



EXTRIANGULATED CATEGORIES, HOVEY TWIN COTORSION PAIRS AND MODEL STRUCTURES.

Hiroyuki Nakaoka and Yann Palu

Résumé. Nous proposons une généralisation simultanée de la notion de catégorie exacte et de celle de catégorie triangulée, qui est bien adaptée à l'étude des paires de cotorsion, et que nous appelons catégorie extriangulée. Les sous-catégories pleines, stables par extension, d'une catégorie triangulée sont également des exemples de catégories extriangulées. Nous montrons une correspondance bijective entre certaines paires de cotorsion, que nous appelons paires de cotorsion jumelles de Hovey, et structures de modèle admissibles, généralisant ainsi la correspondance de Hovey. En passant à la catégorie homotopique, cette approche permet de relier certaines localisations à des quotients par des idéaux. Bien que ces structures de modèle ne soient généralement pas stables, nous montrons que la catégorie homotopique est triangulée. Nous donnons ainsi un cadre naturel pour formuler les notions de réduction et de mutation de paires de cotorsion, s'appliquant à la fois aux catégories exactes et aux catégories triangulées. Ces résultats peuvent être vus comme des arguments corroborant l'hypothèse que les catégories extriangulées sont une axiomatisation agréable pour écrire des démonstrations valables pour les catégories exactes et pour les catégories (stables par extension dans une catégorie) triangulée.

Abstract. We give a simultaneous generalization of exact categories and triangulated categories, which is suitable for considering cotorsion pairs, and which we call extriangulated categories. Extension-closed, full subcategories of triangulated categories are examples of extriangulated categories. We give a bijective correspondence between some pairs of cotorsion pairs which we call Hovey twin cotorsion pairs, and admissible model structures. As a consequence, these model structures relate certain localizations with certain ideal quotients, via the homotopy category which can be given a triangulated structure. This gives a natural framework to formulate reduction and mutation of cotorsion pairs, applicable to both exact categories and triangulated categories. These results can be thought of as arguments towards the view that extriangulated categories are a convenient setup for writing down proofs which apply to both exact categories and (extension-closed subcategories of) triangulated categories.

Keywords. Extriangulated categories, Quillen exact categories, triangulated categories, cotorsion pairs, model structures, localisation, mutation.

Mathematics Subject Classification (2010). 18E10, 18E30, 18G55, 18E35, 18G15, 13F60.

Acknowledgements. Both authors would like to thank Professor Osamu Iyama and Professor Ivo Dell’Ambrogio for inspiring comments on a previous version of the paper. The first author wishes to thank Professor Frederik Marks and Professor Jorge Vitória for their interests, and for giving him several opportunities. ^{1 2}

Contents

1	Introduction and Preliminaries	3
2	Extriangulated category	5
2.1	$\mathbb{E}$ -extensions	6
2.2	Realization of $\mathbb{E}$ -extensions	7
2.3	Definition of extriangulated categories	9

¹The first author is supported by JSPS KAKENHI Grant Numbers JP25800022, JP17K18727.

²The second author is supported by the ANR grant SC³A 15CE40 0004 01.

2.4	Terminology in an extriangulated category	12
3	Fundamental properties	14
3.1	Associated exact sequence	14
3.2	Shifted octahedrons	20
3.3	Relation with triangulated categories	27
3.4	Projectives and injectives	29
4	Cotorsion pairs	34
4.1	Cotorsion pairs	34
4.2	Associated adjoint functors	35
4.3	Concentric twin cotorsion pairs	36
5	Bijjective correspondence with model structures	39
5.1	Hovey twin cotorsion pair	39
5.2	From admissible model structure to Hovey twin cotorsion pair	40
5.3	From Hovey twin cotorsion pair to admissible model structure	42
6	Triangulation of the homotopy category	54
6.1	Shift functor	54
6.2	Connecting morphism	57
6.3	Triangulation	62
7	Reduction and mutation via localization	68
7.1	Happel and Iyama-Yoshino's construction	68
7.2	Mutable cotorsion pairs	69

1. Introduction and Preliminaries

Cotorsion pairs, first introduced in [Sal], are defined on an exact category or a triangulated category, and are related to several homological structures, such as: t -structures [BBD], cluster tilting subcategories [KR, KZ], co- t -structures [Pau], functorially finite rigid subcategories. A careful look reveals that what is necessary to define a cotorsion pair on a category is the existence of an Ext^1 bifunctor with appropriate properties. In this article, we formalize the notion of an *extriangulated category* by extracting those

properties of Ext^1 on exact categories and on triangulated categories that seem relevant from the point-of-view of cotorsion pairs.

The class of extriangulated categories not only contains exact categories and extension-closed subcategories of triangulated categories as examples, but it is also closed under taking some ideal quotients (Proposition 3.30). This will allow us to construct an extriangulated category which is not exact nor triangulated. Moreover, this axiomatization rams down the problem of the *non-existence of a canonical choice of the middle arrow in the axiom (TR3)* to the ambiguity of a representative of realizing sequences (Section 2.2) and the exactness of the associated sequences of natural transformations (Proposition 3.3).

Let us motivate a bit more the use of extriangulated categories. Many results which are homological in nature apply (after suitable adaptation) to both setups: exact categories and triangulated categories. In order to transfer a result known for triangulated categories to a result that applies to exact categories, the usual strategy is the following (non-chronological):

- (1) Specify to the case of stable categories of Frobenius exact categories.
- (2) Lift all definitions and statements from the stable category to the Frobenius category.
- (3) Adapt the proof so that it applies to any exact categories (with suitable assumptions).

Conversely, a result known for exact categories might have an analog for triangulated categories proven as follows:

- (1) Specify to the case of a Frobenius exact category.
- (2) Descend all definitions and statements to the stable category.
- (3) Adapt the proof so that it applies to any triangulated categories (with suitable assumptions).

Even though step (2) might be far from trivial, the main difficulty often lies in step (3) for both cases. The use of extriangulated categories somehow removes that difficulty. It is not that this difficulty has vanished into thin air,

but that it has already been taken care of in the first results on extriangulated categories obtained in Section 3.

The term “extriangulated” stands for *externally triangulated* by means of a bifunctor. It can also be viewed as the mixing of *exact* and *triangulated*, or as an abbreviation of *Ext-triangulated*. The precise definition will be given in Section 2. Fundamental properties including several analogs of the octahedron axiom in an extriangulated category, will be given in Section 3.

On an extriangulated category, we can define the notion of a cotorsion pair, which generalizes that on exact categories [Ho1, Ho2, Liu, S] and on triangulated categories [AN, Na1]. Basic properties are stated in Section 4.

In Section 5, we give a bijective correspondence between *Hovey twin cotorsion pairs* and *admissible model structures*. This result is inspired from [Ho1, Ho2, G], where the case of exact categories is studied in more details. We note that an analog of Hovey’s result in [Ho1, Ho2] has been proven for triangulated categories in [Y]. The use of extriangulated categories allows for a uniform proof.

As a result, we can realize the associated homotopy category by a certain ideal quotient, on which we can give a triangulated structure as in Section 6. This triangulation can be regarded as a simultaneous generalization of those given by Happel’s theorem on stable categories of Frobenius exact categories and of Iyama-Yoshino reductions, and gives a link to the one given by the Verdier quotient. As a consequence, the homotopy category of any exact model structure on a weakly idempotent complete exact category is triangulated. This result was previously known in the case of hereditary exact model structures [G, Proposition 5.2].

With this view, in Section 7, we propose a natural framework to formulate reduction and mutation of cotorsion pairs, applicable to both exact categories and triangulated categories. Indeed, we establish a bijective correspondence between the class of *mutable cotorsion pairs* associated with a Hovey twin cotorsion pair and the class of all cotorsion pairs on the triangulated homotopy category.

2. Extriangulated category

In this section, we abstract the properties of extension-closed subcategory of triangulated or exact category, to formulate it in an internal way by means

of an Ext^1 functor. This gives a simultaneous generalization of triangulated categories and exact categories, suitable for dealing with cotorsion pairs.

2.1 $\mathbb{E}$ -extensions

Throughout this paper, let $\mathcal{C}$ be an additive category.

Definition 2.1. Suppose $\mathcal{C}$ is equipped with a biadditive functor $\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$. For any pair of objects $A, C \in \mathcal{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$ -extension. Thus formally, an $\mathbb{E}$ -extension is a triplet (A, δ, C) .

Remark 2.2. Let (A, δ, C) be any $\mathbb{E}$ -extension. Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C', C)$, we have $\mathbb{E}$ -extensions

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \text{ and } \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We abbreviately denote them by $a_*\delta$ and $c^*\delta$. In this terminology, we have

$$\mathbb{E}(c, a)(\delta) = c^*a_*\delta = a_*c^*\delta \quad \text{in } \mathbb{E}(C', A').$$

Definition 2.3. Let $(A, \delta, C), (A', \delta', C')$ be any pair of $\mathbb{E}$ -extensions. A morphism $(a, c): (A, \delta, C) \rightarrow (A', \delta', C')$ of $\mathbb{E}$ -extensions is a pair of morphisms $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C, C')$ in $\mathcal{C}$, satisfying the equality

$$a_*\delta = c^*\delta'.$$

We simply denote it as $(a, c): \delta \rightarrow \delta'$.

We obtain the category $\mathbb{E}\text{-Ext}(\mathcal{C})$ of $\mathbb{E}$ -extensions, with composition and identities naturally induced from those in $\mathcal{C}$.

Remark 2.4. Let (A, δ, C) be any $\mathbb{E}$ -extension. We have the following.

(1) Any morphism $a \in \mathcal{C}(A, A')$ gives rise to a morphism of $\mathbb{E}$ -extensions

$$(a, \text{id}_C): \delta \rightarrow a_*\delta.$$

(2) Any morphism $c \in \mathcal{C}(C', C)$ gives rise to a morphism of $\mathbb{E}$ -extensions

$$(\text{id}_A, c): c^*\delta \rightarrow \delta.$$

Definition 2.5. For any $A, C \in \mathcal{C}$, the zero element $0 \in \mathbb{E}(C, A)$ is called the *split $\mathbb{E}$ -extension*.

Definition 2.6. Let $\delta = (A, \delta, C), \delta' = (A', \delta', C')$ be any $\mathbb{E}$ -extensions. Let $C \xrightarrow{\iota_C} C \oplus C' \xleftarrow{\iota_{C'}} C'$ and $A \xleftarrow{p_A} A \oplus A' \xrightarrow{p_{A'}} A'$ be coproduct and product in $\mathcal{C}$, respectively. Remark that, by the biadditivity of $\mathbb{E}$, we have a natural isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A').$$

Let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the element corresponding to $(\delta, 0, 0, \delta')$ through this isomorphism.

If $A = A'$ and $C = C'$, then the sum $\delta + \delta' \in \mathbb{E}(C, A)$ of $\delta, \delta' \in \mathbb{E}(C, A)$ is obtained by

$$\delta + \delta' = \mathbb{E}(\Delta_C, \nabla_A)(\delta \oplus \delta'),$$

where $\Delta_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix} : C \rightarrow C \oplus C, \nabla_A = [1 \ 1] : A \oplus A \rightarrow A$.

2.2 Realization of $\mathbb{E}$ -extensions

Let $\mathcal{C}, \mathbb{E}$ be as before.

Definition 2.7. Let $A, C \in \mathcal{C}$ be any pair of objects. Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathcal{C}$ are said to be *equivalent* if there exists an isomorphism $b \in \mathcal{C}(B, B')$ which makes the following diagram commutative.

$$\begin{array}{ccccc} & & B & & \\ & x \nearrow & & \searrow y & \\ A & & & & C \\ & x' \searrow & & \nearrow y' & \\ & & B' & & \end{array}$$

$\circlearrowleft \cong \downarrow b \circlearrowright$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

Definition 2.8.

- (1) For any $A, C \in \mathcal{C}$, we denote as $0 = [A \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} A \oplus C \xrightarrow{[0 \ 1]} C]$.

- (2) For any two classes $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, we denote as $[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ the class $[A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C']$.

Definition 2.9. Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$ -extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$, if it satisfies the following condition (*). In this case, we say that sequence $A \xrightarrow{x} B \xrightarrow{y} C$ *realizes* δ , whenever it satisfies $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.

- (*) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$ -extensions, with $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Then, for any morphism $(a, c) \in \mathbb{E}\text{-Ext}(\mathcal{C})(\delta, \delta')$, there exists $b \in \mathcal{C}(B, B')$ which makes the following diagram commutative.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ a \downarrow & \circlearrowleft & \downarrow b & \circlearrowleft & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array} \quad (1)$$

Remark that this condition does not depend on the choices of the representatives of the equivalence classes. In the above situation, we say that (1) (or the triplet (a, b, c)) *realizes* (a, c) .

Definition 2.10. Let $\mathcal{C}, \mathbb{E}$ be as above. A realization of $\mathbb{E}$ is said to be *additive*, if it satisfies the following conditions.

- (i) For any $A, C \in \mathcal{C}$, the split $\mathbb{E}$ -extension $0 \in \mathbb{E}(C, A)$ satisfies

$$\mathfrak{s}(0) = 0.$$

- (ii) For any pair of $\mathbb{E}$ -extensions $\delta = (A, \delta, C)$ and $\delta' = (A', \delta', C')$,

$$\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta') \text{ holds.}$$

Remark 2.11. If $\mathfrak{s}$ is an additive realization of $\mathbb{E}$, then the following holds.

- (1) For any $A, C \in \mathcal{C}$, if $0 \in \mathbb{E}(C, A)$ is realized by $A \xrightarrow{x} B \xrightarrow{y} C$, then there exist a retraction $r \in \mathcal{C}(B, A)$ of x and a section $s \in \mathcal{C}(C, B)$ of y which give an isomorphism $\begin{bmatrix} r \\ s \end{bmatrix} : B \xrightarrow{\cong} A \oplus C$.

(2) For any $f \in \mathcal{C}(A, B)$, the sequence

$$A \xrightarrow{\begin{bmatrix} 1 \\ -f \end{bmatrix}} A \oplus B \xrightarrow{[f \ 1]} B$$

realizes the split $\mathbb{E}$ -extension $0 \in \mathbb{E}(B, A)$.

2.3 Definition of extriangulated categories

Definition 2.12. We call the pair $(\mathbb{E}, \mathfrak{s})$ an *external triangulation* of $\mathcal{C}$ if it satisfies the following conditions. In this case, we call $\mathfrak{s}$ an $\mathbb{E}$ -*triangulation* of $\mathcal{C}$, and call the triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ an *externally triangulated category*, or for short, *extriangulated category*.

(ET1) $\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$ is a biadditive functor.

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$ -extensions, realized as $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. For any commutative square in $\mathcal{C}$

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ a \downarrow & \circlearrowleft & \downarrow b & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array} \quad (2)$$

there exists a morphism $(a, c): \delta \rightarrow \delta'$ which is realized by (a, b, c) .

(ET3)^{op} Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$ -extensions, realized by $A \xrightarrow{x} B \xrightarrow{y} C$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C'$ respectively. For any commutative square in $\mathcal{C}$

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ & & b \downarrow & \circlearrowleft & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

there exists a morphism $(a, c): \delta \rightarrow \delta'$ which is realized by (a, b, c) .

(ET4) Let (A, δ, D) and (B, δ', F) be $\mathbb{E}$ -extensions respectively realized by $A \xrightarrow{f} B \xrightarrow{f'} D$ and $B \xrightarrow{g} C \xrightarrow{g'} F$. Then there exist an object $E \in \mathcal{C}$, a commutative diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\
 \parallel & \circlearrowleft & \downarrow g & \circlearrowleft & \downarrow d \\
 A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\
 & & \downarrow g' & \circlearrowleft & \downarrow e \\
 & & F & = & F
 \end{array} \tag{3}$$

in $\mathcal{C}$, and an $\mathbb{E}$ -extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities.

- (i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $\mathbb{E}(F, f')(\delta')$,
- (ii) $\mathbb{E}(d, A)(\delta'') = \delta$,
- (iii) $\mathbb{E}(E, f)(\delta'') = \mathbb{E}(e, B)(\delta')$.

By (iii), $(f, e): \delta'' \rightarrow \delta'$ is a morphism of $\mathbb{E}$ -extensions, realized by

$$(f, \text{id}_C, e): [A \xrightarrow{h} C \xrightarrow{h'} E] \rightarrow [B \xrightarrow{g} C \xrightarrow{g'} F].$$

(ET4)^{op} Dual of (ET4) (see Remark 2.22).

Example 2.13. Exact categories (with a condition concerning the smallness) and triangulated categories are examples of extriangulated categories. See also Remark 2.18. We briefly show how an exact category can be viewed as an extriangulated category. As for triangulated categories, see the construction in Proposition 3.22. For the definition and basic properties of an exact category, see [Bu] and [Ke]. Let $A, C \in \mathcal{C}$ be any pair of objects. Remark that, as shown in [Bu, Corollary 3.2], for any morphism of short exact sequences (=conflations in [Ke]) of the form

$$\begin{array}{ccccc}
 A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
 \parallel & \circlearrowleft & \downarrow b & \circlearrowleft & \parallel \\
 A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C
 \end{array},$$

the morphism b in the middle automatically becomes an isomorphism. Consider the same equivalence relation as in Definition 2.7, and define the class $\text{Ext}^1(C, A)$ to be the collection of all equivalence classes of short exact sequences of the form $A \xrightarrow{x} B \xrightarrow{y} C$. We denote the equivalence class by $[A \xrightarrow{x} B \xrightarrow{y} C]$ as before.

This becomes a small set, for example in the following cases.

- $\mathcal{C}$ is skeletally small.
- $\mathcal{C}$ has enough projectives or injectives.

In such a case, we obtain a biadditive functor $\text{Ext}^1: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow Ab$, as stated in [S, Definitions 5.1]. Its structure is given as follows.

- For any $\delta = [A \xrightarrow{x} B \xrightarrow{y} C] \in \text{Ext}^1(C, A)$ and any $a \in \mathcal{C}(A, A')$, take a pushout in $\mathcal{C}$, to obtain a morphism of short exact sequences

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ a \downarrow & \text{PO} & \downarrow & \circlearrowleft & \parallel \\ A' & \xrightarrow{m} & M & \xrightarrow{e} & C \end{array}.$$

This gives $\text{Ext}^1(C, a)(\delta) = a_*\delta = [A' \xrightarrow{m} M \xrightarrow{e} C]$.

- For any $c \in \mathcal{C}(C', C)$, the map $\text{Ext}^1(c, A) = c^*: \text{Ext}^1(C, A) \rightarrow \text{Ext}^1(C', A)$ is defined dually by using pullbacks.
- The zero element in $\text{Ext}^1(C, A)$ is given by the split short exact sequence

$$0 = [A \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} A \oplus C \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} C].$$

For any pair $\delta_1 = [A \xrightarrow{x_1} B_1 \xrightarrow{y_1} C], \delta_2 = [A \xrightarrow{x_2} B_2 \xrightarrow{y_2} C] \in \text{Ext}^1(C, A)$, its sum $\delta_1 + \delta_2$ is given by the Baer sum

$$\Delta_C^*(\nabla_A)_*(\delta_1 \oplus \delta_2) = \Delta_C^*(\nabla_A)_*([A \oplus A \xrightarrow{x_1 \oplus x_2} B_1 \oplus B_2 \xrightarrow{y_1 \oplus y_2} C \oplus C]).$$

This shows (ET1). Define the realization $\mathfrak{s}(\delta)$ of $\delta = [A \xrightarrow{x} B \xrightarrow{y} C]$ to be δ itself. Then (ET2) is trivially satisfied. For (ET3) and (ET4), the following fact is useful.

Fact 2.14. ([Bu, Proposition 3.1]) For any morphism, $(*)$ below, of short exact sequences in $\mathcal{C}$, there exists a commutative diagram $(**)$ whose middle row is also a short exact sequence, the upper-left square is a pushout, and the lower-right square is a pullback.

Remark that this means $a_*[A \xrightarrow{x} B \xrightarrow{y} C] = c^*[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$.

$$\begin{array}{ccc}
 & A \xrightarrow{x} B \xrightarrow{y} C & \\
 (*) \quad \begin{array}{ccccc} a \downarrow & \circlearrowleft & \downarrow b & \circlearrowleft & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array} & & (**) \quad \begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ a \downarrow & \text{PO} & \downarrow & \circlearrowleft & \parallel \\ A' & \xrightarrow{m} & \exists M & \xrightarrow{e} & C \\ \parallel & \circlearrowleft & \downarrow & \text{PB} & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}
 \end{array}$$

By Fact 2.14, (ET3) follows immediately from the universality of cokernel. Similarly, (ET4) follows from [Bu, Lemma 3.5]. Dually for (ET3)^{op} and (ET4)^{op}.

2.4 Terminology in an extriangulated category

To allow an argument with familiar terms, we introduce terminology from both exact categories and triangulated categories (cf. [Ke, Bu, Ne]).

Definition 2.15. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triplet satisfying (ET1) and (ET2).

- (1) A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some $\mathbb{E}$ -extension $\delta \in \mathbb{E}(C, A)$. For the ambiguity of such an $\mathbb{E}$ -extension, see Corollary 3.8.
- (2) A morphism $f \in \mathcal{C}(A, B)$ is called an *inflation* if it admits some conflation $A \xrightarrow{f} B \rightarrow C$. For the ambiguity of such a conflation, see Remark 3.10.
- (3) A morphism $f \in \mathcal{C}(A, B)$ is called a *deflation* if it admits some conflation $K \rightarrow A \xrightarrow{f} B$.

Remark 2.16. Condition (ET4) implies that inflations are closed under composition. Dually, (ET4)^{op} implies the composition-closedness of deflations.

Definition 2.17. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full additive subcategory, closed under isomorphisms. The subcategory $\mathcal{D}$ is said to be *extension-closed* if, for any conflation $A \rightarrow B \rightarrow C$ which satisfies $A, C \in \mathcal{D}$, then $B \in \mathcal{D}$.

The following can be checked in a straightforward way.

Remark 2.18. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, and let $\mathcal{D} \subseteq \mathcal{C}$ be an extension-closed subcategory. If we define $\mathbb{E}_{\mathcal{D}}$ to be the restriction of $\mathbb{E}$ onto $\mathcal{D}^{\text{op}} \times \mathcal{D}$, and define $\mathfrak{s}_{\mathcal{D}}$ by restricting $\mathfrak{s}$, then $(\mathcal{D}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ becomes an extriangulated category.

Definition 2.19. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triplet satisfying (ET1) and (ET2).

- (1) If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an $\mathbb{E}$ -triangle, and write it in the following way.

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \triangleright \quad (4)$$

- (2) Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \triangleright$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} \triangleright$ be any pair of $\mathbb{E}$ -triangles. If a triplet (a, b, c) realizes $(a, c): \delta \rightarrow \delta'$ as in (1), then we write it as

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \triangleright \\ a \downarrow & \circ & \downarrow b & \circ & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} \triangleright \end{array}$$

and call (a, b, c) a *morphism of $\mathbb{E}$ -triangles*.

Caution 2.20. Although the abbreviated expression (4) looks superficially asymmetric, we remark that the definition of an extriangulated category is completely self-dual.

Remark 2.21. (ET3) means that any commutative square (2) bridging $\mathbb{E}$ -triangles can be completed into a morphism of $\mathbb{E}$ -triangles. Dually for $(\text{ET3})^{\text{op}}$.

Condition $(*)$ in Definition 2.9 means that any morphism of $\mathbb{E}$ -extensions can be realized by a morphism of $\mathbb{E}$ -triangles.

In the above terminology, condition $(\text{ET4})^{\text{op}}$ can be stated as follows.

Remark 2.22. (Paraphrase of (ET4)^{op}) Let $D \xrightarrow{f'} A \xrightarrow{f} B \xrightarrow{\delta} \dashrightarrow$ and $F \xrightarrow{g'} B \xrightarrow{g} C \xrightarrow{\delta'} \dashrightarrow$ be $\mathbb{E}$ -triangles. Then there exist an $\mathbb{E}$ -triangle $E \xrightarrow{h'} A \xrightarrow{h} C \xrightarrow{\delta''} \dashrightarrow$ and a commutative diagram in $\mathcal{C}$, as on the left below, satisfying the compatibilities on the right:

$$\begin{array}{ccc}
 D & \xrightarrow{d} & E \xrightarrow{e} F \\
 \parallel & \circlearrowleft & \downarrow h' \quad \circlearrowleft \quad \downarrow g' \\
 D & \xrightarrow{f'} & A \xrightarrow{f} B \\
 & & \downarrow h \quad \circlearrowleft \quad \downarrow g \\
 & & C \xlongequal{\quad} C
 \end{array}
 \quad \begin{array}{l}
 \text{(i) } D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{g'^*\delta} \dashrightarrow \text{ is an } \mathbb{E}\text{-triangle,} \\
 \text{(ii) } \delta' = e_*\delta'', \\
 \text{(iii) } d_*\delta = g^*\delta''.
 \end{array}$$

3. Fundamental properties

3.1 Associated exact sequence

In this section, we will associate exact sequences of natural transformations

$$\mathcal{C}(C, -) \xrightarrow{\mathcal{C}(y, -)} \mathcal{C}(B, -) \xrightarrow{\mathcal{C}(x, -)} \mathcal{C}(A, -) \xrightarrow{\delta^\sharp} \mathbb{E}(C, -) \xrightarrow{\mathbb{E}(y, -)} \mathbb{E}(B, -) \xrightarrow{\mathbb{E}(x, -)} \mathbb{E}(A, -)$$

$$\mathcal{C}(-, A) \xrightarrow{\mathcal{C}(-, x)} \mathcal{C}(-, B) \xrightarrow{\mathcal{C}(-, y)} \mathcal{C}(-, C) \xrightarrow{\delta_\sharp} \mathbb{E}(-, A) \xrightarrow{\mathbb{E}(-, x)} \mathbb{E}(-, B) \xrightarrow{\mathbb{E}(-, y)} \mathbb{E}(-, C)$$

to any $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \dashrightarrow$ in an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ (Corollary 3.12). Here, $\delta_\sharp$ and $\delta^\sharp$ are defined in the following.

Definition 3.1. Assume $\mathcal{C}$ and $\mathbb{E}$ satisfy (ET1). By Yoneda's lemma, any $\mathbb{E}$ -extension $\delta \in \mathbb{E}(C, A)$ induces natural transformations $\delta_\sharp: \mathcal{C}(-, C) \Rightarrow \mathbb{E}(-, A)$ and $\delta^\sharp: \mathcal{C}(A, -) \Rightarrow \mathbb{E}(C, -)$. For any $X \in \mathcal{C}$, these $(\delta_\sharp)_X$ and $\delta_X^\sharp$ are given as follows.

$$(1) \quad (\delta_\sharp)_X: \mathcal{C}(X, C) \rightarrow \mathbb{E}(X, A); \quad f \mapsto f^*\delta.$$

$$(2) \quad \delta_X^\sharp: \mathcal{C}(A, X) \rightarrow \mathbb{E}(C, X); \quad g \mapsto g_*\delta.$$

We abbreviately denote $(\delta_\sharp)_X(f)$ and $\delta_X^\sharp(g)$ by $\delta_\sharp f$ and $\delta^\sharp g$, when there is no confusion.

Lemma 3.2. Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)^{op}. Then for any $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$, the following hold:

- (1) $y \circ x = 0$, (2) $x_*\delta (= \delta^\#x) = 0$ and (3) $y^*\delta (= \delta_\#y) = 0$.

Proof. (1) By (ET2), the conflation $A \xrightarrow{\text{id}} A \rightarrow 0$ realizes $0 \in \mathbb{E}(0, A)$. Applying (ET3) similarly as for triangulated categories gives the result.

(2) Similarly, applying (ET3) we obtain a morphism of $\mathbb{E}$ -extensions $(x, 0): \delta \rightarrow 0$. Especially we have $x_*\delta = 0$. (3) is dual to (2). $\square$

Proposition 3.3. Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2). Then the following are equivalent.

- (1) $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET3) and (ET3)^{op}.
- (2) For any $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$, the following sequences of natural transformations are exact:
- (2-i) $\mathcal{C}(C, -) \xrightarrow{\mathcal{C}(y, -)} \mathcal{C}(B, -) \xrightarrow{\mathcal{C}(x, -)} \mathcal{C}(A, -) \xrightarrow{\delta^\#} \mathbb{E}(C, -) \xrightarrow{\mathbb{E}(y, -)} \mathbb{E}(B, -)$ in $\text{Mod}(\mathcal{C})$. Here $\text{Mod}(\mathcal{C})$ denotes the abelian category of additive functors from $\mathcal{C}$ to Ab .
- (2-ii) $\mathcal{C}(-, A) \xrightarrow{\mathcal{C}(-, x)} \mathcal{C}(-, B) \xrightarrow{\mathcal{C}(-, y)} \mathcal{C}(-, C) \xrightarrow{\delta_\#} \mathbb{E}(-, A) \xrightarrow{\mathbb{E}(-, x)} \mathbb{E}(-, B)$ in $\text{Mod}(\mathcal{C}^{\text{op}})$.

