Journal of Pure and Applied Algebra 213 (2009) 1438–1449

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Grothendieck group and generalized mutation rule for 2-Calabi–Yau triangulated categories

Yann Palu

Université Paris 7 - Denis Diderot, UMR 7586 du CNRS, case 7012, 2 place Jussieu, 75251 Paris Cedex 05, France

ARTICLE INFO

Article history:
Received 28 April 2008
Received in revised form 28 November 2008
Available online 21 January 2009
Communicated by I. Reiten

MSC:
Primary: 18E30
16E20
16G20
secondary: 16G70

ABSTRACT

We compute the Grothendieck group of certain 2-Calabi–Yau triangulated categories appearing naturally in the study of the link between quiver representations and Fomin–Zelevinsky cluster algebras. In this setup, we also prove a generalization of the Fomin–Zelevinsky mutation rule.

0. Introduction

In their study [1] of the connections between cluster algebras (see [2]) and quiver representations, P. Caldero and B. Keller conjectured that a certain antisymmetric bilinear form is well-defined on the Grothendieck group of a cluster-tilted algebra associated with a finite-dimensional hereditary algebra. The conjecture was proved in [3] in the more general context of Hom-finite 2-Calabi–Yau triangulated categories. It was used in order to study the existence of a cluster character on such a category $\mathcal{C}$, by using a formula proposed by Caldero–Keller.

In the present paper, we restrict ourselves to the case where $\mathcal{C}$ is algebraic (i.e. is the stable category of a Frobenius category). We first use this bilinear form to prove a generalized mutation rule for quivers of cluster-tilting subcategories in $\mathcal{C}$. When the cluster-tilting subcategories are related by a single mutation, this shows, via the method of [4], that their quivers are related by the Fomin–Zelevinsky mutation rule. This special case was already proved in [5], without assuming $\mathcal{C}$ to be algebraic.

We also compute the Grothendieck group of the triangulated category $\mathcal{C}$. In particular, this allows us to improve on results by M. Barot, D. Kussin and H. Lenzing: We compare the Grothendieck group of a cluster category $\mathcal{C}_A$ with the group $\overline{\mathrm{K}}_0(\mathcal{C}_A)$. The latter group was defined in [6] by only considering the triangles in $\mathcal{C}_A$ which are induced by those of the derived category. More precisely, we prove that those two groups are isomorphic for any cluster category associated with a finite-dimensional hereditary algebra, with its triangulated structure defined by Keller in [7].

This paper is organized as follows: The first section is dedicated to notation and necessary background from [8,4,9,3]. In Section 2, we compute the Grothendieck group of the triangulated category $\mathcal{C}$. In Section 3, we prove a generalized mutation rule for quivers of cluster-tilting subcategories in $\mathcal{C}$. In particular, this yields a new way to prove, under the restriction that $\mathcal{C}$ is algebraic, that the quiver of the mutation of a cluster-tilting object T is given by the Fomin–Zelevinsky mutation of the quiver of T. We finally show that $\mathrm{K}_0(\mathcal{C}_A) = \overline{\mathrm{K}}_0(\mathcal{C}_A)$ for any finite-dimensional hereditary algebra A.

E-mail address: palu@math.jussieu.fr.

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2008.12.012

1. Notations and background

Let $\mathcal{E}$ be a Frobenius category whose idempotents split and which is linear over a given algebraically closed field k. By a result of Happel [10], its stable category $\mathcal{C} = \underline{\mathcal{E}}$ is triangulated. We assume moreover, that $\mathcal{C}$ is Hom-finite, 2-Calabi–Yau and has a cluster-tilting subcategory (see Section 1.2), and we denote by Σ its suspension functor. Note that we do not assume that $\mathcal{E}$ is Hom-finite.

We write $\mathcal{X}(\ ,\ )$, or $\mathrm{Hom}_{\mathcal{X}}(\ ,\ )$, for the morphisms in a category $\mathcal{X}$ and $\mathrm{Hom}_{\mathcal{X}}(\ ,\ )$ for the morphisms in the category of $\mathcal{X}$-modules. We also denote by $X^\wedge$ the projective $\mathcal{X}$-module represented by X: $X^\wedge = \mathcal{X}(?, X)$.

1.1. Fomin–Zelevinsky mutation for matrices

Let $B = (b_{ij})_{i,j\in I}$ be a finite or infinite matrix, and let k be in I. The Fomin and Zelevinsky mutation of B (see [8]) in direction k is the matrix

$$\mu_k(B) = (b'_{ij})$$

defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{else.} \end{cases}$$

Note that $\mu_k(\mu_k(B)) = B$ and that if B is skew-symmetric, then so is $\mu_k(B)$.

We recall two lemmas of [4], stated for infinite matrices, which will be useful in Section 3. Note that Lemma 7.2 is a restatement of [11, (3.2)]. Let $S = (s_{ij})$ be the matrix defined by

$$s_{ij} = \begin{cases} -\delta_{ij} + \dfrac{|b_{ij}| - b_{ij}}{2} & \text{if } i = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

Lemma 7.1 ([4, Geiss–Leclerc–Schröer]). *Assume that B is skew-symmetric. Then, $S^2 = 1$ and the (i, j)-entry of the transpose of the matrix S is given by*

$$s^t_{ij} = \begin{cases} -\delta_{ij} + \dfrac{|b_{ij}| + b_{ij}}{2} & \text{if } j = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

The matrix S yields a convenient way to describe the mutation of B in the direction k:

Lemma 7.2 ([4, Geiss–Leclerc–Schröer], [11, Berenstein–Fomin–Zelevinsky]). *Assume that B is skew-symmetric. Then we have:*

$$\mu_k(B) = S^t BS.$$

Note that the product is well-defined since the matrix S has a finite number of non-vanishing entries in each column.

1.2. Cluster-tilting subcategories

A cluster-tilting subcategory (see [9]) of $\mathcal{C}$ is a full subcategory $\mathcal{T}$ such that:

(a) $\mathcal{T}$ is a linear subcategory;
(b) for any object X in $\mathcal{C}$, the contravariant functor $\mathcal{C}(?, X)|_{\mathcal{T}}$ is finitely generated;
(c) for any object X in $\mathcal{C}$, we have $\mathcal{C}(X, \Sigma T) = 0$ for all T in $\mathcal{T}$ if and only if X belongs to $\mathcal{T}$.

We now recall some results from [9], which we will use in what follows. Let $\mathcal{T}$ be a cluster-tilting subcategory of $\mathcal{C}$, and denote by $\mathcal{M}$ its preimage in $\mathcal{E}$. In particular $\mathcal{M}$ contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by the projective–injective objects, and we have $\underline{\mathcal{M}} = \mathcal{T}$.

The following proposition will be used implicitly, extensively in this paper.

Proposition ([9, Keller–Reiten]).

(a) *The category* mod $\underline{\mathcal{M}}$ *of finitely presented $\underline{\mathcal{M}}$-modules is abelian.*
(b) *For each object $X \in \mathcal{C}$, there is a triangle*

$$\Sigma^{-1}X \longrightarrow M_1^X \longrightarrow M_0^X \longrightarrow X$$

of $\mathcal{C}$, with M_0^X and M_1^X in $\underline{\mathcal{M}}$.

Recall that the perfect derived category per $\mathcal{M}$ is the full triangulated subcategory of the derived category of $\mathcal{D}\,\mathrm{Mod}\,\mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$-modules.

Proposition (*[9, Keller–Reiten]*).

