CLUSTER CHARACTERS II: A MULTIPLICATION FORMULA

YANN PALU

Abstract. Let $\mathcal{C}$ be a Hom-finite triangulated 2-Calabi–Yau category with a cluster tilting object. Under some constructibility assumptions on $\mathcal{C}$ which are satisfied for instance by cluster categories, by generalized cluster categories and by stable categories of modules over a preprojective algebra of Dynkin type, we prove a multiplication formula for the cluster character associated with any cluster tilting object. This formula generalizes those obtained by Caldero–Keller for representation finite path algebras and by Xiao–Xu for finite-dimensional path algebras. We prove an analogous formula for the cluster character defined by Fu–Keller in the setup of Frobenius categories. It is similar to a formula obtained by Geiss–Leclerc–Schröer in the context of preprojective algebras.

Introduction

In recent years, the link between Fomin–Zelevinsky's cluster algebras [FZ02] and the representation theory of quivers and finite-dimensional algebras has been investigated intensely, cf. for example the surveys [BM06], [GLS08b], [Kel]. In its most tangible form, this link is given by a map taking objects of cluster categories to elements of cluster algebras. Such a map was first constructed by P. Caldero and F. Chapoton [CC06] for cluster categories and cluster algebras associated with Dynkin quivers. Another approach, leading to proofs of several conjectures on cluster algebras in a more general context, can be found in [DWZ08], [DWZ10].

The results of P. Caldero and B. Keller [CK08] yield two multiplication formulae for the Caldero–Chapoton map of cluster categories associated with Dynkin quivers. The first one categorifies the exchange relations of cluster variables and only applies to objects L and M such that $\mathrm{Ext}^1(L, M)$ is of dimension 1. The second one generalizes it to arbitrary dimensions, and yields some new relations in the associated cluster algebras. These relations very much resemble relations in dual Ringel–Hall algebras [Sch08, section 5.5]. Motivated by these results, C. Geiss, B. Leclerc and J. Schröer [GLS07] proved two analogous formulae for module categories over preprojective algebras. In this latter situation, the number of isomorphism classes of indecomposable objects is usually infinite. Generalizations of the first formula were proved in [CK06] for cluster categories associated with any acyclic quiver, and later in [Pal08] for 2-Calabi–Yau triangulated categories. The first generalization of the second multiplication formula, by A. Hubery (see [Huba]), was based on the existence of Hall polynomials which he proved in the affine case [Hubb], generalizing Ringel's result [Rin90] for Dynkin quivers. Staying close to this point of view, J. Xiao and F. Xu proved in [XX] a projective version of Green's formula [Rin96] and applied it to generalize the multiplication formula for acyclic cluster algebras. Another proof of this formula was found by F. Xu in [Xu10], who used the 2-Calabi–Yau property instead of Green's formula. Our aim in this paper is to generalize the second multiplication formula to more general 2-Calabi–Yau categories for the cluster character associated with an arbitrary cluster tilting object. This in particular applies to the generalized cluster categories introduced by C. Amiot [Ami09] and to stable categories of modules over a preprojective algebra.

Assume that the triangulated category $\mathcal{C}$ is the stable category of a Hom-finite Frobenius category $\mathcal{E}$. Then C. Fu and B. Keller defined a cluster character on $\mathcal{E}$, which "lifts" the one on $\mathcal{C}$. We prove that it satisfies the same multiplication formula as the one proved by Geiss–Leclerc–Schröer in [GLS07].

The paper is organized as follows: In the first section, we fix some notations and state our main result: A multiplication formula for the cluster character associated with any cluster tilting object. In section 2, we recall some definitions and prove the 'constructibility of kernels and cokernels' in modules categories. We apply these facts to prove that:

- If the triangulated category has constructible cones (see section 1.4), the sets under consideration in the multiplication formula, and in its proof, are constructible.
- Stable categories of Hom-finite Frobenius categories have constructible cones.
- Generalized cluster categories defined in [Ami09] have constructible cones.

