Automorphisms of low complexity subshifts

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Basic topological notions: Topological dynamical system

Throughout X will be a compact metric space.

$\mathrm{Homeo}(X)$: the group of self homeomorphisms of X.

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(Y, S) is a (topological) factor of (X, T), or (X, T) is an extension of (Y, S), if there exists a continuous surjective $\phi \colon X \to Y$ such that

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Definition

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- Q: How does $\mathrm{Aut}(X, T)$ acts on X? On T-invariant measures?

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A word u is an element of the free monoid A^* generated by A.

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The open sets are unions of cylinders:

$$[u.v] := \{(x_n)_n \in A^{\mathbb{Z}} : x_{-|u|} \dots x_{|v|-1} = uv\}; \qquad u, v \in A^*$$

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For a closed set $X \subset A^{\mathbb{Z}}$, shift invariant ($\sigma(X) = X$), a subshift is the dynamical system $(X, \sigma|_X)$.

Similarly

$$X = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{F} \ \forall m, i\}, \text{ where } \mathcal{F} \subset A^*.$$

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The language

$$\mathcal{L}(X) := \{u \in A^* : u = x_0 \cdots x_{|u|-1} \text{ for some } (x_n)_n \in X\}.$$

$$\mathcal{L}_n(X) := \mathcal{L}(X) \cap A^n.$$

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The system (X, σ) is expansive: $\exists \epsilon > 0$, $x \neq y \in X$,

$$\sup_{n \in \mathbb{Z}} \operatorname{dist}(\sigma^n(x), \sigma^n(y)) > \epsilon.$$

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- The centralizer of $S \subset G$: $C_G(S) := \{g \in G : gs = sg \ \forall s \in S\}$.

- $$C_{\mathrm{Homeo}(X)}(T) = \mathrm{Aut}(X, T).$$

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Giordano-Putnam-Skau (1999): If (X, σ) is minimal (without proper subshift)

$$\mathrm{Out}([[\sigma]]) \simeq \{\phi \in \mathrm{Homeo}(X) : \phi \circ \sigma = \sigma^{\pm} \circ \phi\}/\langle \sigma \rangle.$$

$$\{\phi \in \mathrm{Homeo}(X) : \phi \circ \sigma = \sigma^{\pm} \circ \phi\}_{/\mathrm{Aut}(X,\sigma)} \subset \mathbb{Z}/2\mathbb{Z}.$$

Theorem (Curtis-Hedlund-Lyndon)

An automorphism ϕ *of* (X, σ) *is a sliding block code*,
i.e. *there exists a* block map $\hat{\phi}\colon \mathcal{L}_{2r+1}(X) \to A$ *s.t.*

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$$x = \ldots 01001 \quad 1.10 \quad 10101000111 \ldots$$

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Theorem (Curtis-Hedlund-Lyndon)

An automorphism ϕ of (X, σ) is a sliding block code,
i.e. *there exists a* block map $\hat{\phi}$: $\mathcal{L}_{2r+1}(X) \to A$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \cdots x_{n+r}) \text{ for any } n \in \mathbb{Z}.$$

e.g. $A = \{0, 1\}$, $\hat{\phi}$:

$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

$$x = \quad \ldots 010011.10101010000111 \ldots$$

$$\phi(x) = \quad \ldots 0100111.0101010000111 \ldots = \sigma(x)$$

Theorem (Curtis-Hedlund-Lyndon)

Let ϕ be an automorphism of (X, σ)
There exists a local map $\hat{\phi}\colon \mathcal{L}_{2r+1}(X) \to A$ *s.t.*

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Corollary

$Aut(X, \sigma)$ is countable.
$Aut(X, \sigma)$ is a discrete subgroup of Homeo(X) for the uniform convergence topology.

Complexity restrictions

The complexity $p_X : \mathbb{N} \to \mathbb{N}$,

$$p_X(n) = \#\mathcal{L}_n(X) = \# \text{ words of length } n \text{ in } X.$$

Q: How the growth of the complexity restricts $\mathrm{Aut}(X, \sigma)$?