(a) *For each $X \in \mathcal{E}$, there are conflations*

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow X \longrightarrow M^0 \longrightarrow M^1 \longrightarrow 0$$

in $\mathcal{E}$, with M_0, M_1, M^0 and M^1 in $\mathcal{M}$.

(b) *Let Z be in* mod $\underline{\mathcal{M}}$. *Then Z considered as an $\mathcal{M}$-module lies in the perfect derived category* per $\mathcal{M}$.

1.3. The antisymmetric bilinear form

In Section 3, we will use the existence of the antisymmetric bilinear form $\langle\ ,\ \rangle_a$ on $\mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}})$. We thus recall its definition from [1].

Let $\langle\ ,\ \rangle$ be a truncated Euler form on mod $\underline{\mathcal{M}}$ defined by

$$\langle M, N\rangle = \dim \mathrm{Hom}_{\underline{\mathcal{M}}}(M, N) - \dim \mathrm{Ext}^1_{\underline{\mathcal{M}}}(M, N)$$

for any $M, N \in \mathrm{mod}\,\underline{\mathcal{M}}$. Define $\langle\ ,\ \rangle_a$ to be the antisymmetrization of this form:

$$\langle M, N\rangle_a = \langle M, N\rangle - \langle N, M\rangle.$$

This bilinear form descends to the Grothendieck group $\mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}})$:

Lemma (*[3, Section 3]*). *The antisymmetric bilinear form*

$$\langle , \rangle_a : \mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}}) \times \mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$$

is well-defined.

2. Grothendieck groups of algebraic 2-CY categories with a cluster-tilting subcategory

We fix a cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{C}$, and we denote by $\mathcal{M}$ its preimage in $\mathcal{E}$. In particular $\mathcal{M}$ contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by the projective–injective objects, and we have $\underline{\mathcal{M}} = \mathcal{T}$.

We denote by $\mathcal{H}^b(\mathcal{E})$ and $\mathcal{D}^b(\mathcal{E})$ respectively the bounded homotopy category and the bounded derived category of $\mathcal{E}$. We also denote by $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{E})$, $\mathcal{H}^b(\mathcal{P})$, $\mathcal{H}^b(\mathcal{M})$ and $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ the full subcategories of $\mathcal{H}^b(\mathcal{E})$ whose objects are the $\mathcal{E}$-acyclic complexes, the complexes of projective objects in $\mathcal{E}$, the complexes of objects of $\mathcal{M}$ and the $\mathcal{E}$-acyclic complexes of objects of $\mathcal{M}$, respectively.

2.1. A short exact sequence of triangulated categories

Lemma 1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be thick, full triangulated subcategories of a triangulated category $\mathcal{A}$ and let $\mathcal{B}$ be $\mathcal{A}_1 \cap \mathcal{A}_2$. Assume that for any object X in $\mathcal{A}$ there is a triangle $X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow \Sigma X_1$ in $\mathcal{A}$, with X_1 in $\mathcal{A}_1$ and X_2 in $\mathcal{A}_2$. Then the induced functor $\mathcal{A}_1/\mathcal{B} \longrightarrow \mathcal{A}/\mathcal{A}_2$ is a triangle equivalence.*

Proof. Under these assumptions, denote by F the induced triangle functor from $\mathcal{A}_1/\mathcal{B}$ to $\mathcal{A}/\mathcal{A}_2$. We are going to show that the functor F is a full, conservative, dense functor. Since any full conservative triangle functor is fully faithful, F will then be an equivalence of categories.

We first show that F is full. Let X_1 and X_1' be two objects in $\mathcal{A}_1$. Let f be a morphism from X_1 to X_1' in $\mathcal{A}/\mathcal{A}_2$ and let

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr, "w"] & \\ X_1 & & X_1' \end{tikzcd}$$

be a left fraction which represents f. The morphism w is in the multiplicative system associated with $\mathcal{A}_2$ and thus yields a triangle $\Sigma^{-1}A_2 \to Y \xrightarrow{w} X_1' \to A_2$ where A_2 lies in the subcategory $\mathcal{A}_2$. Moreover, by assumption, there exists a triangle

$Y_1 \to Y \to Y_2 \to \Sigma Y_1$ with Y_i in $\mathcal{A}_i$. Applying the octahedral axiom to the composition $Y_1 \to Y \to X_1'$ yields a commutative diagram whose two middle rows and columns are triangles in $\mathcal{A}$

$$\begin{tikzcd}
 & \Sigma^{-1}A_2 \arrow[r, equal] \arrow[d] & \Sigma^{-1}A_2 \arrow[d] & \\
Y_1 \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & \Sigma Y_1 \arrow[d, equal] \\
Y_1 \arrow[r] & X_1' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma Y_1 \\
 & A_2 \arrow[r, equal] & A_2 & .
\end{tikzcd}$$

Since Y_2 and A_2 belong to $\mathcal{A}_2$, so does Z. And since X_1' and Y_1 belong to $\mathcal{A}_1$, so does Z. This implies, that Z belongs to $\mathcal{B}$. The morphism $Y_1 \to X_1'$ is in the multiplicative system of $\mathcal{A}_1$ associated with $\mathcal{B}$ and the diagram

$$\begin{tikzcd}
 & Y_1 \arrow[dl] \arrow[dr] & \\
X_1 & & X_1'
\end{tikzcd}$$

is a left fraction which represents f. This implies that f is the image of a morphism in $\mathcal{A}_1/\mathcal{B}$. Therefore the functor F is full.

We now show that F is conservative. Let $X_1 \xrightarrow{f} Y_1 \to Z_1 \to \Sigma X_1$ be a triangle in $\mathcal{A}_1$. Assume that Ff is an isomorphism in $\mathcal{A}/\mathcal{A}_2$, which implies that Z_1 is an object of $\mathcal{A}_2$. Therefore, Z_1 is an object of $\mathcal{B}$ and f is an isomorphism in $\mathcal{A}_1/\mathcal{B}$.

We finally show that F is dense. Let X be an object of the category $\mathcal{A}/\mathcal{A}_2$, and let $X_1 \to X \to X_2 \to \Sigma X_1$ be a triangle in $\mathcal{A}$ with X_i in $\mathcal{A}_i$. Since X_2 belongs to $\mathcal{A}_2$, the image of the morphism $X_1 \to X$ in $\mathcal{A}/\mathcal{A}_2$ is an isomorphism. Thus X is isomorphic to the image by F of an object in $\mathcal{A}_1/\mathcal{B}$. $\square$

As a corollary, we have the following:

Lemma 2. *The following sequence of triangulated categories is short exact:*

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M}) \longrightarrow \mathcal{D}^b(\mathcal{E}) \longrightarrow 0.$$

Remark. This lemma remains true if $\mathcal{C}$ is d-Calabi–Yau and $\underline{\mathcal{M}}$ is $(d-1)$-cluster-tilting, using Section 5.4 of [9].

