2$, and $(x_0, \dots, x_n)$ with $E(x_0, \dots, x_n) = \min_{(y_0, \dots, y_n)} E_\omega(y_0, \dots, y_n)$, the inequality $\operatorname{diam}(\{x_k : 0 \leq k \leq n\}) \geq R$ implies that $(x_0, \dots, x_n)$ is strictly monotone and $\sup_{1 \leq k \leq n} |x_k - x_{k-1}| \leq R$.*

Proof. Consider $\omega \in \Omega$, $n \geq 2$, and $(x_0, \dots, x_n)$ realizing the minimum of the energy among all configurations of length n in the environment ω.

Part 1. We show there exists $R' > 0$ (independent from ω and n) such that $|x_1 - x_0| \leq R'$ and $|x_2 - x_1| \leq R'$. Indeed, we have

$$E_\omega(x_0, x_1) \leq E_\omega(x_1, x_1) \quad \text{and} \quad E_\omega(x_0, x_1, x_2) \leq E_\omega(x_2, x_2, x_2),$$

which implies

$$E_\omega(x_0, x_1) \leq \sup_{x \in \mathbb{R}} E_\omega(x, x) \quad \text{and} \quad E_\omega(x_1, x_2) \leq 2 \sup_{x \in \mathbb{R}} E_\omega(x, x) - \inf_{x, y \in \mathbb{R}} E_\omega(x, y).$$

The existence of R' follows then from the coerciveness of L, which is uniform with respect to ω. Similarly, we have $|x_{n-1} - x_{n-2}| \leq R'$ and $|x_n - x_{n-1}| \leq R'$.

Part 2. We show there exists $R'' > 0$ such that, if $(x_0, \dots, x_m)$ is strictly monotone, then $|x_i - x_{i-1}| \leq R''$ for every $1 \leq i \leq m$. It is clear from the definition that, if L is transversally constant with respect to a particular flow box decomposition $\{\tau[B_{r_i} \times \Xi_i]\}$, then L is transversally constant for any flow box decomposition such that its flow boxes are compatible with respect to $\{\tau[B_{r_i} \times \Xi_i]\}$. Therefore, let $\{U_i = \tau[B_{R'} \times \Xi'_i]\}$ be a finite cover of Ω by flow boxes such that $\tau[B_{2R'} \times \Xi'_i]$ is again a flow box and L is transversally constant with respect to $\{\tau[B_{2R'} \times \Xi'_i]\}$. We choose $R'' > 0$ large enough so that every orbit of length R'' meets entirely each $\tau[B_{2R'} \times \Xi'_i]$. Let U_i be a flow box containing $\tau_{x_1}(\omega)$: there exist $|s_1| < R'$ and $\omega' \in \Xi'_i$ such that $\tau_{x_1}(\omega) = \tau_{s_1}(\omega')$. From part 1, we deduce that $\tau[B_{2R'} \times \Xi'_i]$ contains $\{\tau_{x_0}(\omega), \tau_{x_1}(\omega), \tau_{x_2}(\omega)\}$: there exist $|s_0|, |s_2| < 2R'$ such that $\tau_{x_0}(\omega) = \tau_{s_0}(\omega')$ and $\tau_{x_2}(\omega) = \tau_{s_2}(\omega')$. Assume by contradiction $|x_i - x_{i-1}| > R''$. Then, there exists $t \in \mathbb{R}$ such that $(t - 2R', t + 2R') \subset [\min\{x_{i-1}, x_i\}, \max\{x_{i-1}, x_i\}]$ and $\tau_t(\omega) \in \Xi'_i$. Let $z_0 = t + s_0$, $z_1 = t + s_1$ and $z_2 = t + s_2$. Notice that (x_{i-1}, x_i) and (z_0, z_1, z_2) are ordered in the same way. As L is transversally constant on $\tau[B_{2R'} \times \Xi'_i]$, we obtain

$$E_\omega(x_0, x_1, x_2) = E_{\omega'}(s_0, s_1, s_2) = E_{\tau_t(\omega)}(s_0, s_1, s_2) = E_\omega(z_0, z_1, z_2).$$

Aubry crossing lemma applied twice gives

$$\begin{aligned}E_\omega(x_{i-1},x_i)+E_\omega(z_0,z_1,z_2) &> E_\omega(x_{i-1},z_1)+E_\omega(z_0,x_i)+E_\omega(z_1,z_2),\\ &> E_\omega(x_{i-1},z_1,x_i)+E_\omega(z_0,z_2).\end{aligned}$$

As L is transversally constant, $E_\omega(z_0,z_2)=E_\omega(x_0,x_2)$ as above and we obtain

$$E_\omega(x_{i-1},x_i)+E_\omega(x_0,x_1,x_2) > E_\omega(x_{i-1},z_1,x_i)+E_\omega(x_0,x_2).$$

The configuration $(x_0,x_2,\dots,x_{i-1},z_1,x_i,\dots,x_m)$ has a strictly lower energy, which contradicts the fact that $(x_0,\dots,x_m)$ is minimizing. We obtain similarly that, if $(x_m,\dots,x_n)$ is strictly monotone, then $|x_{i-1}-x_i|\le R''$ for every $m+1\le i\le n$.

Part 3. Let R''' be the constant given by lemma 47. Take $R>2R''+4R'''$. If $|x_n-x_0|>R'''$, then $(x_0,\dots,x_n)$ is strictly monotone by lemma 47 and the jumps $|x_i-x_{i-1}|$ are uniformly bounded by R''. The proof is finished.

Assume by contradiction that $|x_n-x_0|\le R'''$. Let $a=\min_{0\le k\le n}x_k$ and $b=\max_{0\le k\le n}x_k$. Since $\mathrm{diam}(\{x_k: 0\le k\le n\})\ge R$, one of the two inequalities $|a-x_0|>R/2$ or $|b-x_0|>R/2$ must be satisfied. Assume to simplify $|b-x_0|>R/2$ (the case $|a-x_0|>R/2$ is done similarly). Hence, $b=x_m$ for some $0<m<n$. Since $(x_0,\dots,x_m)$ and $(x_m,\dots,x_n)$ are minimizing and satisfy $|x_m-x_0|>R'''$ and $|x_m-x_n|>R'''$, these two configurations are strictly monotone. Then, part 2 tells us that the jumps $|x_i-x_{i-1}|$ are uniformly bounded by R''. In particular, $|x_{m+1}-x_m|\le R''$. The configuration $(x_0,\dots,x_{m+1})$ is minimizing and, since $|x_m-x_0|>R''+2R'''$, it satisfies $|x_{m+1}-x_0|>R'''$. By lemma 47, it must be strictly monotone. Thus, $(x_0,\dots,x_n)$ is strictly monotone and $|x_n-x_0|>|x_{m+1}-x_0|>R'''$, which is a contradiction. ☐

The proof of the fact that $|x_k-x_{k-1}|$ is uniformly bounded uses the same ideas as in lemma 3.1 of [13]. The fact that L is transversally constant enables us to translate subconfigurations without modifying the total energy. For a minimizing and strictly monotone configuration, by minimality of the energy, two consecutive points cannot enclose a translated subconfiguration of three points. More precisely, we have the following lemma that extends lemma 3.2 of [13].

Lemma 49. *Let L be a weakly twist Lagrangian which is transversally constant for a flow box decomposition $\{U_i\}_{i\in I}$. Suppose that the flow box $\tau[B_R\times\Xi]$ is compatible with respect to $\{U_i\}_{i\in I}$. Let $(x_0,\dots,x_n)$ be a strictly monotone minimizing configuration for some environment $\omega\in\Omega$. Let $(a-R,a+R)$ and $(b-R,b+R)$ be two disjoint intervals such that $\tau_a(\omega)\in\Xi$ and $\tau_b(\omega)\in\Xi$. Assume that $(a-R,a+R)$ is a subset of $[x_0,x_n]$. Let A be the number of sites $0\le k\le n$ such that x_k belongs to $(a-R,a+R)$ and let B be defined similarly. Then $B\le A+2$. In particular, if $(b-R,b+R)\ \subset[x_0,x_n]$, then $|A-B|\le 2$.*

Proof. To simplify we assume that $(x_0,\dots,x_n)$ is strictly increasing. The proof is done by contradiction by assuming $B\ge A+3$. Denote

$$\begin{aligned}\{y_1,\dots,y_A\} &:= \{x_0,\dots,x_n\}\cap(a-R,a+R) \quad\text{and}\\ \{y'_1,\dots,y'_B\} &:= \{x_0,\dots,x_n\}\cap(b-R,b+R).\end{aligned}$$

Let y_0 be the greatest $x_k \le a - R$ and y_{A+1} be the smallest $x_k \ge a + R$. We write $s_k := y'_k - b$ and $z_k := a + s_k$ for $k = 1, \ldots, B$. The partition into $A + 1$ disjoint intervals $\cup_{k=1}^{A+1} (y_{k-1}, y_k]$ must contain $A+3$ distinct points $\{z_1, \ldots, z_{A+3}\}$. We have therefore to consider two cases.