Thus, all of the 2-Calabi–Yau triangulated categories related to cluster algebras which have been introduced so far have constructible cones. Notably this holds for cluster categories associated with acyclic quivers, and for the stable categories associated with the exact subcategories of module categories over preprojective algebras constructed in [GLS08a] and [BIRS09]. In section 3, we prove the main theorem. In the last section, we consider the setup of Hom-finite Frobenius categories. We prove a multiplication formula for the cluster character defined by Fu–Keller in [FK10].

Acknowledgements

This article is part of my PhD thesis under the supervision of Bernhard Keller. I would like to thank him deeply for his outstanding supervision and for pointing out reference [Joy06] to me. I would like to thank Bernt Tore Jensen for his clear explanations on constructible sets and locally closed sets. The definition of a constructible map has been changed according to his remarks.

Contents

1. Notations and main result

Let k be the field of complex numbers. The only place where we will need more than the fact that k is an algebraically closed field is proposition 2 in section 2.1. See [Joy06, section 3.3] for an explanation, illustrated with an example, of the fact that the theory of constructible functions does not extend to fields of positive characteristic. Let $\mathcal{C}$ be a Hom-finite, 2-Calabi–Yau, Krull–Schmidt k-category which admits a basic cluster tilting object T. In order to prove the main theorem, a constructibility hypothesis will be needed. This hypothesis is precisely stated in section 1.3 and it will always be explicitly stated when it is assumed. Stable categories of Hom-finite Frobenius categories satisfy this constructibility hypothesis, cf. section 2.4, so that the main theorem applies to cluster categories (thanks to the construction in [GLS, Theorem 2.1]), to stable module categories over preprojective algebras... Moreover, the main theorem applies to the generalized cluster categories of [Ami09], cf. section 2.5.

We let B denote the endomorphism algebra of T in $\mathcal{C}$, and we let F denote the covariant functor from $\mathcal{C}$ to $\operatorname{mod} B$ co-represented by T. We denote the image in $\mathbb{Q}(x_1, \ldots, x_n)$ of an object M in $\mathcal{C}$ under the cluster character associated with T (see [Pal08]) by X^T_M. Recall that it is given by the following formula: Let Q_T be the Gabriel quiver of B, and denote by $1, \ldots, n$ its vertices. For each vertex i, denote by S_i (resp. P_i) the corresponding simple (resp. projective) module. Then we have

$$X^T_M = \underline{x}^{-\operatorname{coind} M} \sum_e \chi(\operatorname{Gr}_e FM) \prod_{i=1}^n x_i^{\langle S_i, e\rangle_a},$$

where $\operatorname{coind} M$ denotes the coindex of M and $\langle\ ,\ \rangle_a$ the antisymmetric bilinear form on $K_0(\operatorname{mod} B)$ (for more complete definitions, see sections 1 and 2.1 of [Pal08]). For any two objects L and M in $\mathcal{C}$, and any morphism ε in $\mathcal{C}(L, \Sigma M)$, we denote any object Y appearing in a triangle of the form

$$M \longrightarrow Y \longrightarrow L \xrightarrow{\varepsilon} \Sigma M$$

by $\operatorname{mt}(\varepsilon)$ (the middle term of ε).

1.1. X^T-stratification.

Let L and M be objects in $\mathcal{C}$. If an object Y of $\mathcal{C}$ occurs as $\operatorname{mt}(\varepsilon)$ for some morphism ε in $\mathcal{C}(L, \Sigma M)$, we let $\langle Y \rangle$ denote the set of all isomorphism classes of objects $Y' \in \mathcal{C}$ such that:

- Y' is the middle term of some morphism in $\mathcal{C}(L, \Sigma M)$,
- $\operatorname{coind} Y' = \operatorname{coind} Y$ and
- for all e in $K_0(\operatorname{mod} B)$, we have $\chi\big(\operatorname{Gr}_e(FY')\big) = \chi\big(\operatorname{Gr}_e(FY)\big)$.