Remark 3.4. In the above (2-i), the category $\text{Mod}(\mathcal{C})$ is not locally small in general. The “exactness” of the sequence in (2-i) simply means that

$$\mathcal{C}(C, X) \xrightarrow{\mathcal{C}(y, X)} \mathcal{C}(B, X) \xrightarrow{\mathcal{C}(x, X)} \mathcal{C}(A, X) \xrightarrow{\delta_X^\#} \mathbb{E}(C, X) \xrightarrow{\mathbb{E}(y, X)} \mathbb{E}(B, X)$$

is exact in Ab for any $X \in \mathcal{C}$. Similarly for (2-ii).

Proof of Proposition 3.3. First we assume (1). We only show the exactness of (2-i), since (2-ii) can be shown dually. By Lemma 3.2, composition of any consecutive morphisms in (2-i) is equal to 0. Let us show the exactness of $\mathcal{C}(C, X) \xrightarrow{\mathcal{C}(y, X)} \mathcal{C}(B, X) \xrightarrow{\mathcal{C}(x, X)} \mathcal{C}(A, X) \xrightarrow{\delta_X^\#} \mathbb{E}(C, X) \xrightarrow{\mathbb{E}(y, X)} \mathbb{E}(B, X)$ for any $X \in \mathcal{C}$.

Exactness at $\mathcal{C}(B, X)$ is shown similarly as for triangulated categories.

Exactness at $\mathcal{C}(A, X)$

Let $a \in \mathcal{C}(A, X)$ be any morphism satisfying $\delta_X^\sharp(a) = a_*\delta = 0$. This means that $(a, 0): \delta \rightarrow 0$ is a morphism of $\mathbb{E}$ -extensions. Since $\mathfrak{s}$ realizes $\mathbb{E}$, there exists $b \in \mathcal{C}(B, X)$ which gives the following morphism of $\mathbb{E}$ -triangles.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \gg \\ a \downarrow & \circlearrowleft & \downarrow b & \circlearrowleft & \downarrow \\ X & \xrightarrow{\text{id}_X} & X & \longrightarrow & 0 \xrightarrow{0} \gg \end{array}$$

Especially we have $a = b \circ x = \mathcal{C}(x, X)(b)$.

Exactness at $\mathbb{E}(C, X)$

Let $\theta \in \mathbb{E}(C, X)$ be any $\mathbb{E}$ -extension satisfying $\mathbb{E}(y, X)(\theta) = y^*\theta = 0$. Realize them as $\mathbb{E}$ -triangles $X \xrightarrow{f} Y \xrightarrow{g} C \xrightarrow{\theta} \gg$ and $X \xrightarrow{m} Z \xrightarrow{e} B \xrightarrow{y^*\theta} \gg$. Then the morphism $(\text{id}_X, y): y^*\theta \rightarrow \theta$ can be realized by $(*)$ below

$$\begin{array}{ccc} (*) & \begin{array}{ccccc} X & \xrightarrow{m} & Z & \xrightarrow{e} & B \xrightarrow{y^*\theta} \gg \\ \parallel & \circlearrowleft & \downarrow e' & \circlearrowleft & \downarrow y \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & C \xrightarrow{\theta} \gg \end{array} & (**) \quad \begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \gg \\ & & e' \circ s \downarrow & \circlearrowleft & \parallel \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & C \xrightarrow{\theta} \gg \end{array} \end{array}$$

with some $e' \in \mathcal{C}(Z, Y)$. Since $y^*\theta$ splits by assumption, e has a section s . Applying $(\text{ET3})^{\text{op}}$ to $(**)$ above, we obtain $a \in \mathcal{C}(A, X)$ which gives a morphism $(a, \text{id}_C): \delta \rightarrow \theta$. This means $\theta = a_*\delta = \delta^\sharp a$.

Conversely, let us assume (2) and show (ET3). Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \gg$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} \gg$ be any pair of $\mathbb{E}$ -triangles. Suppose that we are given a commutative diagram (α) below.

$$\begin{array}{ccc} (\alpha) & \begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ a \downarrow & \circlearrowleft & \downarrow b & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array} & (\beta) \quad \begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \gg \\ a \downarrow & \circlearrowleft & \downarrow b' & \circlearrowleft & \downarrow c' \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} \gg \end{array} \end{array}$$

Remark that $\mathbb{E}(-, x) \circ \delta_\sharp = 0$ is equivalent to $x_*\delta = 0$ by Yoneda's lemma.

Similarly, $y'^*\delta'$ vanishes. By the exactness of the sequence $\mathcal{C}(C, C') \xrightarrow{(\delta'_\sharp)^C} \mathbb{E}(C, A') \xrightarrow{\mathbb{E}(C, x')} \mathbb{E}(C, B')$ and the equality $\mathbb{E}(C, x')(a_*\delta) = x'_*a_*\delta = b_*x_*\delta = 0$

there is $c' \in \mathcal{C}(C, C')$ satisfying $a_*\delta = \delta'_\#c' = c'^*\delta'$. Thus $(a, c): \delta \rightarrow \delta'$ is a morphism of $\mathbb{E}$ -extensions. Take its realization as in (β) above.

Then by the exactness of $\mathcal{C}(C, B') \xrightarrow{\mathcal{C}(y, B')} \mathcal{C}(B, B') \xrightarrow{\mathcal{C}(x, B')} \mathcal{C}(A, B')$ and the equality $(b - b') \circ x = x' \circ a - x' \circ a = 0$, there exists $c'' \in \mathcal{C}(C, B')$ satisfying $c'' \circ y = b - b'$. If we put $c = c' + y' \circ c''$, this satisfies

$$\begin{aligned} c \circ y &= c' \circ y + y' \circ c'' \circ y = y' \circ b' + (y' \circ b - y' \circ b') = y' \circ b, \\ c^*\delta' &= c'^*\delta' + c''^*y'^*\delta' = a_*\delta. \end{aligned}$$

Dually, (2) implies (ET3)^{op}. $\square$

The following two corollaries are direct consequences of the long exact sequences of Proposition 3.3.

Corollary 3.5. Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)^{op}. Let

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \triangleright \\ a \downarrow & \circlearrowleft & \downarrow b & \circlearrowleft & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} \triangleright \end{array} \quad (5)$$

be any morphism of $\mathbb{E}$ -triangles. Then a factors through x if and only if $a_*\delta = c^*\delta' = 0$ if and only if c factors through y' .

In particular, in the case $\delta = \delta'$ and $(a, b, c) = (\text{id}, \text{id}, \text{id})$, we obtain

$$x \text{ has a retraction} \Leftrightarrow \delta \text{ splits} \Leftrightarrow y \text{ has a section.}$$

Corollary 3.6. Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)^{op}. The following holds for any morphism (a, b, c) of $\mathbb{E}$ -triangles.

- (1) If a and c are isomorphisms in $\mathcal{C}$ (equivalently, if (a, c) is an isomorphism in $\mathbb{E}\text{-Ext}(\mathcal{C})$ in Definition 2.3), then so is b .
- (2) If a and b are isomorphisms in $\mathcal{C}$, then so is c .
- (3) If b and c are isomorphisms in $\mathcal{C}$, then so is a .

Proposition 3.7. Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)^{op} and let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \triangleright$ be any $\mathbb{E}$ -triangle. If $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C', C)$ are isomorphisms, then $A' \xrightarrow{x \circ a^{-1}} B \xrightarrow{c^{-1} \circ y} C' \xrightarrow{a_*c^*\delta} \triangleright$ is again an $\mathbb{E}$ -triangle.

Proof. Put $\mathfrak{s}(a_*c^*\delta) = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Since $(c^{-1})^*(a_*c^*\delta) = (c \circ c^{-1})^*a_*\delta = a_*\delta$, we see that $(a, c^{-1}): \delta \rightarrow c^*a_*\delta$ is a morphism of $\mathbb{E}$ -extensions. Take a morphism of $\mathbb{E}$ -triangles (a, b, c^{-1}) realizing (a, c^{-1}) . Then b is an isomorphism by Corollary 3.6. Thus $[A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A' \xrightarrow{x \circ a^{-1}} B \xrightarrow{c^{-1} \circ y} C']$. $\square$

Corollary 3.8. Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)^{op}. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \rightarrow$ be any $\mathbb{E}$ -triangle. Then for any $\delta' \in \mathbb{E}(C, A)$, the following are equivalent.

- (1) $\mathfrak{s}(\delta) = \mathfrak{s}(\delta')$.
- (2) $\delta' = a_*\delta$ for some automorphism $a \in \mathcal{C}(A, A)$ satisfying $x \circ a = x$.
- (3) $\delta' = c^*\delta$ for some automorphism $c \in \mathcal{C}(C, C)$ satisfying $c \circ y = y$.
- (4) $\delta' = a_*c^*\delta$ for some automorphisms $a \in \mathcal{C}(A, A)$ and $c \in \mathcal{C}(C, C)$ satisfying $x \circ a = x$ and $c \circ y = y$.

Proof. (2) $\Rightarrow$ (4) is trivial. Similarly for (3) $\Rightarrow$ (4). Proposition 3.7 shows (4) $\Rightarrow$ (1). The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow from Corollary 3.6. $\square$

Definition 3.9. Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)^{op}.

- (1) For an inflation $f \in \mathcal{C}(A, B)$, take a conflation $A \xrightarrow{f} B \rightarrow C$, and denote this C by $\text{Cone}(f)$.
- (2) For a deflation $f \in \mathcal{C}(A, B)$, take a conflation $K \rightarrow A \xrightarrow{f} B$. We denote this K by $\text{CoCone}(f)$.

$\text{Cone}(f)$ is determined uniquely up to isomorphism by the following remark. They are not functorial in general, as the case of triangulated category suggests. Dually for $\text{CoCone}(f)$.

Remark 3.10. Let $f \in \mathcal{C}(A, B)$ be an inflation, and suppose

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} \rightarrow, \quad A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{\delta'} \rightarrow$$

are $\mathbb{E}$ -triangles. Then, by (ET3) applied to $(\text{id}_A, \text{id}_B): f \rightarrow f$, there exists $c \in \mathcal{C}(C, C')$ which gives a morphism of $\mathbb{E}$ -triangles $(\text{id}_A, \text{id}_B, c)$. By Corollary 3.6, this c is an isomorphism.

Proposition 3.11. Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET1),(ET2),(ET3),(ET3)^{op}.

Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \rightarrow$ be an $\mathbb{E}$ -triangle. Then, we have the following.

- (1) If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET4), then $\mathbb{E}(-, A) \xrightarrow{\mathbb{E}(-, x)} \mathbb{E}(-, B) \xrightarrow{\mathbb{E}(-, y)} \mathbb{E}(-, C)$ is exact.
- (2) If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies (ET4)^{op}, then $\mathbb{E}(C, -) \xrightarrow{\mathbb{E}(y, -)} \mathbb{E}(B, -) \xrightarrow{\mathbb{E}(x, -)} \mathbb{E}(A, -)$ is exact.

Proof. (1) $\mathbb{E}(-, y) \circ \mathbb{E}(-, x) = 0$ follows from Lemma 3.2. Let $X \in \mathcal{C}$ be any object. Let $\theta \in \mathbb{E}(X, B)$ be any $\mathbb{E}$ -extension, realized by an $\mathbb{E}$ -triangle $B \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{\theta} \rightarrow$. By (ET4), there exist $E \in \mathcal{C}$, $\theta' \in \mathbb{E}(E, A)$ and a commutative diagram which satisfies

$$\begin{array}{ccc}
 A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
 \parallel & \circlearrowleft & \downarrow f & \circlearrowright & \downarrow d \\
 A & \xrightarrow{h} & Y & \xrightarrow{h'} & E \\
 & & \downarrow g & \circlearrowleft & \downarrow e \\
 & & X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{aligned}
 \mathfrak{s}(y_*\theta) &= [C \xrightarrow{d} E \xrightarrow{e} X], \\
 \mathfrak{s}(\theta') &= [A \xrightarrow{h} Y \xrightarrow{h'} E], \\
 x_*\theta' &= e^*\theta.
 \end{aligned}$$

Thus if $\mathbb{E}(X, y)(\theta) = y_*\theta = 0$, then e has a section $s \in \mathcal{C}(X, E)$. If we put $\rho = s^*\theta'$, then this satisfies $\mathbb{E}(x, X)(\rho) = x_*s^*\theta' = s^*x_*\theta' = s^*e^*\theta = \theta$.

(2) is dual to (1). $\square$

Corollary 3.12. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. For any $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \rightarrow$, the following sequences of natural transformations are exact.

$$\begin{aligned}
 \mathcal{C}(C, -) &\xrightarrow{\mathcal{C}(y, -)} \mathcal{C}(B, -) \xrightarrow{\mathcal{C}(x, -)} \mathcal{C}(A, -) \xrightarrow{\delta^\#} \mathbb{E}(C, -) \xrightarrow{\mathbb{E}(y, -)} \mathbb{E}(B, -) \xrightarrow{\mathbb{E}(x, -)} \mathbb{E}(A, -) \\
 \mathcal{C}(-, A) &\xrightarrow{\mathcal{C}(-, x)} \mathcal{C}(-, B) \xrightarrow{\mathcal{C}(-, y)} \mathcal{C}(-, C) \xrightarrow{\delta_\#} \mathbb{E}(-, A) \xrightarrow{\mathbb{E}(-, x)} \mathbb{E}(-, B) \xrightarrow{\mathbb{E}(-, y)} \mathbb{E}(-, C)
 \end{aligned}$$

Proof. This immediately follows from Propositions 3.3 and 3.11. $\square$

The following lemma shows that the upper-right square of Diagram (3) obtained by (ET4) is a weak pushout.

Lemma 3.13. Let (3) be a commutative diagram in $\mathcal{C}$, where

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \xrightarrow{d^* \delta''} \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \xrightarrow{\delta''} \end{array}, \quad \begin{array}{ccccc} B & \xrightarrow{g} & C & \xrightarrow{g'} & F \xrightarrow{\delta'} \\ D & \xrightarrow{d} & E & \xrightarrow{e} & F \xrightarrow{f'_* \delta'} \end{array}$$

are $\mathbb{E}$ -triangles, which satisfy $e^* \delta' = f_* \delta''$.

Then, for any commutative square $y \circ f' = x \circ g$ in $\mathcal{C}$, there exists $z \in \mathcal{C}(E, Y)$ which makes the following diagram commutative.

$$\begin{array}{ccc} B & \xrightarrow{f'} & D \\ g \downarrow & \circlearrowleft & \downarrow d \\ C & \xrightarrow{h'} & E \\ & \circlearrowleft & \downarrow y \\ & & Y \end{array}$$

$x \searrow$ (from C to Y)
 $z \searrow$ (from E to Y)

Proof. By $(f'_* \delta')^\sharp(y) = y_* f'_* \delta' = x_* g_* \delta' = 0$ and the exactness of

$$\mathcal{C}(E, Y) \xrightarrow{\mathcal{C}(d, Y)} \mathcal{C}(D, Y) \xrightarrow{(f'_* \delta')^\sharp_Y} \mathbb{E}(F, Y) \rightarrow 0,$$

there exists $z_1 \in \mathcal{C}(E, Y)$ satisfying $z_1 \circ d = y$. Then by $(x - z_1 \circ h') \circ g = y \circ f' - z_1 \circ d \circ f' = 0$ and the exactness of

$$\mathcal{C}(F, Y) \xrightarrow{\mathcal{C}(g', Y)} \mathcal{C}(C, Y) \xrightarrow{\mathcal{C}(g, Y)} \mathcal{C}(B, Y),$$

there exists $z_2 \in \mathcal{C}(F, Y)$ satisfying $z_2 \circ g' = x - z_1 \circ h'$. If we put $z = z_1 + z_2 \circ e$, this satisfies the desired commutativity. $\square$

3.2 Shifted octahedrons

Condition (ET4) in Definition 2.12 is an analog of the octahedron axiom (TR4) for a triangulated category. As in the case of a triangulated category, we can make it slightly more rigid as follows.

Lemma 3.14. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Let

$A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta_f} B \xrightarrow{g} C \xrightarrow{g'} F \xrightarrow{\delta_g} A \xrightarrow{h} C \xrightarrow{h_0} E_0 \xrightarrow{\delta_h}$ be any triplet of $\mathbb{E}$ -triangles satisfying $h = g \circ f$. Then there are morphisms d_0, e_0 in $\mathcal{C}$ which make the diagram below commutative, with the following compatibilities:

$$\begin{aligned}
 & \text{(i) } D \xrightarrow{d_0} E_0 \xrightarrow{e_0} F \xrightarrow{f'_*(\delta_g)} \text{ is an } \mathbb{E}\text{-triangle,} \\
 & \text{(ii) } d_0^*(\delta_h) = \delta_f, \\
 & \text{(iii) } f_*(\delta_h) = e_0^*(\delta_g).
 \end{aligned}
 \quad
 \begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\
 \parallel & \circlearrowleft & \downarrow g & \circlearrowright & \downarrow d_0 \\
 A & \xrightarrow{h} & C & \xrightarrow{h_0} & E_0 \\
 & & \downarrow g' & \circlearrowleft & \downarrow e_0 \\
 & & F & \xlongequal{\quad} & F
 \end{array} \quad (6)$$

Proof. By (ET4), there exist an object $E \in \mathcal{C}$, a commutative diagram (3) in $\mathcal{C}$, and an $\mathbb{E}$ -triangle $A \xrightarrow{h} C \xrightarrow{h'} E \xrightarrow{\delta''}$, which satisfy the following compatibilities. (i') $D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{f'_*(\delta_g)}$ is an $\mathbb{E}$ -triangle, (ii') $d^*(\delta'') = \delta_f$, (iii') $f_*(\delta'') = e^*(\delta_g)$. By Remark 3.10, we obtain a morphism of $\mathbb{E}$ -triangles

$$\begin{array}{ccccc}
 A & \xrightarrow{h} & C & \xrightarrow{h'} & E \xrightarrow{\delta''} \\
 \parallel & \circlearrowleft & \parallel & \circlearrowright & \downarrow u \\
 A & \xrightarrow{h} & C & \xrightarrow{h_0} & E_0 \xrightarrow{\delta_h}
 \end{array}$$

in which u is an isomorphism. In particular we have $\delta'' = u^*(\delta_h)$. If we put $d_0 = u \circ d$ and $e_0 = e \circ u^{-1}$, then the commutativity of (6) follows from that of (3). By the definition of the equivalence relation, we have $[D \xrightarrow{d_0} E_0 \xrightarrow{e_0} F] = [D \xrightarrow{d} E \xrightarrow{e} F]$. It is straightforward to check that (i'), (ii'), (iii') imply (i), (ii), (iii). $\square$

Proposition 3.15. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Then the following (and its dual) holds.

Let C be any object, and let $A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \xrightarrow{\delta_1}$, $A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \xrightarrow{\delta_2}$ be any pair of $\mathbb{E}$ -triangles. Then there is a commutative diagram in $\mathcal{C}$

which satisfies:

$$\begin{aligned}
 \mathfrak{s}(y_2^* \delta_1) &= [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2], \\
 \mathfrak{s}(y_1^* \delta_2) &= [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1], \\
 m_{1*} \delta_1 + m_{2*} \delta_2 &= 0.
 \end{aligned}
 \quad
 \begin{array}{ccccc}
 & & A_2 & \xlongequal{\quad} & A_2 \\
 & & \downarrow m_2 & \circlearrowleft & \downarrow x_2 \\
 A_1 & \xrightarrow{m_1} & M & \xrightarrow{e_1} & B_2 \\
 \parallel & \circlearrowleft & \downarrow e_2 & \circlearrowleft & \downarrow y_2 \\
 A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C
 \end{array}
 \quad (7)$$

Proof. By the additivity of $\mathfrak{s}$, we have $\mathfrak{s}(\delta_1 \oplus \delta_2) = [A_1 \oplus A_2 \xrightarrow{x_1 \oplus x_2} B_1 \oplus B_2 \xrightarrow{y_1 \oplus y_2} C \oplus C]$. Let $A_1 \xrightleftharpoons[p_1]{\iota_1} A_1 \oplus A_2 \xrightleftharpoons[p_2]{\iota_2} A_2$ be a biproduct in $\mathcal{C}$. Put $\mu = (\Delta_C)^*(\delta_1 \oplus \delta_2)$ and take $\mathfrak{s}(\mu) = [A_1 \oplus A_2 \xrightarrow{j} M \xrightarrow{k} C]$. Then μ satisfies

$$p_{1*} \mu = \delta_1 \quad \text{and} \quad p_{2*} \mu = \delta_2. \quad (8)$$

Applying (ET4) to $\mathfrak{s}(0) = [A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{p_2} A_2]$ and $\mathfrak{s}(\mu) = [A_1 \oplus A_2 \xrightarrow{j} M \xrightarrow{k} C]$, we obtain $B'_2 \in \mathcal{C}$, $\theta_1 \in \mathbb{E}(B'_2, A_1)$ and a commutative diagram such that:

$$\begin{array}{ccc}
 A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 \xrightarrow{p_2} A_2 \\
 \parallel & \circlearrowleft & \downarrow j \quad \circlearrowleft \quad \downarrow x'_2 \\
 A_1 & \xrightarrow{m_1} & M \xrightarrow{e'_1} B'_2 \\
 & \downarrow k & \circlearrowleft \quad \downarrow y'_2 \\
 & C & \xlongequal{\quad} C
 \end{array}
 \quad
 \begin{aligned}
 \mathfrak{s}(p_{2*} \mu) &= [A_2 \xrightarrow{x'_2} B_2 \xrightarrow{y'_2} C], \\
 x_2^* \theta_1 &= 0, \\
 \mathfrak{s}(\theta_1) &= [A_1 \xrightarrow{m_1} M \xrightarrow{e'_1} B'_2],
 \end{aligned}$$

and $(\iota_1, \text{id}_M, y'_2)$ is a morphism of $\mathbb{E}$ -triangles. Especially, we have an equality $y_2'^* \mu = \iota_{1*} \theta_1$. In particular, we have

$$[A_2 \xrightarrow{x'_2} B_2 \xrightarrow{y'_2} C] = \mathfrak{s}(\delta_2) = [A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C].$$

Thus there is an isomorphism $b_2 \in \mathcal{C}(B_2, B'_2)$ satisfying $b_2 \circ x_2 = x'_2$ and $y'_2 \circ b_2 = y_2$. If we put $e_1 = b_2^{-1} \circ e'_1$, then $\mathfrak{s}(b_2^* \theta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2]$

by Proposition 3.7. Thus we obtain a commutative diagram (a) below

$$(a) \quad \begin{array}{ccccc} A_1 & \xrightarrow{\iota_1} & A_1 \oplus A_2 & \xrightarrow{p_2} & A_2 \\ \parallel & \circlearrowleft & \downarrow j & \circlearrowleft & \downarrow x_2 \\ A_1 & \xrightarrow{m_1} & M & \xrightarrow{e_1} & B_2 \\ & & \downarrow k & \circlearrowleft & \downarrow y_2 \\ & & C & \xlongequal{\quad} & C \end{array} \quad (b) \quad \begin{array}{ccccc} A_2 & \xrightarrow{\iota_2} & A_1 \oplus A_2 & \xrightarrow{p_1} & A_1 \\ \parallel & \circlearrowleft & \downarrow j & \circlearrowleft & \downarrow x_1 \\ A_2 & \xrightarrow{m_2} & M & \xrightarrow{e_2} & B_1 \\ & & \downarrow k & \circlearrowleft & \downarrow y_1 \\ & & C & \xlongequal{\quad} & C \end{array}$$

which satisfies $y_2^* \delta_1 = b_2^* y_2'^* p_1^* \mu = b_2^* p_1^* y_2'^* \mu = b_2^* p_1^* \iota_1^* \theta_1 = b_2^* \theta_1$. Thus we obtain $\mathfrak{s}(y_2^* \delta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2]$. Similarly, from $\mathfrak{s}(0) = [A_2 \xrightarrow{\iota_2} A_1 \oplus A_2 \xrightarrow{p_1} A_1]$ and $\mathfrak{s}(\mu) = [A_1 \oplus A_2 \xrightarrow{j} M \xrightarrow{k} C]$, we obtain a commutative diagram (b) above, which satisfies $\mathfrak{s}(y_1^* \delta_2) = [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1]$. Since $e_2 \circ m_1 = e_2 \circ j \circ \iota_1 = x_1 \circ p_1 \circ \iota_1 = x_1$, $e_1 \circ m_2 = e_1 \circ j \circ \iota_2 = x_2 \circ p_2 \circ \iota_2 = x_2$ and $y_2 \circ e_1 = k = y_1 \circ e_2$, diagram (7) is commutative. Moreover, we have

$$m_{1*} \delta_1 + m_{2*} \delta_2 = j_* (\iota_{1*} \delta_1 + \iota_{2*} \delta_2) = j_* ((\iota_1 \circ p_1)_* + (\iota_2 \circ p_2)_*) (\mu) = j_* \mu = 0$$

by (8) and Lemma 3.2. $\square$

Corollary 3.16. Let $x \in \mathcal{C}(A, B)$, $f \in \mathcal{C}(A, D)$ be any pair of morphisms. If x is an inflation, then so is $\begin{bmatrix} f \\ x \end{bmatrix} \in \mathcal{C}(A, D \oplus B)$. Dually for deflations.