Proof. For any object X in $\mathcal{H}^b(\mathcal{E})$, the existence of an object M in $\mathcal{H}^b(\mathcal{M})$ and of a quasi-isomorphism w from M to X is obtained using the approximation conflations of Keller–Reiten (see Section 1.2). Since the cone of the morphism w belongs to $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{E})$, Lemma 1 applies to the subcategories $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$, $\mathcal{H}^b(\mathcal{M})$ and $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{E})$ of $\mathcal{H}^b(\mathcal{E})$. $\square$

Proposition 3. *The following diagram is commutative with exact rows and columns:*

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \arrow[r, "i_{\mathcal{M}}"] & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r] \arrow[u] & \underline{\mathcal{E}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \arrow[r] \arrow[u, equal] & \mathcal{H}^b(\mathcal{M}) \arrow[r] \arrow[u] & \mathcal{D}^b(\mathcal{E}) \arrow[r] \arrow[u] & 0 \\
 & & \mathcal{H}^b(\mathcal{P}) \arrow[r, equal] \arrow[u] & \mathcal{H}^b(\mathcal{P}) \arrow[r] \arrow[u, "i_{\mathcal{P}}"'] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] & .
\end{tikzcd} \quad (D)$$

Proof. The column on the right side has been shown to be exact in [12] and [13]. The second row is exact by Lemma 2. The subcategories $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ and $\mathcal{H}^b(\mathcal{P})$ of $\mathcal{H}^b(\mathcal{M})$ are left and right orthogonal to each other. This implies that the induced functors $i_{\mathcal{M}}$ and $i_{\mathcal{P}}$ are fully faithful and that taking the quotient of $\mathcal{H}^b(\mathcal{M})$ by those two subcategories either in one order or in the other gives the same category. Therefore the first row is exact. $\square$

2.2. Invariance under mutation

A natural question is then to which extent the diagram (D) depends on the choice of a particular cluster-tilting subcategory. Thus let $\mathcal{T}'$ be another cluster-tilting subcategory of $\mathcal{C}$, and let $\mathcal{M}'$ be its preimage in $\mathcal{E}$. Let $\operatorname{Mod}\mathcal{M}$ (resp. $\operatorname{Mod}\mathcal{M}'$) be the category of $\mathcal{M}$-modules (resp. $\mathcal{M}'$-modules), i.e. of k-linear contravariant functors from $\mathcal{M}$ (resp. $\mathcal{M}'$) to the category of k-vector spaces.

Let X be the $\mathcal{M}'$–$\mathcal{M}$-bimodule which sends the pair of objects (M', M) to the k-vector space $\mathcal{E}(M', M)$. The bimodule X induces a functor $F = ? \otimes_{\mathcal{M}} X : \operatorname{Mod}\mathcal{M} \longrightarrow \operatorname{Mod}\mathcal{M}'$ denoted by T_X in [14, Section 6.1].

Recall that the perfect derived category $\operatorname{per}\mathcal{M}$ is the full triangulated subcategory of the derived category $\mathcal{D}\operatorname{Mod}\mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$-modules.

Proposition 4. *The left derived functor*

$$\mathbb{L}F: \quad \mathcal{D}\operatorname{Mod}\mathcal{M} \quad \longrightarrow \quad \mathcal{D}\operatorname{Mod}\mathcal{M}'$$

is an equivalence of categories.

Proof. Recall that if X is an object in a category $\mathcal{X}$, we denote by $X\hat{}$ the functor $\mathcal{X}(?, X)$ represented by X. By [14, 6.1], it is enough to check the following three properties:

1. For all objects M, N of $\mathcal{M}$, the group $\operatorname{Hom}_{\mathcal{D}\operatorname{Mod}\mathcal{M}'}\left(\mathbb{L}FM\hat{}, \ \mathbb{L}FN\hat{}[n]\right)$ vanishes for $n \neq 0$ and identifies with $\operatorname{Hom}_{\mathcal{M}}(M, N)$ for $n = 0$;
2. for any object M of $\mathcal{M}$, the complex $\mathbb{L}FM\hat{}$ belongs to $\operatorname{per}\mathcal{M}'$;
3. the set $\left\{\mathbb{L}FM\hat{}, \ M \in \mathcal{M}\right\}$ generates $\mathcal{D}\operatorname{Mod}\mathcal{M}'$ as a triangulated category with infinite sums.

Let M be an object of $\mathcal{M}$, and let $M'_1 \rightarrowtail M'_0 \twoheadrightarrow M$ be a conflation in $\mathcal{E}$, with M'_0 and M'_1 in $\mathcal{M}'$, and whose deflation is a right $\mathcal{M}'$-approximation (cf. Section 4 of [9]). The surjectivity of the map $(M'_0)\hat{} \longrightarrow \mathcal{E}(?, M)|_{\mathcal{M}'}$ implies that the complex $P = (\cdots \to 0 \to (M'_1)\hat{} \to (M'_0)\hat{} \to 0 \to \cdots)$ is quasi-isomorphic to $\mathbb{L}FM\hat{} = \mathcal{E}(?, M)|_{\mathcal{M}'}$. Therefore $\mathbb{L}FM\hat{}$ belongs to the subcategory $\operatorname{per}\mathcal{M}'$ of $\mathcal{D}\operatorname{Mod}\mathcal{M}'$. Moreover, we have, for any $n \in \mathbb{Z}$ and any $N \in \mathcal{M}$, the equality

$$\operatorname{Hom}_{\mathcal{D}\operatorname{Mod}\mathcal{M}'}\left(\mathbb{L}FM\hat{}, \mathbb{L}FN\hat{}[n]\right) = \operatorname{Hom}_{\mathcal{H}^{\mathrm{b}}\operatorname{Mod}\mathcal{M}}\left(P, \mathcal{E}(?, N)|_{\mathcal{M}'}[n]\right)$$

where the right-hand side vanishes for $n \neq 0, 1$. In case $n = 1$ it also vanishes, since $\operatorname{Ext}^1_{\mathcal{E}}(M, N)$ vanishes. Now,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{H}^{\mathrm{b}}\operatorname{Mod}\mathcal{M}'}\left(P, \mathcal{E}(?, N)|_{\mathcal{M}'}\right) &\simeq \operatorname{Ker}\left(\mathcal{E}(M'_0, N) \to \mathcal{E}(M'_1, N)\right) \\ &\simeq \mathcal{E}(M, N).\end{aligned}$$

It only remains to be shown that the set $R = \left\{\mathbb{L}FM\hat{}, \ M \in \mathcal{M}\right\}$ generates $\mathcal{D}\operatorname{Mod}\mathcal{M}'$. Denote by $\mathcal{R}$ the full triangulated subcategory with infinite sums of $\mathcal{D}\operatorname{Mod}\mathcal{M}'$ generated by the set R. The set $\left\{(M')\hat{}, \ M' \in \mathcal{M}'\right\}$ generates $\mathcal{D}\operatorname{Mod}\mathcal{M}'$ as a triangulated category with infinite sums. Thus it is enough to show that, for any object M' of $\mathcal{M}'$, the complex $(M')\hat{}$ concentrated in degree 0 belongs to the subcategory $\mathcal{R}$. Let M' be an object of $\mathcal{M}'$, and let $M' \overset{i}{\rightarrowtail} M_0 \overset{p}{\twoheadrightarrow} M_1$ be a conflation of $\mathcal{E}$ with M_0 and M_1 in $\mathcal{M}$. Since $\operatorname{Ext}^1_{\mathcal{E}}(?, M')|_{\mathcal{M}'}$ vanishes, we have a short exact sequence of $\mathcal{M}'$-modules

$$0 \longrightarrow \mathcal{E}(?, M')|_{\mathcal{M}'} \longrightarrow \mathcal{E}(?, M_0)|_{\mathcal{M}'} \longrightarrow \mathcal{E}(?, M_1)|_{\mathcal{M}'} \longrightarrow 0,$$

which yields the triangle

$$(M')\hat{} \longrightarrow \mathbb{L}FM_0\hat{} \longrightarrow \mathbb{L}FM_1\hat{} \longrightarrow \Sigma(M')\hat{}. \quad \square$$

As a corollary of Proposition 4, up to equivalence the diagram (D) does not depend on the choice of a cluster-tilting subcategory. To be more precise: The functor $\mathbb{L}F$ restricts to a functor from $\operatorname{per}\mathcal{M}$ to $\operatorname{per}\mathcal{M}'$. Let G be the functor from $\mathcal{H}^b(\mathcal{M})$ to $\mathcal{H}^b\left(\mathcal{M}'\right)$ induced by this restriction via the Yoneda equivalence.