Case 1. Either some interval $(y_{k-1}, y_k]$ contains three points (z_{i-1}, z_i, z_{i+1}). By Aubry crossing lemma,

$$E_\omega(y_{k-1}, y_k) + E_\omega(z_{i-1}, z_i) > E_\omega(y_{k-1}, z_i) + E_\omega(z_{i-1}, y_k),$$
$$E_\omega(z_{i-1}, y_k) + E_\omega(z_i, z_{i+1}) \ge E_\omega(z_{i-1}, z_{i+1}) + E_\omega(z_i, y_k).$$

Since L is transversally constant on $\tau[B_R \times \Xi]$, we obtain

$$\begin{aligned} E_\omega(y'_{i-1}, y'_i, y'_{i+1}) + E_\omega(y_{k-1}, y_k) &= E_\omega(z_{i-1}, z_i, z_{i+1}) + E_\omega(y_{k-1}, y_k) \\ &> E_\omega(z_{i-1}, z_{i+1}) + E_\omega(y_{k-1}, z_i, y_k) \\ &= E_\omega(y'_{i-1}, y'_{i+1}) + E_\omega(y_{k-1}, z_i, y_k). \end{aligned}$$

We have obtained a configuration $(\ldots, y'_{i-1}, y'_{i+1}, \ldots, y_{k-1}, z_i, y_k, \ldots)$ with strictly lower energy, which contradicts the fact that $(x_0, \ldots, x_n)$ is minimizing.

Case 2. Or there exist two distinct intervals $(y_{k-1}, y_k]$ and $(y_{l-1}, y_l]$, $k < l$, that contain each two points (z_{i-1}, z_i) and (z_{j-1}, z_j), respectively. Notice that we may have $y_k = y_{l-1}$, but we must have $z_i < z_{j-1}$, $z_{i+1} \in (a - R, a + R)$, and possibly $z_{i+1} = z_{j-1}$. We want to obtain a contradiction by showing that one can decrease the sum of energies $E_\omega(y'_{i-1}, \ldots, y'_j) + E_\omega(y_{k-1}, \ldots, y_l)$ while fixing the four boundary points. By changing by a coboundary as in definition 18, we may assume that $E_\omega(x, y)$ is C^2 in x and y.

We perturb the point z_i slightly by a small quantity ϵ and allow an increase of the energy of order ϵ^2. Since (z_{i-1}, z_i, z_{i+1}) is minimizing, we have

$$E_\omega(z_{i-1}, z_i, z_{i+1}) = E_\omega(z_{i-1}, z_i - \epsilon, z_{i+1}) + o(\epsilon^2).$$

By Aubry crossing lemma,

$$\begin{aligned} &E_\omega(y_{k-1}, y_k) + E_\omega(z_{i-1}, z_i - \epsilon) \\ &\qquad = E_\omega(y_{k-1}, z_i - \epsilon) + E_\omega(z_{i-1}, y_k) - \epsilon(z_{i-1} - y_{k-1})\alpha + o(\epsilon), \end{aligned}$$

with $\alpha = \frac{1}{z_{i-1} - y_{k-1}} \int_{y_{k-1}}^{z_{i-1}} \frac{\partial^2 E_\omega}{\partial x \partial y}(x, y_k)\, dx < 0$. Again by Aubry crossing lemma,

$$E_\omega(y_{l-1}, y_l) + E_\omega(z_{j-1}, z_j) \ge E_\omega(y_{l-1}, z_j) + E_\omega(z_{j-1}, y_l),$$

with equality if $z_j = y_l$. Since L is transversally constant, we obtain

$$\begin{aligned} &E_\omega(y'_{i-1}, \ldots, y'_j) + E_\omega(y_{k-1}, \ldots, y_l) \\ &\qquad = E_\omega(z_{i-1}, \ldots, z_j) + E_\omega(y_{k-1}, \ldots, y_l) \\ &\qquad > E_\omega(z_{i-1}, y_k, \ldots, y_{l-1}, z_j) + E_\omega(y_{k-1}, z_i - \epsilon, z_{i+1}, \ldots, z_{j-1}, y_l) \\ &\qquad = E_\omega(y'_{i-1}, w_k, \ldots, w_{l-1}, y'_j) + E_\omega(y_{k-1}, z_i - \epsilon, z_{i+1}, \ldots, z_{j-1}, y_l) \end{aligned}$$

with $t_k := y_k - a$, $w_k := b + t_k$,...,$t_{l-1} := y_{l-1} - a$, $w_{l-1} := b + t_{l-1}$. We have obtained a configuration $(\ldots, y'_{i-1}, w_k, \ldots, w_{l-1}, y'_j, \ldots, y_{k-1}, z_i - \epsilon, z_{i+1}, \ldots, z_{j-1}, y_l, \ldots)$ with strictly lower energy, which contradicts the fact that $(x_0, \ldots, x_n)$ is minimizing. □

It may happen that $E_\omega(x,x) = \bar{E}$ for some $\omega \in \Omega$ and $x \in \mathbb{R}$. Let $x_n = x$ for every n. Then $(x_n)_{n\in\mathbb{Z}}$ is a calibrated configuration in the environment ω and $\delta_{(\tau_x(\omega),0)}$ is a minimizing measure. If L is transversally constant on a flow box $\tau[B_R \times \Xi]$ such that $\tau_x(\omega) \in \Xi$, then $\delta_{(\omega',0)}$ is a minimizing measure for every $\omega' \in \Xi$. The projected Mather set contains Ξ and theorem 19 is proved. We are thus left to understand the case $\inf_{\omega\in\Omega,\ x\in\mathbb{R}} E_\omega(x,x) > \bar{E}$.

Lemma 50. *Let L be a weakly twist Lagrangian for which*

$$\inf_{\omega\in\Omega,\ x\in\mathbb{R}} E_\omega(x,x) > \bar{E}.$$

For $\omega \in \Omega$ and for every n, let $(x_0^n, \dots, x_n^n)$ be a configuration realizing the minimum $E_\omega(x_0^n, \dots, x_n^n) = \min_{x_0,\dots,x_n\in\mathbb{R}} E_\omega(x_0, \dots, x_n)$. Then $\lim_{n\to+\infty} |x_n^n - x_0^n| = +\infty$.

Proof. The proof is done by contradiction. Let $\omega \in \Omega$ and $R > 0$. Assume there exist infinitely many n's for which every configuration $(x_0^n, \dots, x_n^n)$ realizing the minimum of $E_\omega(x_0, \dots, x_n)$ satisfies $|x_n^n - x_0^n| \le R$. Thanks to lemma 44, we can find distinct indices $\{i_0, \dots, i_r\}$ of $\{0, \dots, n\}$ such that $i_0 = 0$, $i_r = n$, $(x_{i_0}^n, \dots, x_{i_r}^n)$ is monotone (possibly not strictly monotone) and

$$E_\omega(x_0^n, \dots, x_n^n) \ge E_\omega(x_{i_0}^n, \dots, x_{i_r}^n) + \sum_{i\notin\{i_0,\dots,i_r\}} E_\omega(x_i^n, x_i^n).$$

Let $\epsilon > 0$ be chosen so that $E_\omega(x,y) \ge \bar{E} + \epsilon$ for every $|y - x| \le \epsilon$. Thus, if θ_n denotes the number of indices $1 \le k \le r$ such that $|x_{i_k}^n - x_{i_{k-1}}^n| > \epsilon$, it is clear that $\theta_n \le R/\epsilon$. Since

$$n\bar{E} \ge E_\omega(x_0^n, \dots, x_n^n) \ge (n - \theta_n)(\bar{E} + \epsilon) + \theta_n \inf_{x,y\in\mathbb{R}} E_\omega(x,y),$$

we obtain a contradiction by letting $n \to +\infty$. □

We now assume that L is transversally constant. We show in the following proposition that a sequence of configurations $(x_0^n, \dots, x_n^n)$ realizing the minimum of the energy $E_\omega(x_0, \dots, x_n)$ among all configurations of length n admits a weak rotation number in the sense that

$$\liminf_{n\to+\infty} \frac{|x_n^n - x_0^n|}{n} > 0. \tag{26}$$

The existence of a rotation number for an infinite minimizing configuration $(x_k)_{k\in\mathbb{Z}}$ has been established in [13]. The following proposition extends partially this result in two directions: the interaction model is more general; we compute the rotation number of a sequence of configurations of increasing length and not the rotation number of a unique infinite configuration.

Proposition 51. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ be a one-dimensional weakly twist quasicrystal interaction model. Assume that*

$$\inf_{\omega\in\Omega,\ x\in\mathbb{R}} E_\omega(x,x) > \bar{E}.$$

For $\omega \in \Omega$ and for every n, let $(x_0^n, \ldots, x_n^n)$ be a configuration realizing the minimum of the energy among all configurations of length n:

$$E_\omega(x_0^n, \ldots, x_n^n) = \min_{x_0, \ldots, x_n} E_\omega(x_0, \ldots, x_n).$$

Then,

- $\bar{E} = \lim_{n\to+\infty} \frac{1}{n} E_\omega(x_0^n, \ldots, x_n^n) = \sup_{n\geq 1} \frac{1}{n} E_\omega(x_0^n, \ldots, x_n^n)$,
- *for n sufficiently large, $(x_0^n, \ldots, x_n^n)$ is strictly monotone,*
- *there is $R > 0$ (independent of ω) such that* $\sup_{n\geq 1} \sup_{1\leq k\leq n} |x_k^n - x_{k-1}^n| \leq R$,
- $\liminf_{n\to+\infty} \frac{1}{n} |x_n^n - x_0^n| > 0$.

Proof. We shall assume that the flow $(\Omega, \{\tau_t\}_{t\in\mathbb{R}})$ is not periodic.

Step 1. The first item has been proved in proposition 14; the limit can be obtained as a supremum because of superadditivity. Moreover, from lemma 50, $|x_n^n - x_0^n| \to +\infty$. From proposition 48, the configuration $(x_0^n, \ldots, x_n^n)$ must be strictly monotone and have uniformly bounded jumps R. We are left to prove the last item of the proposition.