Proof. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$ be an $\mathbb{E}$ -triangle. Realize $f_* \delta$ by an $\mathbb{E}$ -triangle $D \xrightarrow{d} E \xrightarrow{e} C \xrightarrow{f_* \delta} D$. By Proposition 3.15, we obtain a commutative diagram (*), below, made of $\mathbb{E}$ -triangles satisfying $m_* \delta + k_* f_* \delta = 0$. Since $y^* f_* \delta = f_* y^* \delta = 0$, we may assume $M = D \oplus B$, $k = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\ell = \begin{bmatrix} 0 & 1 \end{bmatrix}$,

and take $p \in \mathcal{C}(M, D)$, $i \in \mathcal{C}(B, M)$ which make $D \xrightleftharpoons[p]{k} M \xrightleftharpoons[\ell]{i} B$

a biproduct. By the exactness of $\mathcal{C}(B, M) \xrightarrow{\mathcal{C}(x, M)} \mathcal{C}(A, M) \xrightarrow{\delta^\#} \mathbb{E}(C, M)$ and the equality $\delta^\#(m + k \circ f) = m_* \delta + k_* f_* \delta = 0$, there exists $b \in \mathcal{C}(B, M)$ satisfying $b \circ x = m + k \circ f$. Modifying $A \xrightarrow{m} M \xrightarrow{e} E$ by the automorphism

$$n = \begin{bmatrix} -1 & p \circ b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \circ (\text{id}_M - k \circ p \circ b \circ \ell): M \xrightarrow{\cong} M,$$

we obtain a conflation $A \xrightarrow{n \circ m} D \oplus B \xrightarrow{e \circ n^{-1}} E$. Then, since

$$\begin{aligned} p \circ (n \circ m) &= -p \circ (\text{id}_M - k \circ p \circ b \circ \ell) \circ m \\ &= p \circ k \circ p \circ b \circ \ell \circ m - p \circ m \\ &= p \circ b \circ x - p \circ m = p \circ k \circ f = f \end{aligned}$$

and $\ell \circ (n \circ m) = \ell \circ (\text{id}_M - k \circ p \circ b \circ \ell) \circ m = \ell \circ m = x$, we have $n \circ m = \begin{bmatrix} f \\ x \end{bmatrix}$.

$$(*) \quad \begin{array}{ccccc} & & A & \xlongequal{\quad} & A \\ & & \downarrow m & \circlearrowleft & \downarrow x \\ D & \xrightarrow{k} & M & \xrightarrow{\ell} & B \xrightarrow{y^* f_* \delta} \twoheadrightarrow \\ & \circlearrowleft & \downarrow e & \circlearrowleft & \downarrow y \\ D & \xrightarrow{d} & E & \xrightarrow{e} & C \xrightarrow{f_* \delta} \twoheadrightarrow \\ & & \downarrow e^* \delta & & \downarrow \delta \\ & & \Downarrow & & \Downarrow \end{array}$$

□

Proposition 3.17. Suppose we are given $\mathbb{E}$ -triangles $D \xrightarrow{f} A \xrightarrow{f'} C \xrightarrow{\delta_f} \twoheadrightarrow$, $A \xrightarrow{g} B \xrightarrow{g'} F \xrightarrow{\delta_g} \twoheadrightarrow$, $E \xrightarrow{h} B \xrightarrow{h'} C \xrightarrow{\delta_h} \twoheadrightarrow$ satisfying $h' \circ g = f'$. Then there is an $\mathbb{E}$ -triangle $D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{\theta} \twoheadrightarrow$ which makes diagram (9) commutative in $\mathcal{C}$ and satisfy the following equalities:

$$\begin{aligned} & \text{(i) } d_*(\delta_f) = \delta_h, \\ & \text{(ii) } f_*(\theta) = \delta_g, \\ & \text{(iii) } g'^*(\theta) + h'^*(\delta_f) = 0. \end{aligned} \quad \begin{array}{ccccc} D & \xrightarrow{f} & A & \xrightarrow{f'} & C \\ d \downarrow & \circlearrowleft & \downarrow g & \circlearrowleft & \parallel \\ E & \xrightarrow{h} & B & \xrightarrow{h'} & C \\ e \downarrow & \circlearrowleft & \downarrow g' & & \\ F & \xlongequal{\quad} & F & & \end{array} \quad (9)$$

Proof. By axiom (ET4), we have two $\mathbb{E}$ -triangles $D \xrightarrow{g \circ f} B \xrightarrow{a} G \xrightarrow{\mu} \twoheadrightarrow$ and $C \xrightarrow{b} G \xrightarrow{c} F \xrightarrow{\nu} \twoheadrightarrow$ which make the following diagram commutative

$$\begin{array}{ccccc}
D & \xrightarrow{f} & A & \xrightarrow{f'} & C \\
\parallel & \circlearrowleft & \downarrow g & \circlearrowleft & \downarrow b \\
D & \xrightarrow{g \circ f} & B & \xrightarrow{a} & G \\
& & \downarrow g' & \circlearrowleft & \downarrow c \\
& & F & \xlongequal{\quad} & F
\end{array} \tag{10}$$

and satisfy $f'_*(\delta_g) = \nu$, $b^*\mu = \delta_f$, $c^*(\delta_g) = f_*\mu$. It follows from Lemma 3.2 that $\nu = f'_*(\delta_g) = h'_*g_*(\delta_g) = 0$. Thus, up to equivalence, we may assume $G = C \oplus F$, $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 1 \end{bmatrix}$ from the start. Then $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} : B \rightarrow G = C \oplus F$ satisfies $a_1 \circ g = f'$ and $a_2 = g'$ by the commutativity of (10). Since $h' - a_1 \in \mathcal{C}(B, C)$ satisfies $(h' - a_1) \circ g = f' - f' = 0$, there exists $z \in \mathcal{C}(F, C)$ satisfying $z \circ g' = h' - a_1$. Put $z' = \begin{bmatrix} -z \\ 1 \end{bmatrix}$. Applying the dual of Lemma 3.14 to the following diagram (*) made of $\mathbb{E}$ -triangles,

$$\begin{array}{ccc}
(*) & \begin{array}{ccccc}
& E & & F & \\
& \downarrow h & & \downarrow z' & \\
D & \xrightarrow{g \circ f} B & \xrightarrow{a} G & \xrightarrow{\mu} & \\
& \downarrow h' & \circlearrowleft & \downarrow [1 \ z] & \\
& C & \xlongequal{\quad} & C & \\
& \downarrow \delta_h & & \downarrow 0 & \\
& \Psi & & \Psi &
\end{array} & (**) \quad \begin{array}{ccccc}
D & \xrightarrow{d} E & \xrightarrow{e} F & & \\
\parallel & \circlearrowleft & \downarrow h & \circlearrowleft & \downarrow z' \\
D & \xrightarrow{g \circ f} B & \xrightarrow{a} G & & \\
& \downarrow h' & \circlearrowleft & \downarrow [1 \ z] & \\
& C & \xlongequal{\quad} & C &
\end{array}
\end{array}$$

we obtain an $\mathbb{E}$ -triangle $D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{\theta} D$ which makes diagram (**) above commutative, and satisfies $\theta = z'^*\mu$, $d_*\mu = [1 \ z]^*(\delta_h)$. Then the commutativity of (9) can be checked in a straightforward way. Let us show the equalities (i),(ii),(iii).

(i) follows from $d_*(\delta_f) = d_*b^*\mu = b^*[1 \ z]^*(\delta_h) = ([1 \ z] \circ b)^*(\delta_h) = \delta_h$.

(ii) follows from the injectivity of $\mathbb{E}(c, A) = c^*$ and

$$\begin{aligned}
c^*f_*(\theta) &= c^*f_*z'^*\mu = f_*(z' \circ c)^*\mu \\
&= f_* \begin{bmatrix} 0 & -z \\ 0 & 1 \end{bmatrix}^* \mu = f_*(1 - b \circ [1 \ z])^*\mu \\
&= f_*\mu - f_*[1 \ z]^*(\delta_f) = f_*\mu - [1 \ z]^*f_*(\delta_f) = f_*\mu = c^*(\delta_g).
\end{aligned}$$

(iii) follows from

$$\begin{aligned} g'^*(\theta) + h'^*(\delta_f) &= g'^* z'^* \mu + h'^* b^* \mu = \left(\begin{bmatrix} -z & g' \\ & g' \end{bmatrix} + \begin{bmatrix} h' \\ 0 \end{bmatrix} \right)^* \mu \\ &= \begin{bmatrix} a_1 \\ g' \end{bmatrix}^* \mu = a^* \mu = 0. \end{aligned}$$

□

As in Example 2.13, an exact category (with some smallness assumption) can be regarded as an extriangulated category, whose inflations are monomorphic and whose deflations are epimorphic. Conversely, the following holds.

Corollary 3.18. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, in which any inflation is monomorphic, and any deflation is epimorphic. If we let $\mathcal{S}$ be the class of conflations given by the $\mathbb{E}$ -triangles (see Definition 2.15), then $(\mathcal{C}, \mathcal{S})$ is an exact category in the sense of [Bu].

Proof. By the exact sequences obtained in Proposition 3.3, for any conflation $A \xrightarrow{x} B \xrightarrow{y} C$, the pair (A, x) gives a weak kernel of y . Since x is monomorphic by assumption, it is a kernel of y . Dually (C, y) gives a cokernel of x , and $A \xrightarrow{x} B \xrightarrow{y} C$ becomes a kernel-cokernel pair.

Thus $\mathcal{S}$ consists of some kernel-cokernel pairs. Moreover, it is closed under isomorphisms. Indeed, let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \cdot$ be any $\mathbb{E}$ -triangle, let $A' \xrightarrow{x'} B' \xrightarrow{y'} C'$ be a kernel-cokernel pair, and suppose that there are isomorphisms $a \in \mathcal{C}(A, A')$, $b \in \mathcal{C}(B, B')$ and $c \in \mathcal{C}(C, C')$ satisfying $x' \circ a = b \circ x$ and $y' \circ b = c \circ y$. By Proposition 3.7, we obtain an $\mathbb{E}$ -triangle $A' \xrightarrow{x \circ a^{-1}} B \xrightarrow{c \circ y} C' \xrightarrow{(c^{-1})^* a_* \delta} \cdot$. This gives $\mathfrak{s}(\delta) = [A' \xrightarrow{x \circ a^{-1}} B \xrightarrow{c \circ y} C'] = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, which means that $A' \xrightarrow{x'} B' \xrightarrow{y'} C'$ belongs to $\mathcal{S}$.

Let us confirm conditions [E0],[E1],[E2] in [Bu, Definition 2.1]. Since our assumptions are self-dual, the other conditions [E0^{op}],[E1^{op}],[E2^{op}] can be shown dually.

[E0] For any object $A \in \mathcal{C}$, the split sequence $A \xrightarrow{\text{id}_A} A \rightarrow 0$ belongs to $\mathcal{S}$ by (ET2).

[E1] The class of inflations (= admissible monics) is closed under composition by (ET4), as in Remark 2.16.

[E2] Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$ be any $\mathbb{E}$ -triangle, and let $a \in \mathcal{C}(A, A')$ be any morphism. By Corollary 3.16, there exists a conflation

$$A \xrightarrow{s} B \oplus A' \xrightarrow{\exists [b \ x']} \exists B',$$

where $s = \begin{bmatrix} x \\ -a \end{bmatrix}$. Since it becomes a kernel-cokernel pair by the above argument, it follows that (a) is a pushout square. By the dual of Proposition 3.17, we obtain a commutative diagram (b) made of conflations, which shows that x' is an inflation.

$$(a) \quad \begin{array}{ccc} A & \xrightarrow{x} & B \\ a \downarrow & \circlearrowleft & \downarrow b \\ A' & \xrightarrow{x'} & B' \end{array} \quad (b) \quad \begin{array}{ccccc} & & A' & \xlongequal{\quad} & A' \\ & & \downarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \circlearrowleft & \downarrow x' \\ A & \xrightarrow{s} & B \oplus A' & \xrightarrow{[b \ x']} & B' \\ \parallel & \circlearrowleft & \downarrow [1 \ 0] & \circlearrowleft & \downarrow \exists y' \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C \end{array}$$

□

3.3 Relation with triangulated categories

In this section, let $\mathcal{C}$ be an additive category equipped with an equivalence $[1]: \mathcal{C} \xrightarrow{\sim} \mathcal{C}$, and let $\mathbf{E}^1: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$ be the bifunctor defined by $\mathbf{E}^1 = \text{Ext}^1(-, -) = \mathcal{C}(-, -[1])$.

Remark 3.19. As usual, we use notations like $X[1]$ and $f[1]$ for objects X and morphisms f in $\mathcal{C}$. The n -times composition of $[1]$ is denoted by $[n]$.

We will show that, to give a triangulation of $\mathcal{C}$ with shift functor $[1]$, is equivalent to give an $\mathbf{E}^1$ -triangulation of $\mathcal{C}$ (Proposition 3.22).

Remark 3.20. Let $\mathcal{C}, [1], \mathbf{E}^1$ be as above. Then for any $\delta \in \mathbf{E}^1(C, A) = \mathcal{C}(C, A[1])$, we have the following.

- (1) $\delta_{\#} = \mathcal{C}(-, \delta): \mathcal{C}(-, C) \Rightarrow \mathcal{C}(-, A[1])$.
- (2) $\delta^{\#}$ is given by $\delta_X^{\#}: \mathcal{C}(A, X) \rightarrow \mathcal{C}(C, X[1])$; $f \mapsto (f[1]) \circ \delta$, for any $X \in \mathcal{C}$.

Lemma 3.21. Let $\mathcal{C}, [1], \mathbf{E}^1$ be as above. Suppose that $\mathfrak{s}$ is an $\mathbf{E}^1$ -triangulation of $\mathcal{C}$. Then for any $A \in \mathcal{C}$, the $\mathbf{E}^1$ -extension $\mathbf{1} = \text{id}_{A[1]} \in \mathbf{E}^1(A[1], A) = \mathcal{C}(A[1], A[1])$ can be realized as $\mathfrak{s}(\mathbf{1}) = [A \rightarrow 0 \rightarrow A[1]]$. Namely, $A \rightarrow 0 \rightarrow A[1] \xrightarrow{-1}$ is an $\mathbf{E}^1$ -triangle.

Proof. Put $\mathfrak{s}(\mathbf{1}) = [A \xrightarrow{x} X \xrightarrow{y} A[1]]$. By Proposition 3.3, $\mathcal{C}(-, A) \xrightarrow{\mathcal{C}(-, x)} \mathcal{C}(-, X) \xrightarrow{\mathcal{C}(-, y)} \mathcal{C}(-, A[1]) \xrightarrow{\mathbf{1}_{\#} = \text{id}} \mathcal{C}(-, A[1]) \xrightarrow{\mathcal{C}(-, x[1])} \mathcal{C}(-, X[1])$ is exact. In particular $\text{id}_X \in \mathcal{C}(X, X)$ satisfies $y = (\mathbf{1}_{\#})_X \circ \mathcal{C}(X, y)(\text{id}_X) = 0$. Similarly $x[1] = 0$ implies $x = 0$. Thus $0 \Rightarrow \mathcal{C}(-, X) \Rightarrow 0$ becomes exact, which shows $X = 0$. $\square$

Proposition 3.22. As before, let $\mathcal{C}$ be an additive category equipped with an auto-equivalence $[1]$, and put $\mathbf{E}^1 = \mathcal{C}(-, -[1])$. Then we have the following.

- (1) Suppose $\mathcal{C}$ is a triangulated category with shift functor $[1]$. For any $\delta \in \mathbf{E}^1(C, A) = \mathcal{C}(C, A[1])$, take a distinguished triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$$

and define as $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$. Remark that this $\mathfrak{s}(\delta)$ does not depend on the choice of the distinguished triangle above. With this definition, $(\mathcal{C}, \mathbf{E}^1, \mathfrak{s})$ becomes an extriangulated category.

- (2) Suppose we are given an $\mathbf{E}^1$ -triangulation $\mathfrak{s}$ of $\mathcal{C}$. Define that $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$ is a distinguished triangle if and only if it satisfies $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$. With this class of distinguished triangles, $\mathcal{C}$ becomes a triangulated category.

By construction, distinguished triangles correspond to $\mathbf{E}^1$ -triangles by the above (1) and (2).

Proof. (1) is straightforward. For (2), all the axioms except for (TR2) are easily confirmed. Let us show (TR2).

Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$ be any $\mathbf{E}^1$ -triangle. Applying Proposition 3.15

to $A \rightarrow 0 \rightarrow A[1] \xrightarrow{-1}$ and δ , we obtain

$$\begin{array}{ccc}
 A & \xrightarrow{x} & B \xrightarrow{y} C \\
 \downarrow & \circlearrowleft & \downarrow m' \circlearrowleft \\
 0 & \longrightarrow & \exists M \xrightarrow{e} C \\
 \downarrow & \circlearrowleft & \downarrow e' \\
 A[1] & = & A[1]
 \end{array}
 \quad \text{with} \quad
 \begin{array}{l}
 \text{(i) } [0 \rightarrow M \xrightarrow{e} C] = 0_*\delta = 0, \\
 \text{(ii) } e^*\delta + e'^*\mathbf{1} = 0, \\
 \text{(iii) } \mathfrak{s}(x[1]) = \mathfrak{s}(x_*\mathbf{1}) = [B \xrightarrow{m'} M \xrightarrow{e'} A[1]].
 \end{array}$$

Condition (i) shows that e is an isomorphism, by Remark 2.11 (1). Condition (ii) means $\delta \circ e + e' = 0$ in $\mathcal{C}(M, A[1])$, namely $e' \circ e^{-1} = -\delta$. Thus we have $\mathfrak{s}(x[1]) = [B \xrightarrow{y} C \xrightarrow{-\delta} A[1]]$ by condition (iii), which means that $B \xrightarrow{y} C \xrightarrow{-\delta} A[1] \xrightarrow{x[1]} B[1]$ is a distinguished triangle. This is isomorphic to $B \xrightarrow{y} C \xrightarrow{\delta} A[1] \xrightarrow{-x[1]} B[1]$. $\square$

3.4 Projectives and injectives

If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ has enough “*projectives*”, then the bifunctor $\mathbb{E}$ can be described in terms of them. Throughout this section, let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Duals of the results in this section hold true for “*injectives*”.

Definition 3.23. An object $P \in \mathcal{C}$ is called *projective* if, for any $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ and any morphism c in $\mathcal{C}(P, C)$, there exists $b \in \mathcal{C}(P, B)$ satisfying $y \circ b = c$.

We denote the full subcategory of projective objects in $\mathcal{C}$ by $\text{Proj}(\mathcal{C})$. Dually, the full subcategory of injective objects in $\mathcal{C}$ is denoted by $\text{Inj}(\mathcal{C})$.

Proposition 3.24. An object $P \in \mathcal{C}$ is projective if and only if it satisfies $\mathbb{E}(P, A) = 0$ for any $A \in \mathcal{C}$.

Proof. Sufficiency of $\mathbb{E}(P, A) = 0$ follows from the exact sequence of Proposition 3.3. Conversely, suppose P is projective. Let $A \in \mathcal{C}$ be any object, and let $\delta \in \mathbb{E}(P, A)$ be any element, with $\mathfrak{s}(\delta) = [A \xrightarrow{x} M \xrightarrow{y} P]$. Since P is projective, there exists $m \in \mathcal{C}(P, M)$ which makes the following

diagram commutative.

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \xrightarrow{\text{id}_P} & P \\ & & m \downarrow & \circlearrowleft & \downarrow \text{id}_P \\ A & \xrightarrow{x} & M & \xrightarrow{y} & P \end{array}$$

By (ET3)^{op}, the triplet $(0, m, \text{id}_P)$ realizes the morphism $(0, \text{id}_P): 0 \rightarrow \delta$. Especially we have $\delta = \mathbb{E}(P, 0)(0) = 0$. $\square$

Definition 3.25. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, as before. We say that it *has enough projectives* if, for any object $C \in \mathcal{C}$, there exists an $\mathbb{E}$ -triangle $A \xrightarrow{x} P \xrightarrow{y} C \dashrightarrow^\delta$ satisfying $P \in \text{Proj}(\mathcal{C})$.

Example 3.26. (1) If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an exact category, then these agree with the usual definitions.

- (2) If $(\mathcal{C}, \mathbb{E}^1, \mathfrak{s})$ is a triangulated category as in the previous section, then $\text{Proj}(\mathcal{C})$ consists of zero objects. Moreover it always has enough projectives.
- (3) If $(\mathcal{C}, \mathbb{E}^1, \mathfrak{s})$ is a triangulated category with a rigid subcategory $\mathcal{R}$ (i.e. for all $R_1, R_2 \in \mathcal{R}$, $\text{Ext}^1(R_1, R_2) = 0$), let $\mathcal{D}$ be its full subcategory whose objects are those objects X that satisfy $\text{Ext}^1(R, X) = 0$ for all $R \in \mathcal{R}$. Then $\mathcal{D}$ is an additive and extension-closed subcategory of $\mathcal{C}$, which is thus extriangulated by Remark 2.18. We then have:

- (a) $\mathcal{R} \subseteq \text{Proj}(\mathcal{D})$;
- (b) $\text{Proj}(\mathcal{D}) = \mathcal{R}$ and $\mathcal{D}$ has enough projectives if and only if $\mathcal{R}$ is contravariantly finite.

Corollary 3.27. Assume that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ has enough projectives. For any object $C \in \mathcal{C}$ and any $\mathbb{E}$ -triangle $A \xrightarrow{x} P \xrightarrow{y} C \dashrightarrow^\delta$ with $P \in \text{Proj}(\mathcal{C})$, the sequence $\mathcal{C}(P, -) \xrightarrow{\mathcal{C}(x, -)} \mathcal{C}(A, -) \xrightarrow{\delta^\sharp} \mathbb{E}(C, -) \Rightarrow 0$ is exact. Namely, we have a natural isomorphism $\mathbb{E}(C, -) \cong \text{Cok}(\mathcal{C}(x, -))$.

Proof. This immediately follows from Propositions 3.3 and 3.24. $\square$

The isomorphism in Corollary 3.27 is natural in C , in the following sense.

Remark 3.28. Assume that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ has enough projectives. Let $c: C \rightarrow C'$ be any morphism, and let $A \xrightarrow{x} P \xrightarrow{y} C \xrightarrow{\delta} \succ$, $A' \xrightarrow{x'} P' \xrightarrow{y'} C' \xrightarrow{\delta'} \succ$ be any pair of $\mathbb{E}$ -triangles satisfying $P, P' \in \text{Proj}(\mathcal{C})$. By the projectivity of P and $(\text{ET3})^{\text{op}}$, we obtain a morphism of $\mathbb{E}$ -triangles, and thus a morphism of exact sequences, as follows:

$$\begin{array}{ccc} A \xrightarrow{x} P \xrightarrow{y} C \xrightarrow{\delta} \succ & \mathcal{C}(P, -) \xrightarrow{\mathcal{C}(x, -)} \mathcal{C}(A, -) \xrightarrow{\delta^\#} \mathbb{E}(C, -) \Rightarrow 0 \\ \downarrow a \quad \circ \quad \downarrow p \quad \circ \quad \downarrow c & \mathcal{C}(p, -) \uparrow \quad \circ \mathcal{C}(a, -) \uparrow \quad \circ \quad \uparrow \mathbb{E}(c, -) \\ A' \xrightarrow{x'} P' \xrightarrow{y'} C' \xrightarrow{\delta'} \succ & \mathcal{C}(P', -) \xrightarrow{\mathcal{C}(x', -)} \mathcal{C}(A', -) \xrightarrow{\delta'^\#} \mathbb{E}(C', -) \Rightarrow 0. \end{array}$$

Lemma 3.29. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \succ$ be any $\mathbb{E}$ -triangle, and let i in $\mathcal{C}(A, I)$ be any morphism with $I \in \text{Inj}(\mathcal{C})$. Write p_C for the projection $C \oplus I \rightarrow C$. Then the $\mathbb{E}$ -extension $p_C^* \delta$ is realized by an $\mathbb{E}$ -triangle of the form

$$A \xrightarrow{x_I} B \oplus I \xrightarrow{y_I} C \oplus I \xrightarrow{p_C^* \delta} \succ, \quad (x_I = \begin{bmatrix} x \\ i \end{bmatrix}, y_I = \begin{bmatrix} y & * \\ * & * \end{bmatrix}). \quad (11)$$

Proof. By Corollary 3.16, we have an $\mathbb{E}$ -triangle $A \xrightarrow{x_I} B \oplus I \xrightarrow{d} D \xrightarrow{\nu} \succ$. By the dual of Proposition 3.17, we obtain the following commutative diagram (α) made of $\mathbb{E}$ -triangles

$$\begin{array}{ccc} (\alpha) & \begin{array}{c} I \xlongequal{\quad} I \\ \downarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \circ \quad \downarrow e \\ A \xrightarrow{x_I} B \oplus I \xrightarrow{d} D \xrightarrow{\nu} \succ \\ \parallel \quad \circ \quad \downarrow [1 \ 0] \quad \circ \quad \downarrow f \\ A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \succ \\ \downarrow 0 \quad \downarrow \exists \theta \\ \Downarrow \quad \Downarrow \end{array} & (\beta) & \begin{array}{c} A \xrightarrow{x_I} B \oplus I \xrightarrow{n^{-1} \circ d} C \oplus I \xrightarrow{n^* \nu} \succ \\ \parallel \quad \circ \quad \downarrow [1 \ 0] \quad \circ \quad \downarrow [1 \ 0] = p_C \\ A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \succ \end{array} \end{array}$$

satisfying $f^* \delta = \nu$. Since $I \in \text{Inj}(\mathcal{C})$, we have $\theta = 0$. Thus there is some isomorphism $n: C \oplus I \xrightarrow{\cong} D$ satisfying $n \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e$ and $f \circ n = [1 \ 0]$. Then for $p_C = [1 \ 0]: C \oplus I \rightarrow C$, diagram (β) above is a morphism of $\mathbb{E}$ -triangles. Then $n^{-1} \circ d$ satisfies $p_C \circ n^{-1} \circ d \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = y \circ [1 \ 0] \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = y$, and thus is of the form $\begin{bmatrix} y & * \\ * & * \end{bmatrix}$. $\square$

The following construction gives extriangulated categories which are not exact nor triangulated in general.

Proposition 3.30. Let $\mathcal{I} \subseteq \mathcal{C}$ be a full additive subcategory, closed under isomorphisms. If $\mathcal{I}$ satisfies $\mathcal{I} \subseteq \text{Proj}(\mathcal{C}) \cap \text{Inj}(\mathcal{C})$, then the ideal quotient $\mathcal{C}/\mathcal{I}$ has the structure of an extriangulated category, induced from that of $\mathcal{C}$. In particular, we can associate a “*reduced*” extriangulated category $\mathcal{C}' = \mathcal{C}/(\text{Proj}(\mathcal{C}) \cap \text{Inj}(\mathcal{C}))$ satisfying $\text{Proj}(\mathcal{C}') \cap \text{Inj}(\mathcal{C}') = 0$, to any extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

Proof. Put $\overline{\mathcal{C}} = \mathcal{C}/\mathcal{I}$. Let us confirm conditions (ET1),(ET2),(ET3),(ET4). The other conditions $(\text{ET3})^{\text{op}}, (\text{ET4})^{\text{op}}$ can be shown dually.

(ET1) Since $\mathbb{E}(\mathcal{I}, \mathcal{C}) = \mathbb{E}(\mathcal{C}, \mathcal{I}) = 0$, one can define the biadditive functor $\overline{\mathbb{E}}: \overline{\mathcal{C}}^{\text{op}} \times \overline{\mathcal{C}} \rightarrow \text{Ab}$ given by

- $\overline{\mathbb{E}}(C, A) = \mathbb{E}(C, A) \quad (\forall A, C \in \mathcal{C}),$
- $\overline{\mathbb{E}}(\overline{c}, \overline{a}) = \mathbb{E}(c, a) \quad (\forall a \in \mathcal{C}(A, A'), c \in \mathcal{C}(C, C')).$ Here, $\overline{a}$ and $\overline{c}$ denote the images of a and c in $\mathcal{C}/\mathcal{I}$.

(ET2) For any $\overline{\mathbb{E}}$ -extension $\delta \in \overline{\mathbb{E}}(C, A) = \mathbb{E}(C, A)$, define

$$\overline{\mathfrak{s}}(\delta) = \overline{\mathfrak{s}(\delta)} = [A \xrightarrow{\overline{x_0}} B \xrightarrow{\overline{y_0}} C],$$

using $\mathfrak{s}(\delta) = [A \xrightarrow{x_0} B \xrightarrow{y_0} C]$. Let us show that $\overline{\mathfrak{s}}$ is an additive realization of $\overline{\mathbb{E}}$.

Let $(\overline{a}, \overline{c}): \delta = (A, \delta, C) \rightarrow \delta' = (A', \delta', C')$ be any morphism of $\overline{\mathbb{E}}$ -extensions. By definition, this is equivalent to that $(a, c): \delta \rightarrow \delta'$ is a morphism of $\mathbb{E}$ -extensions. Put $\overline{\mathfrak{s}}(\delta) = [A \xrightarrow{\overline{x}} B \xrightarrow{\overline{y}} C]$, $\overline{\mathfrak{s}}(\delta') = [A' \xrightarrow{\overline{x'}} B' \xrightarrow{\overline{y'}} C']$. Since the condition in Definition 2.9 does not depend on the representatives of the equivalence classes of sequences in $\overline{\mathcal{C}}$, we may assume $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Then there is $b \in \mathcal{C}(B, B')$ with which (a, b, c) realizes (a, c) . It follows that $(\overline{a}, \overline{b}, \overline{c})$ realizes $(\overline{a}, \overline{c})$.

As for the additivity, the equality $\overline{\mathfrak{s}}(0) = 0$ is trivially satisfied. Since $\overline{\mathfrak{s}}(\delta) \oplus \overline{\mathfrak{s}}(\delta')$ only depends on the equivalence classes $\overline{\mathfrak{s}}(\delta)$ and $\overline{\mathfrak{s}}(\delta')$, the equality $\overline{\mathfrak{s}}(\delta \oplus \delta') = \overline{\mathfrak{s}}(\delta) \oplus \overline{\mathfrak{s}}(\delta')$ follows from $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.

(ET3) Suppose we are given $\bar{\mathfrak{s}}(\delta) = [A \xrightarrow{\bar{x}} B \xrightarrow{\bar{y}} C]$, $\bar{\mathfrak{s}}(\delta') = [A' \xrightarrow{\bar{x}'} B' \xrightarrow{\bar{y}'} C']$, and morphisms $\bar{a} \in \overline{\mathcal{C}}(A, A')$, $\bar{b} \in \overline{\mathcal{C}}(B, B')$ satisfying $\bar{x}' \circ \bar{a} = \bar{b} \circ \bar{x}$. As in the proof of (ET2), we may assume $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. By $\bar{x}' \circ \bar{a} = \bar{b} \circ \bar{x}$, there exist $I \in \mathcal{I}$, $i \in \mathcal{C}(A, I)$, $j \in \mathcal{C}(I, B')$ which satisfy $x' \circ a = b \circ x + j \circ i$. By Lemma 3.29, we obtain an $\mathbb{E}$ -triangle (11). This gives the following isomorphism of $\overline{\mathbb{E}}$ -triangles.

$$\begin{array}{ccccccc} A & \xrightarrow{\bar{x}_I} & B \oplus I & \xrightarrow{\bar{y}_I} & C \oplus I & \xrightarrow{p_C^* \delta} & \\ \parallel & \circ & \downarrow \bar{p}_B & \circ & \downarrow \bar{p}_C & & \\ A & \xrightarrow{\bar{x}} & B & \xrightarrow{\bar{y}} & C & \xrightarrow{\delta} & \end{array} \quad (12)$$

On the other hand, by axiom (ET3) for $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, we have a morphism of $\mathbb{E}$ -triangles as follows.

$$\begin{array}{ccccccc} A & \xrightarrow{x_I} & B \oplus I & \xrightarrow{y_I} & C \oplus I & \xrightarrow{p_C^* \delta} & \\ \parallel & \circ & \downarrow [b \ j] & \circ & \downarrow c & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{\delta'} & \end{array} \quad (13)$$

From the two diagrams (12) and (13), we obtain a morphism of $\overline{\mathbb{E}}$ -extensions $(\bar{a}, \bar{c} \circ \bar{p}_C^{-1}): \delta \rightarrow \delta'$ which satisfies $(\bar{c} \circ \bar{p}_C^{-1}) \circ \bar{y} = \bar{y}' \circ \bar{b}$.