The equality of classes $\langle Y \rangle = \langle Y' \rangle$ yields an equivalence relation on the 'set' of middle terms of morphisms in $\mathcal{C}(L, \Sigma M)$. Fix a set $\mathcal{Y}$ of representatives for this relation. Further, we denote the set of all ε with $\operatorname{mt}(\varepsilon) \in \langle Y \rangle$ by $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$, and the set of $\varepsilon' \in \mathcal{C}(L, \Sigma M)$ such that $X^T_{\operatorname{mt}(\varepsilon')} = X^T_{\operatorname{mt}(\varepsilon)}$ by $\langle \varepsilon \rangle$. It will be proven in section 2.3 that if the cylinders of the morphisms $L \to \Sigma M$ are constructible with respect to T in the sense of section 1.3 below, then the sets $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$ are constructible, and the set $\mathcal{Y}$ is finite.

Remark that if Y' belongs to $\langle Y \rangle$, then $X^T_{Y'} = X^T_Y$. Hence the fibers of the map sending ε to $X^T_{\operatorname{mt}(\varepsilon)}$ are finite unions of sets $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$. Therefore, the sets $\langle \varepsilon \rangle$ are constructible, we have

$$\mathcal{C}(L, \Sigma M) = \coprod_{\varepsilon \in \mathcal{R}} \langle \varepsilon \rangle$$

for some finite set $\mathcal{R} \subset \mathcal{C}(L, \Sigma M)$, and

$$\mathcal{C}(L, \Sigma M) = \coprod_{Y \in \mathcal{Y}} \mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$$

is a refinement of the previous decomposition.

1.2. **The variety** $\operatorname{rep}_d BQ$**.** Let V be a finite dimensional k-vector space. We denote by $\operatorname{rep}'_B(V)$ the set of morphisms of k-algebras from B^{op} to $\operatorname{End}_k(V)$. Since B is finitely generated, the set $\operatorname{rep}'_B(V)$ is a closed subvariety of some finite product of copies of $\operatorname{End}_k(V)$.

Let Q be a finite quiver, and let $d = (d_i)_{i \in Q_0}$ be a tuple of non-negative integers. A *d-dimensional matrix representation* of Q in $\operatorname{mod} B$ is given by

- a right B-module structure on k^{d_i} for each vertex i of Q and
- a B-linear map $k^{d_i} \to k^{d_j}$ for each arrow $\alpha : i \to j$ of Q.

Clearly, for fixed d, the d-dimensional matrix representations of Q in $\operatorname{mod} B$ form an affine variety $\operatorname{rep}_d BQ$ on which the group $GL(d) = \prod_{i \in Q_0} GL_{d_i}(k)$ acts by changing the bases in the spaces k^{d_i}. We write $\operatorname{rep}_d BQ/GL(d)$ for the set of orbits.

1.3. **Constructible cones.** Let $\overrightarrow{A_4}$ be the quiver: $1 \to 2 \to 3 \to 4$. Let T, L and M be objects of $\mathcal{C}$. Let $d_{\max}$ be the 4-tuple of integers

$$(\dim FM, \dim FM + \dim FL, \dim FL, \dim F\Sigma M).$$

Let $\Phi_{L,M}$ be the map from $\mathcal{C}(L, \Sigma M)$ to

$$\coprod_{d \leq d_{\max}} \operatorname{rep}_d(B\overrightarrow{A_4})/GL(d)$$

sending a morphism ε to the orbit of the exact sequence of B-modules

$$\mathcal{C}(T, M) \xrightarrow{Fi} \mathcal{C}(T, Y) \xrightarrow{Fp} \mathcal{C}(T, L) \xrightarrow{F\varepsilon} \mathcal{C}(T, \Sigma M),$$

where $M \xrightarrow{i} Y \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ is a triangle in $\mathcal{C}$. The cylinders over the morphisms $L \to \Sigma M$ are *constructible with respect to* T if the map $\Phi_{L,M}$ lifts to a constructible map

$$\varphi_{L,M} : \mathcal{C}(L, \Sigma M) \longrightarrow \coprod_{d \leq d_{\max}} \operatorname{rep}_d(B\overrightarrow{A_4})$$

(see section 2.1). The category $\mathcal{C}$ is said to have constructible cones if this holds for arbitrary objects L,M and T.