Corollary 5. *The following diagram is commutative*

$$\begin{tikzcd}
 & \mathcal{D}\operatorname{Mod}\mathcal{M} \arrow[rrr, "\mathbb{L}F"] & & & \mathcal{D}\operatorname{Mod}\mathcal{M}' \\
\mathcal{H}^b(\mathcal{M}) \arrow[ur, hook] \arrow[rrr, "G"] \arrow[ddd] & & & \mathcal{H}^b(\mathcal{M}') \arrow[ur, hook] \arrow[ddd] & \\
 & & & & \\
 & \mathcal{H}^b(\mathcal{P}) \arrow[uuu, hook] \arrow[uul, hook] \arrow[rrr, equal] \arrow[dl, hook] & & & \mathcal{H}^b(\mathcal{P}) \arrow[uuu, hook] \arrow[uul, hook] \arrow[dl, hook] \\
\mathcal{D}^b(\mathcal{E}) \arrow[rrr, equal] & & & \mathcal{D}^b(\mathcal{E}) &
\end{tikzcd}$$

and the functor G is an equivalence of categories.

We denote by $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$ the full subcategory of $\operatorname{per} \mathcal{M}$ whose objects are the complexes with homologies in $\operatorname{mod} \underline{\mathcal{M}}$. The following lemma will allow us to compute the Grothendieck group of $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$ in Section 2.3:

Lemma 6. *The canonical t-structure on $\mathcal{D} \operatorname{Mod} \mathcal{M}$ restricts to a t-structure on $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$, whose heart is $\operatorname{mod} \underline{\mathcal{M}}$.*

Proof. By [15], it is enough to show that for any object $M^\bullet$ of $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$, its truncation $\tau_{\leq 0}M^\bullet$ in $\mathcal{D} \operatorname{Mod} \mathcal{M}$ belongs to $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$. Since $M^\bullet$ is in $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$, $\tau_{\leq 0}M^\bullet$ is bounded, and is thus formed from the complexes $\mathrm{H}^i(M^\bullet)$ concentrated in one degree by taking iterated extensions. But, for any i, the $\mathcal{M}$-module $\mathrm{H}^i(M^\bullet)$ actually is an $\underline{\mathcal{M}}$-module. Therefore, by [9] (see Section 1.2), it is perfect as an $\mathcal{M}$-module and it lies in $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$. □

The next lemma already appears in [16]. For the convenience of the reader, we include a proof.

Lemma 7. *The Yoneda equivalence of triangulated categories $\mathcal{H}^b(\mathcal{M}) \longrightarrow \operatorname{per} \mathcal{M}$ induces a triangle equivalence $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \longrightarrow \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$.*

Proof. We first show that the cohomology groups of an $\mathcal{E}$-acyclic bounded complex M vanish on $\mathcal{P}$. Let P be a projective object in $\mathcal{E}$ and let E be a kernel in $\mathcal{E}$ of the map $M^n \longrightarrow M^{n+1}$. Since M is $\mathcal{E}$-acyclic, such an object exists, and moreover, it is an image of the map $M^{n-1} \longrightarrow M^n$. Any map from P to M^n whose composition with $M^n \to M^{n+1}$ vanishes factors through the kernel $E \rightarrowtail M^n$. Since P is projective, this factorization factors through the deflation $M^{n-1} \twoheadrightarrow E$.

$$\begin{tikzcd}
& & P \arrow[dll, dashed] \arrow[ddl, dashed] \arrow[d] \arrow[dr, "0"] & \\
M^{n-1} \arrow[rr] \arrow[dr] & & M^n \arrow[r] & M^{n+1} \\
& E \arrow[ur, tail] & &
\end{tikzcd}$$

Therefore, we have $\mathrm{H}^n(M^\wedge)(P) = 0$ for all projective objects P, and $\mathrm{H}^n(M^\wedge)$ belongs to $\operatorname{mod} \underline{\mathcal{M}}$. Thus the Yoneda functor induces a fully faithful functor from $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ to $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$. To prove that it is dense, it is enough to prove that any object of the heart $\operatorname{mod} \underline{\mathcal{M}}$ of the t-structure on $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$ is in its essential image.

But this was proved in [9, Section 4] (see Section 1.2). □

Proposition 8. *There is a triangle equivalence of categories*

$$\operatorname{per}_{\underline{\mathcal{M}'}} \mathcal{M}' \xrightarrow{\simeq} \operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}.$$

Proof. Since the categories $\mathcal{H}^b(\mathcal{P})$ and $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}')$ are left–right orthogonal in $\mathcal{H}^b(\mathcal{M}')$, this is immediate from Corollary 5 and Lemma 7. □

2.3. Grothendieck groups

For a triangulated (resp. additive, resp. abelian) category $\mathcal{A}$, we denote by $\mathrm{K}_0^{tri}(\mathcal{A})$ or simply $\mathrm{K}_0(\mathcal{A})$ (resp. $\mathrm{K}_0^{add}(\mathcal{A})$, resp. $\mathrm{K}_0^{ab}(\mathcal{A})$) its Grothendieck group (with respect to the mentioned structure of the category). For an object A in $\mathcal{A}$, we also denote by $[A]$ its class in the Grothendieck group of $\mathcal{A}$.

The short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \underline{\mathcal{E}} \longrightarrow 0$$

given by Proposition 3 induces an exact sequence in the Grothendieck groups

$$(*) \quad \mathrm{K}_0\left(\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})\right) \longrightarrow \mathrm{K}_0\left(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\right) \longrightarrow \mathrm{K}_0\left(\underline{\mathcal{E}}\right) \longrightarrow 0.$$

Lemma 9. *The exact sequence $(*)$ is isomorphic to an exact sequence*

$$(**) \quad \mathrm{K}_0^{ab}\left(\operatorname{mod} \underline{\mathcal{M}}\right) \xrightarrow{\varphi} \mathrm{K}_0^{add}\left(\underline{\mathcal{M}}\right) \longrightarrow \mathrm{K}_0^{tri}\left(\underline{\mathcal{E}}\right) \longrightarrow 0.$$

Proof. First, note that, by [16], see also Lemma 7, we have an isomorphism between the Grothendieck groups $\mathrm{K}_0\left(\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})\right)$ and $\mathrm{K}_0\left(\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}\right)$. The t-structure on $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$ whose heart is $\mathrm{mod}\,\underline{\mathcal{M}}$, see Lemma 6, in turn yields an isomorphism between the Grothendieck groups $\mathrm{K}_0^{tri}\left(\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}\right)$ and $\mathrm{K}_0^{ab}\left(\mathrm{mod}\,\underline{\mathcal{M}}\right)$. Next, we show that the canonical additive functor $\underline{\mathcal{M}} \xrightarrow{\alpha} \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ induces an isomorphism between the Grothendieck groups $\mathrm{K}_0^{add}\left(\underline{\mathcal{M}}\right)$ and $\mathrm{K}_0^{tri}\left(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\right)$. For this, let us consider the canonical additive functor $\underline{\mathcal{M}} \xrightarrow{\beta} \mathcal{H}^b\left(\underline{\mathcal{M}}\right)$ and the triangle functor $\mathcal{H}^b(\mathcal{M}) \xrightarrow{\gamma} \mathcal{H}^b\left(\underline{\mathcal{M}}\right)$. The following diagram describes the situation:

$$\begin{tikzcd}
\mathcal{H}^b\left(\underline{\mathcal{M}}\right) & \mathcal{H}^b(\mathcal{M}) \arrow[l, "\gamma"'] \arrow[d] \\
\underline{\mathcal{M}} \arrow[u, "\beta"] \arrow[r, "\alpha"] & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[ul, dashed, "\gamma"']
\end{tikzcd}$$