(ET4) Let $A \xrightarrow{\bar{f}} B \xrightarrow{\bar{f}'} D \xrightarrow{\delta}$ and $B \xrightarrow{\bar{g}} C \xrightarrow{\bar{g}'} F \xrightarrow{\delta'}$ be $\overline{\mathbb{E}}$ -triangles. As in the above arguments, we may assume $A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta}$ and $B \xrightarrow{g} C \xrightarrow{g'} F \xrightarrow{\delta'}$ are $\mathbb{E}$ -triangles. Then by (ET4) for $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, we obtain a commutative diagram (3) made of conflations, satisfying $\mathfrak{s}(f'_* \delta') = [D \xrightarrow{d} E \xrightarrow{e} F]$, $d^* \delta'' = \delta$ and $f_* \delta'' = e^* \delta'$. The image of this diagram in $\overline{\mathcal{C}}$ shows (ET4) for $(\overline{\mathcal{C}}, \overline{\mathbb{E}}, \bar{\mathfrak{s}})$. $\square$

Remark 3.31. Proposition 3.30 applied to an exact category, together with Corollary 3.18, gives another proof³ of [DI, Theorem 3.5].

Corollary 3.32. Let $\mathcal{I} \subseteq \mathcal{C}$ be a strictly full additive subcategory, satisfying $\mathbb{E}(\mathcal{I}, \mathcal{I}) = 0$. Let $\mathcal{Z} \subseteq \mathcal{C}$ be the full subcategory of those $Z \in \mathcal{C}$ satisfying $\mathbb{E}(Z, \mathcal{I}) = \mathbb{E}(\mathcal{I}, Z) = 0$. Then, $\mathcal{Z}/\mathcal{I}$ is extriangulated.

³The first author wishes to thank Professor Osamu Iyama for informing this to him.

Proof. This follows from Remark 2.18 and Proposition 3.30. $\square$

4. Cotorsion pairs

In the rest of this article, let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

4.1 Cotorsion pairs

Definition 4.1. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$ be a pair of full additive subcategories, closed under isomorphisms and direct summands. The pair $(\mathcal{U}, \mathcal{V})$ is called a *cotorsion pair* on $\mathcal{C}$ if it satisfies the following conditions.

- (1) $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$.
- (2) For any $C \in \mathcal{C}$, there exists a conflation $V^C \rightarrow U^C \rightarrow C$ satisfying $U^C \in \mathcal{U}, V^C \in \mathcal{V}$.
- (3) For any $C \in \mathcal{C}$, there exists a conflation $C \rightarrow V_C \rightarrow U_C$ satisfying $U_C \in \mathcal{U}, V_C \in \mathcal{V}$.

Definition 4.2. Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{C}$ be any pair of full subcategories closed under isomorphisms. Define full subcategories $\text{Cone}(\mathcal{X}, \mathcal{Y})$ and $\text{CoCone}(\mathcal{X}, \mathcal{Y})$ of $\mathcal{C}$ as follows. These are closed under isomorphisms.

- (i) $C \in \mathcal{C}$ belongs to $\text{Cone}(\mathcal{X}, \mathcal{Y})$ if and only if it admits a conflation $X \rightarrow Y \rightarrow C$ satisfying $X \in \mathcal{X}, Y \in \mathcal{Y}$.
- (ii) $C \in \mathcal{C}$ belongs to $\text{CoCone}(\mathcal{X}, \mathcal{Y})$ if and only if it admits a conflation $C \rightarrow X \rightarrow Y$ satisfying $X \in \mathcal{X}, Y \in \mathcal{Y}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are additive subcategories of $\mathcal{C}$, then so are $\text{Cone}(\mathcal{X}, \mathcal{Y})$ and $\text{CoCone}(\mathcal{X}, \mathcal{Y})$, by condition (ET2).

Remark 4.3. In the case of exact categories, cotorsion pairs satisfying (2) and (3) are often called complete cotorsion pairs. Since all cotorsion pairs considered in this article are complete, this adjective is omitted. Also remark that completeness is equivalent to the equalities $\mathcal{C} = \text{Cone}(\mathcal{V}, \mathcal{U}) = \text{CoCone}(\mathcal{V}, \mathcal{U})$, in the notation of Definition 4.2.

Remark 4.4. Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. By Remark 2.11, for any $C \in \mathcal{C}$: $C \in \mathcal{U} \Leftrightarrow \mathbb{E}(C, \mathcal{V}) = 0$ and $C \in \mathcal{V} \Leftrightarrow \mathbb{E}(\mathcal{U}, C) = 0$.

Corollary 4.5. Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Let $C \in \mathcal{C}$, $U \in \mathcal{U}$ be any pair of objects. If there exists a section $C \rightarrow U$ or a retraction $U \rightarrow C$, then C also belongs to $\mathcal{U}$. Similarly for $\mathcal{V}$.

Proof. In either case, there are $s \in \mathcal{C}(C, U)$ and $r \in \mathcal{C}(U, C)$ satisfying $r \circ s = \text{id}_C$. This gives the following commutative diagram of natural transformations.

$$\begin{array}{ccc} & \mathbb{E}(U, -) & \\ \mathbb{E}(r, -) \nearrow & & \searrow \mathbb{E}(s, -) \\ \mathbb{E}(C, -) & \circlearrowright & \mathbb{E}(C, -) \\ & \mathbb{E}(\text{id}_C, -) = \text{id} & \end{array}$$

Thus $\mathbb{E}(U, \mathcal{V}) = 0$ implies $\mathbb{E}(C, \mathcal{V}) = 0$, and thus $C \in \mathcal{U}$ by Remark 4.4. $\square$

Remark 4.6. Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on $\mathcal{C}$. By Proposition 3.11, the subcategories $\mathcal{U}$ and $\mathcal{V}$ are extension-closed in $\mathcal{C}$.

Remark 4.7. By Proposition 3.24 and Remark 4.4, we have: $(\mathcal{X}, \mathcal{C})$ is a cotorsion pair for some subcategory $\mathcal{X} \subseteq \mathcal{C}$ if and only if $(\text{Proj}(\mathcal{C}), \mathcal{C})$ is a cotorsion pair; if and only if $\mathcal{C}$ has enough projectives. A dual remark concerning injectives holds.

4.2 Associated adjoint functors

Definition 4.8. For a cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{C}$, put $\mathcal{I} = \mathcal{U} \cap \mathcal{V}$, and call it the *core* of $(\mathcal{U}, \mathcal{V})$. For any full additive subcategory $\mathcal{X} \subseteq \mathcal{C}$ containing $\mathcal{I}$, let $\mathcal{X}/\mathcal{I}$ denote the ideal quotient. The image of a morphism f in the ideal quotient is denoted by $\overline{f}$.

Lemma 4.9. For any cotorsion pair $(\mathcal{U}, \mathcal{V})$, we have $(\mathcal{C}/\mathcal{I})(\mathcal{U}/\mathcal{I}, \mathcal{V}/\mathcal{I}) = 0$. Namely, for any $U \in \mathcal{U}$ and $V \in \mathcal{V}$, any morphism $f \in \mathcal{C}(U, V)$ factors through some $I \in \mathcal{I}$.

Proof. By Proposition 3.3 a proof similar as that for triangulated categories apply. $\square$

Proposition 4.10. Let $C \in \mathcal{C}$ be any object, and let λ be an $\mathbb{E}$ -extension with $\mathfrak{s}(\lambda) = [V^C \xrightarrow{v^C} U^C \xrightarrow{u^C} C]$, where $U^C \in \mathcal{U}$ and $V^C \in \mathcal{V}$. Then for any $U \in \mathcal{U}$, the map $\bar{u}^C \circ -$ is bijective:

$$\bar{u}^C \circ -: (\mathcal{C}/\mathcal{I})(U, U^C) \rightarrow (\mathcal{C}/\mathcal{I})(U, C) \quad (14)$$

Proof. By exactness of $\mathcal{C}(U, U^C) \xrightarrow{\mathcal{C}(U, u^C)} \mathcal{C}(U, C) \xrightarrow{(\lambda_\#)^U} \mathbb{E}(U, V) = 0$, the map $\mathcal{C}(U, U^C) \rightarrow \mathcal{C}(U, C)$ is surjective. This implies the surjectivity of (14). Let us show the injectivity of (14). Let $g \in \mathcal{C}(U, U^C)$ be any morphism which satisfies $\bar{u}^C \circ g = u^C \circ g = 0$. By definition, there exist $I \in \mathcal{I}$, $i_1 \in \mathcal{C}(U, I)$ and $i_2 \in \mathcal{C}(I, C)$ which makes the following diagram commutative.

$$\begin{array}{ccccc} U & \xrightarrow{i_1} & I & & \\ g \downarrow & & \circ & & \downarrow i_2 \\ V^C & \xrightarrow[v^C]{} & U^C & \xrightarrow[u^C]{} & C \end{array}$$

Since $\mathbb{E}(I, V^C) = 0$, $\mathcal{C}(I, V^C) \xrightarrow{\mathcal{C}(I, v^C)} \mathcal{C}(I, U^C) \xrightarrow{\mathcal{C}(I, u^C)} \mathcal{C}(I, C) \rightarrow 0$ is exact. Thus there exists $j \in \mathcal{C}(I, U^C)$ satisfying $u^C \circ j = i_2$. Then by $u^C \circ (g - j \circ i_1) = 0$, we obtain $h \in \mathcal{C}(U, V^C)$ satisfying $v^C \circ h = g - j \circ i_1$.

By Lemma 4.9, this h factors through some $I' \in \mathcal{I}$. It follows that $g = v^C \circ h + j \circ i_1$ factors through $I \oplus I' \in \mathcal{I}$. $\square$

Proposition 4.10 means that $(U^C, \bar{u}^C)$ is coreflection of $C \in \mathcal{C}/\mathcal{I}$ along the inclusion functor $\mathcal{U}/\mathcal{I} \hookrightarrow \mathcal{C}/\mathcal{I}$ ([Bo, Definition 3.1.1]). We thus obtain:

Corollary 4.11. The inclusion functor $\mathcal{U}/\mathcal{I} \hookrightarrow \mathcal{C}/\mathcal{I}$ has a right adjoint $\omega_{\mathcal{U}}: \mathcal{C}/\mathcal{I} \rightarrow \mathcal{U}/\mathcal{I}$, which assigns $\omega_{\mathcal{U}}(C) = U^C$ for any $C \in \mathcal{C}/\mathcal{I}$, where $V^C \xrightarrow{v^C} U^C \xrightarrow{u^C} C$ is a conflation with $U^C \in \mathcal{U}, V^C \in \mathcal{V}$. Moreover $\varepsilon_{\mathcal{U}} = \{\bar{u}^C\}_{C \in \text{Ob}(\mathcal{C}/\mathcal{I})}$ gives the counit of this adjoint pair.

4.3 Concentric twin cotorsion pairs

Definition 4.12. Let $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ be cotorsion pairs on $\mathcal{C}$. Then the pair $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is called a *twin cotorsion pair* if it satisfies $\mathbb{E}(\mathcal{S}, \mathcal{V}) = 0$. (Pairs of cotorsion pairs are considered in abelian/exact categories in [Ho1, Ho2], and in triangulated categories in [Na2, Na3].)

If moreover it satisfies $\mathcal{S} \cap \mathcal{T} = \mathcal{U} \cap \mathcal{V} (= \mathcal{I})$, then $\mathcal{P}$ is called a *concentric twin cotorsion pair* similarly as in the triangulated case [Na4]. In this case, we put $\mathcal{Z} = \mathcal{T} \cap \mathcal{U}$.

Remark 4.13. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a concentric twin cotorsion pair. By Corollary 4.11, the inclusion $\mathcal{U}/\mathcal{I} \hookrightarrow \mathcal{C}/\mathcal{I}$ has a right adjoint $\omega_{\mathcal{U}}$. Dually the inclusion $\mathcal{T}/\mathcal{I} \hookrightarrow \mathcal{C}/\mathcal{I}$ has a left adjoint $\sigma_{\mathcal{T}}$.

These restrict to yield the following.

- Left adjoint σ of the inclusion $\mathcal{Z}/\mathcal{I} \hookrightarrow \mathcal{U}/\mathcal{I}$.
- Right adjoint ω of the inclusion $\mathcal{Z}/\mathcal{I} \hookrightarrow \mathcal{T}/\mathcal{I}$.

Definition 4.14. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair on $\mathcal{C}$. We define full subcategories $\mathcal{N}^i, \mathcal{N}^f$ of $\mathcal{C}$ as follows.

$$\mathcal{N}^i = \text{Cone}(\mathcal{V}, \mathcal{S}) \text{ and } \mathcal{N}^f = \text{CoCone}(\mathcal{V}, \mathcal{S}).$$

Remark 4.15. The notation $\mathcal{N}^i, \mathcal{N}^f$ is motivated by Section 5. Morally, an object X belongs to $\mathcal{N}^i$ if and only if the morphism $0 \rightarrow X$ from the initial object is a weak equivalence. For a precise statement, see Proposition 5.7.

Remark 4.16. If $\mathcal{P}$ is concentric, then for any $C \in \mathcal{C}$, we have:

$$C \in \mathcal{N}^i \Leftrightarrow \omega_{\mathcal{U}}(C) \in \mathcal{S}/\mathcal{I} \text{ and } C \in \mathcal{N}^f \Leftrightarrow \sigma_{\mathcal{T}}(C) \in \mathcal{V}/\mathcal{I}.$$

Remark 4.17. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair. Then the following holds:

$$\mathcal{S} \subseteq \mathcal{N}^i, \mathcal{V} \subseteq \mathcal{N}^f, \mathcal{U} \cap \mathcal{N}^i = \mathcal{S} \text{ and } \mathcal{T} \cap \mathcal{N}^f = \mathcal{V}.$$

If moreover $\mathcal{P}$ is concentric, then $\mathcal{S} \subseteq \mathcal{N}^f$ and $\mathcal{V} \subseteq \mathcal{N}^i$.

Lemma 4.18. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a concentric twin cotorsion pair. Then the following holds:

$$(1) \text{Cone}(\mathcal{V}, \mathcal{N}^i) \subseteq \mathcal{N}^i \text{ and } (2) \text{CoCone}(\mathcal{N}^f, \mathcal{S}) \subseteq \mathcal{N}^f.$$

Proof. We only prove (1) since (2) is dual. By definition, $C \in \text{Cone}(\mathcal{V}, \mathcal{N}^i)$ admits a conflation $V \rightarrow N \rightarrow C$, where $V \in \mathcal{V}$ and $N \in \mathcal{N}^i$. Resolve N by a conflation $V^N \rightarrow S^N \rightarrow N$ with $S^N \in \mathcal{S}$, $V^N \in \mathcal{V}$.

By (ET4)^{op}, we obtain a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc}
 V^N = V^N & & & & \\
 \downarrow & \circlearrowleft & \downarrow & & \\
 \exists E \rightarrow S^N \rightarrow C & & & & \\
 \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \parallel \\
 V \rightarrow N \rightarrow C & & & &
 \end{array}$$

in which $V^N \rightarrow E \rightarrow V$ and $E \rightarrow S^N \rightarrow C$ are conflations. Since $\mathcal{V} \subseteq \mathcal{C}$ is extension-closed, it follows that $E \in \mathcal{V}$. This means $C \in \mathcal{N}^i$. $\square$

Lemma 4.19. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a concentric twin cotorsion pair. Let $U \in \mathcal{U}$ be any object. Assume there is a conflation $M \rightarrow U \rightarrow S$ satisfying $M \in \mathcal{N}^f$ and $S \in \mathcal{S}$. Then U belongs to $\mathcal{N}^f$.

Dually, if $T \in \mathcal{T}$ appears in a conflation $V \rightarrow T \rightarrow N$ satisfying $V \in \mathcal{V}, N \in \mathcal{N}^i$, then T belongs to $\mathcal{N}^i$.

Proof. By definition, M admits a conflation $M \rightarrow V_M \rightarrow S_M$ with $V_M \in \mathcal{V}, S_M \in \mathcal{S}$. By Proposition 3.15, we obtain a commutative diagram

$$\begin{array}{ccccc}
 M \rightarrow U \rightarrow S & & & & \\
 \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \parallel \\
 V_M \rightarrow \exists X \rightarrow S & & & & \\
 \downarrow & \circlearrowleft & \downarrow & & \\
 S_M = S_M & & & &
 \end{array}$$

consisting of conflations. Since $\mathcal{U}$ is extension-closed, it follows that $X \in \mathcal{U}$. Since $\mathbb{E}(S, V_M) = 0$, the $\mathbb{E}$ -extension realized by $V_M \rightarrow X \rightarrow S$ splits. Especially V_M is a direct summand of X , and thus it follows that $V_M \in \mathcal{U} \cap \mathcal{V} = \mathcal{I}$. By the extension-closedness of $\mathcal{S}$, we obtain $X \in \mathcal{S}$. Thus $U \in \mathcal{N}^f$ follows from Remark 4.17 and Lemma 4.18 (2). $\square$

Lemma 4.20. Let $\mathcal{P}$ be as in Lemma 4.19. Let $T \in \mathcal{T}, M \in \mathcal{N}^f$ be any pair of objects. If there is a section $T \rightarrow M$ or a retraction $M \rightarrow T$, then T belongs to $\mathcal{V}$.

Proof. In either case, we have morphisms $s \in \mathcal{C}(T, M)$ and $r \in \mathcal{C}(M, T)$ satisfying $r \circ s = \text{id}$. By definition, M admits a conflation $M \xrightarrow{v} V \rightarrow S$. By $\mathbb{E}(S, T) = 0$, the morphism r factors through v . Then $v \circ s \in \mathcal{C}(T, V)$ becomes a section, and thus it follows from Corollary 4.5 that $T \in \mathcal{V}$. $\square$

5. Bijective correspondence with model structures

In the rest, let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

In this section, we give a bijective correspondence between *Hovey twin cotorsion pairs* and *admissible model structures* which we will soon define. This gives a unification of the following preceding works.

- For an abelian category, Hovey has shown their correspondence in [Ho1, Ho2] (*abelian model structure*). This has been generalized to an exact category by Gillespie [G] (*exact model structure*), and investigated by Šťovíček [S].
- For a triangulated category, Yang [Y] has introduced an analogous notion of *triangulated model structure* and showed its correspondence with cotorsion pairs.

5.1 Hovey twin cotorsion pair

We recall that $\mathcal{N}^i$ (resp. $\mathcal{N}^f$) is the collection of all objects $X \in \mathcal{C}$ which are part of a conflation $V \rightarrow S \rightarrow X$ (resp. $X \rightarrow V \rightarrow S$), for some $V \in \mathcal{V}$ and $S \in \mathcal{S}$.

Definition 5.1. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair. We call $\mathcal{P}$ a *Hovey twin cotorsion pair* if it satisfies $\mathcal{N}^f = \mathcal{N}^i$. We denote this subcategory by $\mathcal{N}$.

Remark 5.2. Any *Hovey twin cotorsion pair* is *concentric*. In fact, we have $\mathcal{U} \cap \mathcal{V} = \mathcal{U} \cap (\mathcal{N}^f \cap \mathcal{T}) = (\mathcal{U} \cap \mathcal{N}^i) \cap \mathcal{T} = \mathcal{S} \cap \mathcal{T}$ by Remark 4.17.

For any Hovey twin cotorsion pair, the subcategory $\mathcal{N} \subseteq \mathcal{C}$ is extension-closed. More strongly, it satisfies the following.

Proposition 5.3. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a Hovey twin cotorsion pair. For any conflation $A \xrightarrow{x} B \xrightarrow{y} C$, if two out of A, B, C belong to $\mathcal{N}$, then so does the third.

Proof. We prove (1) $A, C \in \mathcal{N} \Rightarrow B \in \mathcal{N}$, (2) $A, B \in \mathcal{N} \Rightarrow C \in \mathcal{N}$ and (3) $B, C \in \mathcal{N} \Rightarrow A \in \mathcal{N}$.

(1) Resolve A, C by conflations $A \rightarrow V_A \rightarrow S_A$ and $V^C \rightarrow S^C \rightarrow C$ with $S_A, S^C \in \mathcal{S}$, $V_A, V^C \in \mathcal{V}$ respectively. By Proposition 3.15, we obtain the two commutative diagrams on the left of:

$$\begin{array}{ccccc}
 & V^C = V^C & & A \rightarrow X \rightarrow S^C & & A \rightarrow B \rightarrow C \\
 & \downarrow \circ \downarrow & & \downarrow \circ \downarrow \parallel & & \downarrow \circ \downarrow \parallel \\
 A \rightarrow \exists X \rightarrow S^C & , & V_A \rightarrow \exists Y \rightarrow S^C & , & V_A \rightarrow \exists G \rightarrow C \\
 \parallel \circ \downarrow \circ \downarrow & & \downarrow \circ \downarrow & & \downarrow \circ \downarrow \\
 A \rightarrow B \rightarrow C & & S_A = S_A & & S_A = S_A
 \end{array}$$

which are made of conflations. Since $\mathbb{E}(S^C, V_A) = 0$, the conflation $V_A \rightarrow Y \rightarrow S^C$ realizes the split $\mathbb{E}$ -extension. It follows that $Y \cong V_A \oplus S^C \in \mathcal{N}$. We obtain $X \in \mathcal{N}$ by Lemma 4.18 (2), and thus $B \in \mathcal{N}$ by Lemma 4.18 (1).

(2) Resolve A by a conflation $A \rightarrow V_A \rightarrow S_A$ with $S_A \in \mathcal{S}$, $V_A \in \mathcal{V}$. Then by Proposition 3.15, we obtain the rightmost commutative diagram above, made of conflations. Since $B, S_A \in \mathcal{N}$, we have $G \in \mathcal{N}$ by (1). From Lemma 4.18 (1), it follows that $C \in \mathcal{N}$.

(3) is dual to (2). $\square$

Remark 5.4. As a corollary, $\mathcal{N}$ becomes an extriangulated category by Remark 2.18. Almost by definition, the pair $(\mathcal{S}, \mathcal{V})$ is a cotorsion pair on $\mathcal{N}$.

5.2 From admissible model structure to Hovey twin cotorsion pair

Throughout this section, let $\mathcal{M} = (\text{Fib}, \text{Cof}, \mathbb{W})$ be a model structure on $\mathcal{C}$, where $\text{Fib}, \text{Cof}, \mathbb{W}$ are the classes of fibrations, cofibrations, and weak equivalences. Let $w\text{Fib} = \text{Fib} \cap \mathbb{W}$ and $w\text{Cof} = \text{Cof} \cap \mathbb{W}$ denote the classes of acyclic fibrations and acyclic cofibrations, respectively. Associate full subcategories $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$ as follows.

$$\begin{aligned}
 C \in \mathcal{S} &\Leftrightarrow (0 \rightarrow C) \in w\text{Cof}, \\
 C \in \mathcal{T} &\Leftrightarrow (C \rightarrow 0) \in \text{Fib}, \\
 C \in \mathcal{U} &\Leftrightarrow (0 \rightarrow C) \in \text{Cof}, \\
 C \in \mathcal{V} &\Leftrightarrow (C \rightarrow 0) \in w\text{Fib}.
 \end{aligned}$$

Remark that these are full additive subcategories of $\mathcal{C}$, closed under isomorphisms and direct summands. In particular, the definition below makes sense.

Definition 5.5. $\mathcal{M}$ is called an *admissible model structure* if it satisfies the following conditions for any morphism $f \in \mathcal{C}(A, B)$.

- (1) $f \in wCof$ if and only if it is an inflation with $\text{Cone}(f) \in \mathcal{S}$.
- (2) $f \in Fib$ if and only if it is a deflation with $\text{CoCone}(f) \in \mathcal{T}$.
- (3) $f \in Cof$ if and only if it is an inflation with $\text{Cone}(f) \in \mathcal{U}$.
- (4) $f \in wFib$ if and only if it is a deflation with $\text{CoCone}(f) \in \mathcal{V}$.

We note that the model structures which might appear in [Pal] are not admissible.

Proposition 5.6. Let $\mathcal{M}$ be an admissible model structure. Then $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is a twin cotorsion pair on $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

Proof. $\mathcal{S} \subseteq \mathcal{U}$ is obvious from the definition. Since a similar argument works for $(\mathcal{S}, \mathcal{T})$, we show that $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair. Let us confirm the conditions (1) and (2) in Definition 4.1 since (3) is dual to (2).

(1) Let $(U, V) \in \mathcal{U} \times \mathcal{V}$ be any pair of objects, and let $\delta \in \mathbb{E}(U, V)$ be any element. Realize it as an $\mathbb{E}$ -triangle $V \xrightarrow{v} B \xrightarrow{u} U \xrightarrow{\delta} \rightarrow$. Since $U \in \mathcal{U}$ and $u \in wFib$, there exists a section $s \in \mathcal{C}(U, B)$ of u . Thus δ splits by Corollary 3.5.

(2) Let $C \in \mathcal{C}$ be any object. Factorize the zero morphism $0: 0 \rightarrow C$ as a cofibration $i \in Cof$ followed by an acyclic fibration $u \in wFib$. Since $\mathcal{M}$ is admissible, we have conflations $0 \xrightarrow{i} D \xrightarrow{j} U$ and $V \rightarrow D \xrightarrow{u} C$ with $U \in \mathcal{U}, V \in \mathcal{V}$. This shows that j is an isomorphism, and thus we obtain a conflation $V \rightarrow U \rightarrow C$. $\square$

Proposition 5.7. Let $\mathcal{M}$ be an admissible model structure as above. Then the associated twin cotorsion pair $\mathcal{P}$ obtained in Proposition 5.6 is a Hovey twin cotorsion pair. Indeed, if we let $\mathcal{N}^i, \mathcal{N}^f \subseteq \mathcal{C}$ be as in Definition 4.14, then the following are equivalent for any object $N \in \mathcal{C}$:

- (1) $N \in \mathcal{N}^i$; (2) $(0 \rightarrow N) \in \mathbb{W}$; (3) $(N \rightarrow 0) \in \mathbb{W}$; (4) $N \in \mathcal{N}^f$.

Proof. (1) $\Rightarrow$ (2) If $N \in \mathcal{N}^i$, there is a conflation $V \rightarrow S \xrightarrow{s} N$ with $V \in \mathcal{V}, S \in \mathcal{S}$ by definition. Thus $0 \rightarrow N$ can be factorized as an acyclic cofibration $0 \rightarrow S$ followed by an acyclic fibration $S \xrightarrow{s} N$.

It follows that $(0 \rightarrow N) \in wFib \circ wCof = \mathbb{W}$.

(2) $\Rightarrow$ (1) Factorize $0 \rightarrow N$ as an acyclic cofibration $0 \xrightarrow{i} D$ followed by an acyclic fibration $D \xrightarrow{u} N$. A similar argument as in the proof (2) of Proposition 5.6 gives a conflation $V \rightarrow S \rightarrow N$.

(2) $\Leftrightarrow$ (3) follows from the 2-out-of-3 property of $\mathbb{W}$.

(3) $\Leftrightarrow$ (4) is dual to (1) $\Leftrightarrow$ (2). $\square$

5.3 From Hovey twin cotorsion pair to admissible model structure

Throughout this section, let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a Hovey twin cotorsion pair on $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. In addition, we assume the following condition, analogous to the weak idempotent completeness ([Bu, Proposition 7.6]).

Condition 5.8 (WIC). *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Consider the following conditions.*

- (1) *Let $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$ be any composable pair of morphisms. If $g \circ f$ is an inflation, then so is f .*
- (2) *Let $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$ be any composable pair of morphisms. If $g \circ f$ is a deflation, then so is g .*

With the assumption of Condition (WIC), we have the following analog of the nine lemma.