1.4. **Main result.** Let f be a constructible function from an algebraic variety over k to any abelian group, and let C be a constructible subset of this variety. Then one defines "the integral of f on C with respect to the Euler characteristic" to be

$$\int_C f = \sum_{x \in f(C)} \chi\big(C \cap f^{-1}(x)\big)x,$$

cf. for example the introduction of [Lus97]. Our aim in this paper is to prove the following:

Theorem 1. *Let T be any cluster tilting object in $\mathcal{C}$. Let L and M be two objects such that the cylinders over the morphisms $L \to \Sigma M$ and $M \to \Sigma L$ are constructible with respect to T. Then we have:*

$$\chi(\mathbb{P}\mathcal{C}(L, \Sigma M)) X_L^T X_M^T = \int_{[\varepsilon] \in \mathbb{P}\mathcal{C}(L, \Sigma M)} X_{\mathrm{mt}(\varepsilon)}^T + \int_{[\varepsilon] \in \mathbb{P}\mathcal{C}(M, \Sigma L)} X_{\mathrm{mt}(\varepsilon)}^T,$$

where $[\varepsilon]$ denotes the class in $\mathbb{P}\mathcal{C}(L, \Sigma M)$ of a non zero morphism ε in $\mathcal{C}(L, \Sigma M)$.

The statement of the theorem is inspired from [GLS07], cf. also [XX]. We will prove it in section 3. Our proof is inspired from that of P. Caldero and B. Keller in [CK08]. Note that in contrast with the situation considered there, in the above formula, an infinite number of isomorphism classes of objects $\mathrm{mt}(\varepsilon)$ may appear.

2. Constructibility

2.1. **Definitions.** Let X be a topological space. A *locally closed subset* of X is the intersection of a closed subset with an open one. A *constructible subset* is a finite (disjoint) union of locally closed subsets. The family of constructible subsets is the smallest one containing all open (equivalently: closed) subsets of X and stable under taking finite intersections and complements. A function f from X to an abelian group is *constructible* if it is a finite $\mathbb{Z}$-linear combination of characteristic functions of constructible subsets of X. Equivalently, f is constructible if it takes a finite number of values and if its fibers are constructible subsets of X.

For an algebraic variety X, the ring of constructible functions from X to $\mathbb{Z}$ is denoted by $CF(X)$. The following proposition will be used, as in [XX], in order to prove lemma 5 of section 2.3.

Proposition 2. [Dim04, Proposition 4.1.31] *Associated with any morphism of complex algebraic varieties $f : X \longrightarrow Y$, there is a well-defined push-forward homomorphism $CF(f) : CF(X) \longrightarrow CF(Y)$. It is determined by the property*

$$CF(f)(1_Z)(y) = \chi(f^{-1}(y) \cap Z)$$

for any closed subvariety Z in X and any point $y \in Y$.

Let X and Y be algebraic varieties. A map $f : X \longrightarrow Y$ is said to be *constructible* if there exists a decomposition of X into a finite disjoint union of locally closed subsets $X_i, i \in I$, such that the restriction of f to each X_i is a morphism of algebraic varieties. Note that the composition of two constructible maps is constructible, and that the composition of a constructible function with a constructible map is again a constructible function.

2.2. **Kernels and cokernels are constructible.** In section 2.1 of [Xu10], it is shown that the kernel and cokernel of a morphism of modules over a path algebra $\mathbb{C}Q$ are constructible. In this section, we give direct proofs in the more general case where $\mathbb{C}Q$ is replaced by a finite dimensional algebra B.

Let L and M be two finite dimensional vector spaces over the field k, of respective dimensions n and m. Let N be a linear subspace of M. Define E_N to be the set of all morphisms $f \in \mathrm{Hom}_k(L, M)$ such that $\mathrm{Im}\, f \oplus N = M$.

Lemma 3. *The set E_N is a locally closed subset of* $\mathrm{Hom}_k(L, M)$.

Proof. Let $(u_1, \ldots, u_n)$ be a basis of L, and let $(v_1, \ldots, v_m)$ be a basis of M whose p first vectors form a basis of N. Let r be such that $r + p = m$. Let $f : L \longrightarrow M$ be a k-linear map, and denote by $A = (a_{ij})$ its matrix in the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$. Denote by A_1 the submatrix of A formed by its first p rows and by A_2 the one formed by its last r rows. For $t \leq n$, let $P(t, n)$ be the set of all subsets of $\{1, \ldots, n\}$ of cardinality t.