Step 2. By definition of a quasicrystal, L is transversally constant with respect to some flow box decomposition $\{U_i\}_{i\in I}$ (definition 15). Let $\{F_\alpha\}_{\alpha\in A}$ be a Kakutani-Rohlin tower that is compatible with respect to $\{U_i\}_{i\in I}$ (definition 39) and let $\Sigma = \cup_{\alpha\in A}\Sigma_\alpha$ be its basis. We may assume that $\min_{\alpha\in A} H_\alpha$ is as large as we want and, in particular, larger than R (see the construction (25)). We also assume that n is sufficiently large so that every tower F_α of basis Σ_α is completely cut by the trajectory $\tau_t(\omega)$ for $t \in (\min\{x_0^n, x_n^n\}, \max\{x_0^n, x_n^n\})$. We consider ν the transverse measure to Σ (as defined in lemma 41) and we denote $\nu_\alpha := \nu(\Sigma_\alpha)$.

Step 3. Let $S^n < T^n$ be the two return times to Σ (namely, $\tau_{S^n}(\omega) \in \Sigma$ and $\tau_{T^n}(\omega) \in \Sigma$) that are chosen so that $[S^n, T^n)$ is the smallest interval containing the sequence $(x_k^n)_{k=0}^n$. From the definition of a Kakutani-Rohlin tower, $[S^n, T^n)$ can be written as a disjoint union of intervals of type $I_{\alpha,i} := [t_{\alpha,i}, t_{\alpha,i} + H_\alpha)$, where the list $\{t_{\alpha,i}\}_i$, $i = 1, \ldots, C_\alpha^n$, denotes the successive return times to Σ_α between S^n and T^n. We distinguish two exceptional intervals among this list: the two intervals which contain x_0^n and x_n^n. If $x_0^n < x_n^n$, then $N_{\alpha,i}^n$ denotes the number of points $(x_k^n)_{k=1}^n$ belonging to $I_{\alpha,i}$ and N_α^n denotes the maximum of $N_{\alpha,i}^n$. If $x_n^n < x_0^n$, then $N_{\alpha,i}^n$ and N_α^n are defined similarly by considering in this case $(x_k^n)_{k=0}^{n-1}$. From lemma 49, we obtain $N_\alpha^n - 2 \leq N_{\alpha,i}^n \leq N_\alpha^n$ for every nonexceptional interval $I_{\alpha,i}$. We show that $\sup_{n\geq 1} N_\alpha^n < +\infty$ for every $\alpha \in A$. The proof is done by contradiction.

Let $E_{\alpha,i}^n$ be the energy of the configuration localized in $I_{\alpha,i}$. More precisely, assume first $x_0^n < x_n^n$; index the part of $(x_k^n)_{k=1}^n$ in $I_{\alpha,i}$ by $(x_{k,\alpha,i}^n)_{k=1}^N$ with $N = N_{\alpha,i}^n$; denote by $x_{0,\alpha,i}^n$ the nearest point strictly smaller than $x_{1,\alpha,i}^n$ and define the partial energy $E_{\alpha,i}^n := E_\omega(x_{0,\alpha,i}^n, \ldots, x_{N,\alpha,i}^n)$. If $x_n^n < x_0^n$, the part of $(x_k^n)_{k=0}^{n-1}$ in $I_{\alpha,i}$ is indexed by $(x_{k,\alpha,i}^n)_{k=0}^{N-1}$ with $N = N_{\alpha,i}^n$; denote by $x_{N,\alpha,i}^n$ the nearest point strictly larger than $x_{N-1,\alpha,i}^n$ and define $E_{\alpha,i}^n$ similarly.

Thanks to the hypothesis $\inf_{x\in\mathbb{R}} E_\omega(x, x) > \bar{E}$, one can choose $\epsilon > 0$ such that $E_\omega(x, y) \geq \bar{E} + \epsilon$ as soon as $|y - x| \leq \epsilon$. Let $\bar{H} := \max_{\alpha\in A} H_\alpha$. Then, if $\theta_{\alpha,i}^n$ denotes the number of consecutive points $x_{k,\alpha,i}^n$ in $I_{\alpha,i}$ satisfying $|x_{k,\alpha,i}^n - x_{k-1,\alpha,i}^n| > \epsilon$, obviously $\theta^n_{\alpha,i} \leq \bar H/\epsilon$. Thus, since $n = \sum_{\alpha\in A}\ \sum_{1\leq i\leq C^n_\alpha} N^n_{\alpha,i}$, we have that

$$\begin{aligned} n\bar E \geq E_\omega(x^n_0,\ldots,x^n_n) &= \sum_{\alpha\in A}\ \sum_{1\leq i\leq C^n_\alpha} E^n_{\alpha,i} \\ &\geq \sum_{\alpha\in A}\ \sum_{1\leq i\leq C^n_\alpha} \Big[\theta^n_{\alpha,i} \inf_{x,y\in\mathbb{R}} E_\omega(x,y) + \big(N^n_{\alpha,i} - \theta^n_{\alpha,i}\big)(\bar E+\epsilon)\Big] \\ &= n(\bar E+\epsilon) + \sum_{\alpha\in A}\sum_{1\leq i\leq C^n_\alpha} \theta^n_{\alpha,i}\underline{E} \ \geq\ n(\bar E+\epsilon) + \sum_{\alpha\in A} C^n_\alpha \frac{\bar H}{\epsilon}\underline{E}, \end{aligned}$$

where $\underline{E} := (\inf_{x,y\in\mathbb{R}} E_\omega(x,y) - \bar E - \epsilon) < 0$. Among the intervals $(I_{\alpha,i})_i$, $i = 1,\ldots,C^n_\alpha$, at most two of them are exceptional; the other intervals satisfy $N^n_{\alpha,i} \geq N^n_\alpha - 2$. We thus get $n \geq \sum_{\alpha\in A}(C^n_\alpha - 2)(N^n_\alpha - 2)$. For n sufficiently large, we have

$$\begin{gathered} \frac{C^n_\alpha}{T^n - S^n} \leq (1+\epsilon)\nu_\alpha, \quad \frac{C^n_\alpha - 2}{T^n - S^n} \geq (1-\epsilon)\nu_\alpha \quad \text{and} \\ \frac{1}{n}\sum_{\alpha\in A} C^n_\alpha \leq \frac{(1+\epsilon)\sum_{\alpha\in A}\nu_\alpha}{(1-\epsilon)\sum_{\alpha\in A}\nu_\alpha(N^n_\alpha - 2)}. \end{gathered}$$

If $N^n_\alpha \to +\infty$ for some α and a subsequence $n\to+\infty$, then $\frac{1}{n}\sum_{\alpha\in A} C^n_\alpha \to 0$ and we obtain a contradiction with the previous inequality.

Step 4. For every α, $I_{\alpha,i} \subset [x^n_0, x^n_n]$ except maybe for at most two of them. Then

$$\frac{|x^n_n - x^n_0|}{n} \geq \frac{\sum_{\alpha\in A}(C^n_\alpha - 2)H_\alpha}{\sum_{\alpha\in A} C^n_\alpha N^n_\alpha}.$$

Denote $\bar N_\alpha := \limsup_{n\to+\infty} N^n_\alpha$. From step 3 we know that $\bar N_\alpha < +\infty$. By dividing by $(T^n - S^n)$ and by letting $n\to+\infty$, we obtain

$$\liminf_{n\to+\infty} \frac{|x^n_n - x^n_0|}{n} \geq \frac{\sum_{\alpha\in A}\nu_\alpha H_\alpha}{\sum_{\alpha\in A}\nu_\alpha \bar N_\alpha} = \frac{1}{\sum_{\alpha\in A}\nu_\alpha\bar N_\alpha} > 0.$$

□

Proof of theorem 19. We assume that $(\Omega, \{\tau_t\}_{t\in\mathbb{R}}, L)$ is a one-dimensional weakly twist quasicrystal interaction model. We discuss two cases.

Case 1. Either $\inf_{\omega\in\Omega}\inf_{x\in\mathbb{R}} E_\omega(x,x) = \bar E$. Then $E_{\omega_*}(x_*,x_*) = \bar E$ for some ω_* and x_*. By hypothesis, L is transversally constant with respect to a flow box decomposition $\{U_i = \tau[B_{R_i}\times\Xi_i]\}_{i\in I}$. Let $i\in I$ be such that $\tau_{x_*}(\omega_*) \in U_i$. Let be $|t_i| < R_i$ and $\omega_i \in \Xi_i$ such that $\tau_{x_*}(\omega_*) = \tau_{t_i}(\omega_i)$. Then

$$\bar E = E_{\omega_*}(x_*,x_*) = E_{\omega_i}(t_i,t_i) = E_\omega(t_i,t_i), \quad \forall \omega\in\Xi_i.$$

We have just proved that $\delta_{(\tau_{t_i}(\omega),0)}$ is a minimizing measure for every $\omega\in\Xi_i$. The projected Mather set contains $\tau_{t_i}(\Xi_i)$. By minimality of the flow, we have $\Omega = \tau[B_R\times\Xi_i]$ thanks to item 1 of lemma 37. The projected Mather set thus meets every sufficiently long orbit of the flow.