Lemma 5.9. *Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category satisfying Condition (WIC). Let*

$$\begin{array}{ccccc}
 K & & K' & & \\
 \downarrow k & & \downarrow k' & & \\
 A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \triangleright \\
 \downarrow a & \circlearrowleft & \downarrow b & & \\
 A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} \triangleright \\
 \downarrow \kappa & & \downarrow \kappa' & & \\
 \Psi & & \Psi & &
 \end{array}$$

be a diagram made of $\mathbb{E}$ -triangles. Then for some $X \in \mathcal{C}$, we obtain $\mathbb{E}$ -triangles

$$K \xrightarrow{m} K' \xrightarrow{n} X \xrightarrow{\nu} \text{ and } X \xrightarrow{i} C \xrightarrow{c} C' \xrightarrow{\tau} \text{ and } X \xrightarrow{i} C \xrightarrow{c} C' \xrightarrow{\tau} \text{ and } X \xrightarrow{i} C \xrightarrow{c} C' \xrightarrow{\tau}$$

which make the following diagram commutative,

$$\begin{array}{ccccc}
 K & \xrightarrow{m} & K' & \xrightarrow{n} & X & \xrightarrow{\nu} & \triangleright \\
 \downarrow k & \circlearrowleft & \downarrow k' & \circlearrowleft & \downarrow i & & \\
 A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \triangleright \\
 \downarrow a & \circlearrowleft & \downarrow b & \circlearrowleft & \downarrow c & & \\
 A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{\delta'} & \triangleright \\
 \downarrow \kappa & & \downarrow \kappa' & & \downarrow \tau & & \\
 \Psi & & \Psi & & \Psi & &
 \end{array} \tag{15}$$

in which, those (k, k', i) , (a, b, c) , (m, x, x') , (n, y, y') are morphisms of $\mathbb{E}$ -triangles.

Proof. By $(\text{ET4})^{\text{op}}$, we obtain an $\mathbb{E}$ -triangle $E \xrightarrow{f} B' \xrightarrow{y' \circ b} C' \xrightarrow{\theta} \triangleright$ and a commutative diagram (α) as in:

$$\begin{array}{ccc}
 K' & \xrightarrow{d} & E \xrightarrow{e} A' \\
 \parallel & \circlearrowleft & \downarrow f \quad \circlearrowleft \quad \downarrow x' \\
 K' & \xrightarrow{k'} & B \xrightarrow{b} B' \\
 & & \downarrow y' \circ b \quad \circlearrowleft \quad \downarrow y' \\
 & & C' = C'
 \end{array} \tag{\alpha}$$

$$\begin{array}{ccc}
 A & & \\
 \searrow g & \circlearrowleft & \searrow x \\
 & E \xrightarrow{f} B & \\
 \searrow a & \circlearrowleft & \searrow b \\
 & A' \xrightarrow{x'} B' &
 \end{array} \tag{\beta}$$

in $\mathcal{C}$, satisfying the following compatibilities:

(i) $K' \xrightarrow{d} E \xrightarrow{e} A' \xrightarrow{x' \circ \kappa'} \triangleright$ is an $\mathbb{E}$ -triangle, (ii) $\delta' = e_* \theta$, and (iii) $d_* \kappa' = y'^* \theta$.

By the dual of Lemma 3.13, the upper-right square $x' \circ e = b \circ f$ is a weak pullback. Thus there exists a morphism $g \in \mathcal{C}(A, E)$ which makes the diagram (β) commutative. By Condition (WIC), this g becomes an inflation. Complete it into an $\mathbb{E}$ -triangle $A \xrightarrow{g} E \xrightarrow{h} X \xrightarrow{\mu} \triangleright$. By Lemma 3.14, we

obtain a commutative diagram (γ)

$$\begin{array}{ccc}
 A & \xrightarrow{g} & E \xrightarrow{h} X \\
 \parallel & \circlearrowleft & f \downarrow \quad \circlearrowleft \downarrow i \\
 A & \xrightarrow{x} & B \xrightarrow{y} C \\
 & & y'ob \downarrow \quad \circlearrowleft \downarrow c \\
 & & C' = C'
 \end{array}
 \quad
 \begin{array}{ccc}
 K & \xrightarrow{k} & A \xrightarrow{a} A' \\
 m \downarrow & \circlearrowleft & g \downarrow \quad \circlearrowleft \downarrow \\
 K' & \xrightarrow{d} & E \xrightarrow{e} A' \\
 n \downarrow & \circlearrowleft & h \downarrow \\
 X & = & X
 \end{array}
 \quad (\delta)$$

made of conflations, which satisfies:

(iv) $X \xrightarrow{i} C \xrightarrow{c} C' \xrightarrow{h_*\theta}$ is an $\mathbb{E}$ -triangle, (v) $\mu = i^*\delta$, (vi) $g_*\delta = c^*\theta$.

By Proposition 3.17, we obtain an $\mathbb{E}$ -triangle $K \xrightarrow{m} K' \xrightarrow{n} X \xrightarrow{\nu}$ which makes the diagram (δ) commutative in $\mathcal{C}$, and satisfies:

(vii) $m_*\kappa = x'^*\kappa'$, (viii) $\mu = k_*\nu$, (ix) $h^*\nu + e^*\kappa = 0$.

Put $\tau = h_*\theta$. It is straightforward to show that the diagram (15) is indeed commutative. Moreover, (v) and (viii) show $k_*\nu = i^*\delta$, (ii) and (vi) show $a_*\delta = c^*\delta'$, (vii) shows $m_*\kappa = x'^*\kappa'$, (iii) shows $n_*\kappa' = y'^*\tau$. $\square$

Remark 5.10. In the proof of Lemma 5.9, we have obtained an extra compatibility (ix). This can be interpreted by the following analog of Ext^2 -group.

Let $A, D \in \mathcal{C}$ be any pair of objects. We denote triplet of $X \in \mathcal{C}$, $\sigma \in \mathbb{E}(D, X)$, $\tau \in \mathbb{E}(X, A)$ by (σ, X, τ) . For any pair of such triplets (σ, X, τ) and (σ', X', τ') , we write as

$$(\sigma, X, \tau) \rightsquigarrow_x (\sigma', X', \tau') \quad (\text{or simply } (\sigma, X, \tau) \rightsquigarrow (\sigma', X', \tau'))$$

if and only if there exists $x \in \mathcal{C}(X, X')$ satisfying $x_*\sigma = \sigma'$ and $\tau = x^*\tau'$.

Let $\sim$ be the equivalence relation generated by $\rightsquigarrow$, and denote the equivalence class of (σ, X, τ) by $\tau \circ_X \sigma$. Let us denote their collection by

$$\mathbb{E}^2(D, A) = \left(\coprod_{X \in \mathcal{C}} \mathbb{E}(D, X) \times \mathbb{E}(X, A) \right) / \sim.$$

The proof of Lemma 5.9 shows

$$(\quad (\delta', A', -\kappa) \quad \rightsquigarrow$$

E, h

Definition 5.11. Define classes of morphisms Fib , $wFib$, Cof , $wCof$ and $\mathbb{W}$ in $\mathcal{C}$ as follows.

- (1) $f \in Fib$ if it is a deflation with $\text{CoCone}(f) \in \mathcal{T}$.
- (2) $f \in wFib$ if it is a deflation with $\text{CoCone}(f) \in \mathcal{V}$.
- (3) $f \in Cof$ if it is an inflation with $\text{Cone}(f) \in \mathcal{U}$.
- (4) $f \in wCof$ if it is an inflation with $\text{Cone}(f) \in \mathcal{S}$.
- (5) $\mathbb{W} = wFib \circ wCof$.

Claim 5.12.

- (1) If a conflation $A \xrightarrow{f} B \rightarrow N$ satisfies $N \in \mathcal{N}$, then f belongs to $\mathbb{W}$.
- (2) If a conflation $N \rightarrow A \xrightarrow{f} B$ satisfies $N \in \mathcal{N}$, then f belongs to $\mathbb{W}$.

Proof. This follows from Proposition 3.15. $\square$

Proposition 5.13. Fib , $wFib$, Cof , $wCof$ are closed under composition.

Proof. For Fib , this follows from (ET4) and the extension-closedness of $\mathcal{T}$. Similarly for the others. $\square$

Proposition 5.14. We have the following.

- (1) $wCof$ satisfies the left lifting property against Fib .
- (2) $wFib$ satisfies the right lifting property against Cof .

Proof. (1) Suppose we are given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ f \downarrow & \circlearrowleft & \downarrow g \\ B & \xrightarrow{b} & D \end{array} \quad (16)$$

in $\mathcal{C}$, satisfying $f \in wCof$ and $g \in Fib$. By definition, there are $\mathbb{E}$ -triangles $A \xrightarrow{f} B \xrightarrow{s} S \dashrightarrow^{\delta}$ and $T \xrightarrow{t} C \xrightarrow{g} D \dashrightarrow^{\kappa}$. By Corollary 3.12,

$$\mathcal{C}(B, T) \xrightarrow{\mathcal{C}(f, T)} \mathcal{C}(A, T) \rightarrow 0 \rightarrow \mathbb{E}(B, T) \xrightarrow{\mathbb{E}(f, T)} \mathbb{E}(A, T), \quad (17)$$

$$\mathcal{C}(A, T) \xrightarrow{\mathcal{C}(A, t)} \mathcal{C}(A, C) \xrightarrow{\mathcal{C}(A, g)} \mathcal{C}(A, D), \quad (18)$$

$$\mathcal{C}(B, C) \xrightarrow{\mathcal{C}(B, g)} \mathcal{C}(B, D) \xrightarrow{(\kappa_\#)_B} \mathbb{E}(B, T) \quad (19)$$

are exact. By the commutativity of (16), we have $\mathbb{E}(f, T)(b^*\kappa) = f^*b^*\kappa = a^*g^*\kappa = 0$ by Lemma 3.2. Exactness of (17) shows $\kappa_\#b = b^*\kappa = 0$. Thus by the exactness of (19), there exists $c \in \mathcal{C}(B, C)$ satisfying $g \circ c = b$. Then $a - c \circ f \in \mathcal{C}(A, C)$ satisfies $g \circ (a - c \circ f) = g \circ a - b \circ f = 0$. By the exactness of (18), there is $c' \in \mathcal{C}(A, T)$ satisfying $t \circ c' = a - c \circ f$. By the exactness of (17), there is $c'' \in \mathcal{C}(A, T)$ satisfying $c'' \circ f = c'$. If we put $h = c + t \circ c'' \in \mathcal{C}(B, C)$, it satisfies $h \circ f = c \circ f + t \circ c'' \circ f = c \circ f + t \circ c' = a$ and $g \circ h = g \circ c + g \circ t \circ c'' = b$.

(2) is dual to (1). $\square$

Proposition 5.15. $\text{Mor}(\mathcal{C}) = wFib \circ Cof = Fib \circ wCof$.

Proof. We only show $\text{Mor}(\mathcal{C}) = wFib \circ Cof$. Let $f \in \mathcal{C}(A, B)$ be any morphism. Resolve A by a conflation $A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A$ with $U_A \in \mathcal{U}$, $V_A \in \mathcal{V}$, and put $f' = \begin{bmatrix} f \\ v_A \end{bmatrix} : A \rightarrow B \oplus V_A$. By Corollary 3.16, it admits some conflation

$$A \xrightarrow{f'} B \oplus V_A \rightarrow C.$$

Resolve C by a conflation $V^C \rightarrow U^C \rightarrow C$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Then by Proposition 3.15, we obtain a diagram made of conflations as follows.

$$\begin{array}{ccccc} & & V^C & \xlongequal{\quad} & V^C \\ & & \downarrow & \circlearrowleft & \downarrow \\ A & \xrightarrow{m} & M & \longrightarrow & U^C \\ \parallel & \circlearrowleft & \downarrow e & \circlearrowleft & \downarrow \\ A & \xrightarrow{f'} & B \oplus V_A & \longrightarrow & C \end{array}$$

We have $m \in Cof$. Moreover, for $p_B = [1 \ 0] \in \mathcal{C}(B \oplus V_A, B)$, we have $p_B \circ e \in wFib \circ wFib = wFib$. Thus $f = (p_B \circ e) \circ m$ gives the desired factorization. $\square$

Proposition 5.16. *Fib, wFib, Cof, wCof* are closed under retraction.

Proof. We only show the result for *Fib*. Suppose we are given a commutative diagram

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & & \curvearrowright & & \\
 A & \xrightarrow{a} & C & \xrightarrow{c} & A \\
 \downarrow f & & \downarrow g & & \downarrow f \\
 B & \xrightarrow{b} & D & \xrightarrow{d} & B \\
 & & \text{id} & &
 \end{array}$$

in $\mathcal{C}$, satisfying $g \in \text{Fib}$. By definition, there is an $\mathbb{E}$ -triangle $T \xrightarrow{t} C \xrightarrow{g} D \xrightarrow{\theta} T$ with $T \in \mathcal{T}$. By Condition (WIC), $d \circ b = \text{id}$ implies that d is a deflation. Thus $d \circ g$ becomes a deflation by (ET4)^{op}. Again by Condition (WIC), it follows that f is a deflation. Thus there exists an $\mathbb{E}$ -triangle $X \xrightarrow{x} A \xrightarrow{f} B \xrightarrow{\delta} X$. By (ET3)^{op}, we obtain the following two morphisms of $\mathbb{E}$ -triangles (on the left below).

$$\begin{array}{ccccc}
 X \xrightarrow{x} A \xrightarrow{f} B \xrightarrow{\delta} X & \xrightarrow{\ell} & T \xrightarrow{t} C \xrightarrow{g} D \xrightarrow{\theta} T & \xrightarrow{\ell \circ k} & X \xrightarrow{x} A \xrightarrow{f} B \xrightarrow{\delta} X \\
 k \downarrow \quad \circ \quad \downarrow a \quad \circ \quad \downarrow b & & \ell \downarrow \quad \circ \quad \downarrow c \quad \circ \quad \downarrow d & & \ell \circ k \downarrow \quad \circ \quad \downarrow \text{id} \quad \circ \quad \downarrow \text{id} \\
 T \xrightarrow{t} C \xrightarrow{g} D \xrightarrow{\theta} T & \xrightarrow{\ell} & X \xrightarrow{x} A \xrightarrow{f} B \xrightarrow{\delta} X & & X \xrightarrow{x} A \xrightarrow{f} B \xrightarrow{\delta} X
 \end{array}$$

Composing them, we obtain a morphism of $\mathbb{E}$ -triangles (on the right above). By Corollary 3.6, it follows that $\ell \circ k$ is an isomorphism. Especially k is a section, and thus $X \in \mathcal{T}$. This means that f belongs to *Fib*. $\square$

Lemma 5.17. Suppose that two morphisms $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ satisfy $f \in \text{wCof}$, $g \in \text{Fib}$ and $g \circ f \in \text{wCof}$. Then g belongs to *wFib*.

Proof. Let $h = g \circ f$. By assumption, there are conflations $A \xrightarrow{f} B \xrightarrow{s_1} S_1$, $T \xrightarrow{t} B \xrightarrow{g} C$, $A \xrightarrow{h} C \xrightarrow{s_2} S_2$ where $S_1 \in \mathcal{S}$, $T \in \mathcal{T}$ and $S_2 \in \mathcal{S}$. By the dual of Lemma 3.17, we obtain the following commutative diagram

made of conflations.

$$\begin{array}{ccccc}
 & & T & \xlongequal{\quad} & T \\
 & & \downarrow t & \circlearrowleft & \downarrow \\
 A & \xrightarrow{f} & B & \xrightarrow{s_1} & S_1 \\
 \parallel & \circlearrowleft & \downarrow g & \circlearrowleft & \downarrow \\
 A & \xrightarrow{h} & C & \xrightarrow{s_2} & S_2
 \end{array}$$

By Lemma 4.18 (2) and Remark 4.17, we obtain $T \in \mathcal{T} \cap \mathcal{N} = \mathcal{V}$. This means $g \in wFib$. $\square$

Proposition 5.18. $\mathbb{W}$ is closed under composition.

Proof. It suffices to show that $wCof \circ wFib \subseteq \mathbb{W}$. Let $a \in wFib$ and $x' \in wCof$. By Proposition 5.15, there are some $x \in wCof$ and $b \in Fib$ such that $b \circ x = x' \circ a$. It is thus enough to show that b belongs to $wFib$. By definition, there is a commutative diagram of $\mathbb{E}$ -triangles:

$$\begin{array}{ccccccc}
 V & & T & & & & \\
 \downarrow k & & \downarrow k' & & & & \\
 A & \xrightarrow{x} & B & \xrightarrow{y} & S & \xrightarrow{\delta} & \triangleright \\
 \downarrow a & \circlearrowleft & \downarrow b & & & & \\
 A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & S' & \xrightarrow{\delta'} & \triangleright \\
 \downarrow \kappa & & \downarrow \kappa' & & & & \\
 \Psi & & \Psi & & & &
 \end{array}$$

with $V \in \mathcal{V}$, $T \in \mathcal{T}$ and $S, S' \in \mathcal{S}$. Applying Lemma 5.9 gives some $X \in \mathcal{C}$ and two conflations $X \xrightarrow{i} S \xrightarrow{c} S'$ and $V \xrightarrow{m} T \xrightarrow{n} X$. The existence of the first conflation (and Lemma 4.18(2)) shows that X belongs to $\mathcal{N}$; that of the latter conflation and the dual of Lemma 4.19 imply that T belongs to $\mathcal{V}$, and therefore that $b \in wFib$. $\square$

Lemma 5.19. Suppose that two morphisms $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ satisfy $f \in wCof$, $g \in Fib$ and $g \circ f \in wFib$. Then g belongs to $wFib$.

Proof. Let $h = g \circ f$. By assumption, there are conflations $A \xrightarrow{f} B \xrightarrow{s} S$, $T \xrightarrow{t} B \xrightarrow{g} C$, $V \xrightarrow{v} A \xrightarrow{h} C$ with $S \in \mathcal{S}$, $T \in \mathcal{T}$ and $V \in \mathcal{V}$.

By Proposition 3.17, we obtain a conflation $V \rightarrow T \rightarrow S$. Thus from Proposition 5.3 and Remark 4.17, it follows $T \in \mathcal{T} \cap \mathcal{N} = \mathcal{V}$. This means $g \in wFib$. $\square$

Lemma 5.20. Suppose that two morphisms $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ satisfy $f \in wFib$, $g \in Fib$ and $g \circ f \in wFib$. Then g belongs to $wFib$.

Proof. Let $h = g \circ f$. By assumption, there are conflations $V_f \rightarrow A \xrightarrow{f} B$, $T \rightarrow B \xrightarrow{g} C$, $V_h \rightarrow A \xrightarrow{h} C$ with $V_f \in \mathcal{V}$, $T \in \mathcal{T}$ and $V_h \in \mathcal{V}$. By (ET4)^{op}, we obtain a conflation $V_f \rightarrow V_h \rightarrow T$. Thus from Lemma 4.18 (1) and Remark 4.17, it follows that $T \in \mathcal{T} \cap \mathcal{N} = \mathcal{V}$. This means $g \in wFib$. $\square$

Proposition 5.21. Suppose that $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ satisfy $f \in \mathbb{W}$ and $g \circ f \in \mathbb{W}$. Then g also belongs to $\mathbb{W}$.

Proof. Let $h = g \circ f$. By definition and the dual of Proposition 5.15, the morphisms f, g, h can be factorized as

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ f_1 \searrow & \circlearrowleft & \nearrow f_2 \\ & X_f & \end{array}, \quad \begin{array}{ccc} B & \xrightarrow{g} & C \\ g_1 \searrow & \circlearrowleft & \nearrow g_2 \\ & X_g & \end{array}, \quad \begin{array}{ccc} A & \xrightarrow{h} & C \\ h_1 \searrow & \circlearrowleft & \nearrow h_2 \\ & X_h & \end{array},$$

with $f_1, g_1, h_1 \in wCof$, $f_2, h_2 \in wFib$ and $g_2 \in Fib$.

By Proposition 5.18, the morphism $g_1 \circ f_2$ belongs to $\mathbb{W}$ and can thus be factorized as an acyclic cofibration $w_1: X_f \rightarrow X$ followed by an acyclic fibration $w_2: X \rightarrow X_g$. By Proposition 5.14, there is $k \in \mathcal{C}(X_h, X)$ which makes (*) commutative in $\mathcal{C}$. By Proposition 5.15, we can factorize k as an acyclic cofibration $k_1: X_h \rightarrow X_k$ followed by a fibration $k_2: X_k \rightarrow X$. Thus we obtain the following commutative diagram on the right

$$(*) \quad \begin{array}{ccc} A & \xrightarrow{w_1 \circ f_1} & X \\ h_1 \downarrow & \circlearrowleft & \nearrow k \\ & X_h & \end{array} \quad \begin{array}{ccc} A & \xrightarrow{w_1 \circ f_1} & X \\ h_1 \downarrow & \circlearrowleft & \nearrow k_2 \\ & X_k & \end{array} \quad \begin{array}{ccc} X & \xrightarrow{w_2} & X_g \\ & \circlearrowleft & \downarrow g_2 \\ X_h & \xrightarrow{h_2} & C \end{array} \quad \begin{array}{l} (k_1, h_1, w_1 \circ f_1 \in wCof) \\ (w_2, h_2 \in wFib) \\ (k_2, g_2 \in Fib) \end{array}$$

Lemma 5.17 shows $k_2 \in wFib$. On the other hand, Lemma 5.19 shows $g_2 \circ w_2 \circ k_2 \in wFib$. Thus Lemma 5.20 shows $g_2 \in wFib$. $\square$

Corollary 5.22. The class $\mathbb{W}$ satisfies the 2-out-of-3 condition.

Proof. This follows from Proposition 5.18, Proposition 5.21 and its dual. $\square$

When a category has enough pull-backs or enough push-outs, the fact that weak equivalences are stable under retracts follows from the other axioms (e.g. [Joy, Proposition E.1.3], attributed to Joyal–Tierney). However, that proof does not carry over to the setup of extriangulated categories⁴. The following lemma will thus be used for proving that the class $\mathbb{W}$ is closed under retracts.

Lemma 5.23. Let $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow^\delta$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C' \dashrightarrow^{\delta'}$ be $\mathbb{E}$ -triangles. Suppose that $b \in \mathcal{C}(B, B')$ belongs to $\mathbb{W}$ and satisfies $b \circ x = x'$. Then there is $c \in \mathcal{C}(C, C')$ which belongs to $\mathbb{W}$ and gives a morphism of $\mathbb{E}$ -triangles as follows.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \dashrightarrow^\delta \\ \parallel & \circlearrowleft & \downarrow b & \circlearrowleft & \downarrow c \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \dashrightarrow^{\delta'} \end{array}$$

Proof. By definition, b can be factorized as $b = v \circ s$, using $\mathbb{E}$ -triangles $B \xrightarrow{s} P \rightarrow S \dashrightarrow^\theta$ and $V \rightarrow P \xrightarrow{v} B' \dashrightarrow^\tau$ with $S \in \mathcal{S}, V \in \mathcal{V}$. By (ET4), and then by the dual of Proposition 3.17, we obtain the following commutative diagrams made of $\mathbb{E}$ -triangles,

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \dashrightarrow^\delta \\ \parallel & \circlearrowleft & \downarrow s & \circlearrowleft & \downarrow \exists c_1 \\ A & \xrightarrow{s \circ x} & P & \xrightarrow{\exists p} & \exists Q \dashrightarrow^{\exists \nu} \\ & & \downarrow & \circlearrowleft & \downarrow \\ & & S & \xlongequal{\quad} & S \\ & & \downarrow \theta & & \downarrow y_* \theta \\ & & \mathbb{V} & & \mathbb{V} \end{array} \quad , \quad \begin{array}{ccccc} A & \xlongequal{\quad} & A \\ \downarrow s \circ x & \circlearrowleft & \downarrow x' \\ V & \xrightarrow{\quad} & P & \xrightarrow{v} & B' \dashrightarrow^\tau \\ \parallel & \circlearrowleft & \downarrow p & \circlearrowleft & \downarrow y' \\ V & \xrightarrow{\quad} & Q & \xrightarrow{\exists c_2} & C' \dashrightarrow^{\delta'} \\ & & \downarrow \nu & & \downarrow \delta' \\ & & \mathbb{V} & & \mathbb{V} \end{array}$$

⁴It turns out that mere existence of finite products and finite coproducts is enough, as follows from [Eg] and the fact that weak equivalences are precisely the morphisms that become isomorphisms in the localisation $\mathcal{C}[\mathbb{W}^{-1}]$. This is explained in detail in Pierre Cagne’s PhD Thesis [Ca, Section 2.2]. We nonetheless include a different proof.

in which $c_1^* \nu = \delta$ holds. Then $c = c_2 \circ c_1$ belongs to $\mathbb{W}$, satisfies $c \circ y = c_2 \circ p \circ s = y' \circ v \circ s = y' \circ b$ and $c^* \delta' = c_1^* c_2^* \delta' = c_1^* \nu = \delta$. $\square$

Proposition 5.24. The class $\mathbb{W}$ is closed under retracts.

Proof. Suppose we are given a commutative diagram (a) in $\mathcal{C}$

$$(a) \quad \begin{array}{ccccc} & & \text{id} & & \\ & & \circlearrowleft & & \\ A & \xrightarrow{a} & C & \xrightarrow{c} & A \\ f \downarrow & \circlearrowleft & \downarrow g & \circlearrowleft & \downarrow f \\ B & \xrightarrow{b} & D & \xrightarrow{d} & B \\ & & \text{id} & & \end{array} \quad (b) \quad \begin{array}{ccccc} A & \xrightarrow{a} & C & \xrightarrow{c} & A \\ i \downarrow & \circlearrowleft & \downarrow j & \circlearrowleft & \downarrow i \\ M & \xrightarrow{m} & N & \xrightarrow{n} & M \\ x \downarrow & \circlearrowleft & \downarrow y & \circlearrowleft & \downarrow x \\ B & \xrightarrow{b} & D & \xrightarrow{d} & B \end{array}$$

in which $g \in \mathbb{W}$. Let us show that $f \in \mathbb{W}$. If we decompose f and g as

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ i \searrow & \circlearrowleft & \nearrow x \\ & M & \end{array}, \quad \begin{array}{ccc} C & \xrightarrow{g} & D \\ j \searrow & \circlearrowleft & \nearrow y \\ & N & \end{array} \quad \left(\begin{array}{l} i \in \text{Cof}, j \in \text{wCof}, \\ x, y \in \text{wFib} \end{array} \right)$$

by Proposition 5.15, then there exist morphisms m, n which make diagram (b) commutative by Proposition 5.14.

By Corollary 5.22 applied to the lower half, it follows $n \circ m \in \mathbb{W}$. By definition of Cof and wCof , there are $\mathbb{E}$ -triangles $A \xrightarrow{i} M \xrightarrow{p} U \xrightarrow{\rho} \rightarrow$, $C \xrightarrow{j} N \xrightarrow{q} S \xrightarrow{\tau} \rightarrow$ with $U \in \mathcal{U}, S \in \mathcal{S}$. It suffices to show $U \in \mathcal{S}$. Realize $c_* \tau$ by an $\mathbb{E}$ -triangle $A \xrightarrow{j'} N' \xrightarrow{q'} S \xrightarrow{c_* \tau} \rightarrow$. Put $c' = \begin{bmatrix} -c \\ j \end{bmatrix}: C \rightarrow A \oplus N$.

Then by an argument similar to that of the proof of Corollary 3.16, we can find a morphism of $\mathbb{E}$ -triangles as on the left of (20) below which gives an $\mathbb{E}$ -triangle $C \xrightarrow{c'} A \oplus N \xrightarrow{[j' \ n_1]} N' \xrightarrow{q'^* \tau} \rightarrow$ (cf. [LN, Proposition 1.20]). Since we have $[i \ n] \circ c' = n \circ j - i \circ c = 0$, there is $n' \in \mathcal{C}(N', M)$ which satisfies $n' \circ [j' \ n_1] = [i \ n]$, namely $n' \circ j' = i$ and $n' \circ n_1 = n$.