The map f belongs to E_N if and only if:

a) There exists $\underline{j}$ in $P(r, n)$ such that the submatrix $(a_{ij})_{i>p, j\in\underline{j}}$ has a non-zero determinant and
b) if the last r entries of a linear combination of columns of A vanish, then the combination itself vanishes.

Condition b) is equivalent to the inclusion $\mathrm{Ker}\, A_2 \subseteq \mathrm{Ker}\, A_1$ and so to the inclusion $\mathrm{Im}(A_1^{\mathrm{t}}) \subseteq \mathrm{Im}(A_2^{\mathrm{t}})$. Therefore, condition b) can be restated as condition b'):

b') For all $i_0 \leq p$, and all $\underline{l} \in P(r+1,n)$, the determinant of the submatrix of A obtained by taking lines in $\{i_0, p+1, \ldots, m\}$ and columns in $\underline{l}$ vanishes.

Let $\Omega_{\underline{j}}$ be the set of all maps that satisfy condition a) with respect to the index set $\underline{j}$, and let F be the set of all maps that satisfy condition b'). For all $\underline{j} \in P(r,n)$, the set $\Omega_{\underline{j}}$ is an open subset of $\mathrm{Hom}_k(L,M)$ and the set F is a closed subset of $\mathrm{Hom}_k(L,M)$. Since we have the equality:

$$E_N = \Big(\bigcup_{\underline{j} \in P(r,n)} \Omega_{\underline{j}}\Big) \cap F,$$

the set E_N is locally closed in $\mathrm{Hom}_k(L,M)$. □

Let $\overrightarrow{A_2}$ be the quiver: $1 \to 2$.

Lemma 4. *Let B be a finite dimensional algebra, and let L and M be finitely generated B-modules of dimensions n and m respectively. The map c from $\mathrm{Hom}_B(L,M)$ to $\coprod_{d \leq m} \mathrm{rep}_{(m,d)}(B\overrightarrow{A_2})/GL(m,d)$ which sends a morphism l to the orbit of the representation $M \twoheadrightarrow \mathrm{Coker}\, l$ lifts to a constructible map from $\mathrm{Hom}_B(L,M)$ to $\coprod_{d \leq m} \mathrm{rep}_{(m,d)}(B\overrightarrow{A_2})$.*

Dually, the map from $\mathrm{Hom}_B(L,M)$ to $\coprod_{d \leq n} \mathrm{rep}_{(d,n)}(B\overrightarrow{A_2})/GL(d,n)$ which sends a morphism l to the orbit of the representation $\mathrm{Ker}\, l \rightarrowtail L$ lifts to a constructible map from $\mathrm{Hom}_B(L,M)$ to $\coprod_{d \leq n} \mathrm{rep}_{(d,n)}(B\overrightarrow{A_2})$.

Proof. Let us prove the first assertion. We keep the notations of the proof of lemma 3. For a subset $\underline{i}$ of $\{1,\ldots,m\}$, let $N_{\underline{i}}$ be the linear subspace of M generated by $(v_i)_{i \in \underline{i}}$. Then $\mathrm{Hom}_B(L,M)$ is the union of its intersections with each $E_{N_{\underline{i}}}$, for $\underline{i} \subseteq \{1,\ldots,m\}$. It is thus enough to consider the restriction of the map c to E_N, where $N \overset{i_N}{\rightarrowtail} M$ is a given linear subspace of M. Since the set E_N is the union of the locally closed subsets $\Omega_{\underline{j}} \cap F$, for $\underline{j} \in P(r,n)$, we can fix such a $\underline{j}$ and only consider the restriction of c to $\Omega_{\underline{j}} \cap F$. Let f be a morphism in $\mathrm{Hom}_B(L,M)$ and assume that f is in $\Omega_{\underline{j}} \cap F$. Then the cokernel of the k-linear map f is N and the projection p_f of M onto N along $\mathrm{Im}\, f$ is given by the $n \times p$ matrix $\begin{pmatrix} 1 & -CD^{-1} \end{pmatrix}$, where C is the submatrix $(a_{ij})_{i \leq p, j \in \underline{j}}$ and D is the submatrix $(a_{ij})_{i > p, j \in \underline{j}}$. Moreover, if we denote by $\rho^M \in \mathrm{rep}'_B(M)$ the structure of B-module of M, then the structure of B-module ρ of N induced by f is given by $\rho(b) = p_f \circ \rho^M(b) \circ i_N$, for all $b \in B$. □