The functor γ vanishes on the full subcategory $\mathcal{H}^b(\mathcal{P})$, thus inducing a triangle functor, still denoted by γ, from $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ to $\mathcal{H}^b\left(\underline{\mathcal{M}}\right)$. Furthermore, the functor β induces an isomorphism at the level of Grothendieck groups, whose inverse $\mathrm{K}_0(\beta)^{-1}$ is given by

$$\begin{array}{ccc}
\mathrm{K}_0^{tri}\left(\mathcal{H}^b\left(\underline{\mathcal{M}}\right)\right) & \longrightarrow & \mathrm{K}_0^{add}\left(\underline{\mathcal{M}}\right) \\
{[M]} & \longmapsto & \displaystyle\sum_{i\in\mathbb{Z}}(-1)^i[M^i].
\end{array}$$

As the group $\mathrm{K}_0^{tri}\left(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\right)$ is generated by objects concentrated in degree 0, it is straightforward to check that the morphisms $\mathrm{K}_0(\alpha)$ and $\mathrm{K}_0(\beta)^{-1}\,\mathrm{K}_0(\gamma)$ are inverse to each other. □

As a consequence of the exact sequence $(**)$, we have an isomorphism between $\mathrm{K}_0^{tri}\left(\underline{\mathcal{E}}\right)$ and $\mathrm{K}_0^{add}\left(\underline{\mathcal{M}}\right)/\,\mathrm{Im}\,\varphi$. In order to compute $\mathrm{K}_0^{tri}\left(\underline{\mathcal{E}}\right)$, the map φ has to be made explicit. We first recall some results from Iyama–Yoshino [17] which generalize results from [18]: For any indecomposable M of $\underline{\mathcal{M}}$ not in $\mathcal{P}$, there exists M^* unique up to isomorphism such that (M, M^*) is an exchange pair, i.e.

(a) M^* is an indecomposable object, not isomorphic to M and
(b) the full additive subcategory of $\mathcal{C}$ generated by M^* and $\underline{\mathcal{M}}/M$ is cluster-tilting.

Moreover, there exist two (non-split) exchange triangles

$$M^* \to B_M \to M \to \Sigma M^* \quad \text{and} \quad M \to B_{M^*} \to M^* \to \Sigma M.$$

We may now state the following:

Theorem 10. *The Grothendieck group of the triangulated category $\underline{\mathcal{E}}$ is the quotient of that of the additive subcategory $\underline{\mathcal{M}}$ by all relations $[B_{M^*}] - [B_M]$:*

$$\mathrm{K}_0^{tri}\left(\underline{\mathcal{E}}\right) \simeq \mathrm{K}_0^{add}\left(\underline{\mathcal{M}}\right)/([B_{M^*}] - [B_M])_M.$$

Proof. We denote by S_M the simple $\underline{\mathcal{M}}$-module associated to the indecomposable object M. This means that $S_M(M')$ vanishes for all indecomposable objects M' in $\underline{\mathcal{M}}$ not isomorphic to M and that $S_M(M)$ is isomorphic to k. The abelian group $\mathrm{K}_0^{ab}\left(\mathrm{mod}\,\underline{\mathcal{M}}\right)$ is generated by all classes $[S_M]$. In view of Lemma 9, it is sufficient to prove that the image of the class $[(S_M)^{\oplus d}]$ under φ is $[B_{M^*}] - [B_M]$, where d is the dimension of $\underline{\mathcal{E}}(M, \Sigma M^*)$. First note that the $\mathcal{M}$-module $\mathrm{Ext}^1_{\mathcal{E}}(?, M^*)|_{\mathcal{M}}$ vanishes on the projectives; it can thus be viewed as an $\underline{\mathcal{M}}$-module, and as such, is isomorphic to $(S_M)^{\oplus d}$. After replacing B_M and $B_{M'}$ by isomorphic objects of $\underline{\mathcal{E}}$, we can assume that the exchange triangles $M^* \to B_M \to M \to \Sigma M^*$ and $M \to B_{M^*} \to M^* \to \Sigma M$ come from conflations $M^* \rightarrowtail B_M \twoheadrightarrow M$ and $M \rightarrowtail B_{M^*} \twoheadrightarrow M^*$. The spliced complex

$$(\cdots \to 0 \to M \to B_{M^*} \to B_M \to M \to 0 \to \cdots)$$

denoted by $C^\bullet$, is then an $\mathcal{E}$-acyclic complex, and it is the image of $(S_M)^{\oplus d}$ under the functor $\mathrm{mod}\,\underline{\mathcal{M}} \subset \mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M} \simeq \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$. Indeed, we have two long exact sequences induced by the conflations above:

$$0 \to \mathcal{M}(?, M) \to \mathcal{M}(?, B_{M^*}) \to \mathcal{E}(?, M^*)|_{\mathcal{M}} \to \mathrm{Ext}^1_{\mathcal{E}}(?, M)|_{\mathcal{M}} = 0 \quad \text{and}$$

$$0 \to \mathcal{E}(?, M^*)|_{\mathcal{M}} \to \mathcal{M}(?, B_M) \to \mathcal{M}(?, M) \to \mathrm{Ext}^1_{\mathcal{E}}(?, M^*)|_{\mathcal{M}} \to \mathrm{Ext}^1_{\mathcal{E}}(?, B_M)|_{\mathcal{M}}.$$

Since B_M belongs to $\mathcal{M}$, the functor $\mathrm{Ext}^1_{\mathcal{E}}(?, B_M)$ vanishes on $\mathcal{M}$, and the complex:

$$(C\hat{}): \quad (\cdots \to 0 \to M\hat{} \to (B_{M^*})\hat{} \to (B_M)\hat{} \to M\hat{} \to 0 \to \cdots)$$

is quasi-isomorphic to $(S_M)^{\oplus d}$.

Now, in the notations of the proof of Lemma 9, $\varphi\,(d[S_M])$ is the image of the class of the $\mathcal{E}$-acyclic complex complex $C^\bullet$ under the morphism $\mathrm{K}_0(\beta)^{-1}\,\mathrm{K}_0(\gamma)$. This is $[M] - [B_M] + [B_{M^*}] - [M]$ which equals $[B_{M^*}] - [B_M]$ as claimed. □

3. The generalized mutation rule

Let $\mathcal{T}$ and $\mathcal{T}'$ be two cluster-tilting subcategories of $\mathcal{C}$. Let Q and Q' be the quivers obtained from their Auslander–Reiten quivers by removing all loops and oriented 2-cycles.

Our aim, in this section, is to give a rule relating Q' to Q, and to prove that it generalizes the Fomin–Zelevinsky mutation rule.

Remark. . Assume that $\mathcal{C}$ has cluster-tilting objects. Then it is proved in [5, Theorem I.1.6], without assuming that $\mathcal{C}$ is algebraic, that the Auslander–Reiten quivers of two cluster-tilting objects having all but one indecomposable direct summands in common (up to isomorphism) are related by the Fomin–Zelevinsky mutation rule.

. To prove that the generalized mutation rule actually generalizes the Fomin–Zelevinsky mutation rule, we use the ideas of Section 7 of [4].