Put $m' = n_1 \circ m$. This satisfies $n' \circ m' = n \circ m \in \mathbb{W}$ and $m' \circ i = j'$. Resolve N' by an $\mathbb{E}$ -triangle $T' \rightarrow S' \xrightarrow{s'} N' \xrightarrow{\theta} \rightarrow$, with $S' \in \mathcal{S}, T' \in \mathcal{T}$. Then by the dual of Corollary 3.16, the morphism $[m' \ s']: M \oplus S' \rightarrow N'$ can be completed into an $\mathbb{E}$ -triangle $\exists L \rightarrow M \oplus S' \xrightarrow{[m' \ s']} N' \xrightarrow{\theta} \rightarrow$. By the dual

of Proposition 3.17, we obtain the following commutative diagram (right) made of conflations.

$$\begin{array}{ccccc}
 C & \xrightarrow{j} & N & \xrightarrow{q} & S \xrightarrow{\tau} \triangleright \\
 \downarrow c & \circlearrowleft & \downarrow n_1 & \circlearrowleft & \parallel \\
 A & \xrightarrow{j'} & N' & \xrightarrow{q'} & S \xrightarrow{c_*} \triangleright
 \end{array}
 \quad
 \begin{array}{ccccc}
 & & A & \xlongequal{\quad} & A \\
 & & \downarrow \begin{bmatrix} i \\ 0 \end{bmatrix} & \circlearrowleft & \downarrow j' \\
 L & \longrightarrow & M \oplus S' & \xrightarrow{[m' \ s']} & N' \\
 \parallel & \circlearrowleft & \downarrow p \oplus \text{id} & \circlearrowleft & \downarrow q' \\
 L & \xrightarrow{\exists \ell} & U \oplus S' & \xrightarrow{\exists k} & S
 \end{array}
 \quad (20)$$

If we put $m_0 = n' \circ [m' \ s']$, then since $n' \circ m' = m_0 \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we see that $m_0 \in \mathbb{W}$ follows from $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in wCof$ and $n' \circ m' \in \mathbb{W}$ by Corollary 5.22. Applying Lemma 5.23 to

$$\begin{array}{ccccc}
 A & \xrightarrow{\begin{bmatrix} i \\ 0 \end{bmatrix}} & M \oplus S' & \xrightarrow{p \oplus \text{id}} & U \oplus S' \xrightarrow{\quad} \triangleright \\
 \parallel & \circlearrowleft & \downarrow m_0 & & \\
 A & \xrightarrow{i} & M & \xrightarrow{p} & U \xrightarrow{\quad} \triangleright
 \end{array}$$

we obtain $u \in \mathcal{C}(U \oplus S', U)$ which belongs to $\mathbb{W}$ satisfying $u \circ (p \oplus \text{id}_{S'}) = p \circ m_0$. Then since $u \circ \ell = 0$, we see that u factors through S , in the bottom $\mathbb{E}$ -triangle in (20). Thus if we apply the functor $\sigma: \mathcal{U}/\mathcal{I} \rightarrow \mathcal{Z}/\mathcal{I}$, it follows $\sigma(\underline{u}) = 0$. On the other hand, it can be easily seen that $u \in \mathbb{W}$ implies that u can be written as composition of $U \oplus S' \xrightarrow{u'} U \oplus I \xrightarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} U$, with $u' \in wCof$ and $I \in \mathcal{I}$. By (ET4), we can show that $\sigma(\underline{u}')$ is an isomorphism, and thus $\sigma(\underline{u})$ is an isomorphism. This means $\sigma(U) = 0$ in $\mathcal{Z}/\mathcal{I}$, which shows $U \in \mathcal{U} \cap \text{CoCone}(\mathcal{I}, S) \subseteq \mathcal{U} \cap \mathcal{N} = S$. $\square$

By the argument so far, admissible model structures and Hovey twin cotorsion pairs on $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ correspond bijectively. Remark that, a model structure induces an equivalence

$$\mathcal{C}_{cf}/\sim \xrightarrow{\simeq} \mathcal{C}[\mathbb{W}^{-1}].$$

Here, the right hand side is the localization $\ell: \mathcal{C} \rightarrow \mathcal{C}[\mathbb{W}^{-1}]$. The left hand side is the category of fibrant-cofibrant objects modulo homotopies. Let us describe it in terms of the corresponding Hovey twin cotorsion pair $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$.

- $X \in \mathcal{C}$ is fibrant if and only if $(X \rightarrow 0) \in \text{Fib}$, if and only if $X \in \mathcal{T}$. Dually, X is cofibrant if and only if $X \in \mathcal{U}$. Thus the full subcategory of fibrant-cofibrant objects in $\mathcal{C}$ agrees with $\mathcal{Z} \subseteq \mathcal{C}$.
- For any $X, Y \in \mathcal{Z}$, morphisms $f, g \in \mathcal{Z}(X, Y)$ satisfy $f \sim g$ if and only if $f - g$ factors through some object $I \in \mathcal{I}$.

Thus we have $\mathcal{C}_{cf}/\sim = \mathcal{Z}/\mathcal{I}$. In summary, we obtain the following. This gives an explanation for the equivalence in [Na4, Proposition 6.10] and [IYa, Theorem 4.1].

Corollary 5.25. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, and let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a Hovey twin cotorsion pair. Then for $\mathbb{W} = w\text{Fib} \circ w\text{Cof}$ defined as above, we have an equivalence $\mathcal{Z}/\mathcal{I} \xrightarrow{\sim} \mathcal{C}[\mathbb{W}^{-1}]$ which makes the following diagram commutative up to natural isomorphism.

$$\begin{array}{ccc}
 \mathcal{Z} & \xrightarrow{\text{inclusion}} & \mathcal{C} \\
 \downarrow \text{ideal quotient} & \circlearrowleft & \downarrow \ell \\
 \mathcal{Z}/\mathcal{I} & \xrightarrow{\sim} & \mathcal{C}[\mathbb{W}^{-1}]
 \end{array}$$

In particular, the map

$$(\mathcal{Z}/\mathcal{I})(X, Y) \rightarrow \mathcal{C}[\mathbb{W}^{-1}](X, Y) ; \bar{f} \mapsto \ell(f)$$

is an isomorphism for any $X, Y \in \mathcal{Z}$.

Remark 5.26. By the generality of a model structure, we can also deduce that

$$(\mathcal{C}/\mathcal{I})(U, T) \rightarrow \mathcal{C}[\mathbb{W}^{-1}](U, T) ; \bar{f} \mapsto \ell(f)$$

is an isomorphism for any $U \in \mathcal{U}$ and $T \in \mathcal{T}$. (This also follows from the adjoint property given in Remark 4.13.)

Remark 5.27. If $\mathcal{C}$ is abelian, then we can show easily that $\mathbb{W}$ agrees with the class of morphisms f satisfying $\text{Ker}(f) \in \mathcal{N}$ and $\text{Cok}(f) \in \mathcal{N}$. Remark that $\mathcal{N} \subseteq \mathcal{C}$ becomes a Serre subcategory only when $\mathcal{N} = \mathcal{C}$. Indeed, if $\mathcal{N} \subseteq \mathcal{C}$ is a Serre subcategory, then the localization $\ell: \mathcal{C} \rightarrow \mathcal{C}[\mathbb{W}^{-1}]$ becomes an exact functor between abelian categories. Since any $C \in \mathcal{C}$ admits an inflation $C \rightarrow V$ to some $V \in \mathcal{V}$, this shows that $C = 0$ holds in $\mathcal{C}[\mathbb{W}^{-1}]$, which means $C \in \mathcal{N}$.

6. Triangulation of the homotopy category

In this section, we assume $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ to be a Hovey twin cotorsion pair. Put $\tilde{\mathcal{C}} = \mathcal{C}[\mathbb{W}^{-1}]$ and let $\ell: \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ be the localization functor. We will show that $\tilde{\mathcal{C}}$ is triangulated (Theorem 6.20).

Remark 6.1. *We note that the category $\mathcal{C}$ is usually not complete and co-complete, so that the model structure is not stable. However, axiom (ET4) gives specific choices of weak bicartesian squares which will compensate for the lack of stability.*

6.1 Shift functor

We first aim at defining a shift functor on the category $\tilde{\mathcal{C}}$.

Definition 6.2. Let us fix a choice, for any object $A \in \mathcal{C}$, of an $\mathbb{E}$ -triangle $A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A \xrightarrow{\rho_A} \rightarrow$, with $V_A \in \mathcal{V}$ and $U_A \in \mathcal{U}$. The functor $[1]: \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ is defined on objects by $A[1] = U_A$, and on morphisms as follows: Let $A \xrightarrow{f} B$ be a morphism in $\mathcal{C}$. Then there exists $u_f \in \mathcal{C}(U_A, U_B)$ which gives a morphism of $\mathbb{E}$ -extensions $(f, u_f): \rho_A \rightarrow \rho_B$. Indeed, since $\mathbb{E}(U_A, V_B) = 0$, one shows, by using the long exact sequence of Proposition 3.3 that there is a morphism $V_A \xrightarrow{v_f} V_B$ such that $v_f \circ v_A = v_B \circ f$. There is an induced morphism of $\mathbb{E}$ -triangles as follows.

$$\begin{array}{ccccccc} A & \xrightarrow{v_A} & V_A & \xrightarrow{u_A} & U_A & \xrightarrow{\rho_A} & \rightarrow \\ f \downarrow & \circlearrowleft & \downarrow v_f & \circlearrowleft & \downarrow u_f & & \\ B & \xrightarrow{v_B} & V_B & \xrightarrow{u_B} & U_B & \xrightarrow{\rho_B} & \rightarrow \end{array}$$

Define $f[1]$ to be the image $\ell(u_f)$ of u_f in $\tilde{\mathcal{C}}$. Since $f_*\rho_A = u_f^*\rho_B$, the morphism u_f is uniquely defined in $\mathcal{C}/\mathcal{V}$ by Corollary 3.5. This implies that $f[1] = \ell(u_f)$ is well-defined.

Claim 6.3. *The functor $[1]$ does not essentially depend on the choices made.*

More precisely, fix any other choice of $\mathbb{E}$ -triangles $A \xrightarrow{v'_A} V'_A \xrightarrow{u'_A} U'_A \xrightarrow{\rho'_A} \rightarrow$ and let $\{1\}$ be the functor defined as above by means of these $\mathbb{E}$ -triangles. Then $[1]$ and $\{1\}$ are naturally isomorphic.

Proof. The identity on A induces two morphisms of $\mathbb{E}$ -extensions

$$(\mathrm{id}, \exists t_A): \rho_A \rightarrow \rho'_A \quad \text{and} \quad (\mathrm{id}, \exists t'_A): \rho'_A \rightarrow \rho_A.$$

Then, since both $(\mathrm{id}, t'_A \circ t_A)$, $(\mathrm{id}, \mathrm{id}): \rho_A \rightarrow \rho_A$ are morphisms of $\mathbb{E}$ -extensions, we have $\ell(t'_A \circ t_A) = \ell(\mathrm{id}) = \mathrm{id}$ as in the argument in Definition 6.2. Similarly we have $\ell(t_A \circ t'_A) = \mathrm{id}$, and thus $\ell(t_A)$ is an isomorphism. Put $\tau_A = \ell(t_A)$, and let us show the naturality of $\tau = \{\tau_A \in \tilde{\mathcal{C}}(A[1], A\{1\})\}_{A \in \mathcal{C}}$. Let $f \in \mathcal{C}(A, B)$ be any morphism. By definition, $f[1] = \ell(u)$ and $f\{1\} = \ell(u')$ are given by morphisms of $\mathbb{E}$ -extensions $(f, u): \rho_A \rightarrow \rho_B$ and $(f', u'): \rho'_A \rightarrow \rho'_B$. Then, since both $(f, t_B \circ u)$ and $(f, u' \circ t_A)$ are morphisms $\rho_A \rightarrow \rho'_B$, we obtain $\ell(t_B \circ u) = \ell(u' \circ t_A)$, namely $\tau_B \circ f[1] = f\{1\} \circ \tau_A$. $\square$

We would like to show that $[1]: \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ induces an endofunctor of $\tilde{\mathcal{C}}$. For this, it is enough to show that $[1]$ sends weak equivalences to isomorphisms. Since weak equivalences are compositions of a morphism in $wCof$ followed by a morphism in $wFib$, it is enough to show that $[1]$ inverts all morphisms in $wCof$ and in $wFib$.

Lemma 6.4. Let $j \in wCof$. Then $j[1]$ is an isomorphism in $\tilde{\mathcal{C}}$.

Proof. Let $A \xrightarrow{j} B$ be a morphism in $wCof$. There is an $\mathbb{E}$ -triangle $A \xrightarrow{j} B \rightarrow S \dashrightarrow$, with $S \in \mathcal{S}$. Then (ET4) gives morphisms $(*)$ of $\mathbb{E}$ -triangles:

$$\begin{array}{ccc}
 A \xrightarrow{j} B \longrightarrow S \dashrightarrow & & A \xrightarrow{v'_A} V_B \xrightarrow{u'_A} C \xrightarrow{\rho'_A} \gg \\
 \parallel \quad \circ \quad v_B \downarrow \quad \circ \quad \downarrow & & j \downarrow \quad \circ \quad \parallel \quad \circ \quad \downarrow w \\
 A \xrightarrow{v'_A} V_B \xrightarrow{u'_A} C \xrightarrow{\rho'_A} \gg & & B \xrightarrow{v_B} V_B \xrightarrow{u_B} U_B \xrightarrow{\rho_B} \gg \\
 \quad \quad \quad u_B \downarrow \quad \circ \quad \downarrow w & & \\
 \quad \quad \quad U_B = U_B & & \\
 \quad \quad \quad \downarrow \rho_B \quad \downarrow & & \\
 \quad \quad \quad \Downarrow \quad \Downarrow & &
 \end{array}
 \quad (**)$$

Since S and U_B belong to $\mathcal{U}$, so does C . Moreover, w is a weak equivalence by Claim 5.12 (2). Claim 6.3 allows to conclude that $j[1]$ is an isomorphism in $\tilde{\mathcal{C}}$, since $(**)$ is a morphism of $\mathbb{E}$ -triangles. $\square$

Lemma 6.5. Let $q \in wFib$. Then $q[1]$ is an isomorphism in $\tilde{\mathcal{C}}$.

Proof. Let $X \xrightarrow{q} Y$ be a morphism in $wFib$. It induces a morphism of $\mathbb{E}$ -triangles:

$$\begin{array}{ccccc} X & \xrightarrow{v_X} & V_X & \xrightarrow{u_X} & U_X \xrightarrow{\rho_X} \rightarrow \\ q \downarrow & \circlearrowleft & \downarrow n & \circlearrowleft & \downarrow q[1] \\ Y & \xrightarrow{v_Y} & V_Y & \xrightarrow{u_Y} & U_Y \xrightarrow{\rho_Y} \rightarrow \end{array}$$

Since $V_X \rightarrow 0$ and $V_Y \rightarrow 0$ are acyclic fibrations, in particular they are weak equivalences. By the 2-out-of-3 property, this implies that $V_X \xrightarrow{n} V_Y$ is a weak equivalence. Factor n as an acyclic cofibration j followed by an acyclic fibration p . Then (ET4) gives a diagram (α) of $\mathbb{E}$ -triangles:

$$\begin{array}{ccc} \begin{array}{ccccc} X & \xrightarrow{v_X} & V_X & \xrightarrow{u_X} & U_X \\ \parallel & \circlearrowleft & \downarrow j & \circlearrowleft & \downarrow \\ X & \xrightarrow{j \circ v_X} & B & \longrightarrow & C \\ & & \downarrow & \circlearrowleft & \downarrow \\ & & S & = & S \end{array} & (\alpha) & \begin{array}{ccccc} V & \longrightarrow & V' & \longrightarrow & D \dashrightarrow \\ \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\ X & \xrightarrow{j \circ v_X} & B & \longrightarrow & C \dashrightarrow \\ q \downarrow & \circlearrowleft & \downarrow p & \circlearrowleft & \downarrow q' \\ Y & \xrightarrow{v_Y} & V_Y & \xrightarrow{u_Y} & U_Y \xrightarrow{\rho_Y} \rightarrow \\ \downarrow & & \downarrow & & \downarrow \\ \downarrow & & \downarrow & & \downarrow \end{array} \end{array} \quad (\beta)$$

where $S \in \mathcal{S}$. We have $B \in \mathcal{N}$ (since $V_X, S \in \mathcal{N}$) and $C \in \mathcal{U}$ (since $U_X, S \in \mathcal{U}$). The nine Lemma 5.9 gives morphisms of $\mathbb{E}$ -triangles as in (β) , with $V, V' \in \mathcal{V}$, and thus $B \in \mathcal{V}$. This implies $D \in \mathcal{N}$ and thus q' is a weak equivalence by Claim 5.12 (2). We conclude that $q[1]$ is an isomorphism in $\tilde{\mathcal{C}}$ in the same manner as in the end of the proof of Lemma 6.4. $\square$

Corollary 6.6. The functor $[1]: \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ induces an endofunctor of $\tilde{\mathcal{C}}$, which we denote by the same symbol $[1]$. Thus $f[1]$ for a morphism f in $\mathcal{C}$ will be denoted also by $\ell(f)[1]$ in the rest.

Proof. By Lemma 6.4 and Lemma 6.5, the functor $[1]: \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ inverts all weak equivalences. By the universal property of the localization of a category, it induces a functor $[1]: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}$. $\square$

Definition 6.7. By using $\mathbb{E}$ -triangles $T^C \xrightarrow{t^C} S^C \xrightarrow{s^C} C \xrightarrow{\lambda^C} 0$ one can define dually a functor $[-1]: \mathcal{C} \rightarrow \mathcal{C}$ with $C[-1] = T^C$, which induces an endofunctor $[-1]$ of $\mathcal{C}$.

6.2 Connecting morphism

Define the bifunctor $\tilde{\mathbb{E}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ by $\tilde{\mathbb{E}} = \mathcal{C}(-, -[1])$. In this section, we will construct a homomorphism $\tilde{\ell} = \tilde{\ell}_{C,A}: \mathbb{E}(C, A) \rightarrow \tilde{\mathbb{E}}(C, A)$ for each pair $A, C \in \mathcal{C}$. For any $A \in \mathcal{C}$, we continue to use the $\mathbb{E}$ -triangle

$$A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A \xrightarrow{\rho_A} 0 \quad (21)$$

chosen to define the shift functor $[1]$.

Lemma 6.8. Let $A, C \in \mathcal{C}$ be any pair of objects. If $f_1, f_2 \in \mathcal{C}(X, U_A)$ satisfy $f_1^* \rho_A = f_2^* \rho_A$, then $\ell(f_1) = \ell(f_2)$ holds.

Proof. This immediately follows from the exactness of

$$\mathcal{C}(X, V_A) \xrightarrow{u_A \circ -} \mathcal{C}(X, U_A) \xrightarrow{(\rho_A)^\#} \mathbb{E}(X, A) \text{ shown in Proposition 3.3.} \quad \square$$

Definition 6.9. For any $\mathbb{E}$ -extension $\delta \in \mathbb{E}(C, A)$, define $\tilde{\ell}(\delta) \in \tilde{\mathbb{E}}(C, A)$ by the following. Take a span of morphisms $(C \xleftarrow{w} D \xrightarrow{d} U_A)$ from some $D \in \mathcal{C}$, which satisfy

$$w \in w\text{Fib} \quad \text{and} \quad w^* \delta = d^* \rho_A. \quad (22)$$

Then, define as $\tilde{\ell}(\delta) = \ell(d) \circ \ell(w)^{-1}$.

Claim 6.10. For any $\delta \in \mathbb{E}(C, A)$, the morphism $\tilde{\ell}(\delta)$ in Definition 6.9 is well-defined. More precisely, the following holds.

- (1) Take any $D \in \mathcal{C}$, $w \in \mathcal{C}(D, C)$. If both $d_1, d_2 \in \mathcal{C}(D, U_A)$ satisfy $w^* \delta = d_i^* \rho_A$ ($i = 1, 2$), then $\ell(d_1) = \ell(d_2)$ holds.
- (2) If both spans $(C \xleftarrow{w_1} D_1 \xrightarrow{d_1} U_A)$ and $(C \xleftarrow{w_2} D_2 \xrightarrow{d_2} U_A)$ satisfy (22), then $\ell(d_1) \circ \ell(w_1)^{-1} = \ell(d_2) \circ \ell(w_2)^{-1}$ holds.
- (3) There exists at least one span $(C \xleftarrow{w} D \xrightarrow{d} U_A)$ satisfying (22).

Proof. (1) This immediately follows from Lemma 6.8.

(2) Let $V_i \xrightarrow{v_i} D_i \xrightarrow{w_i} C$ ($i = 1, 2$) be conflations. By Proposition 3.15, we have a commutative diagram made of conflations as follows.

$$\begin{array}{ccccc}
 & & V_2 & \xlongequal{\quad} & V_2 \\
 & & \downarrow m_2 & \circlearrowleft & \downarrow v_2 \\
 V_1 & \xrightarrow{m_1} & \exists D & \xrightarrow{e_1} & D_2 \\
 \parallel & & \downarrow e_2 & \circlearrowleft & \downarrow w_2 \\
 V_1 & \xrightarrow{v_1} & D_1 & \xrightarrow{w_1} & C
 \end{array}$$

If we put $w = w_1 \circ e_2 = w_2 \circ e_1$, then we have $w \in wFib \circ wFib = wFib$.

If we put $k_1 = d_2 \circ e_1$ and $k_2 = d_1 \circ e_2$, then they give

$$\begin{aligned}
 \ell(k_1) \circ \ell(w)^{-1} &= \ell(d_2 \circ e_1) \circ \ell(w_2 \circ e_1)^{-1} = \ell(d_2) \circ \ell(w_2)^{-1}, \\
 \ell(k_2) \circ \ell(w)^{-1} &= \ell(d_1 \circ e_2) \circ \ell(w_1 \circ e_2)^{-1} = \ell(d_1) \circ \ell(w_1)^{-1}.
 \end{aligned}$$

Since both k_1, k_2 satisfy $k_1^* \delta = e_1^* d_2^* \delta = e_1^* w_2^* \delta = w^* \delta$, $k_2^* \delta = e_2^* d_1^* \delta = e_2^* w_1^* \delta = w^* \delta$, we obtain $\ell(k_1) = \ell(k_2)$ by (1). Thus it follows that $\ell(d_2) \circ \ell(w_2)^{-1} = \ell(d_1) \circ \ell(w_1)^{-1}$.

(3) Realize δ by an $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$. Then Proposition 3.15 gives a commutative diagram made of $\mathbb{E}$ -triangles

$$\begin{array}{ccccc}
 A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} A \\
 \downarrow v_A & \circlearrowleft & \downarrow & \circlearrowleft & \parallel \\
 V_A & \longrightarrow & \exists D & \xrightarrow{w} & C \xrightarrow{(v_A)_* \delta} V_A \\
 \downarrow u_A & \circlearrowleft & \downarrow e & & \\
 U_A & \xlongequal{\quad} & U_A & & \\
 \downarrow \rho_A & & \downarrow x_* \rho_A & & \\
 \Psi & & \Psi & &
 \end{array}$$

satisfying $w^* \delta + e^* \rho_A = 0$. Thus the span $(C \xleftarrow{w} D \xrightarrow{-e} U_A)$ satisfies (22). $\square$

Proposition 6.11. For any $A, C \in \mathcal{C}$, the map $\tilde{\ell}: \mathbb{E}(C, A) \rightarrow \tilde{\mathbb{E}}(C, A)$ is an additive homomorphism.

Proof. Let $\delta_1, \delta_2 \in \mathbb{E}(C, A)$ be any pair of elements. By (3) of Claim 6.10, we can find spans $(C \xleftarrow{w_1} D_1 \xrightarrow{d_1} U_A)$, $(C \xleftarrow{w_2} D_2 \xrightarrow{d_2} U_A)$ which give $\tilde{\ell}(\delta_1)$, $\tilde{\ell}(\delta_2)$. As in the proof of (2) in Claim 6.10, replacing (w_i, D_i) by a common (w, D) , we may assume $D_1 = D_2 = D$ and $w_1 = w_2 = w$ from the start. Then by $w^*\delta_1 = d_1^*\rho_A$ and $w^*\delta_2 = d_2^*\rho_A$, we have $w^*(\delta_1 + \delta_2) = (d_1 + d_2)^*\rho_A$. Thus the span $(C \xleftarrow{w} D \xrightarrow{d_1+d_2} U_A)$ satisfies (22) for $\delta_1 + \delta_2$. This shows $\tilde{\ell}(\delta_1 + \delta_2) = \ell(d_1 + d_2) \circ \ell(w)^{-1} = \ell(d_1) \circ \ell(w)^{-1} + \ell(d_2) \circ \ell(w)^{-1} = \tilde{\ell}(\delta_1) + \tilde{\ell}(\delta_2)$. $\square$

Lemma 6.12. For any $U \in \mathcal{U}$ and $T \in \mathcal{T}$, the map $\tilde{\ell}: \mathbb{E}(U, T) \rightarrow \tilde{\mathbb{E}}(U, T)$ is monomorphic.

Proof. Let $\delta \in \mathbb{E}(U, T)$ be any $\mathbb{E}$ -extension. Realize it by an $\mathbb{E}$ -triangle $T \xrightarrow{x} A \xrightarrow{y} U \xrightarrow{\delta} \gg$. Let $T \xrightarrow{v_T} V_T \xrightarrow{u_T} U_T \xrightarrow{\rho_T} \gg$ be the chosen $\mathbb{E}$ -triangle, as before. By Proposition 3.15, we obtain a diagram made of $\mathbb{E}$ -triangles

$$\begin{array}{ccccc}
T & \xrightarrow{x} & A & \xrightarrow{y} & U \xrightarrow{\delta} \gg \\
v_T \downarrow & \circlearrowleft & \downarrow m & \circlearrowleft & \parallel \\
V_T & \xrightarrow{m'} & M & \xrightarrow{e'} & U \xrightarrow{(v_T)_*\delta} \gg \\
u_T \downarrow & \circlearrowleft & \downarrow e & & \\
U_T & = & U_T & & \\
\rho_T \downarrow & & \downarrow x_*(\rho_T) & & \\
\Downarrow & & \Downarrow & &
\end{array}$$

satisfying $e^*(\rho_T) + e'^*\delta = 0$. As the proof of (3) of Claim 6.10 suggests, we have $\tilde{\ell}(\delta) = -\ell(e) \circ \ell(e')^{-1}$. By $\mathbb{E}(U, V_T) = 0$, we have $(v_T)_*\delta = 0$. Thus, replacing M by an isomorphic object, we may assume

$$M = V_T \oplus U, \quad m' = \iota_{V_T}: V_T \rightarrow M, \quad e' = p_U: M \rightarrow U,$$

where

$$V_T \xrightleftharpoons[p_{V_T}]{\iota_{V_T}} V_T \oplus U \xrightleftharpoons[p_U]{\iota_U} U$$

is a biproduct. Put $q = -e \circ \iota_U: U \rightarrow U_T$. Then we have $e = u_T \circ p_{V_T} - q \circ e'$. Since $\ell(p_{V_T}) = 0$, this gives $\ell(e) = -\ell(q) \circ \ell(e')$, namely

$$\tilde{\ell}(\delta) = \ell(q). \tag{23}$$

On the other hand, since the morphism e'^* is a monomorphism, the equality

$$e'^*\delta = -e^*(\rho_T) = -p_{V_T}^*u_T^*(\rho_T) + e'^*q^*(\rho_T) = e'^*q^*(\rho_T)$$

implies

$$\delta = q^*(\rho_T). \quad (24)$$

By (23) and (24), it suffices to show $\ell(q) = 0 \Rightarrow q^*(\rho_T) = 0$. Assume $\ell(q) = 0$. Take $\mathbb{E}$ -triangles $U_T \xrightarrow{z} Z \rightarrow S \dashrightarrow$ ($Z \in \mathcal{Z}$, $S \in \mathcal{S}$), $T' \rightarrow I \xrightarrow{i} Z \dashrightarrow$ ($I \in \mathcal{I}$, $T' \in \mathcal{T}$). Then by $\ell(z \circ q) = \ell(z) \circ 0 = 0$, the morphism $z \circ q \in \mathcal{C}(U, Z)$ factors through i by Remark 5.26. Namely, there exists $k \in \mathcal{C}(U, I)$ such that $z \circ q = i \circ k$. By (ET4)^{op}, we obtain a diagram (a) made of conflations in which, N belongs to $\mathcal{N}$ by Lemma 4.18 (2). By the dual of Lemma 3.13, we obtain $j \in \mathcal{C}(U, N)$ which makes diagram (b) commutative. By $N \in \mathcal{N} = \text{Cone}(\mathcal{V}, \mathcal{S})$ and $\mathbb{E}(U, \mathcal{V}) = 0$, this j factors through some $S_0 \in \mathcal{S}$, as in (c).

(a)
$$\begin{array}{ccccc} T' & \longrightarrow & \exists N & \xrightarrow{\exists p} & U_T \\ \parallel & \circlearrowleft \exists n \downarrow & \circlearrowleft & \downarrow z & \\ T' & \longrightarrow & I & \xrightarrow{i} & Z \\ & & \downarrow & \circlearrowleft & \downarrow \\ & & S & = & S \end{array}$$

(b)
$$\begin{array}{ccccc} U & \xrightarrow{q} & & & \\ & \searrow j & \circlearrowleft & \searrow p & \\ & & N & \longrightarrow & U_T \\ & \swarrow k & \circlearrowleft \downarrow n & \circlearrowleft \downarrow z & \\ & & I & \xrightarrow{i} & Z \end{array}$$

(c)
$$\begin{array}{ccccc} U & \xrightarrow{q} & & & \\ & \searrow j & \circlearrowleft & \searrow p & \\ & & N & \longrightarrow & U_T \\ & \swarrow \exists r & \circlearrowleft \downarrow & \circlearrowleft & \\ & & S_0 & \longrightarrow & N \end{array}$$

By $\mathbb{E}(S_0, T) = 0$, the morphism $p \circ r$ factors through u_T . This implies $q^*(\rho_T) = 0$ by Lemma 3.2. $\square$

Lemma 6.13. Let $(A, \delta, C), (A', \delta', C')$ be $\mathbb{E}$ -extensions, and let $(a, c): \delta \rightarrow \delta'$ be a morphism of $\mathbb{E}$ -triangles. Then $(\ell(a), \ell(c)): \widetilde{\ell}(\delta) \rightarrow \widetilde{\ell}(\delta')$ is a morphism of $\widetilde{\mathbb{E}}$ -extensions. Namely,

$$\begin{array}{ccc} C & \xrightarrow{\widetilde{\ell}(\delta)} & U_A = A[1] \\ \ell(c) \downarrow & \circlearrowleft & \downarrow \ell(a)[1] \\ C' & \xrightarrow{\widetilde{\ell}(\delta')} & U_{A'} = A'[1] \end{array}$$

is commutative in $\widetilde{\mathcal{C}}$.