2.3. **Constructibility of $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$.** Let k, $\mathcal{C}$ and T be as in section 1. Recall that B denotes the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$. This algebra is the path algebra of a quiver Q_T with ideal of relations I. Recall that we denote by $1,\ldots,n$ the vertices of Q_T.

The following lemma is a particular case of [Dim04, Proposition 4.1.31], and was already stated in [XX] for hereditary algebras.

Lemma 5. *For any two dimension vectors e and d with $e \leq d$, the function*

$$\begin{array}{rcl} \mu_e : \mathrm{rep}_d(Q_T, I) & \longrightarrow & \mathbb{Z} \\ M & \longmapsto & \chi(\mathrm{Gr}_e\, M) \end{array}$$

is constructible.

Proof. Let $\mathrm{Gr}_e(d)$ be the closed subset of

$$\mathrm{rep}_d(Q_T, I) \times \prod_{i \in Q_0} \mathrm{Gr}_{e_i}(k^{d_i})$$

formed by those pairs (ρ, W) for which the subspaces $W_i \subseteq k^{d_i}$, $i \in Q_0$, form a subrepresentation. Apply proposition 2 to the first projection $f : \mathrm{Gr}_e(d) \to \mathrm{rep}_d(Q_T, I)$ and remark that $\mu_e = CF(f)(1_{\mathrm{Gr}_e(d)})$.

□

Corollary 6. *Let L and M be objects in $\mathcal{C}$, and let $e \leq \underline{\dim} FL + \underline{\dim} FM$ be in $K_0(\mathrm{mod}\, B)$. Assume that the cylinders over the morphisms $L \to \Sigma M$ are constructible. Then the function*

$$\begin{aligned} \lambda_e : \mathcal{C}(L, \Sigma M) &\longrightarrow \mathbb{Z} \\ \varepsilon &\longmapsto \chi(\mathrm{Gr}_e\, F\, \mathrm{mt}(\varepsilon)) \end{aligned}$$

is constructible.

Proof. By our hypothesis, the map sending $\varepsilon \in \mathcal{C}(L, \Sigma M)$ to the image of its middle term in $\coprod \mathrm{rep}_d(Q_T, I)/GL(d)$, where the union is over the dimension vectors d not greater than $\underline{\dim} FL + \underline{\dim} FM$, lifts to a constructible map from $\mathcal{C}(L, \Sigma M)$ to $\coprod \mathrm{rep}_d(Q_T, I)$. The claim therefore follows from lemma 5. □

Let $M \xrightarrow{i} Y \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ be a triangle in $\mathcal{C}$, and denote by g the class of $\mathrm{Ker}\, Fi$ in the Grothendieck group $K_0(\mathrm{mod}\, B)$.

Lemma 7. *We have:*

$$\mathrm{coind}\, Y = \mathrm{coind}(L \oplus M) - \sum_{i=1}^{n} \langle S_i, g \rangle_a [P_i].$$

Proof. Let $K \in \mathcal{C}$ lift $\mathrm{Ker}\, Fi$. Using respectively proposition 2.2, lemma 2.1.(2), lemma 7 and section 3 of [Pal08], we have the following equalities:

$$\begin{aligned} \mathrm{coind}\, Y &= \mathrm{coind}\, L + \mathrm{coind}\, M - \mathrm{coind}\, K - \mathrm{coind}\, \Sigma K \\ &= \mathrm{coind}(L \oplus M) + \mathrm{ind}\, K - \mathrm{coind}\, K \\ &= \mathrm{coind}(L \oplus M) - \sum_{i=1}^{n} \langle S_i, FK \rangle_a [P_i] \\ &= \mathrm{coind}(L \oplus M) - \sum_{i=1}^{n} \langle S_i, g \rangle_a [P_i]. \end{aligned}$$