Case 2. Or $\inf_{\omega\in\Omega}\inf_{x\in\mathbb{R}} E_\omega(x,x) > \bar{E}$. Proposition 51 shows that, if $\omega_* \in \Omega$ has been fixed, if for every $n \geq 1$ a sequence $(x_k^n)_{0\leq k<n}$ of points of $\mathbb{R}$ realizing the minimum $E_{\omega_*}(x_0^n,\ldots,x_n^n) = \min_{x_0,\ldots,x_n} E_{\omega_*}(x_0,\ldots,x_n)$ has been fixed, then
– $\bar{E} = \lim_{n\to+\infty} \frac{1}{n} E_{\omega_*}(x_0^n,\ldots,x_n^n)$,
– $(x_k^n)_{0\leq k<n}$ is strictly monotone for n large enough,
– there is $R > 0$ (independent of ω_*) such that $\sup_{n\geq1}\ \sup_{1\leq k\leq n} |x_k^n - x_{k-1}^n| < 2R$,
– $\rho := \liminf_{n\to+\infty} \frac{1}{n}|x_n^n - x_0^n| > 0$.

Let μ_{n,ω_*} be the probability measure on $\Omega\times\mathbb{R}$ defined by

$$\mu_{n,\omega_*} := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{(\tau_{x_k^n}(\omega_*),\, x_{k+1}^n - x_k^n)}.$$

Notice that $\int L\, d\mu_{n,\omega_*} = \frac{1}{n}E_{\omega_*}(x_0^n,\ldots,x_n^n)$. Since the consecutive jumps of x_k^n are uniformly bounded, the sequence of measures $\{\mu_{n,\omega_*}\}_{n\geq1}$ is tight. By taking a subsequence, we may assume that $\mu_{n,\omega_*} \to \mu_\infty$ with respect to the weak topology. Moreover, μ_∞ is holonomic and minimizing. Let $\Xi \subset \Omega$ be a transverse section of a flow box $\tau[B_R \times \Xi]$. Let $\mathcal{R}_\Xi(\omega_*)$ be the set of return times to Ξ as defined in lemma 41. Let $pr^1 : \Omega\times\mathbb{R} \to \Omega$ be the first projection. Then

$$\begin{aligned} pr^1_*(\mu_{n,\omega_*})(\tau[B_R\times\Xi]) &= \frac{1}{n}\#\big\{k : x_k^n \in \cup_{t\in\mathcal{R}_\Xi(\omega_*)} B_R(t)\big\} \\ &\geq \frac{1}{n}\#(B_{T_n}(c_n)\cap\mathcal{R}_\Xi(\omega_*)), \end{aligned}$$

with $T_n := \frac{1}{2}|x_n^n - x_0^n|$ and $c_n := \frac{1}{2}(x_0^n + x_n^n)$. The previous inequality comes from the fact that the intervals $B_R(t)$ are disjoints and contain at least one x_k^n. Then

$$pr^1_*(\mu_{n,\omega_*})(\tau[B_R\times\Xi]) \geq \frac{2T_n}{n}\ \frac{\#(B_{T_n}(0)\cap\mathcal{R}_\Xi(\tau_{c_n}(\omega_*))}{\mathrm{Leb}(B_{T_n}(0))}.$$

By taking the limit as $n\to+\infty$, one obtains $pr^1_*(\mu_\infty)(\overline{\tau[B_R\times\Xi]}) \geq \rho\nu_\Xi(\Xi) > 0$. Therefore, since Ξ is arbitrary, every orbit of the flow of length $2R$ meets the projected Mather set. $\square$

5 Lax-Oleinik operators

The Lax-Oleinik operator is a tool used in PDE's to solve Hamilton-Jacobi equations. The Frenkel-Kontorova model appears naturally by discretization in time of these equations. The solutions of the Lax-Oleinik operator are called viscosity solutions or weak KAM solutions in the continuous time setting. We will call them here sub-actions.

Definition 52. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model. We call backward Lax-Oleinik operator the (nonlinear) operator acting on the space of Borel measurable functions by*

$$T_-[u](\omega) := \inf_{t\in\mathbb{R}^d}\big[u\circ\tau_{-t}(\omega) + L(\tau_{-t}(\omega),t)\big].$$

Similarly, we call forward Lax-Oleinik operator the operator

$$T_+[u](\omega) := \sup_{t\in\mathbb{R}^d} \big[u\circ\tau_t(\omega) - L(\omega,t)\big].$$

We will see that these Lax-Oleinik operators are less regularizing than the usual operators used in discrete weak KAM theory [14] (or in discrete dynamic programming [15]), when they are defined for a specific choice of an environment. For the usual definition of $T_\pm$, for a particular choice of E, see Appendix A, definition 59. From now on, we denote by $\mathcal{L}^\infty(X)$ the space of bounded Borel measurable functions on a topological space X.

Definition 53. *A measurable function u is called a sub-action (at the level $\bar L = \bar E$) if one of the following conditions is satisfied*

$$\begin{gathered}\forall\,\omega\in\Omega,\ \forall\,t\in\mathbb{R}^d,\quad u\circ\tau_t(\omega)\le u(\omega)+L(\omega,t)-\bar L\\ \iff\ u+\bar L\le T_-[u]\ \iff\ u-\bar L\ge T_+[u].\end{gathered}$$

There are then two possibilities for calibration: a sub-action u is said to be

$$\begin{aligned}&\textit{backward calibrated}\quad \textit{if}\quad T_-[u]=u+\bar L,\\ &\textit{forward calibrated}\quad \textit{if}\quad T_+[u]=u-\bar L.\end{aligned}$$

Continuous calibrated sub-actions do exist in the periodic setting. The main problem we are facing is that bounded measurable sub-actions may not exist in the almost periodic setting. We recall that $\bar L = \bar E$ may be computed using four formulas, given by definition 6, and propositions 10, 13 and 14.

As in definition 59, one may introduce two Lax-Oleinik operators $T_{\omega\pm}$, associated to the interaction E_ω for any $\omega\in\Omega$, each one acting on measurable functions as follows

$$T_{\omega-}[u](y) := \inf_{x\in\mathbb{R}^d}\big[u(x)+E_\omega(x,y)\big], \tag{27}$$

$$T_{\omega+}[u](x) := \sup_{y\in\mathbb{R}^d}\big[u(y)-E_\omega(x,y)\big]. \tag{28}$$

Notice that, if u is a solution of $T_-[u]=u+\bar L$ or $T_+[u]=u-\bar L$, then, for every $\omega\in\Omega$, $u_\omega(x):=u\circ\tau_x(\omega)$ is a solution of $T_{\omega-}[u_\omega]=u_\omega+\bar E$ or $T_{\omega+}[u_\omega]=u_\omega-\bar E$.

The main result in this section is about the existence of a bounded calibrated sub-action provided an obvious obstruction is removed. The following result is similar to Gottschalk-Hedlund theorem. We denote by $C_b^{usc}(\Omega)$ and $C_b^{lsc}(\Omega)$ the spaces of bounded upper semi-continuous and bounded lower semi-continuous functions, respectively.

Theorem 54. *Let $(\Omega,\{\tau_t\}_{t\in\mathbb{R}^d},L)$ be an almost periodic interaction model. Assume that L is C^0 coercive. Then, the following conditions are equivalent:*

1. $\exists\,u\in C_b^{lsc}(\Omega),\quad T_-[u]=u+\bar L,$
2. $\exists\,u\in C_b^{usc}(\Omega),\quad T_+[u]=u-\bar L,$

3. $\forall\,\omega \in \Omega,\ \sup_{n\geq 0} |T_-^n[0](\omega) - n\bar{L}| < +\infty,$

4. $\forall\,\omega \in \Omega,\ \sup_{n\geq 0} |T_+^n[0](\omega) + n\bar{L}| < +\infty,$

5. $\exists\,\omega \in \Omega,\ \exists\,u \in \mathcal{L}^\infty(\mathbb{R}^d), \quad T_{\omega-}[u] = u + \bar{E},$

6. $\exists\,\omega \in \Omega,\ \exists\,u \in \mathcal{L}^\infty(\mathbb{R}^d), \quad T_{\omega+}[u] = u - \bar{E}.$

(As usual, $T_\pm^n$ denotes the n^{th} iterate of $T_\pm$.) Moreover, any bounded measurable solution of $T_{\omega-}[u] = u + \bar{E}$ or $T_{\omega+}[u] = u - \bar{E}$ is actually uniformly continuous.

The backward and forward calibrated solutions are two very different objects obtained by reversing the group action. Define

$$\check{\tau}_t := \tau_{-t}, \quad \rho(\omega, t) = (\tau_{-t}(\omega), t), \quad \text{and} \quad \check{L} := L \circ \rho. \tag{29}$$

The family of interactions associated to $\check{L}$ reads

$$\check{E}_\omega(x, y) := \check{L}(\check{\tau}_x(\omega), y - x) = E_\omega(-y, -x). \tag{30}$$

Notice that coerciveness and superlinearity are preserved by changing L to $\check{L}$. For every probability measure μ, we associate the reversed measure

$$\check{\mu} := \rho_*^{-1}(\mu). \tag{31}$$

Then μ is holonomic for $\{\tau_t\}_t$ if, and only if, $\check{\mu}$ is holonomic for $\{\check{\tau}_t\}_t$, and μ is minimizing for L if, and only if, $\check{\mu}$ is minimizing for $\check{L}$. In particular, L and $\check{L}$ have the same ground energy. For every measurable function u, we associate the reversed function

$$\check{u} := -u, \quad \text{then} \quad T_+[u] = -\check{T}_-[\check{u}]. \tag{32}$$

This duality between T_- and $\check{T}_+$ implies readily

$$u + \bar{L} \leq T_-[u] \iff u - \bar{L} \geq T_+[u] \iff \check{u} + \bar{L} \leq \check{T}_-[\check{u}], \tag{33}$$

$$u - \bar{L} = T_+[u] \iff \check{u} + \bar{L} = \check{T}_-[\check{u}]. \tag{34}$$

The second equivalence means that u is forward calibrated for L if, and only if, $\check{u}$ is backward calibrated for $\check{L}$.