Proof. Take spans $(C \xleftarrow{w} D \xrightarrow{d} U_A)$ and $(C' \xleftarrow{w'} D' \xrightarrow{d'} U_{A'})$ satisfying $w, w' \in wFib$, $w^*\delta = d^*\rho_A$, $w'^*\delta' = d'^*\rho_{A'}$. By definition, we have $\tilde{\ell}(\delta) = \ell(d) \circ \ell(w)^{-1}$, $\tilde{\ell}(\delta') = \ell(d') \circ \ell(w')^{-1}$. Remark that $\ell(a)[1] = \ell(u)$ is given by a morphism of $\mathbb{E}$ -triangles (α) below.

$$\begin{array}{ccc}
 A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A \xrightarrow{\rho_A} & & V' \rightarrow \exists D'' \xrightarrow{\exists w''} C \xrightarrow{c^* \nu'} \\
 (\alpha) \quad \begin{array}{ccccc} a \downarrow & \circ & \downarrow v & \circ & \downarrow u \\ A' & \xrightarrow{v_{A'}} V_{A'} & \xrightarrow{u_{A'}} U_{A'} & \xrightarrow{\rho_{A'}} \end{array} & & (\beta) \quad \begin{array}{ccccc} \parallel & \circ \exists f \downarrow & \circ & \downarrow c \\ V' & \rightarrow D' & \xrightarrow{w'} C' & \xrightarrow{\nu'} \end{array}
 \end{array}$$

Since $w' \in wFib$, there exists an $\mathbb{E}$ -triangle $V' \rightarrow D' \xrightarrow{w'} C' \xrightarrow{\nu'}$ with $V' \in \mathcal{V}$. By realizing $c^*\nu'$, we obtain a morphism (β) of $\mathbb{E}$ -triangles. Then both spans $(C \xleftarrow{w} D \xrightarrow{u \circ d} U_{A'})$ and $(C \xleftarrow{w''} D'' \xrightarrow{d' \circ f} U_{A'})$ satisfy

$$\begin{aligned}
 w^*(c^*\delta') &= w^*a_*\delta = a_*w^*\delta \\
 &= a_*d^*\rho_A = d^*u^*\rho_{A'} = (u \circ d)^*\rho_{A'}, \\
 w''^*(c^*\delta') &= f^*w'^*\delta' = f^*d'^*\rho_{A'} = (d' \circ f)^*\rho_{A'}.
 \end{aligned}$$

Thus by Claim 6.10 (2), we obtain

$$\begin{aligned}
 \ell(u \circ d) \circ \ell(w)^{-1} &= \ell(d' \circ f) \circ \ell(w'')^{-1} \\
 &= \ell(d') \circ (\ell(f) \circ \ell(w'')^{-1}) = \ell(d') \circ (\ell(w')^{-1} \circ \ell(c)),
 \end{aligned}$$

which means $\ell(u) \circ \tilde{\ell}(\delta) = \tilde{\ell}(\delta') \circ \ell(c)$. $\square$

Proposition 6.14. The functor $[1]: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ is an auto-equivalence, with quasi-inverse $[-1]$.

Proof. By the definitions of $[-1]$ and $[1]$, for each $C \in \mathcal{C}$, there are $\mathbb{E}$ -triangles $C[-1] \rightarrow S^C \rightarrow C \xrightarrow{\lambda^C}$ and $C[-1] \rightarrow V_{C[-1]} \rightarrow (C[-1])[1] \xrightarrow{\rho_{C[-1]}}$, where $S^C \in \mathcal{S}$, $V_{C[-1]} \in \mathcal{V}$. Then by Proposition 3.15, we have a commutative diagram $(*)$ made of conflations

$$\begin{array}{ccc}
 C[-1] \longrightarrow S^C \longrightarrow C & & C \xrightarrow{\tilde{\ell}(\lambda^C)} (C[-1])[1] \\
 \downarrow \quad \circ \quad \downarrow & & \downarrow \ell(f) \quad \circ \quad \downarrow \ell(g)[1] = (\ell(f)[-1])[1] \\
 V_{C[-1]} \longrightarrow \exists D \xrightarrow{w} C & & C' \xrightarrow{\tilde{\ell}(\lambda^{C'})} (C'[-1])[1] \\
 \downarrow \quad \circ \quad \downarrow e & & \\
 (C[-1])[1] = (C[-1])[1] & &
 \end{array}$$

which gives $\tilde{\ell}(\lambda^C) = -\ell(e) \circ \ell(w)^{-1}$ as in the proof of (3) in Claim 6.10. Since $e \in \mathbb{W}$, it follows that $\tilde{\ell}(\lambda^C)$ is an isomorphism in $\tilde{\mathcal{C}}$. Let us show the naturality of $\{\tilde{\ell}(\lambda^C): C \rightarrow (C[-1])[1]\}_{C \in \tilde{\mathcal{C}}}$. For this purpose, it suffices to show the naturality with respect to the morphisms in $\mathcal{C}$. For any morphism $f \in \mathcal{C}(C, C')$, the morphism $\ell(f)[-1] = \ell(g) \in \mathcal{C}(C[-1], C'[-1])$ is given by a morphism of $\mathbb{E}$ -extensions $(g, f): \lambda^C \rightarrow \lambda^{C'}$, dually to Definition 6.2. Thus by Lemma 6.13, the diagram $(**)$ becomes commutative. This shows $[1] \circ [-1] \cong \text{Id}$. The isomorphism $[-1] \circ [1] \cong \text{Id}$ can be shown dually. $\square$

6.3 Triangulation

Definition 6.15. For an $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$, its associated *standard triangle* in $\tilde{\mathcal{C}}$ is defined to be $A \xrightarrow{\ell(x)} B \xrightarrow{\ell(y)} C \xrightarrow{\tilde{\ell}(\delta)} A[1]$. A *distinguished triangle* in $\tilde{\mathcal{C}}$ is a triangle isomorphic to some standard triangle.

Proposition 6.16. Any morphism of $\mathbb{E}$ -triangles (left, below) gives the following morphism between standard triangles (right, below):

$$\begin{array}{ccc} A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} & \xrightarrow{\quad} & A \xrightarrow{\ell(x)} B \xrightarrow{\ell(y)} C \xrightarrow{\ell(\delta)} A[1] \\ a \downarrow \quad \circ \quad b \downarrow \quad \circ \quad c \downarrow & & \ell(a) \downarrow \quad \circ \quad \ell(b) \downarrow \quad \circ \quad \ell(c) \downarrow \quad \circ \quad \ell(a)[1] \downarrow \\ A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} & \xrightarrow{\quad} & A' \xrightarrow{\ell(x')} B' \xrightarrow{\ell(y')} C' \xrightarrow{\ell(\delta')} A'[1] \end{array}$$

Proof. This immediately follows from Lemma 6.13. $\square$

This gives a cofibrant replacement of a standard triangle, as follows.

Corollary 6.17. Assume $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies Condition (WIC) as before. Any standard triangle is isomorphic to a standard triangle associated to an $\mathbb{E}$ -triangle $U \rightarrow U' \rightarrow U'' \dashrightarrow$ whose terms satisfy $U, U', U'' \in \mathcal{U}$.

Proof. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ be any $\mathbb{E}$ -triangle. Resolve A by an $\mathbb{E}$ -triangle $V \xrightarrow{v} U \xrightarrow{a} A \xrightarrow{\lambda}$ satisfying $U \in \mathcal{U}$ and $V \in \mathcal{V}$. By Proposition 5.15, we have $x \circ a \in wFib \circ Cof$. Namely, there are $\mathbb{E}$ -triangles $U \xrightarrow{x'} B' \xrightarrow{y'} U_0 \xrightarrow{\delta'}$ and $V_0 \xrightarrow{v'} B' \xrightarrow{b} B \dashrightarrow$ satisfying $U_0 \in \mathcal{U}$, $V_0 \in \mathcal{V}$ and $x \circ a = b \circ x'$.

Since $\mathcal{U}$ is extension-closed, it follows that $B' \in \mathcal{U}$. Moreover, by Lemma 5.9 and Lemma 4.18 (1), we obtain a morphism (α) of $\mathbb{E}$ -triangles

$$(\alpha) \quad \begin{array}{ccccc} U & \xrightarrow{x'} & B' & \xrightarrow{y'} & U_0 \xrightarrow{\delta'} \rightarrow \\ a \downarrow & \circlearrowleft & \downarrow b & \circlearrowleft & \downarrow c \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \rightarrow \end{array} \quad (\beta) \quad \begin{array}{ccccc} U & \xrightarrow{\ell(x')} & B' & \xrightarrow{\ell(y')} & U_0 \xrightarrow{\tilde{\ell}(\delta')} \rightarrow U[1] \\ \ell(a) \downarrow \cong & \circlearrowleft & \downarrow \ell(b) & \circlearrowleft & \downarrow \ell(c) \circlearrowleft \cong \\ A & \xrightarrow{\ell(x)} & B & \xrightarrow{\ell(y)} & C \xrightarrow{\tilde{\ell}(\delta)} \rightarrow A[1] \end{array}$$

which admits an $\mathbb{E}$ -triangle $N \rightarrow U_0 \xrightarrow{c} C \dashrightarrow$ with some $N \in \mathcal{N}$. By Claim 5.12, it follows that $c \in \mathbb{W}$. By Proposition 6.16, we obtain an isomorphism (β) of standard triangles in $\tilde{\mathcal{C}}$. $\square$

Remark 6.18. Similarly, for any $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$, we can construct a morphism of $\mathbb{E}$ -triangles

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \rightarrow \\ a \downarrow & \circlearrowleft & \downarrow b & \circlearrowleft & \downarrow c \\ T_A & \longrightarrow & T_B & \longrightarrow & T_C \dashrightarrow \end{array}$$

which satisfies $T_A, T_B, T_C \in \mathcal{T}$ and $a, b, c \in \mathbb{W}$.

Lemma 6.19. Let $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ be any $\mathbb{E}$ -triangle. From the $\mathbb{E}$ -triangle $A \xrightarrow{v_A} V_A \xrightarrow{u_A} U_A \dashrightarrow$, we obtain an $\mathbb{E}$ -triangle

$$A \xrightarrow{\begin{bmatrix} x \\ v_A \end{bmatrix}} B \oplus V_A \rightarrow X \dashrightarrow \quad (25)$$

by Corollary 3.16. Then the standard triangles associated to the $\mathbb{E}$ -triangles (x, y, δ) , and (25) are isomorphic. Moreover we have $V_A \in \mathcal{I}$ if $A \in \mathcal{U}$.

Proof. By the dual of Proposition 3.17, we obtain a commutative diagram made of $\mathbb{E}$ -triangles, as follows.

$$\begin{array}{ccccc} & & V_A & \xlongequal{\quad} & V_A \\ & & \downarrow & \circlearrowleft & \downarrow \\ A & \xrightarrow{\begin{bmatrix} x \\ v_A \end{bmatrix}} & B \oplus V_A & \rightarrow & X \dashrightarrow \\ \parallel & \circlearrowleft & \downarrow [1 \ 0] \circlearrowleft & & \downarrow \exists_e \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C \dashrightarrow \\ & & \downarrow 0 & & \downarrow 0 \\ & & \downarrow & & \downarrow \end{array}$$

Remark that $\ell([1 \ 0]): B \rightarrow B \oplus V_A$ and $\ell(e): C \rightarrow Z$ are isomorphisms in $\tilde{\mathcal{C}}$. Thus Lemma 6.19 follows from Proposition 6.16. $\square$

Theorem 6.20. *The shift functor in Definition 6.2 and the class of distinguished triangles in Definition 6.15 give a triangulation of $\tilde{\mathcal{C}}$.*

Proof. (TR1) By definition, the class of distinguished triangles is closed under isomorphisms. From the $\mathbb{E}$ -triangle $A \xrightarrow{\text{id}_A} A \rightarrow 0 \dashrightarrow$, we obtain a distinguished triangle $A \xrightarrow{\text{id}_A} A \rightarrow 0 \rightarrow A[1]$.

Let $\alpha \in \mathcal{C}(A, B)$ be any morphism, and let us show the existence of a distinguished triangle of the form $A \xrightarrow{\alpha} B \rightarrow C \rightarrow A[1]$. Up to isomorphism in $\tilde{\mathcal{C}}$, we may assume $A \in \mathcal{U}, B \in \mathcal{T}$ from the start. As in Remark 5.26, then there is a morphism $f \in \mathcal{C}(A, B)$ satisfying $\ell(f) = \alpha$.

By Corollary 3.16, we have an $\mathbb{E}$ -triangle $A \rightarrow B \oplus V_A \xrightarrow{g} C \dashrightarrow$, where the first morphism is $\begin{bmatrix} f \\ v_A \end{bmatrix}$, which gives a standard triangle compatible with $\ell(f)$ as follows.

$$\begin{array}{ccccc} A & \xrightarrow{\begin{bmatrix} \ell(f) \\ \ell(v_A) \end{bmatrix}} & B \oplus V_A & \xrightarrow{\ell(g)} & C \xrightarrow{\tilde{\ell}(\delta)} A[1] \\ & \searrow \ell(f) & \downarrow \cong & & \\ & & B & & \end{array}$$

(TR2) It suffices to show this axiom for standard triangles. Let $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ be an $\mathbb{E}$ -triangle, and let $A \xrightarrow{\ell(x)} B \xrightarrow{\ell(y)} C \xrightarrow{\tilde{\ell}(\delta)} A[1]$ be its associated standard triangle. Let us show that $B \xrightarrow{\ell(y)} C \xrightarrow{\tilde{\ell}(\delta)} A[1] \xrightarrow{-\ell(x)[1]} B[1]$ is distinguished. By Proposition 3.15, we obtain a commutative diagram made of $\mathbb{E}$ -triangles

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \dashrightarrow \delta \gg \\ v_A \downarrow & \circlearrowleft & \downarrow m & \circlearrowleft & \parallel \\ V_A & \longrightarrow & \exists D & \xrightarrow{w} & C \xrightarrow{(v_A)_* \delta} \\ u_A \downarrow & \circlearrowleft & \downarrow e & & \\ U_A & = & U_A & & \\ \rho_A \downarrow & & \downarrow x_* \rho_A & & \\ \Psi & & \Psi & & \end{array}$$

satisfying $w^*\delta + e^*\rho_A = 0$. In particular, we obtain a distinguished triangle $B \xrightarrow{\ell(m)} D \xrightarrow{\ell(e)} A[1] \xrightarrow{\tilde{\ell}(x_*\rho_A)} B[1]$. Remark that we have the following equality $\tilde{\ell}(\delta) = -\ell(e) \circ \ell(w)^{-1}$, as the proof of (3) of Claim 6.10 suggests. Thus it remains to show the commutativity in $\tilde{\mathcal{C}}$ of the right-most square of (a).

$$(a) \quad \begin{array}{ccccc} B & \xrightarrow{\ell(m)} & D & \xrightarrow{\ell(e)} & A[1] \xrightarrow{\tilde{\ell}(x_*\rho_A)} B[1] \\ \parallel & \circ & \cong \downarrow \ell(w) \circ & \cong \downarrow -1 \circ & \parallel \\ B & \xrightarrow{\ell(y)} & C & \xrightarrow{\tilde{\ell}(\delta)} & A[1] \xrightarrow{-\ell(x)[1]} B[1] \end{array} \quad \begin{array}{ccccc} A & \xrightarrow{v_A} & V_A & \xrightarrow{u_A} & U_A \xrightarrow{-\rho_A} > \\ x \downarrow & \circ & \downarrow v & \circ & \downarrow u \\ B & \xrightarrow{v_B} & V_B & \xrightarrow{u_B} & U_B \xrightarrow{-\rho_B} > \end{array} \quad (b)$$

Let (b) be a morphism of $\mathbb{E}$ -triangles, which gives $\ell(x)[1] = \ell(u)$. Then, since the span $(U_A \xleftarrow{\text{id}} U_A \xrightarrow{u} U_B)$ satisfies $\text{id}^*(x_*\rho_A) = x_*\rho_A = u^*\rho_B$, it follows that $\tilde{\ell}(x_*\rho_A) = \ell(u) = \ell(x)[1]$.

(TR3) Up to isomorphism of triangles, it suffices to show this axiom for standard triangles. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$, $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} A'$ be $\mathbb{E}$ -triangles, and suppose we are given a commutative diagram $(*)$

$$(*) \quad \begin{array}{ccc} A & \xrightarrow{\ell(x)} & B \\ \alpha \downarrow & \circ & \downarrow \beta \\ A' & \xrightarrow{\ell(x')} & B' \end{array} \quad (**) \quad \begin{array}{ccc} A & \xrightarrow{x_0} & B \oplus V_A \\ a \downarrow & \circ & \downarrow [b \ j] \\ A' & \xrightarrow{x'} & B' \end{array}$$

in $\tilde{\mathcal{C}}$. By Corollary 6.17 and Remark 6.18, we may assume $A, B, C \in \mathcal{U}$ and $A', B', C' \in \mathcal{T}$ from the start. In that case, α and β can be written as $\alpha = \ell(a)$, $\beta = \ell(b)$ for some $a \in \mathcal{C}(A, A')$ and $b \in \mathcal{C}(B, B')$ by Remark 5.26. Moreover, the commutativity of $(*)$ means that $b \circ x - x' \circ a$ factors through some $I \in \mathcal{I}$. By exactness of $\mathcal{C}(V_A, I) \xrightarrow{-\circ v_A} \mathcal{C}(A, I) \rightarrow \mathbb{E}(U_A, I) = 0$, this shows that there exists $j \in \mathcal{C}(V_A, B')$ which makes $(**)$ commutative in $\mathcal{C}$, where $x_0 = [x \ v_A]$. By Lemma 6.19, replacing $A \xrightarrow{x} B$ by $A \xrightarrow{x_0} B \oplus V_A$, we may assume $b \circ x = x' \circ a$ in $\mathcal{C}$ from the start. Now (TR3) follows from (ET3) and Proposition 6.16.

(TR4) Let $A \xrightarrow{\alpha} B \rightarrow D \rightarrow A[1]$, $B \xrightarrow{\beta} C \rightarrow F \rightarrow B[1]$ and $A \xrightarrow{\gamma} C \rightarrow E \rightarrow A[1]$ be any distinguished triangles in $\tilde{\mathcal{C}}$ satisfying $\beta \circ \alpha = \gamma$. In a similar way as in the proof of (TR3), we may assume that they are standard triangles associated to $\mathbb{E}$ -triangles $A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta_f} A$,

$B \xrightarrow{g} C \xrightarrow{g'} F \xrightarrow{\delta_g} A \xrightarrow{h} C \xrightarrow{h'} E \xrightarrow{\delta_h}$, satisfying $g \circ f = h$. By Lemma 3.14, we obtain a commutative diagram (α) made of $\mathbb{E}$ -triangles:

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{f'} & D \xrightarrow{\delta_f} > \\
 \parallel & \circlearrowleft & \downarrow g & \circlearrowleft & \downarrow \exists d \\
 A & \xrightarrow{h} & C & \xrightarrow{h'} & E \xrightarrow{\delta_h} > \\
 & & \downarrow g' & \circlearrowleft & \downarrow \exists e \\
 & & F & \xlongequal{\quad} & F \\
 & & \downarrow \delta_g & & \downarrow f'_* \delta_g
 \end{array} & (\alpha) &
 \begin{array}{ccccc}
 A & \xrightarrow{\ell(f)} & B & \xrightarrow{\ell(f')} & D \xrightarrow{\tilde{\ell}(\delta_f)} A[1] \\
 \parallel & \circlearrowleft & \downarrow \ell(g) & \circlearrowleft & \downarrow \ell(d) \circlearrowleft \\
 A & \xrightarrow{\ell(h)} & C & \xrightarrow{\ell(h')} & E \xrightarrow{\tilde{\ell}(\delta_h)} A[1] \\
 & & \downarrow \ell(g') & \circlearrowleft & \downarrow \ell(e) \circlearrowleft \\
 & & F & \xlongequal{\quad} & F \xrightarrow{\tilde{\ell}(\delta_g)} B[1] \\
 & & \downarrow \tilde{\ell}(\delta_g) & \circlearrowleft & \downarrow \tilde{\ell}(f'_* \delta_g) \\
 & & B[1] & \xrightarrow{\ell(x')[1]} & D[1]
 \end{array} & (\beta)
 \end{array}$$

Thus by Proposition 6.16, we obtain a diagram (β) made of distinguished triangles, as desired. $\square$

The following argument ensures the dual arguments concerning distinguished triangles, in the following sections. Recall that the endofunctor $[-1]$ of $\tilde{\mathcal{C}}$ induced from the chosen $\mathbb{E}$ -triangle $T^C \xrightarrow{t^C} S^C \xrightarrow{s^C} C \xrightarrow{\lambda^C}$ with $S^C \in \mathcal{S}$, $T^C \in \mathcal{T}$, $T^C = C[-1]$, for each C , gives a quasi-inverse of $[1]$ by Proposition 6.14. Its proof shows that the isomorphisms $\tilde{\ell}(\lambda^C): C \rightarrow T^C[1]$ give a natural isomorphism $\text{Id} \xrightarrow{\cong} [1] \circ [-1]$.

The dual construction of Definition 6.9 goes as follows.

Definition 6.21. For any $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$, take a cospan of morphisms

$$(T^C \xrightarrow{m} E \xleftarrow{n} A) \quad (26)$$

to some $T \in \mathcal{C}$ satisfying

$$n \in w\text{Cof} \quad \text{and} \quad n_* \delta = m_* \lambda^C. \quad (27)$$

Then, $\ell^\dagger(\delta) = \ell(n)^{-1} \circ \ell(m) \in \tilde{\mathcal{C}}(C[-1], A)$ is well-defined.

With this definition, we can give a triangulation of $\tilde{\mathcal{C}}$ by requiring the diagram $T^C \xrightarrow{\ell^\dagger(\delta)} A \xrightarrow{\ell(x)} B \xrightarrow{\ell(y)} C$ to be a left triangle. The following

proposition (and its dual) shows that the resulting triangulation is the same as that defined in Definition 6.15.

Proposition 6.22. For any $\mathbb{E}$ -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$, the diagram $T^C \xrightarrow{\ell^\dagger(\delta)} A \xrightarrow{\ell(x)} B \xrightarrow{\tilde{\ell}(\lambda^C) \circ \ell(y)} T^C[1]$ becomes a distinguished triangle in $\tilde{\mathcal{C}}$, with respect to the triangulation given in Definition 6.15.

Proof. Take the standard triangle $A \xrightarrow{\ell(x)} B \xrightarrow{\ell(y)} C \xrightarrow{\tilde{\ell}(\delta)} A[1]$. Since $\tilde{\mathcal{C}}$ is triangulated, by the converse of (TR2), it suffices to show the commutativity of the following diagram.

$$\begin{array}{ccccc}
 A & \xrightarrow{\ell(x)} & B & \xrightarrow{\tilde{\ell}(\lambda^C) \circ \ell(y)} & T^C[1] \xrightarrow{-\ell^\dagger(\delta)[1]} A[1] \\
 & & \searrow \ell(y) & \circlearrowleft \cong \uparrow \tilde{\ell}(\lambda^C) & \nearrow \tilde{\ell}(\delta) \\
 & & & C &
 \end{array}$$

As $\ell^\dagger(\delta)$ does not depend on the choice of a cospan (26), we may take it in the following way: By Proposition 3.15, we obtain a commutative diagram made of $\mathbb{E}$ -triangles (a) below

$$\begin{array}{ccc}
 \begin{array}{c} (a) \\ \begin{array}{ccccc} T^C & = & T^C \\ \downarrow k & \circlearrowleft & \downarrow t^C \\ A & \xrightarrow{n} \exists E & \longrightarrow & S^{C(s^C)^*\delta} & \xrightarrow{\quad} \\ \parallel & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow s^C \\ A & \xrightarrow{x} B & \xrightarrow{y} & C & \xrightarrow{\delta} A \\ & \downarrow y^* \lambda^C & & \downarrow \lambda^C & \\ & \Psi & & \Psi & \end{array} \end{array} & & \begin{array}{c} (b) \\ \begin{array}{ccccc} C & = & C & = & C \\ \downarrow \tilde{\ell}(\lambda^C) & \circlearrowleft & \downarrow \tilde{\ell}(\theta) & \circlearrowleft & \downarrow \tilde{\ell}(\delta) \\ T^C[1] & \xrightarrow{\ell(k)[1]} E[1] & \xleftarrow{\ell(n)[1]} & A[1] & \end{array} \end{array}
 \end{array}$$

satisfying $n_*\delta + k_*\lambda^C = 0$. Then the cospan $(T^C \xrightarrow{-k} E \xleftarrow{n} A)$ satisfies the desired property (27), and thus gives $\ell^\dagger(\delta) = -\ell(n)^{-1} \circ \ell(k)$. If we put $\theta = n_*\delta = -k_*\lambda^C$, then $(n, \text{id}_C): \delta \rightarrow \theta$ and $(-k, \text{id}_C): \lambda^C \rightarrow \theta$ are morphisms of $\mathbb{E}$ -extensions. Thus (b) becomes commutative by Lemma 6.13. This shows $(\ell^\dagger(\delta)[1]) \circ \tilde{\ell}(\lambda^C) = (-\ell(n)^{-1}[1] \circ \ell(k)[1]) \circ \tilde{\ell}(\lambda^C) = \tilde{\ell}(\delta)$. $\square$

7. Reduction and mutation via localization

7.1 Happel and Iyama-Yoshino's construction

Definition 7.1. An extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is said to be *Frobenius* if it satisfies the following conditions.

- (1) $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ has enough injectives and enough projectives.
- (2) $\text{Proj}(\mathcal{C}) = \text{Inj}(\mathcal{C})$.

Example 7.2. (1) If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an exact category, then this agrees with the usual definition ([Ha, section I.2]).

- (2) Suppose that $\mathcal{T}$ is a triangulated category and $(\mathcal{Z}, \mathcal{Z})$ is an $\mathcal{I}$ -mutation pair in the sense of [IYo, Definition 2.5]. Then $\mathcal{Z}$ becomes a Frobenius extriangulated category, with the extriangulated structure given in Remark 2.18.

Remark 7.3. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, as before. By Remark 4.7, $((\mathcal{I}, \mathcal{C}), (\mathcal{C}, \mathcal{I}))$ is a twin cotorsion pair for some subcategory $\mathcal{I} \subseteq \mathcal{C}$ if and only if $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is Frobenius. Moreover, in that case $\mathcal{I} = \text{Proj}(\mathcal{C}) = \text{Inj}(\mathcal{C})$.

The following can be regarded as a generalization of the constructions by Happel [Ha] and Iyama-Yoshino [IYo]. See also [Li] for the triangulated case.

Corollary 7.4. Let $(\mathcal{F}, \mathbb{E}, \mathfrak{s})$ be a Frobenius extriangulated category satisfying Condition (WIC), with $\mathcal{I} = \text{Inj}(\mathcal{F})$. Then its stable category, namely the ideal quotient $\mathcal{F}/\mathcal{I}$, becomes triangulated.

Proof. Since $((\mathcal{I}, \mathcal{F}), (\mathcal{F}, \mathcal{I}))$ becomes a Hovey twin cotorsion pair with $\text{Cone}(\mathcal{I}, \mathcal{I}) = \text{CoCone}(\mathcal{I}, \mathcal{I}) = \mathcal{I}$, this follows from Corollary 5.25 and Theorem 6.20. $\square$

Remark 7.5. A direct proof for Corollary 7.4 is not difficult either, by imitating the proofs by [Ha] or [IYo], even without assuming Condition (WIC).

Corollary 7.6. Let $\mathcal{C}$ be a category. Then $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is triangulated, as in Proposition 3.22 if and only if $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a Frobenius extriangulated category and $\text{Proj}(\mathcal{C}) = \text{Inj}(\mathcal{C}) = 0$.