□

Corollary 8. *Let L and M be two objects such that the cylinders over the morphisms $L \to \Sigma M$ are constructible. The map $\lambda : \mathcal{C}(L, \Sigma M) \longrightarrow K_0(\mathrm{proj}\, B)$ which sends ε to the coindex (or to the index) of its middle term Y is constructible.*

Proof. Note that g is at most the sum of the dimension vectors of FL and FM, so that by lemma 7 the map λ takes a finite number of values. By our hypothesis and lemma 4, there exists a constructible map:

$$\mathcal{C}(L, \Sigma M) \longrightarrow \coprod_{d \leq \dim FM} \mathrm{rep}'_B(k^d)$$

which lifts the map sending ε to the isomorphism class of the structure of B-module on $\mathrm{Ker}\, Fi$. Moreover, the map sending a module ρ in $\bigcup_{d \leq \dim FM} \mathrm{rep}'_B(k^d)$ to $\sum_{i=1}^{n} \langle S_i, \rho \rangle_a [P_i]$ in $K_0(\mathrm{proj}\, B)$ only depends on the dimension vector of ρ and thus is constructible. Therefore, the map λ is constructible. □

Proposition 9. *Let $L, M \in \mathcal{C}$ be such that the cylinders over the morphisms $L \to \Sigma M$ are constructible. Then the sets $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$ are constructible subsets of $\mathcal{C}(L, \Sigma M)$. Moreover, the set $\mathcal{C}(L, \Sigma M)$ is a finite disjoint union of such constructible subsets.*

Proof. Fix a triangle $M \xrightarrow{i} Y \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ in $\mathcal{C}$. Then $\varepsilon' \in \mathcal{C}(L, \Sigma M)$ is in $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$ if and only if

- $\lambda(\varepsilon') = \lambda(\varepsilon)$ and
- For all $e \leq \underline{\dim} FY$, $\lambda_e(\varepsilon') = \lambda_e(\varepsilon)$.

Therefore, the claim follows from corollary 6 and corollary 8. □

2.4. **Stable categories have constructible cones.** In this section, we assume moreover that $\mathcal{C}$ is the stable category of a Hom-finite, Frobenius, Krull–Schmidt category $\mathcal{E}$, which is linear over the algebraically closed field k. Our aim is to prove that such a category has constructible cones.

Let $\mathcal{P}$ denote the ideal in $\mathcal{E}$ of morphisms factoring through a projective-injective object. Let T, L and M be objects of the category $\mathcal{C}$. Fix a k-linear section s of the projection $\mathrm{Ext}^1_{\mathcal{E}}(L, M) \twoheadrightarrow \mathcal{C}(L, \Sigma M)$ induced by the canonical functor $\mathcal{E} \xrightarrow{\Pi} \mathcal{C}$. Fix a conflation $M \rightarrowtail IM \twoheadrightarrow \Sigma M$ in $\mathcal{E}$, with IM being projective-injective in $\mathcal{E}$, and, for any ε in $\mathcal{C}(L, \Sigma M)$, consider its pull-back via $s\varepsilon$:

$$\begin{array}{ccccc} M & \overset{\iota}{\rightarrowtail} & Y & \overset{\pi}{\twoheadrightarrow} & L \\ \| & & \downarrow & & \downarrow{\scriptstyle s\varepsilon} \\ M & \rightarrowtail & IM & \twoheadrightarrow & \Sigma M. \end{array}$$

Via Π, this diagram induces a triangle $M \xrightarrow{i} Y \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ in $\mathcal{C}$.