We will use the following regularity along every orbit of the action.

Definition 55. *A function $u \in \mathcal{L}^\infty(\Omega)$ is said to be equicontinuous along the group action if*

$$\lim_{\epsilon\to 0+} \sup_{\omega\in\Omega} \sup_{\|t\|\leq\epsilon} |u \circ \tau_t(\omega) - u(\omega)| = 0.$$

Lemma 56. *Assume that L is C^0 coercive.*

1. *If u is lower semi-continuous and finite everywhere, then $T_-[u] \in \mathcal{L}^\infty(\Omega)$. If u is upper semi-continuous and finite everywhere, then $T_+[u] \in \mathcal{L}^\infty(\Omega)$. If u is a finite everywhere sub-action which is either lower semi-continuous or upper semi-continuous, then $u \in \mathcal{L}^\infty(\Omega)$.*

2. *If $u \in C^0(\Omega)$, then $T_-[u] \in C^0(\Omega)$.*

3. *If $u \in \mathcal{L}^\infty(\Omega)$, then $T_-[u] \in \mathcal{L}^\infty(\Omega)$ and is equicontinuous along the group action. Moreover, the modulus of equicontinuity is uniform over $\|u\|_\infty \leq R$, that is,*

$$\forall R > 0, \quad \lim_{\epsilon \to 0+} \sup_{\|u\|_\infty \leq R} \sup_{\omega \in \Omega} \sup_{\|t\| \leq \epsilon} |T_-[u] \circ \tau_t(\omega) - T_-[u](\omega)| = 0.$$

4. *If $\{u_n\}_{n \geq 0}$ is a nondecreasing sequence of lower semi-continuous functions such that $\sup_{n\geq 0} \|u_n\|_\infty < +\infty$, then*

$$\sup_{n \geq 0} T_-[u_n] = T_-\big[\sup_{n \geq 0} u_n\big].$$

If $\{u_n\}_{n \geq 0}$ is any sequence of measurable functions, then

$$\inf_{n \geq 0} T_-[u_n] = T_-\big[\inf_{n \geq 0} u_n\big].$$

5. *If $u \in C_b^{lsc}(\Omega)$, then $T_-[u] \in C_b^{lsc}(\Omega)$. If $u \in C_b^{usc}(\Omega)$, then $T_-[u] \in C_b^{usc}(\Omega)$.*

Proof. Part 1. Let $F_N := \{\omega \in \Omega : u(\omega) \leq N\}$. As u is lower semi-continuous, F_N is closed; as u is finite everywhere, $\Omega = \cup_{N \in \mathbb{Z}} F_n$. By Baire's theorem, there exists $N(u)$ such that $F_{N(u)}$ has nonempty interior. By minimality, on may find $D > 0$ such that, for every $\omega \in \Omega$, there exists $\|t\| \leq D$ with $\tau_{-t}(\omega) \in F_{N(u)}$. By the definition of the backward Lax-Oleinik operator, $T_-[u](\omega) \leq u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t)$. We obtain the uniform upper bound:

$$\sup_{\omega \in \Omega} T_-[u](\omega) \leq N(u) + \sup_{\omega \in \Omega, \ \|t\| \leq D} L(\tau_{-t}(\omega), t).$$

By the lower semi-continuity of u, we obtain the following uniform lower bound

$$\inf_{\omega \in \Omega} T_-[u](\omega) \geq \inf_{\omega \in \Omega} u(\omega) + \inf_{\omega \in \Omega, \ t \in \mathbb{R}^d} L(\omega, t).$$

We have just proved that $T_-[u]$ is bounded. If u is upper semi-continuous, $\check{u}$ is lower semi-continuous and $T_+[u] = -\check{T}_-[\check{u}]$ is bounded by the previous proof.

If u is a lower semi-continuous and finite everywhere sub-action, then $u \leq T_-[u] - \bar{L}$. As $T_-|u]$ is bounded, u is bounded from above, being bounded from bellow by semi-continuity. Similarly, from upper semi-continuity and $u \geq T_+[u] + \bar{L}$, one obtains that $u \in \mathcal{L}^\infty(\Omega)$.

Part 2. We first notice that, if $u \in \mathcal{L}^\infty(\Omega)$, then an optimal translation $t \in \mathbb{R}^d$ given in the definition of $T_-[u]$ is uniformly bounded from above by a constant $D > 0$, which is obtained from the coerciveness of L:

$$\inf_{\omega \in \Omega, \ \|t\| \geq D} \big[u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t)\big] > \sup_{\omega \in \Omega} \big[u(\omega) + L(\omega, 0)\big].$$

The family of continuous functions $\big\{\omega \in \Omega \mapsto u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t)\big\}_{\|t\| \leq D}$ is equicontinuous and, by the compactness of Ω, the infimum $T_-[u]$ is also continuous.

Part 3. For $R > 0$, choose as in part 2 a constant $D_R > 0$ so that, for every $\|u\|_\infty \leq R$,

$$\forall\, \omega \in \Omega, \quad T_-[u](\omega) = \inf_{\|t\| \leq D_R} \big[u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t) \big].$$

(Notice that we can choose $\|t\| \leq D_R$ uniformly over the set $\{u : \|u\|_\infty \leq R\}$ for every R.) Then, given $\eta > 0$, there exists $\|t\| \leq D_R$ such that, for all $\omega \in \Omega$ and $s \in \mathbb{R}^d$,

$$\begin{aligned} &T_-[u](\tau_s(\omega)) - T_-[u](\omega) \leq \\ &\qquad \leq \big[u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t+s) \big] - \big[u \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t) \big] + \eta \leq \\ &\qquad\qquad \leq L(\tau_{-t}(\omega), t+s) - L(\tau_{-t}(\omega), t) + \eta. \end{aligned}$$

Taking first suprema and letting then $\eta \to 0$, one obtains

$$\sup_{\omega \in \Omega,\ \|s\| \leq \epsilon} \big| T_-[u](\tau_s(\omega)) - T_-[u](\omega) \big| \leq \sup_{\omega \in \Omega,\ \|t\| \leq D_R,\ \|s\| \leq \epsilon} |L(\omega, t+s) - L(\omega, t)|.$$

The right hand side goes to 0 as $\epsilon \to 0$ by the uniform continuity of L on compact sets. We have proved that $\{T_-[u]\}_{\|u\|_\infty \leq R}$ is equicontinuous along the group action.

Part 4. Since the set $\{u_n\}_n$ is uniformly bounded in $\mathcal{L}^\infty(\Omega)$, the infimum on t in the definition of $T_-[u_n]$ can be realized over $\|t\| \leq D_R$, for some $D_R > 0$, uniformly in ω and $n \geq 0$. Define

$$f_n(\omega, t) := u_n \circ \tau_{-t}(\omega) + L(\tau_{-t}(\omega), t).$$

Then $f_n : \Omega \times \{\|t\| \leq D_R\} \to \mathbb{R}$ is lower semi-continuous and nondecreasing in n. The following lemma 57 shows that, for every ω fixed,

$$\sup_{n \geq 0} \inf_{\|t\| \leq D_R} f_n(\omega, t) = \inf_{\|t\| \leq D_R} \sup_{n \geq 0} f_n(\omega, t) \ \Leftrightarrow\ \sup_{n \geq 0} T_-[u_n](\omega) = T_-\Big[\sup_{n \geq 0} u_n \Big](\omega).$$

For any sequence $\{u_n\}_n$, the property $\inf_n T_-[u_n] = T_-[\inf_n u_n]$ is obtained by simply permuting the two infima.

Part 5. Let $u \in C_b^{lsc}(\Omega)$. There exists a nondecreasing sequence of continuous functions u_n such that $\sup_{n \geq 0} u_n = u$. Part 4 implies that $\sup_{n \geq 0} T_-[u_n] = T_-[u]$. Moreover, $T_-[u_n]$ is continuous by part 2, which shows that $T_-[u]$ is lower semi-continuous. Besides, $T_-[u]$ is bounded by part 3. If $u \in C_b^{usc}(\Omega)$, then there exists a nonincreasing sequence of continuous functions u_n such that $u = \inf_n u_n$. One gets by part 4 that $\inf_n T_-[u_n] = T_-[u]$ is upper semi-continuous and by part 3 that $T_-[u]$ is bounded. □

We have used the following basic lemma.