Proof. This follows from Corollary 7.4. $\square$

7.2 Mutable cotorsion pairs

Lemma 7.7.

- (1) For any weak equivalence $f \in \mathcal{C}(U, U')$ between $U, U' \in \mathcal{U}$, there exist $I \in \mathcal{I}$ and $i \in \mathcal{C}(U, I)$, with which $\begin{bmatrix} f \\ i \end{bmatrix} : U \rightarrow U' \oplus I$ becomes an acyclic cofibration.
- (2) Dually, for any weak equivalence $g \in \mathcal{C}(T, T')$ between $T, T' \in \mathcal{T}$, there exist $J \in \mathcal{I}$ and $j \in \mathcal{C}(J, T')$, with which $[g \ j] : T \oplus J \rightarrow T'$ becomes an acyclic fibration.

Proof. We only show (1). Since $f \in \mathbb{W} = wFib \circ wCof$, there are $\mathbb{E}$ -triangles

$$U \xrightarrow{m} E \rightarrow S \dashrightarrow, \quad (28)$$

$$V \rightarrow E \xrightarrow{e} U' \dashrightarrow^{\delta} \quad (29)$$

satisfying $S \in \mathcal{S}$, $V \in \mathcal{V}$ and $e \circ m = f$. By $\mathbb{E}(U', V) = 0$, we have $\delta = 0$. Thus we may assume $E = U' \oplus V$ and $e = [1 \ 0]$ in (29), from the start.

By the extension-closedness of $\mathcal{U} \subseteq \mathcal{C}$, the $\mathbb{E}$ -triangle (28) gives $U' \oplus V = E \in \mathcal{U}$, which implies $V \in \mathcal{I}$. Moreover by $e \circ m = f$, the acyclic cofibration $m : U \rightarrow U' \oplus V$ should be of the form $m = \begin{bmatrix} f \\ i \end{bmatrix}$, with some $i \in \mathcal{C}(U, V)$. $\square$

The following is an immediate consequence of the existence of the model structure.

Remark 7.8. For any morphism $f \in \mathcal{C}(A, B)$ in $\mathcal{C}$, we have $f \in \mathbb{W}$ if and only if $\ell(f)$ is an isomorphism in $\tilde{\mathcal{C}}$.

For any extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, let $\mathfrak{CP}(\mathcal{C})$ denote the class of cotorsion pairs on $\mathcal{C}$. Since $\tilde{\mathcal{C}}$ is triangulated as shown in Theorem 6.20, we may use the usual notation $\text{Ext}_{\tilde{\mathcal{C}}}^1$ for $\mathbb{E}$.

Definition 7.9. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a Hovey twin cotorsion pair on $\mathcal{C}$ and let $\ell : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ be the associated localization functor as before. Define the class of *mutable cotorsion pairs on $\mathcal{C}$ with respect to $\mathcal{P}$* by

$$\mathfrak{M}_{\mathcal{P}} = \left\{ (A, B) \in \mathfrak{CP}(\mathcal{C}) \mid \begin{array}{l} \mathcal{S} \subseteq A \subseteq \mathcal{U} \\ \mathcal{V} \subseteq B \subseteq \mathcal{T} \end{array}, \text{Ext}_{\tilde{\mathcal{C}}}^1(\ell(A), \ell(B)) = 0 \right\}.$$

Here, $\ell(\mathcal{A}), \ell(\mathcal{B}) \subseteq \tilde{\mathcal{C}}$ denote the essential images of $\mathcal{A}, \mathcal{B}$ under ℓ . Remark that $\mathcal{S} \subseteq \mathcal{A}$ is equivalent to $\mathcal{B} \subseteq \mathcal{T}$, and $\mathcal{A} \subseteq \mathcal{U}$ is equivalent to $\mathcal{V} \subseteq \mathcal{B}$, for any $(\mathcal{A}, \mathcal{B}) \in \mathfrak{CP}(\mathcal{C})$.

Theorem 7.10. *For any Hovey twin cotorsion pair $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on $\mathcal{C}$, we have mutually inverse bijective correspondences*

$$\mathbb{R} = \mathbb{R}_{\mathcal{P}}: \mathfrak{M}_{\mathcal{P}} \rightarrow \mathfrak{CP}(\tilde{\mathcal{C}}), \quad \mathbb{I} = \mathbb{I}_{\mathcal{P}}: \mathfrak{CP}(\tilde{\mathcal{C}}) \rightarrow \mathfrak{M}_{\mathcal{P}}$$

given by

$$\mathbb{R}((\mathcal{A}, \mathcal{B})) = (\ell(\mathcal{A}), \ell(\mathcal{B})), \quad \mathbb{I}((\mathcal{L}, \mathcal{R})) = (\mathcal{U} \cap \ell^{-1}(\mathcal{L}), \mathcal{T} \cap \ell^{-1}(\mathcal{R})).$$

Proof. It suffices to show the following. (1) For any mutable $(\mathcal{A}, \mathcal{B}) \in \mathfrak{M}_{\mathcal{P}}$, we have $\mathbb{R}((\mathcal{A}, \mathcal{B})) \in \mathfrak{CP}(\tilde{\mathcal{C}})$. (2) For any $(\mathcal{L}, \mathcal{R}) \in \mathfrak{CP}(\tilde{\mathcal{C}})$, we have $\mathbb{I}((\mathcal{L}, \mathcal{R})) \in \mathfrak{M}_{\mathcal{P}}$. (3) $\mathbb{I} \circ \mathbb{R} = \text{id}$. (4) $\mathbb{R} \circ \mathbb{I} = \text{id}$.

To distinguish, in this proof, let $\ell(X) \in \tilde{\mathcal{C}}$ denote the image under ℓ of an object $X \in \mathcal{C}$.

(1) Since $\ell(\mathcal{A})$ is the essential image of $\mathcal{A}$ under ℓ , it is closed under isomorphisms and finite direct sums. The equality $\mathcal{C} = \text{Cone}(\mathcal{B}, \mathcal{A})$ implies $\tilde{\mathcal{C}} = \ell(\mathcal{A}) * \ell(\mathcal{B})[1]$, by Definition 6.15. Moreover $\text{Ext}_{\tilde{\mathcal{C}}}^1(\ell(\mathcal{A}), \ell(\mathcal{B})) = 0$ follows from the definition of $\mathfrak{M}_{\mathcal{P}}$.

It remains to show that $\ell(\mathcal{A}), \ell(\mathcal{B}) \subseteq \tilde{\mathcal{C}}$ are closed under direct summands. To show that $\ell(\mathcal{A}) \subseteq \tilde{\mathcal{C}}$ is closed under direct summands, it suffices to show $\ell(\mathcal{A}) = {}^{\perp}\ell(\mathcal{B})[1]$. Take any $X \in \mathcal{C}$, and suppose it satisfies $\text{Ext}_{\tilde{\mathcal{C}}}^1(\ell(X), \ell(\mathcal{B})) = 0$.

Let us show $\ell(X) \in \ell(\mathcal{A})$. By a cofibrant replacement, we may assume X belongs to $\mathcal{U}$. Resolve X by an $\mathbb{E}$ -triangle in $\mathcal{C}$: $B \rightarrow A \rightarrow X \xrightarrow{\delta}$, with $A \in \mathcal{A}, B \in \mathcal{B}$. Since $\ell(\delta) = 0$ by assumption, we obtain $\delta = 0$ by Lemma 6.12. Thus X is a direct summand of A , which implies that X itself belongs to $\mathcal{A}$. Similarly for $\ell(\mathcal{B}) \subseteq \tilde{\mathcal{C}}$.

(2) Put $\mathcal{A} = \mathcal{U} \cap \ell^{-1}(\mathcal{L}), \mathcal{B} = \mathcal{T} \cap \ell^{-1}(\mathcal{R})$. Since both $\mathcal{U}$ and $\ell^{-1}(\mathcal{L})$ are closed under isomorphisms, finite direct sums and direct summands, so is their intersection $\mathcal{A}$. Similarly for $\mathcal{B}$. By $\ell(\mathcal{S}) \subseteq \ell(\mathcal{N}) = 0$, we have $\mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{U}$. By Lemma 6.12, $\text{Ext}_{\tilde{\mathcal{C}}}^1(\ell(\mathcal{A}), \ell(\mathcal{B})) = 0$ implies $\mathbb{E}(\mathcal{A}, \mathcal{B}) = 0$. It remains to show that $\mathcal{C} = \text{Cone}(\mathcal{B}, \mathcal{A}) = \text{CoCone}(\mathcal{B}, \mathcal{A})$. Let us show that $\mathcal{C} = \text{Cone}(\mathcal{B}, \mathcal{A})$.

Let $X \in \mathcal{C}$ be any object. By assumption, there exist $R \in \ell^{-1}(\mathcal{R})$, $L \in \ell^{-1}(\mathcal{L})$ and a distinguished triangle $\ell(R) \rightarrow \ell(L) \rightarrow \ell(X) \rightarrow \ell(R)[1]$ in $\tilde{\mathcal{C}}$. By definition, it is isomorphic to the standard triangle associated to an $\mathbb{E}$ -triangle, which we may assume to be of the form

$$R_0 \xrightarrow{x} L_0 \xrightarrow{y} Z \xrightarrow{\delta} \quad (30)$$

satisfying $R_0, L_0, Z \in \mathcal{Z}$, by a fibrant-cofibrant replacement (Corollary 6.17 and Remark 6.18). Thus we have an $\mathbb{E}$ -triangle (30) satisfying $R_0 \in \mathcal{Z} \cap \ell^{-1}(\mathcal{R})$, $L_0 \in \mathcal{Z} \cap \ell^{-1}(\mathcal{L})$ and $Z \in \mathcal{Z}$, together with an isomorphism $\zeta: \ell(Z) \xrightarrow{\cong} \ell(X)$ in $\tilde{\mathcal{C}}$. Resolve X by an $\mathbb{E}$ -triangle $X \xrightarrow{t_X} T_X \xrightarrow{s_X} S_X \xrightarrow{\rho_X} \rightarrow$, with $T_X \in \mathcal{T}$, $S_X \in \mathcal{S}$. Then there exists a morphism $z \in \mathcal{C}(Z, T_X)$ which satisfies $\zeta = \ell(t_X)^{-1} \circ \ell(z)$. Since ζ is an isomorphism, it follows that $z \in \mathbb{W}$. By Lemma 7.7 (2), there exists an $\mathbb{E}$ -triangle $V \rightarrow Z \oplus I \xrightarrow{[z \ i]} T_X \rightarrow$, with $V \in \mathcal{V}$, $I \in \mathcal{I}$. On the other hand by (ET2), we have an $\mathbb{E}$ -triangle $R_0 \xrightarrow{x_0} L_0 \oplus I \xrightarrow{y_0} Z \oplus I \rightarrow$ from (30), where $x_0 = \begin{bmatrix} x \\ 0 \end{bmatrix}$, $y_0 = y \oplus \text{id}_I$. Thus by (ET4)^{op}, we obtain a diagram (*)

$$\begin{array}{ccc}
 R_0 & \xrightarrow{\exists_r} \exists E & \longrightarrow V \\
 \parallel & \circ \quad \exists_e \downarrow & \circ \quad \downarrow \\
 (*) \quad R_0 & \xrightarrow{x_0} L_0 \oplus I & \xrightarrow{y_0} Z \oplus I \\
 & d \downarrow & \circ \quad \downarrow [z \ i] \\
 & T_X & = T_X
 \end{array}
 \quad (**) \quad
 \begin{array}{ccc}
 E & \longrightarrow \exists F & \longrightarrow X \\
 \parallel & \circ \quad \exists_f \downarrow & \circ \quad \downarrow t_X \\
 E & \xrightarrow{e} L_0 \oplus I & \xrightarrow{d} T_X \\
 & \downarrow & \circ \quad \downarrow s_X \\
 & S_X & = S_X
 \end{array}$$

made of conflations, where $d = [z \ i] \circ y_0 = [z \circ y \ i]$. Since $\ell(r)$ is an isomorphism in $\tilde{\mathcal{C}}$, we have $E \in \ell^{-1}(\mathcal{R})$. Besides, $R_0, V \in \mathcal{T}$ implies $E \in \mathcal{T}$. By (ET4)^{op}, we obtain a diagram (**) made of conflations. Since $\ell(f)$ is an isomorphism in $\tilde{\mathcal{C}}$, this shows $F \in \ell^{-1}(\mathcal{L})$. Resolve F by an $\mathbb{E}$ -triangle $V^F \rightarrow U^F \rightarrow F \rightarrow$, with $U^F \in \mathcal{U}$, $V^F \in \mathcal{V}$. By (ET4)^{op}, we obtain a diagram (A), below, made of conflations. Then in the $\mathbb{E}$ -triangle $G \rightarrow U^F \rightarrow X \rightarrow$, we have the equalities $U^F \in \mathcal{U} \cap \ell^{-1}(\mathcal{L}) = \mathcal{A}$ and $G \in \mathcal{T} \cap \ell^{-1}(\mathcal{R}) = \mathcal{B}$.

$$(A) \quad \begin{array}{ccccc} V^F & \longrightarrow & \exists G & \longrightarrow & E \\ \parallel & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\ V^F & \longrightarrow & U^F & \longrightarrow & F \\ & & \downarrow & \circlearrowleft & \downarrow \\ & & X & = & X \end{array}$$

(3) For any $(\mathcal{A}, \mathcal{B}) \in \mathfrak{M}_{\mathcal{P}}$, we have

$$\mathbb{I} \circ \mathbb{R}((\mathcal{A}, \mathcal{B})) = (\mathcal{U} \cap \ell^{-1}(\ell(\mathcal{A})), \mathcal{T} \cap \ell^{-1}(\ell(\mathcal{B}))).$$

Obviously, $\mathcal{A} \subseteq \mathcal{U} \cap \ell^{-1}(\ell(\mathcal{A}))$ and $\mathcal{B} \subseteq \mathcal{T} \cap \ell^{-1}(\ell(\mathcal{B}))$ hold. Since both $(\mathcal{A}, \mathcal{B})$ and $\mathbb{I} \circ \mathbb{R}((\mathcal{A}, \mathcal{B}))$ are cotorsion pairs, this means $(\mathcal{A}, \mathcal{B}) = \mathbb{I} \circ \mathbb{R}((\mathcal{A}, \mathcal{B}))$.

(4) For any $(\mathcal{L}, \mathcal{R}) \in \mathfrak{CP}(\tilde{\mathcal{C}})$, we have

$$\mathbb{R} \circ \mathbb{I}((\mathcal{L}, \mathcal{R})) = (\ell(\mathcal{U} \cap \ell^{-1}(\mathcal{L})), \ell(\mathcal{T} \cap \ell^{-1}(\mathcal{R}))),$$

which obviously satisfies $\ell(\mathcal{U} \cap \ell^{-1}(\mathcal{L})) \subseteq \mathcal{L}$ and $\ell(\mathcal{T} \cap \ell^{-1}(\mathcal{R})) \subseteq \mathcal{R}$. Similarly as in (3), it follows that $(\mathcal{L}, \mathcal{R}) = \mathbb{R} \circ \mathbb{I}((\mathcal{L}, \mathcal{R}))$. $\square$

Claim 7.11. *For any $(\mathcal{A}, \mathcal{B}) \in \mathfrak{CP}(\mathcal{C})$ satisfying $\mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{U}$ (or equivalently $\mathcal{V} \subseteq \mathcal{B} \subseteq \mathcal{T}$), we have*

$$\begin{aligned} \mathcal{U} \cap \ell^{-1} \ell(\mathcal{A}) &= \mathcal{U} \cap \text{CoCone}(\mathcal{A}, \mathcal{S}), \\ \mathcal{T} \cap \ell^{-1} \ell(\mathcal{B}) &= \mathcal{T} \cap \text{Cone}(\mathcal{V}, \mathcal{B}). \end{aligned}$$

Proof. We only show $\mathcal{U} \cap \ell^{-1} \ell(\mathcal{A}) = \mathcal{U} \cap \text{CoCone}(\mathcal{A}, \mathcal{S})$.

$\mathcal{U} \cap \text{CoCone}(\mathcal{A}, \mathcal{S}) \subseteq \mathcal{U} \cap \ell^{-1} \ell(\mathcal{A})$ is obvious. For the converse, let $U \in \mathcal{U}$ be any object satisfying $\ell(U) \cong \ell(A)$ in $\tilde{\mathcal{C}}$ for some $A \in \mathcal{A}$. Resolve U by an $\mathbb{E}$ -triangle $U \rightarrow Z \rightarrow S \dashrightarrow$, with $Z \in \mathcal{Z}$, $S \in \mathcal{S}$. Then $\ell(A) \cong \ell(Z)$ holds in $\tilde{\mathcal{C}}$. Since $A \in \mathcal{U}$, $Z \in \mathcal{T}$, there is a morphism $f \in \mathcal{C}(A, Z)$ which gives the isomorphism $\ell(f): \ell(A) \rightarrow \ell(Z)$ by Remark 5.26. Factorize this $f \in \mathbb{W}$ as $f = h \circ g$, with $g \in wCof$, $h \in wFib$. By definition, we have $\mathbb{E}$ -triangles $V_0 \rightarrow E \xrightarrow{h} Z \dashrightarrow$ and $A \xrightarrow{g} E \rightarrow S_0 \dashrightarrow$, where $V_0 \in \mathcal{V}$, $S_0 \in \mathcal{S}$. Since $\mathbb{E}(Z, V_0) = 0$, it follows that $V_0 \oplus Z \cong E \in \mathcal{A}$, which implies $Z \in \mathcal{A}$. $\square$

The class $\mathfrak{M}_{\mathcal{P}}$ can be rewritten as follows.

Corollary 7.12. Let $\mathcal{P}$ be a Hovey twin cotorsion pair on $\mathcal{C}$. For any $(\mathcal{A}, \mathcal{B}) \in \mathfrak{EP}(\mathcal{C})$ satisfying $\mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{U}$ (or equivalently $\mathcal{V} \subseteq \mathcal{B} \subseteq \mathcal{T}$), the following are equivalent.

- (1) $(\mathcal{A}, \mathcal{B}) \in \mathfrak{M}_{\mathcal{P}}$ i.e., it satisfies $\text{Ext}_{\mathcal{C}}^1(\ell(\mathcal{A}), \ell(\mathcal{B})) = 0$.
- (2) $\mathcal{U} \cap \ell^{-1}\ell(\mathcal{A}) = \mathcal{A}$. (2') $\mathcal{U} \cap \ell^{-1}\ell(\mathcal{A}) \subseteq \mathcal{A}$.
- (3) $\mathcal{T} \cap \ell^{-1}\ell(\mathcal{B}) = \mathcal{B}$. (3') $\mathcal{T} \cap \ell^{-1}\ell(\mathcal{B}) \subseteq \mathcal{B}$.

Thus by Claim 7.11, we have

$$\begin{aligned} \mathfrak{M}_{\mathcal{P}} &= \{(\mathcal{A}, \mathcal{B}) \in \mathfrak{EP}(\mathcal{C}) \mid \mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{U}, \mathcal{U} \cap \text{CoCone}(\mathcal{A}, \mathcal{S}) \subseteq \mathcal{A}\} \\ &= \{(\mathcal{A}, \mathcal{B}) \in \mathfrak{EP}(\mathcal{C}) \mid \mathcal{V} \subseteq \mathcal{B} \subseteq \mathcal{T}, \mathcal{T} \cap \text{Cone}(\mathcal{V}, \mathcal{B}) \subseteq \mathcal{B}\}. \end{aligned}$$

Proof. We only show $(1) \Leftrightarrow (2) \Leftrightarrow (2)'$. The implication $(1) \Rightarrow (2)$ follows from Theorem 7.10 and $(2) \Rightarrow (2)'$ is obvious. Suppose that $(2)'$ is satisfied. Let us show $\text{Ext}_{\mathcal{C}}^1(\ell(A), \ell(B)) = 0$ for any pair of objects $A \in \mathcal{A}$, $B \in \mathcal{B}$. Resolve A by an $\mathbb{E}$ -triangle $T^A \xrightarrow{t} S^A \rightarrow A \dashrightarrow$, with $T^A \in \mathcal{T}$, $S^A \in \mathcal{S}$ and T_A by $V^T \rightarrow Z^T \xrightarrow{z} T^A \dashrightarrow$, with $Z^T \in \mathcal{Z}$, $V^T \in \mathcal{V}$. Then we have $\ell(A)[-1] \cong \ell(T^A) \cong \ell(Z^T)$, and thus

$$\text{Ext}_{\mathcal{C}}^1(\ell(A), \ell(B)) \cong \tilde{\mathcal{C}}(\ell(Z^T), \ell(B)) \cong (\mathcal{C}/\mathcal{I})(Z^T, B)$$

by Remark 5.26. Let us show that $(\mathcal{C}/\mathcal{I})(Z^T, B) = 0$. Factorize $t \circ z$ as $t \circ z = h \circ g$, where $g \in \text{Cof}$, $h \in \text{wFib}$, to obtain a diagram

$$\begin{array}{ccccc} V^T & & \exists V_0 & & \\ \downarrow & & \downarrow & & \\ Z^T & \xrightarrow{g} & \exists E & \longrightarrow & \exists U_0 \\ \downarrow z & \circlearrowleft & \downarrow h & & \\ T^A & \xrightarrow{t} & S^A & \longrightarrow & A \end{array} \quad (U_0 \in \mathcal{U}, V_0 \in \mathcal{V})$$

made of conflations. Since $\mathbb{E}(S^A, V_0) = 0$, we have $E \cong S^A \oplus V_0$. Besides, by the extension-closedness of $\mathcal{U} \subseteq \mathcal{C}$, we have $E \in \mathcal{U}$, which shows that $V_0 \in \mathcal{V} \cap \mathcal{U} = \mathcal{I}$. Thus it follows that $E \in \mathcal{S}$.

By Lemma 5.9, we obtain $X \in \mathcal{C}$ and conflations $V^T \rightarrow V_0 \rightarrow X$ and $X \rightarrow U_0 \xrightarrow{u} A$. By Lemma 4.18 (1), we have $X \in \mathcal{N}$, and thus $u \in \mathbb{W}$ by Claim 5.12. This shows $\ell(U_0) \cong \ell(A)$, which means that $U_0 \in \mathcal{U} \cap \ell^{-1}\ell(\mathcal{A}) \subseteq \mathcal{A}$ by assumption. Thus we obtain an exact sequence

$$\mathcal{C}(E, B) \xrightarrow{\mathcal{C}(g, B)} \mathcal{C}(Z^T, B) \rightarrow \mathbb{E}(U_0, B) = 0.$$

This shows that any morphism $f \in \mathcal{C}(Z^T, B)$ factors through $E \in \mathcal{S}$, and thus $\bar{f} = 0$ holds in $(\mathcal{C}/\mathcal{I})(Z^T, B)$. $\square$

The above arguments allow us to define the mutation of cotorsion pairs as follows.

Definition 7.13. Let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a Hovey twin cotorsion pair on $\mathcal{C}$. We define *mutation with respect to* $(\mathcal{S}, \mathcal{V})$ as a $\mathbb{Z}$ -action on $\mathfrak{M}_{\mathcal{P}}$ given, for any $n \in \mathbb{Z}$, by $\mu_n = \mathbb{I} \circ [n] \circ \mathbb{R}$, i.e.,

$$\mu_n: \mathfrak{M}_{\mathcal{P}} \rightarrow \mathfrak{M}_{\mathcal{P}}; (\mathcal{A}, \mathcal{B}) \mapsto (\mathcal{U} \cap \ell^{-1}(\ell(\mathcal{A})[n]), \mathcal{T} \cap \ell^{-1}(\ell(\mathcal{B})[n])).$$

References

- [AN] Abe, N.; Nakaoka, H.: *General heart construction on a triangulated category (II): associated homological functor*, Appl. Categ. Structures, **20** (2012) no.2, 161–174.
- [BBD] Beilinson, A. A.; Bernstein, J.; Deligne, P.: *Faisceaux pervers* (French) [Perverse sheaves] Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, **100**, Soc. Math. France, Paris, 1982.
- [Bo] Borceux, F.: *Handbook of categorical algebra 1, Basic category theory*, Encyclopedia of Mathematics and its Applications, **50**. Cambridge University Press, Cambridge, 1994. xvi+345 pp.
- [Bu] Bühler, T.: *Exact Categories*, Expo. Math. **28** (2010) 1–69.
- [Ca] Cagne, P.: *Towards a homotopical algebra of dependent types*, PhD Thesis (2018), Université Paris Diderot.

- [DI] Demonet, L.; Iyama, O.: *Lifting preprojective algebras to orders and categorifying partial flag varieties*, Algebra Number Theory, **10** (2016) no. 7, 1527–1580.
- [Eg] Egger, J.M.: *Quillen model categories without equalisers or co-equalisers*, arXiv:0609808v1.
- [G] Gillespie, J.: *Model structures on exact categories*. J. Pure Appl. Algebra **215** (2011) no. 12, 2892–2902.
- [Ha] Happel, D.: *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, **119**. Cambridge University Press, Cambridge, 1988. x+208 pp.
- [Ho1] Hovey, M.: *Cotorsion pairs, model category structures, and representation theory*, Math. Z. **241** (2002) no. 3, 553–592.
- [Ho2] Hovey, M.: *Cotorsion pairs and model categories. Interactions between homotopy theory and algebra*, 277–296, Contemp. Math., **436**, Amer. Math. Soc., Providence, RI, 2007.
- [IYa] Iyama, O.; Yang, D.: *Silting reduction and Calabi–Yau reduction of triangulated categories*, Trans. Amer. Math. Soc., **370** (2018) no. 11, 7861–7898.
- [IYo] Iyama, O.; Yoshino, Y.: *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. **172** (2008) no. 1, 117–168.
- [Joy] Joyal, A.: *The theory of quasi-categories and its applications*, available online at <https://web.archive.org/web/20110706214636/http://www.crm.cat/HigherCategories/hc2.pdf>.
- [Ke] Keller, B.: *Derived categories and their uses*. Handbook of algebra, Vol. 1, 671–701, North-Holland, Amsterdam, 1996.
- [KR] Keller, B.; Reiten, I.: *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007) no. 1, 123–151.

- [KZ] Koenig, S.; Zhu, B.: *From triangulated categories to abelian categories: cluster tilting in a general framework*, Math. Z. **258** (2008) no. 1, 143–160.
- [Li] Li, Z.: *The triangulation of the subfactor categories of additive categories with suspensions*, arXiv:1510.02258.
- [Liu] Liu, Y.: *Hearts of twin cotorsion pairs on exact categories*, J. Algebra, **394** (2013) 245–284.
- [LN] Liu, Y; Nakaoka, H.: *Hearts of twin Cotorsion pairs on extriangulated categories*. arXiv:1702.00244.
- [MP] Marsh, R.; Palu, Y.: *Nearly Morita equivalences and rigid objects*, Nagoya Math. J., **225** (2017), 64–99.
- [Na1] Nakaoka, H.: *General heart construction on a triangulated category (I): unifying t -structures and cluster tilting subcategories*, Appl. Categ. Structures, **19** (2011) no.6, 879–899.
- [Na2] Nakaoka, H.: *General heart construction for twin torsion pairs on triangulated categories*, J. Algebra, **374** (2013) 195–215.
- [Na3] Nakaoka, H.: *Equivalence of hearts of twin cotorsion pairs on triangulated categories*, Comm. Algebra, **44** (2016) no. 12, 5105–5148.
- [Na4] Nakaoka, H.: *A simultaneous generalization of mutation and recollement on a triangulated category*, Appl. Categ. Structures, **26** (2018) no. 3, 491–544.
- [Ne] Neeman, A.: *Triangulated categories*. Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ, 2001. viii+449 pp.
- [Pal] Palu, Y.: *From triangulated categories to module categories via homotopical algebra*, arXiv:1412.7289.
- [Pau] Pauksztello, D.: *Compact corigid objects in triangulated categories and co- t -structures*. Cent. Eur. J. Math. **6** (2008) no. 1, 25–42.

- [Sal] Salce, L.: *Cotorsion theories for abelian groups* in Symposia Mathematica, Vol. XXIII (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977) (1979), 11–32.
- [S] Šťovíček, J.: *Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves*. Advances in representation theory of algebras, 297–367, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013.
- [Y] Yang, X.: *Model structures on triangulated categories*. Glasg. Math. J. **57** (2015) no. 2, 263–284.

Hiroiyuki Nakaoka

Research and Education Assembly, Science and Engineering Area, Research Field in Science

Kagoshima University

1-21-35 Korimoto

Kagoshima, 890-0065 (Japan)

nakaoka@sci.kagoshima-u.ac.jp

Yann Palu

Laboratoire Amienois de Mathématique Fondamentale et Appliquée

Université de Picardie Jules Verne

33 rue Saint-Leu

80000 Amiens (France)