3.1. The rule

As in Section 2, we fix a cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{C}$, and write $\mathcal{M}$ for its preimage in $\mathcal{E}$, so that $\mathcal{T} = \underline{\mathcal{M}}$. Define Q to be the quiver obtained from the Auslander–Reiten quiver of $\underline{\mathcal{M}}$ by deleting its loops and its oriented 2-cycles. Its vertex corresponding to an indecomposable object L will also be labelled by L. We denote by a_{LN} the number of arrows from vertex L to vertex N in the quiver Q. Let $B_{\mathcal{M}}$ be the matrix whose entries are given by $b_{LN} = a_{LN} - a_{NL}$.

Let $R_{\mathcal{M}}$ be the matrix of $\langle\ ,\ \rangle_a : \mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}}) \times \mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$ in the basis given by the classes of the simple modules.

Lemma 11. *The matrices $R_{\mathcal{M}}$ and $B_{\mathcal{M}}$ are equal: $R_{\mathcal{M}} = B_{\mathcal{M}}$.*

Proof. Let L and N be two non-projective indecomposable objects in $\mathcal{M}$. Then $\dim \mathrm{Hom}(S_L, S_N) - \dim \mathrm{Hom}(S_N, S_L) = 0$ and we have:

$$\langle [S_L], [S_N] \rangle_a = \dim \mathrm{Ext}^1(S_N, S_L) - \dim \mathrm{Ext}^1(S_L, S_N) = b_{L,N}. \quad \Box$$

Let $\mathcal{T}'$ be another cluster-tilting subcategory of $\mathcal{C}$, and let $\mathcal{M}'$ be its preimage in the Frobenius category $\mathcal{E}$. Let $(M'_i)_{i\in I}$ (resp. $(M_j)_{j\in J}$) be representatives for the isoclasses of non-projective indecomposable objects in $\mathcal{M}'$ (resp. $\mathcal{M}$). The equivalence of categories $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M} \xrightarrow{\sim} \mathrm{per}_{\underline{\mathcal{M}'}}\,\mathcal{M}'$ of Proposition 8 induces an isomorphism between the Grothendieck groups $\mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}})$ and $\mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}'})$ whose matrix, in the bases given by the classes of the simple modules, is denoted by S. The equivalence of categories $\mathcal{D}\,\mathrm{Mod}\,\mathcal{M} \xrightarrow{\sim} \mathcal{D}\,\mathrm{Mod}\,\mathcal{M}'$ restricts to the identity on $\mathcal{H}^b(\mathcal{P})$, so that it induces an equivalence $\mathrm{per}\,\mathcal{M}/\,\mathrm{per}\,\mathcal{P} \xrightarrow{\sim} \mathrm{per}\,\mathcal{M}'/\,\mathrm{per}\,\mathcal{P}$. Denote by $\mathrm{proj}\,\mathcal{P}$ (resp. $\mathrm{proj}\,\mathcal{M}$, resp. $\mathrm{proj}\,\mathcal{M}'$) the full subcategory of $\mathrm{mod}\,\mathcal{P}$ (resp. $\mathrm{Mod}\,\mathcal{M}$, resp. $\mathrm{Mod}\,\mathcal{M}'$) whose objects are the representable functors. Let T be the matrix of the induced isomorphism from $\mathrm{K}_0(\mathrm{proj}\,\mathcal{M})/\,\mathrm{K}_0(\mathrm{proj}\,\mathcal{P})$ to $\mathrm{K}_0(\mathrm{proj}\,\mathcal{M}')/\,\mathrm{K}_0(\mathrm{proj}\,\mathcal{P})$, in the bases given by the classes $[\mathcal{M}(?, M_j)]$, $j \in J$, and $[\mathcal{M}'(?, M'_i)]$, $i \in I$. The matrix T is much easier to compute than the matrix S. Its entries t_{ij} are given by the approximation triangles of Keller and Reiten in the following way: For all j, there exists a triangle of the form

$$\Sigma^{-1} M_j \longrightarrow \bigoplus_i \beta_{ij} M'_i \longrightarrow \bigoplus_i \alpha_{ij} M'_i \longrightarrow M_j.$$

Then, we have:

Theorem 12. (a) *(Generalized mutation rule). The following equalities hold:*

$$t_{ij} = \alpha_{ij} - \beta_{ij}$$

and

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^t.$$

(b) *The category $\mathcal{C}$ has a cluster-tilting object if and only if all its cluster-tilting subcategories have a finite number of pairwise non-isomorphic indecomposable objects.*

(c) *All cluster-tilting objects of $\mathcal{C}$ have the same number of indecomposable direct summands (up to isomorphism).*

Note that point (c) was shown in [19, 5.3.3(1)] (see also [5, I.1.8]) and, in a more general context, in [20]. Note also that, for the generalized mutation rule to hold, the cluster-tilting subcategories do not need to be related by a sequence of mutation.

Proof. Assertions (b) and (c) are consequences of the existence of an isomorphism between the Grothendieck groups $\mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}})$ and $\mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}'})$. Let us prove the equalities (a). Recall from [3, Section 3.3], that the antisymmetric bilinear form $\langle\ ,\ \rangle_a$ on $\mathrm{mod}\,\underline{\mathcal{M}}$ is induced by the usual Euler form $\langle\ ,\ \rangle_E$ on $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$. The following commutative diagram

$$\begin{tikzcd}
\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M} \times \mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M} \arrow[rr, "\simeq"] \arrow[dr, "\langle\,,\,\rangle_E"'] & & \mathrm{per}_{\underline{\mathcal{M}'}}\,\mathcal{M}' \times \mathrm{per}_{\underline{\mathcal{M}'}}\,\mathcal{M}' \arrow[dl, "\langle\,,\,\rangle_E"] \\
& \mathbb{Z} &
\end{tikzcd},$$

thus induces a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}}) \times \mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}}) \arrow[rr, "S\times S"] \arrow[dr, "\langle\,,\,\rangle_a"'] & & \mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}'}) \times \mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}'}) \arrow[dl, "\langle\,,\,\rangle_a"] \\
& \mathbb{Z} &
\end{tikzcd}.$$

This proves the equality $R_{\mathcal{M}} = S^t R_{\mathcal{M}'} S$, or, by Lemma 11,

$$(1) \quad B_{\mathcal{M}} = S^t B_{\mathcal{M}'} S.$$

Any object of $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M}$ becomes an object of $\mathrm{per}\,\mathcal{M}/\,\mathrm{per}\,\mathcal{P}$ through the composition $\mathrm{per}_{\underline{\mathcal{M}}}\,\mathcal{M} \hookrightarrow \mathrm{per}\,\mathcal{M} \twoheadrightarrow \mathrm{per}\,\mathcal{M}/\,\mathrm{per}\,\mathcal{P}$. Let M and N be two non-projective indecomposable objects in $\mathcal{M}$. Since S_N vanishes on $\mathcal{P}$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{per}\,\mathcal{M}/\,\mathrm{per}\,\mathcal{P}}\,(\mathcal{M}(?,M), S_N) &= \mathrm{Hom}_{\mathrm{per}\,\mathcal{M}}\,(\mathcal{M}(?,M), S_N) \\
&= \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{M}}\,(\mathcal{M}(?,M), S_N) \\
&= S_N(M).
\end{aligned}$$