Lemma 57. *Let X be a compact metric space and $u_n : X \to \mathbb{R}$ be a nondecreasing sequence of lower semi-continuous functions. Suppose that $\sup_n u_n(x) < +\infty$ for every $x \in X$. Then $\sup_n \inf_{x \in X} u_n(x) = \inf_{x \in X} \sup_n u_n(x)$.*

Proof. On the one hand, it is clear that

$$\inf_{x\in X}\sup_{n\geq 0} u_n(x) \geq \sup_{n\geq 0}\inf_{x\in X} u_n(x).$$

On the other hand, since u_n is lower semi-continuous, the minimum of every u_n is attained: let $x_n \in X$ be such that $\inf_{x\in X} u_n(x) = u_n(x_n)$. By compactness of X, let x_∞ be an accumulation point of $\{x_n\}_n$. Let $u = \sup_n u_n$, which is finite by assumption. For $\epsilon > 0$, choose N such that $u_N(x_\infty) > u(x_\infty) - \epsilon$. Since u_N is lower semi-continuous, choose a neighborhood U of x_∞ so that $u_N(x) > u(x_\infty) - 2\epsilon$ for every $x \in U$. Since $\{u_n\}_n$ is nondecreasing, we have that, for $n \geq N$ sufficiently large, $x_n \in U$ and $u_n(x_n) \geq u_N(x_n) > u(x_\infty) - 2\epsilon$, from which we obtain $\sup_{n\geq 0}\inf_{x\in X} u_n(x) > u(x_\infty) - 2\epsilon$. Letting $\epsilon \to 0$, we have just proved that $\sup_{n\geq 0}\inf_{x\in X} u_n(x) \geq \inf_{x\in X}\sup_{n\geq 0} u_n(x)$. □

We will also need to recall the notions of lower semi-continuous envelope u_{lsc} and upper semi-continuous envelope u_{usc} of a bounded function u, namely,

$$\forall\,\omega \in \Omega, \quad u_{lsc}(\omega) := \sup\{\phi(\omega) : \phi \leq u \text{ and } \phi \in C^0(\Omega)\}, \tag{35}$$

$$\forall\,\omega \in \Omega, \quad u_{usc}(\omega) := \inf\{\phi(\omega) : u \leq \phi \text{ and } \phi \in C^0(\Omega)\}. \tag{36}$$

We have then a key lemma.

Lemma 58. *Let $u \in \mathcal{L}^\infty(\Omega)$.*

1. *If $v := T_-[u]$, then $v_{lsc} = T_-[u_{lsc}]$ and $v_{usc} \leq T_-[u_{usc}]$.*
2. *If $v := T_+[u]$, then $v_{usc} = T_+[u_{usc}]$ and $v_{lsc} \geq T_+[u_{lsc}]$.*
3. *If $u + \bar L \leq T_-[u]$, then $u_{lsc} + \bar L \leq T_-[u_{lsc}]$ and $u_{usc} + \bar L \leq T_-[u_{usc}]$.*
4. *If $u - \bar L \geq T_+[u]$, then $u_{lsc} - \bar L \geq T_+[u_{lsc}]$ and $u_{usc} - \bar L \geq T_+[u_{usc}]$.*
5. *If $u + \bar L = T_-[u]$, then $u_{lsc} + \bar L = T_-[u_{lsc}]$.*
6. *If $u - \bar L = T_+[u]$, then $u_{usc} - \bar L = T_+[u_{usc}]$.*

Proof. Even items may be derived immediately from respective odd items simply by reversing the group action and using, in particular, relation (32). So we only prove the odd items of the lemma.

Part 1. Let $\phi \in C^0(\Omega)$ be such that $\phi \leq v$. Then, for all ω and t, $\phi(\tau_t(\omega)) \leq u(\omega) + L(\omega,t) - \bar L$. For a fixed t, $\phi(\tau_t(\omega)) - L(\omega,t) + \bar L$ is continuous in ω. By definition of the envelope, $\phi(\tau_t(\omega)) \leq u_{lsc}(\omega) + L(\omega,t) - \bar L$ for all ω and t. By taking the supremum on ϕ, we obtain $v_{lsc}(\tau_t(\omega)) \leq u_{lsc}(\omega) + L(\omega,t) - \bar L$ or $v_{lsc} \leq T_-[u_{lsc}]$. Conversely, $u_{lsc} \leq u$ implies $T_-[u_{lsc}] \leq T_-[u]$. By lemma 56, part 5, $T_-[u_{lsc}]$ is lower semi-continuous. We thus obtain that $T_-[u_{lsc}] \leq v_{lsc}$. Hence, $T_-[u_{lsc}] = v_{lsc}$.

Let $\{\phi_n\}_n \subset C^0(\Omega)$ be a nonincreasing sequence such that $\inf_n \phi_n = u_{usc}$. By lemma 56, part 4, $T_-[u_{usc}] = \inf_n T_-[\phi_n] \geq T_-[u] = v$. By lemma 56, part 5, $T_-[u_{usc}]$ is upper semi-continuous. We have obtained that $T_-[u_{usc}] \geq v_{usc}$.

Part 3. If $u + \bar{L} \leq T_-[u]$, by taking the semi-continuous envelope of both parts of the inequality and by using the first part of this lemma, we obtain $u_{lsc} + \bar{L} \leq T_-[u_{lsc}]$. Moreover, $u + \bar{L} \leq T_-[u_{usc}]$. By lemma 56, part 5, $T_-[u_{usc}]$ is upper semi-continuous. In particular, $u_{usc} + \bar{L} \leq T_-[u_{usc}]$.

Part 5. If $u + \bar{L} = T_-[u]$, then $u_{lsc} + \bar{L} \leq T_-[u_{lsc}]$ by part 3. Let $\{\phi_n\}_n$ be a nondecreasing sequence of continuous functions such that $u_{lsc} = \sup_n \phi_n$. Then $\phi_n \leq u$, $T_-[\phi_n] \leq T_-[u] = u + \bar{L}$, $T_-[\phi_n]$ is continuous, $T_-[\phi_n] \leq u_{lsc} + \bar{L}$, and, by lemma 56, part 4, we obtain $T_-[u_{lsc}] \leq u_{lsc} + \bar{L}$. Thus, $T_-[u_{lsc}] = u_{lsc} + \bar{L}$. $\square$

Proof of theorem 54. It is clear by reversing the direction of the group action as in (29) and (30) that *item 1* $\Leftrightarrow$ *item 2*, *item 3* $\Leftrightarrow$ *item 4*, and *item 5* $\Leftrightarrow$ *item 6*. It is also clear that *item 1* $\Rightarrow$ *item 5* using lemma 56 (item 3) to show that $u_\omega \in C^0_b(\mathbb{R}^d)$.

Part 1. We prove that *item 5* $\Rightarrow$ *item 3*. Notice first that

$$T^n_-[0](\bar{\omega}) = \inf \big\{ E_{\bar{\omega}}(x_{-n}, \ldots, x_{-1}, x_0) : x_0 = 0 \text{ and } x_{-k} \in \mathbb{R}^d \big\}, \quad \forall \bar{\omega} \in \Omega.$$

By assumption, there exist $\omega \in \Omega$ and $u \in \mathcal{L}^\infty(\mathbb{R}^d)$ such that

$$\forall\, y \in \mathbb{R}^d, \quad u(y) = \inf_{x \in \mathbb{R}^d} \big\{ u(x) + E_\omega(x, y) - \bar{E} \big\}.$$

On the one hand, we have that

$$\forall\, t \in \mathbb{R}^d,\ \forall\, x_{-n}, \ldots, x_0 \in \mathbb{R}^d, \quad E_{\tau_t(\omega)}(x_{-n}, \ldots, x_0) \geq u(x_0 + t) - u(x_{-n} + t) + n\bar{E}.$$

Since $\bar{E} = \bar{L}$, by minimality of the interaction model, we obtain thus

$$\inf_{\bar{\omega} \in \Omega} \inf_{n \geq 0} \big[T^n_-[0](\bar{\omega}) - n\bar{L} \big] \geq -2\|u\|_\infty.$$

On the other hand, for all $t \in \mathbb{R}^d$, there are $x^t_{-n}, \ldots, x^t_0 \in \mathbb{R}^d$, with $x^t_0 = 0$, such that

$$E_{\tau_t(\omega)}(x^t_{-n}, \ldots, x^t_0) \leq u(x^t_0 + t) - u(x^t_{-n} + t) + n\bar{E} + \sum_{k=0}^{n-1} \frac{1}{2^k},$$

which yields

$$\forall\, n \geq 1,\ \forall\, t \in \mathbb{R}^d, \quad T^n_-[0](\tau_t(\omega)) - n\bar{L} \leq 2(\|u\|_\infty + 1),$$

and an upper bound also follows from the minimality of the action.

Part 2. We prove that *item 3* $\Rightarrow$ *item 1*. We claim that it is enough to show the existence of $v_0 \in \mathcal{L}^\infty(\Omega)$ such that

$$v_0 + \bar{L} \leq T_-[v_0] \ \text{ and } \ \sup_{n \geq 0} \big\| T^n_-[v_0] - n\bar{L} \big\|_\infty < +\infty. \tag{37}$$

Indeed, we may first assume that $v_0 \in C^{lsc}_b(\Omega)$ since by lemma 58, part 3,

$$(v_0)_{lsc} + \bar{L} \leq T_-[(v_0)_{lsc}],$$
$$-\|v_0\|_\infty \leq (v_0)_{lsc} \leq T^n_-[(v_0)_{lsc}] - n\bar{L} \leq T^n_-[v_0] - n\bar{L} \leq \|T^n_-[v_0] - n\bar{L}\|_\infty.$$

From now on, suppose that v_0 is lower semi-continuous and bounded. Let $v_n := T_-^n[v_0] - n\bar{L}$. Then v_n is lower semi-continuous by lemma 56, part 5, $v_{n+1} \geq v_n$ by the sub-action property, $\sup_n \|v_n\|_\infty < +\infty$ by the claim, and $T_-[v_n] = v_{n+1} + \bar{L}$ by construction. By lemma 56, part 4, if $v = \sup_n v_n$, then

$$v \in C_b^{lsc}(\Omega) \quad \text{and} \quad T_-[v] = T_-[\lim_{n\to+\infty} v_n] = \lim_{n\to+\infty} T_-[v_n] = v + \bar{L}.$$