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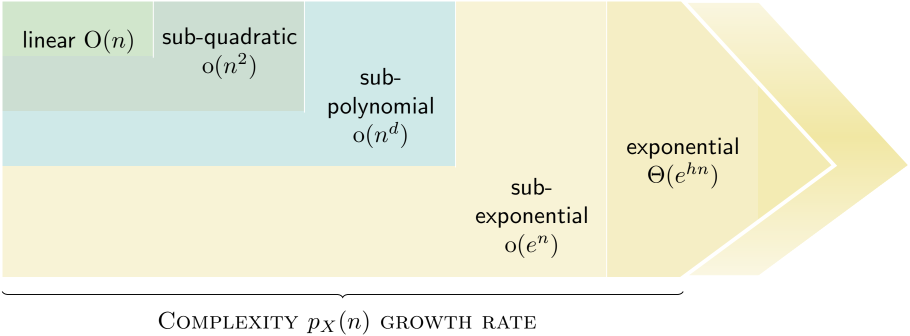

Plan

1. Automorphism of SFT
2. Automorphism of classical minimal systems
   a) Linear complexity case
   b) Toeplitz subshifts case
3. Automorphism for sub-exponential complexity subshifts and restrictions on automorphisms groups.

Automorphism of SFT

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In this case:

$\mathrm{Aut}(X, \sigma)$ is not finitely generated, not amenable.

Embedding free products

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Study the action of $\langle \phi_1, \phi_2, \phi_3 \rangle$ on the the point $\cdots 000 * 000 \cdots$.
See it generates a group isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.
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Basic algebra shows it contains the free group on 2 generators, hence the free group with countably many generators.

Automorphism of SFT

A group G is residually finite if for any $g_1 \neq g_2 \in G$ there is a homomorphism $\pi\colon G \to G_0$ onto a finite group G_0 such that $\pi(g_1) \neq \pi(g_2)$.

Ex: finite group, $\mathbb{Z}^d$, free group, finitely generated linear group,...

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Since $\bigcup_n \mathrm{Per}_n$ is dense in X,

$$\pi_n(\phi_1) = \pi_n(\phi_2),\ \forall n \Rightarrow \phi_1 = \phi_2.$$

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For an irreducible SFT, the group $\mathrm{Aut}(X, \sigma)$ *is residually finite.*

Corollary

For an irreducible SFT, $\mathrm{Aut}(X, \sigma)$ *does not contains a divisible subgroup: For any* $\phi \in \mathrm{Aut}(X, \sigma) \setminus \{\mathrm{Id}\}$*, there exists* $n \in \mathbb{N}$ *s.t. the equation*

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has no solution $\psi \in \mathrm{Aut}(X, \sigma)$*.*

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Ex: $\mathrm{Aut}(X, \sigma)$ does not contains $\mathbb{Q}$.
Open problem: is $\mathbb{Z}[1/p]$ contained in $\mathrm{Aut}(X, \sigma)$ for any prime p?

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Theorem (BLR)

For an SFT, the group $\mathrm{Aut}(X, \sigma)$ *contains no finitely generated group with unsolvable word problem.*

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For an SFT, the group $\mathrm{Aut}(X,\sigma)$ contains no finitely generated group with unsolvable word problem.

Proof. Given $\phi_1, \ldots, \phi_\ell \in \mathrm{Aut}(X,\sigma)$, find a finite procedure to decide if

$$\psi = \phi_{i_1}^{\pm} \circ \cdots \circ \phi_{i_r}^{\pm} = \mathrm{Id}, \qquad i_1, \ldots, i_r \in \{1, \ldots, \ell\}.$$

By Curtys-Hedlund-Lyndon Theorem, it is enough to check if the block map of ψ with range $r_\psi = \mathrm{O}(r)$ satisfies

$$\hat{\psi}(x_{-r_\psi} \cdots x_{r_\psi}) = x_0.$$

Automorphism of $\mathbb{Z}^d$- SFT, $d \geq 2$

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Let (X, σ) be a $\mathbb{Z}^d$-SFT with $h(X, \sigma) > 0$ then

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- If moreover, minimal orbits are dense (e.g. periodic orbits), then $\mathrm{Aut}(X, \sigma)$ contains a copy of $\mathrm{Aut}(\{1, \ldots, n\}^{\mathbb{Z}}, \sigma)$ $\forall n$.

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Open problem: $\mathrm{Aut}(\{0,1\}^{\mathbb{Z}^d}, \sigma) < \mathrm{Aut}(\{0,1\}^{\mathbb{Z}^m}, \sigma)$?

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