Thus $\dim \mathrm{Hom}_{\mathrm{per}\,\mathcal{M}/\,\mathrm{per}\,\mathcal{P}}\,(\mathcal{M}(?,M), S_N) = \delta_{MN}$, and the commutative diagram

$$\begin{tikzcd}
\mathrm{per}\,\mathcal{M}/\,\mathrm{per}\,\mathcal{P} \times \mathrm{per}\,\mathcal{M}/\,\mathrm{per}\,\mathcal{P} \arrow[rr, "\simeq"] \arrow[dr, "R\,\mathcal{H}\mathrm{om}"'] & & \mathrm{per}\,\mathcal{M}'/\,\mathrm{per}\,\mathcal{P} \times \mathrm{per}\,\mathcal{M}'/\,\mathrm{per}\,\mathcal{P} \arrow[dl, "R\,\mathcal{H}\mathrm{om}"] \\
& \mathrm{per}\,k &
\end{tikzcd},$$

induces a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_0(\mathrm{proj}\,\mathcal{M})/\,\mathrm{K}_0(\mathrm{proj}\,\mathcal{P}) \times \mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}}) \arrow[rr, "T\times S"] \arrow[dr, "\mathrm{Id}"'] & & \mathrm{K}_0(\mathrm{proj}\,\mathcal{M}')/\,\mathrm{K}_0(\mathrm{proj}\,\mathcal{P}) \times \mathrm{K}_0(\mathrm{mod}\,\underline{\mathcal{M}'}) \arrow[dl, "\mathrm{Id}"] \\
& \mathbb{Z} &
\end{tikzcd}.$$

In other words, the matrix S is the inverse of the transpose of T:

$$(2) \quad S = T^{-t}.$$

Equalities (1) and (2) imply what was claimed, that is

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^t.$$

Let us compute the matrix T: Let M be indecomposable non-projective in $\mathcal{M}$, and let

$$\Sigma^{-1} M \longrightarrow M_1' \longrightarrow M_0' \longrightarrow M$$

be a Keller–Reiten approximation triangle of M with respect to $\mathcal{M}'$, which we may assume to come from a conflation in $\mathcal{E}$. This conflation yields a projective resolution

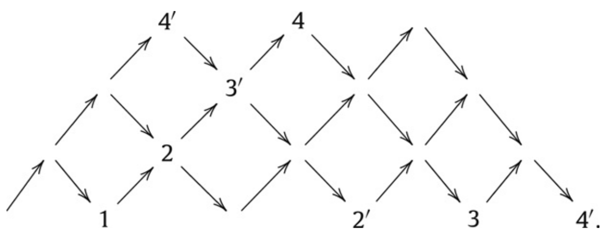
$$0 \longrightarrow (M_1')\hat{} \longrightarrow (M_0')\hat{} \longrightarrow \mathcal{E}(?,M)|_{\mathcal{M}'} \longrightarrow \mathrm{Ext}^1_{\mathcal{E}}(?,M_1')|_{\mathcal{M}'} = 0$$

so that T sends the class of $M\hat{}$ to $[(M_0')\hat{}] - [(M_1')\hat{}]$. Therefore, t_{ij} equals $\alpha_{ij} - \beta_{ij}$. $\square$

3.2. Examples

3.2.1

As a first example, let $\mathcal{C}$ be the cluster category associated with the quiver of type A_4: $1 \to 2 \to 3 \to 4$. Its Auslander–Reiten quiver is the Moebius strip:



Let $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$, where the indecomposable M_i corresponds to the vertex labelled by i in the picture. Let also $M' = M'_1 \oplus M'_2 \oplus M'_3 \oplus M'_4$, where $M'_1 = M_1$, and where the indecomposable M'_i corresponds to the vertex labelled by i' if $i \neq 1$. One easily computes the following Keller–Reiten approximation triangles:

$$\Sigma^{-1}M_1 \longrightarrow \ 0 \ \longrightarrow M'_1 \longrightarrow M_1,$$
$$\Sigma^{-1}M_2 \longrightarrow M'_2 \longrightarrow M'_1 \longrightarrow M_2,$$
$$\Sigma^{-1}M_3 \longrightarrow M'_4 \longrightarrow \ 0 \ \longrightarrow M_4 \text{ and}$$
$$\Sigma^{-1}M_4 \longrightarrow M'_4 \longrightarrow M'_3 \longrightarrow M_4;$$

so that the matrix T is given by:

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

We also have

$$B_{M'} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let maple compute

$$T^{-1}B_{M'}T^{-t} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

which is B_M.

3.2.2

Let us look at a more interesting example, where one cannot easily read the quiver of M' from the Auslander–Reiten quiver of $\mathcal{C}$. Let $\mathcal{C}$ be the cluster category associated with the quiver Q:

$$\begin{tikzcd} & 1 \\ 0 \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & \\ & 2. \end{tikzcd}$$

For $i = 0, 1, 2$, let M_i be (the image in $\mathcal{C}$ of) the projective indecomposable (right) kQ-module associated with vertex i. Their dimension vectors are respectively [1, 0, 0], [2, 1, 0] and [2, 0, 1]. Let M be the direct sum $M_0 \oplus M_1 \oplus M_2$. Let M' be the direct sum $M'_0 \oplus M'_1 \oplus M'_2$, where M'_0, M'_1 and M'_2 are (the images in $\mathcal{C}$ of) the indecomposable regular kQ-modules with dimension vectors [1, 2, 0], [0, 1, 0] and [2, 4, 1] respectively. As one can check, using [21], M and M' are two cluster-tilting objects of $\mathcal{C}$. Computation of Keller–Reiten approximation triangles, amounts to computing projective resolutions in mod kQ, viewed as mod $\operatorname{End}_{\mathcal{C}}(M)$. One easily computes these projective resolutions, by considering dimension vectors:

$$0 \longrightarrow 8M_0 \longrightarrow M_2 \oplus 4M_1 \longrightarrow M'_2 \longrightarrow 0,$$
$$0 \longrightarrow 2M_0 \longrightarrow M_1 \longrightarrow M'_1 \longrightarrow 0 \text{ and}$$
$$0 \longrightarrow 3M_0 \longrightarrow 2M_1 \longrightarrow M'_0 \longrightarrow 0.$$

By applying the generalized mutation rule, one gets the following quiver

$$\begin{tikzcd} & 1 \arrow[dl, "(6)"'] \\ 0 \arrow[dr, "(2)"'] & \\ & 2, \arrow[uu, "(4)"'] \end{tikzcd}$$

which is therefore the quiver of $\operatorname{End}_{\mathcal{C}}(M')$ since by [22], there are no loops or 2-cycles in the quiver of the endomorphism algebra of a cluster-tilting object in a cluster category.

3.3. Back to the mutation rule

We assume in this section that the Auslander–Reiten quiver of $\underline{\mathcal{M}}$ has no loops or 2-cycles. Under the notations of Section 3.1, let k be in I and let (M_k, M'_k) be an exchange pair (see Section 2.3). We choose $\underline{\mathcal{M}}'$ to be the cluster-tilting subcategory of $\mathcal{C}$ obtained from $\underline{\mathcal{M}}$ by replacing M_k by M'_k, so that $M'_i = M_i$ for all $i \neq k$. Recall that T is the matrix of the isomorphism $\mathrm{K}_0(\mathrm{proj}\,\mathcal{M})/\,\mathrm{K}_0(\mathrm{proj}\,\mathcal{P}) \longrightarrow \mathrm{K}_0(\mathrm{proj}\,\mathcal{M}')/\,\mathrm{K}_0(\mathrm{proj}\,\mathcal{P})$.