It remains just to prove the existence of $v_0 \in \mathcal{L}^\infty(\Omega)$ verifying (37). Define then $v_0 := \inf_{k\geq 0}[T_-^k[0] - k\bar{L}]$. Notice that v_0 is finite everywhere by assumption and satisfies $v_0 + \bar{L} \leq T_-[v_0]$ by the following inequalities

$$v_0(\omega) = \inf_{n\geq 0} \inf_{x_{-n},\ldots,x_{-1},x_0=0} \big[E_\omega(x_{-n},\ldots,x_0) - n\bar{L}\big],$$
$$\forall\, \omega \in \Omega,\ \forall\, t \in \mathbb{R}^d, \quad v_0(\tau_t(\omega)) \leq v_0(\omega) + E_\omega(0,t) - \bar{L}.$$

Moreover, v_0 is upper semi-continuous and, by lemma 56, part 1, v_0 is bounded. Notice that

$$v_n := T_-^n[v_0] - n\bar{L} = \inf_{k\geq n}[T_-^k[0] - k\bar{L}].$$

is a nondecreasing sequence. Define

$$u_n := \sup_{k\geq n}[T_-^k[0] - k\bar{L}].$$

Then u_0 is finite everywhere by hypothesis, and lower semi-continuous. By lemma 56, part 1, $T_-[u_0]$ is bounded. Since

$$T_-[u_0] - \bar{L} \geq u_1 \geq u_n \geq v_n \geq v_1 \geq v_0,$$

we finally obtain that $\sup_n \|v_n\|_\infty = \sup_n \|T_-^n[v_0] - n\bar{L}\|_\infty < +\infty$.

□

Appendices

A Minimizing configurations for general interaction

The existence of a semi-infinite minimizing configuration without asking it to be calibrated at the level $\bar{E}$ is easier to guarantee and requires few hypothesis. We consider, in the first part of this appendix, a unique interaction energy $E(x, y)$ that will be supposed to be superlinear (7), translation bounded (5) and translation uniformly continuous (6). By adapting a point of view proposed by Zavidovique [25, Appendix], we will show that there always exists a semi-infinite minimizing configuration $\{x_n\}_{n=-\infty}^0$ with bounded jumps. The configuration will actually be calibrated at some level $\bar{c}$, which has no reason to be equal to $\bar{E}$. We consider, in the second part of this appendix, an almost periodic interaction model and show the existence of a bi-infinite calibrated configuration for some $E_{\bar{\omega}}$. We do not describe the set of such environments $\bar{\omega}$.

The main problem for a general interaction energy is to obtain an *a priori* bound on the jumps $\|x_{n+1} - x_n\|$ of any finite minimizing configuration. The main tool is to construct a discrete weak KAM solution (or a calibrated sub-action as in [14]). We will say that $u : \mathbb{R}^d \to \mathbb{R}$ is *Lipschitz in the large* if

$$\sup_{x,y\in\mathbb{R}^d} \frac{|u(y) - u(x)|}{\|y - x\| + 1} < +\infty. \tag{38}$$

Definition 59. *We call backward Lax-Oleinik operator the (nonlinear) operator T_- acting on continuous functions $u : \mathbb{R}^d \to \mathbb{R}$ by*

$$\forall\, y \in \mathbb{R}^d, \quad T_-[u](y) := \inf\{u(x) + E(x, y) : x \in \mathbb{R}^d\}.$$

We say that u is a calibrated sub-action for E at the level $c \in \mathbb{R}$ if $T_-[u] = u + c$.

For translation periodic interaction energy E, it was shown in [14] that the interaction energy $E_\lambda(x, y) = E(x, y) - \langle\lambda, y - x\rangle$ admits a periodic calibrated sub-action u_λ at the level $\bar{E}(\lambda)$. Notice then that $u(x) := u_\lambda(x) + \langle\lambda, x\rangle$ becomes calibrated for $E = E_0$ at the level $\bar{E}(\lambda)$. It was also shown there that $\lambda \mapsto -\bar{E}(\lambda)$ is convex and superlinear. These two simple observations implies that the equation $T_-[u] = u + c$ admits a solution Lipschitz in the large for all values c in $(-\infty, \sup_\lambda \bar{E}(\lambda)]$.

For general interaction energies as discussed in this appendix, we do not have an *a priori* growth on calibrated sub-actions. An important observation in [25] is that translation boundedness implies Lipschitz in the large and superlinearity implies sublinearity and compactness. Let

$$\bar{c} := \sup_{u\in C^0(\mathbb{R}^d)} \inf_{x,y\in\mathbb{R}^d} [E(x, y) + u(x) - u(y)]. \tag{39}$$

Proposition 60. *Let $E : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a C^0 superlinear, translation bounded and translation uniformly continuous interaction energy. Then there exists a uniformly continuous function $\bar{u} : \mathbb{R}^d \to \mathbb{R}$ which solves the Lax-Oleinik equation $T_-[\bar{u}] = \bar{u} + \bar{c}$. In particular, there exists a backward calibrated configuration $\{x_{-k}\}_{k=0}^{+\infty}$ at the level $\bar{c}$ with uniformly bounded jumps $\sup_{k\geq 1} \|x_{-k+1} - x_{-k}\| < +\infty$.*

Proposition 61. *Let $(\Omega, \{\tau_t\}_{t\in\mathbb{R}^d}, L)$ be an almost periodic interaction model. Suppose L is superlinear. Then*

$$\bar{c} := \sup_{u\in C^0(\mathbb{R}^d)} \inf_{x,y\in\mathbb{R}^d} [E_\omega(x, y) + u(x) - u(y)]$$

is independent of ω and, for a certain $\bar{\omega} \in \Omega$, there exists a (bi-infinite) calibrated configuration for $E_{\bar{\omega}}$ at the level $\bar{c}$.

As we noticed above, the constant $\bar{c}$ may not be equal to $\bar{E}$ if we do not assume any growth at infinity on u. It is not clear that calibrated configurations exist for any environment ω.

The first two lemmas exhibit *a priori* compactness for the Lax-Oleinik operator. Let

$$c_0 := \inf_{x,y\in\mathbb{R}^d} E(x, y) \quad \text{and} \quad K_0 := \sup_{\|y-x\|\leq 1} E(x, y) - c_0. \tag{40}$$

Notice that $c_0 \leq \bar{c} \leq \sup_x E(x, x)$ and that $K_0 < +\infty$ thanks to the translation boundedness. Then, we have the following lemma.

Lemma 62. *Let $c_0 \le c \le \bar{c}$ and $u \in C^0(\mathbb{R}^d)$ be such that $u(y) - u(x) \le E(x,y) - c$ for every $x, y \in \mathbb{R}^d$. Then u is Lipschitz in the large with constant K_0,*

$$\forall\, x, y \in \mathbb{R}^d, \quad |u(y) - u(x)| \le K_0\big(\|y - x\| + 1\big).$$

Proof. Let $n \ge 1$ be the unique integer satisfying $n - 1 < \|y - x\| \le n$. Define $x_k := x + \frac{k}{n}(y - x)$, for $k = 0, \ldots, n$. Then

$$u(x_{k+1}) - u(x_k) \le E(x_k, x_{k+1}) - c, \quad |u(x_{k+1}) - u(x_k)| \le K_0,$$
$$|u(y) - u(x)| \le nK_0 \le K_0\big(\|y - x\| + 1\big). \qquad \square$$

Notice that T_- is a monotone operator, $u \le v \Rightarrow T_-[u] \le T_-[v]$, commutes with the constants, $T_-[u + \lambda] = u + \lambda$, $\forall\, \lambda \in \mathbb{R}$, and is concave, $T_-[\lambda u + (1 - \lambda)v] \ge \lambda T_-[u] + (1 - \lambda)T_-[v]$, $\forall\, \lambda \in [0, 1]$. Notice also that $u + c \le T_-[u]$ is equivalent to $u(y) - u(x) \le E(x, y) - c$, $\forall\, x, y \in \mathbb{R}^d$. Define the semi-norm

$$\|u\|_{Lip} := \sup_{0 < \|y - x\| \le R_0} \frac{|u(y) - u(x)|}{\|y - x\| + \sup\limits_{\|x-z\| \vee \|y-z\| \le 2R_0} |E(z, y) - E(z, x)|}, \tag{41}$$

where $R_0 > 0$ is a constant chosen *a priori* and given explicitly by the formula

$$R_0 := \frac{1}{K_0}\Big(K_0 + B_0 + \sup_{x \in \mathbb{R}^d} E(x, x)\Big), \tag{42}$$

with $B_0 > 0$ defined by the superlinearity:

$$\forall\, x, y \in \mathbb{R}^d, \quad E(x, y) \ge 2K_0\|y - x\| - B_0. \tag{43}$$

We equip $C^0(\mathbb{R}^d)$ with the topology of the uniform convergence on any compact sets. Then $C^0(\mathbb{R}^d)$ becomes a Frechet space. Let

$$\mathcal{H}_c := \big\{u \in C^0(\mathbb{R}^d) : u(0) = 0,\ u + c \le T_-[u] \text{ and } \|u\|_{Lip} \le 1\big\}. \tag{44}$$

Define $\tilde{T}_-[u] := T_-[u] - T_-[u](0)$. Notice that the case $c_0 = \bar{c}$ occurs if, and only if, $u \equiv 0$ satisfies the inequality $u + \bar{c} \le T_-[u]$. For the general situation, we point out the following lemma.