Lemma 13. *Then, the (i, j)-entry of the matrix T is given by*

$$t_{ij} = \begin{cases} -\delta_{ij} + \dfrac{|b_{ij}| + b_{ij}}{2} & \textit{if } j = k \\ \delta_{ij} & \textit{else.} \end{cases}$$

Proof. Let us apply Theorem 12 to compute the matrix T. For all $j \neq k$, the triangle $\Sigma^{-1}M_j \to 0 \to M'_j = M_j$ is a Keller–Reiten approximation triangle of M_j with respect to $\mathcal{M}'$. We thus have $t_{ij} = \delta_{ij}$ for all $j \neq k$. There is a triangle unique up to isomorphism

$$M'_k \longrightarrow B_{M_k} \longrightarrow M_k \longrightarrow \Sigma M'_k$$

where $B_{M_k} \longrightarrow M_k$ is a right $\underline{\mathcal{M}} \cap \underline{\mathcal{M}}'$-approximation. Since the Auslander–Reiten quiver of $\underline{\mathcal{M}}$ has no loops and no 2-cycles, B_{M_k} is isomorphic to the direct sum: $\bigoplus_{i \in I} (M'_j)^{a_{ik}}$. We thus have $t_{ik} = -\delta_{ik} + a_{ik}$, which equals $\frac{|b_{ik}|+b_{ik}}{2}$. Remark that, by Lemma 7.1 of [4], as stated in Section 1.1, we have $T^2 = Id$, so that $S = T^{\mathrm{t}}$ and

$$s_{ij} = \begin{cases} -\delta_{ij} + \dfrac{|b_{ij}| - b_{ij}}{2} & \text{if } i = k \\ \delta_{ij} & \text{else.} \end{cases} \qquad \square$$

Theorem 14. *The matrix $B_{\mathcal{M}'}$ is obtained from the matrix $B_{\mathcal{M}}$ by the Fomin–Zelevinsky mutation rule in the direction M.*

Proof. By [11] (see Section 1.1), and by Lemma 13, we know that the mutation of the matrix $B_{\mathcal{M}}$ in direction M is given by $TB_{\mathcal{M}'}T^t$, which is $B_{\mathcal{M}}$, by the generalized mutation rule (Theorem 12). $\square$

3.4. Cluster categories

In [6], the authors study the Grothendieck group of the cluster category $\mathcal{C}_A$ associated with an algebra A which is either hereditary or canonical, endowed with any admissible triangulated structure. A triangulated structure on the category $\mathcal{C}_A$ is said to be admissible in [6] if the projection functor from the bounded derived category $\mathcal{D}^b\,(\mathrm{mod}\,A)$ to $\mathcal{C}_A$ is exact (triangulated). They define a Grothendieck group $\overline{\mathrm{K}}_0(\mathcal{C}_A)$ with respect to the triangles induced by those of $\mathcal{D}^b\,(\mathrm{mod}\,A)$, and show that it coincides with the usual Grothendieck group of the cluster category in many cases:

Theorem 15 (*Barot–Kussin–Lenzing*). *We have $\mathrm{K}_0(\mathcal{C}_A) = \overline{\mathrm{K}}_0(\mathcal{C}_A)$ in each of the following three cases:*

(i) *A is canonical with weight sequence $(p_1, \ldots, p_t)$ having at least one even weight.*
(ii) *A is tubular,*
(iii) *A is hereditary of finite representation type.*

Under some restriction on the triangulated structure of $\mathcal{C}_A$, we have the following generalization of case (iii) of Theorem 15:

Theorem 16. *Let A be a finite-dimensional hereditary algebra, and let $\mathcal{C}_A$ be the associated cluster category with its triangulated structure defined in [7]. Then we have $\mathrm{K}_0(\mathcal{C}_A) = \overline{\mathrm{K}}_0(\mathcal{C}_A)$.*

Proof. By Lemma 3.2 in [6], this theorem is a corollary of the following lemma. $\square$

Lemma 17. *Under the assumptions of Section 3.1, and if moreover $\underline{\mathcal{M}}$ has a finite number n of non-isomorphic indecomposable objects, then we have an isomorphism $\mathrm{K}_0(\mathcal{C}) \simeq \mathbb{Z}^n/\,\mathrm{Im}\,B_{\mathcal{M}}$.*

Proof. This is a restatement of Theorem 10. $\square$

Acknowledgement

This article is part of my Ph.D. thesis, under the supervision of Professor B. Keller. I would like to thank him deeply for introducing me to the subject and for his infinite patience.

References

[1] Philippe Caldero, Bernhard Keller, From triangulated categories to cluster algebras, Invent. Math. 172 (1) (2008) 169–211.
[2] Andrei Zelevinsky, What is . . . a cluster algebra?, Notices Amer. Math. Soc. 54 (11) (2007) 1494–1495.
[3] Yann Palu, Cluster characters for triangulated 2-Calabi–Yau categories, Ann. Inst. Fourier 58 (6) (2008) 2221–2248.
[4] Christof Geiß, Bernard Leclerc, Jan Schröer, Rigid modules over preprojective algebras, Invent. Math. 165 (3) (2006) 589–632.
[5] Aslak Bakke Buan, Osamu Iyama, Idun Reiten, Jeanne Scott, Cluster structures for 2-Calabi–Yau categories and unipotent groups, preprint arXiv:math/0701557 [math.RT].
[6] M. Barot, D. Kussin, H. Lenzing, The Grothendieck group of a cluster category, J. Pure Appl. Algebra 212 (1) (2008) 33–46.
[7] Bernhard Keller, On triangulated orbit categories, Doc. Math. 10 (2005) 551–581 (electronic).
[8] Sergey Fomin, Andrei Zelevinsky, Cluster algebras. I. Foundations, J. Amer. Math. Soc. 15 (2) (2002) 497–529 (electronic).
[9] Bernhard Keller, Idun Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi–Yau, Adv. Math. 211 (1) (2007) 123–151.
[10] Dieter Happel, Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras, in: London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.
[11] Arkady Berenstein, Sergey Fomin, Andrei Zelevinsky, Cluster algebras. III. Upper bounds and double Bruhat cells, Duke Math. J. 126 (1) (2005) 1–52.
[12] Bernhard Keller, Dieter Vossieck, Sous les catégories dérivées, C. R. Acad. Sci. Paris Sér. I Math. 305 (6) (1987) 225–228.
[13] Jeremy Rickard, Derived equivalences as derived functors, J. London Math. Soc. (2) 43 (1) (1991) 37–48.
[14] Bernhard Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. (4) 27 (1) (1994) 63–102.
[15] B. Keller, D. Vossieck, Aisles in derived categories, Bull. Soc. Math. Belg. Sér. A 40 (2) (1988) 239–253. Deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987).
[16] Gonçalo Tabuada, On the structure of Calabi–Yau categories with a cluster tilting subcategory, Doc. Math. 12 (2007) 193–213 (electronic).
[17] Osamu Iyama, Yuji Yoshino, Mutation in triangulated categories and rigid Cohen–Macaulay modules, Invent. Math. 172 (1) (2008) 117–168.
[18] Aslak Bakke Buan, Robert Marsh, Markus Reineke, Idun Reiten, Gordana Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204 (2) (2006) 572–618.
[19] Osamu Iyama, Auslander correspondence, Adv. Math. 210 (1) (2007) 51–82.
[20] Raika Dehy, Bernhard Keller, On the combinatorics of rigid objects in 2-Calabi–Yau categories, Int. Math. Res. Not. (11) (2008) 17 pages. Art. ID rnn029.
[21] Bernhard Keller, Quiver mutation in Java, Java applet available at B. Keller's home page.
[22] Aslak Bakke Buan, Robert J. Marsh, Idun Reiten, Cluster mutation via quiver representations, Comment. Math. Helv. 83 (1) (2008) 143–177.