Lemma 63. *For every $c_0 < c < \bar{c}$, $\mathcal{H}_c$ is a nonempty compact convex set of $C^0(\mathbb{R}^d)$, $\tilde{T}_-[\mathcal{H}_c] \subseteq \mathcal{H}_c$, and $\tilde{T}_-$ is a continuous map restricted to $\mathcal{H}_c$.*

Proof. Define

$$\tilde{\mathcal{H}}_c := \big\{u \in C^0(\mathbb{R}^d) : u(0) = 0 \text{ and } u + c \le T_-[u]\big\}.$$

Because of the monotonicity and concavity of T_-, $\tilde{\mathcal{H}}_c$ is a closed convex subset of $C^0(\mathbb{R}^d)$ invariant by $\tilde{T}_-$. By the choice of c, $\tilde{\mathcal{H}}_c$ is nonempty. By Ascoli theorem, $\mathcal{H}_c$ is compact in $C^0(\mathbb{R}^d)$. We prove that $\tilde{T}_-[\tilde{\mathcal{H}}_c] \subseteq \mathcal{H}_c$ and that $\tilde{T}_- : \tilde{\mathcal{H}}_c \to C^0(\mathbb{R}^d)$ is continuous.

We first prove that $\|T_-[u]\|_{Lip} \le 1$ for every $u \in \tilde{\mathcal{H}}_c$. We claim that an optimal point x_{opt} in the definition of $T_-[u](x)$ is at a uniform distance from x. Indeed, notice that we have $T_-[u](x) = u(x_{opt}) + E(x_{opt}, x) \le u(x) + E(x, x)$, and then

$$\begin{aligned} 2K_0\|x - x_{opt}\| - B_0 \le E(x_{opt}, x) &\le u(x) - u(x_{opt}) + E(x, x) \\ &\le K_0(\|x - x_{opt}\| + 1) + E(x, x), \end{aligned}$$

from which it follows that

$$\|x - x_{opt}\| \le R_0.$$

We show now that $\|T_-[u]\|_{Lip} \le 1$. For $\|y - x\| \le R_0$, we obtain that

$$\begin{aligned} T_-[u](x) &= u(x_{opt}) + E(x_{opt}, x), \\ T_-[u](y) &\le u(x_{opt}) + E(x_{opt}, y), \\ T_-[u](y) - T_-[u](x) &\le \sup_{\|x-z\| \vee \|y-z\| \le 2R_0} |E(z, y) - E(z, x)|, \\ \|T_-[u]\|_{Lip} &\le 1. \end{aligned}$$

We next show the T_- restricted to $\tilde{\mathcal{H}}_c$ is continuous. For $u, v \in \tilde{\mathcal{H}}_c$ and $R > 0$, notice that

$$\begin{aligned} T_-[u](x) &= u(x_{opt}) + E(x_{opt}, x), \\ T_-[v](x) &\le v(x_{opt}) + E(x_{opt}, x), \\ \sup_{\|x\| \le R} |T_-[v](x) - T_-[u](x)| &\le \sup_{\|x\| \le R + K_0} |v(x) - u(x)|. \end{aligned}$$

Then T_- and therefore $\tilde{T}_-$ are continuous for the topology of the uniform convergence on compact sets. □

Proof of proposition 60. The set $\mathcal{H}_{\bar{c}} = \cap_{c_0 < c < \bar{c}} \mathcal{H}_c$ is a nonempty compact convex subset of the Hausdorff topological vector space $C^0(\mathbb{R}^d)$ and $\tilde{T}_- : \mathcal{H}_{\bar{c}} \to \mathcal{H}_{\bar{c}}$ is a continuous map. By Schauder theorem (see [5] for a recent reference), $\tilde{T}_-$ admits a fixed point $\bar{u} \in \mathcal{H}_{\bar{c}}$. Let $\underline{c} := \tilde{T}_-[\bar{u}](0)$, then $T_-[\bar{u}] = \bar{u} + \underline{c}$. Since $\bar{u} \in \mathcal{H}_{\bar{c}}$, we have, on the one hand, $\bar{u} + \bar{c} \le T_-[\bar{u}] = \bar{u} + \underline{c}$ and therefore $\bar{c} \le \underline{c}$. On the other hand,

$$\bar{c} \ge \inf_{x,y} \big[E(x, y) + \bar{u}(x) - \bar{u}(y) \big] = \inf_{y} \big[T_-[\bar{u}](y) - \bar{u}(y) \big] = \underline{c}.$$

We have just shown that there exists $\bar{u} \in C^0(\mathbb{R}^d)$, uniformly Lipschitz in the large, with $\|\bar{u}\|_{Lip} \le 1$, such that $T_-[\bar{u}] = \bar{u} + \bar{c}$, where $\bar{c}$ is given by (39). We construct by induction a backward calibrated configuration using the identity

$$\forall\, k \ge 1, \quad \bar{u}(x_{-k+1}) = \bar{u}(x_{-k}) + E(x_{-k}, x_{-k+1}) - \bar{c}.$$ □

Proof of proposition 61. Let

$$\bar{c}(\omega) := \sup_{u \in C^0(\mathbb{R}^d)} \inf_{x,y \in \mathbb{R}^d} [E_\omega(x, y) + u(x) - u(y)].$$

The conclusion of the proof of proposition 60 asserts that the supremum in $\bar c(\omega)$ can be realized on a smaller space which may be defined independently of ω. Let

$$C^0_{Lip}(\mathbb{R}^d) := \Big\{ u \in C^0(\mathbb{R}^d) \ : u(0) = 0, \ \|u\|_{Lip} \leq 1 \Big\},$$

where the new semi-norm $\|u\|_{Lip}$ is given by

$$\|u\|_{Lip} := \sup_{\|y-x\| \geq \bar R} \frac{|u(y)-u(x)|}{2\bar K \|y-x\|} \bigvee \\ \bigvee \sup_{0<\|y-x\|\leq \bar R} \ \inf_{\substack{\|x-z\|\leq 2\bar R \\ \|y-z\|\leq 2\bar R}} \frac{|u(y)-u(x)|}{\|y-x\| + \sup_{\omega\in\Omega} |E_\omega(z,y) - E_\omega(z,x)|},$$

with $\bar K$, $\bar R$ given as in (40), (42) and (43):

$$\begin{gathered} \bar K := \sup_{\omega\in\Omega, \ \|y-x\|\leq 1} E_\omega(x,y) - \inf_{\omega\in\Omega, \ x,y\in\mathbb{R}^d} E_\omega(x,y), \\ \bar R := \frac{1}{\bar K}\Big(\bar K + \bar B + \sup_{\omega\in\Omega, \ x\in\mathbb{R}^d} E_\omega(x,x)\Big), \\ \forall\, x,y \in \mathbb{R}^d, \quad \inf_{\omega\in\Omega} E_\omega(x,y) \geq 2\bar K\|y-x\| - \bar B. \end{gathered}$$

Then

$$\bar c(\omega) := \max_{u\in C^0_{Lip}(\mathbb{R}^d)} \ \inf_{x,y\in\mathbb{R}^d} [E_\omega(x,y) + u(x) - u(y)].$$

For every $u \in C^0_{Lip}(\mathbb{R}^d)$, the infimum is a continuous function of ω thanks to the uniform superlinearity of E_ω. In particular, $\omega \mapsto \bar c(\omega)$ is lower semi-continuous. By the topological stationarity (10) of E_ω, $\omega \mapsto \bar c(\omega)$ is constant along any orbit $\{\tau_t(\omega)\}_{t\in\mathbb{R}^d}$. The set $\{\omega : \bar c(\omega) \leq \inf \bar c\}$ is closed, nonempty, and invariant. By minimality, $\bar c$ is a constant function.

We now prove the existence of a calibrated configuration at the level $\bar c$. Let $\omega\in\Omega$ be fixed. By proposition 60, there exists $u_\omega \in C^0_{Lip}(\mathbb{R}^d)$ such that

$$\forall\, y \in \mathbb{R}^d, \quad u_\omega(y) = \min_{x\in\mathbb{R}^d, \ \|y-x\|\leq \bar R} \big[u_\omega(x) + E_\omega(x,y) - \bar c \big].$$

Let $n \geq 1$. We construct by induction a backward configuration $\{x_{-k}\}_{k=0}^{k=2n}$ starting at $x_0 = 0$ and satisfying

$$\forall\, 1 \leq k \leq 2n, \quad u_\omega(x_{-k+1}) = u_\omega(x_{-k}) + E_\omega(x_{-k}, x_{-k+1}) - \bar c.$$

By shifting by the same amount the environment $\omega_n = \tau_{x_{-n}}(\omega)$ and the configuration $x^n_k := x_{k-n} - x_{-n}$, we obtain a finite configuration $\{x^n_k\}_{k=-n}^n$ centered at the origin $x^n_0 = 0$ and calibrated for E_{ω_n} at the level $\bar c$. Thanks to the fact that the successive jumps are uniformly bounded, by a diagonal extraction procedure, one can find a subsequence of integers $\{n'\}$, $\bar\omega \in \Omega$, and a bi-infinite configuration $\{\bar x_k\}_{k=-\infty}^{+\infty}$ so that $\omega_n \to \bar\omega$ and $x^n_k \to \bar x_k$ for every $k \in \mathbb{Z}$ along the subsequence $\{n'\}$. Since the calibration property passes to the limit, $\{\bar x_k\}_{k=-\infty}^{+\infty}$ is a calibrated configuration for $E_{\bar\omega}$ at the level $\bar c$. $\square$

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