For any $X \in \mathcal{E}$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}(X, M) & \xrightarrow{\mathcal{E}(X,\iota)} & \mathcal{E}(X, Y) & \xrightarrow{\mathcal{E}(X,\pi)} & \mathcal{E}(X, L) \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \mathcal{E}(X, s\varepsilon)} \\ 0 & \longrightarrow & \mathcal{E}(X, M) & \longrightarrow & \mathcal{E}(X, IM) & \longrightarrow & \mathcal{E}(X, \Sigma M). \end{array}$$

Fix $X' \in \mathcal{E}$ and a morphism $X' \to X$. Denote by C the endomorphism algebra of $X' \to X$ in the category of morphisms of $\mathcal{E}$, and by $\mathcal{D}'$ the set of dimension vectors $d = (d_1, d_2, d_3, d_4)$ such that $d_1 = \dim \mathcal{E}(X, M)$, $d_3 = \dim \mathcal{E}(X, L)$, $d_2 \leq d_1 + d_3$ and $d_4 = \dim \mathcal{E}(X, \Sigma M)$.

Lemma 10. *There exists a constructible map*

$$\mu : \mathcal{C}(L, \Sigma M) \longrightarrow \coprod_{d \in \mathcal{D}'} \mathrm{rep}_d\, C\overrightarrow{A_4}$$

which lifts the map sending ε to the orbit of the matrix representation of $\overrightarrow{A_4}$ in $\mathrm{mod}\, C$ *given by* $\mathcal{E}(X, M) \xrightarrow{\mathcal{E}(X,\iota)} \mathcal{E}(X, Y) \xrightarrow{\mathcal{E}(X,\pi)} \mathcal{E}(X, L) \xrightarrow{\mathcal{E}(X,s\varepsilon)} \mathcal{E}(X, \Sigma M)$.

Proof. By definition of a pull-back, the map $\mathcal{E}(X, Y) \longrightarrow \mathcal{E}(X, IM) \oplus \mathcal{E}(X, L)$ is a kernel for the map $\mathcal{E}(X, IM) \oplus \mathcal{E}(X, L) \longrightarrow \mathcal{E}(X, \Sigma M)$. Moreover, the morphism $\mathcal{E}(X, M) \xrightarrow{\mathcal{E}(X,\iota)} \mathcal{E}(X, Y)$ is a kernel for $\mathcal{E}(X, \pi)$. Therefore, lemma 4 in section 2.2 applies and such a constructible map μ exists. □

Denote by $\mathcal{D}$ the set of dimension vectors $d = (d_1, d_2, d_3, d_4)$ such that:
$d_1 = \dim \mathcal{C}(T, M)$, $d_3 = \dim \mathcal{C}(T, L)$, $d_2 \leq d_1 + d_3$ and $d_4 = \dim \mathcal{C}(T, \Sigma M)$.

Proposition 11. *There exists a constructible map*

$$\varphi : \mathcal{C}(L, \Sigma M) \longrightarrow \coprod_{d \in \mathcal{D}} \mathrm{rep}_d\, B\overrightarrow{A_4}$$

which lifts the map sending ε to the orbit of the representation

$$\mathcal{C}(T,M) \xrightarrow{Fi} \mathcal{C}(T,Y) \xrightarrow{Fp} \mathcal{C}(T,L) \xrightarrow{F\varepsilon} \mathcal{C}(T,\Sigma M)\ .$$

Proof. Let $T \rightarrowtail IT$ be an inflation from T to a projective-injective object in $\mathcal{E}$. This inflation induces a commutative diagram $(*)$ of modules over the endomorphism algebra $\widetilde{B}$ of $T \rightarrowtail IT$ in the Frobenius category of inflations of $\mathcal{E}$:

$$(*)\qquad \begin{array}{ccccccc} \mathcal{E}(IT,M) & \longrightarrow & \mathcal{E}(IT,Y) & \longrightarrow & \mathcal{E}(IT,L) & \longrightarrow & \mathcal{E}(IT,\Sigma M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}(T,M) & \longrightarrow & \mathcal{E}(T,Y) & \longrightarrow & \mathcal{E}(T,L) & \longrightarrow & \mathcal{E}(T,\Sigma M). \end{array}$$

The map which sends ε to the orbit of the diagram $(*)$ lifts to a constructible one. This is proved by repeating the proof of lemma 10 for the functor

$$\mathcal{E} \longrightarrow \operatorname{mod} \widetilde{B},\ U \longmapsto \big(\mathcal{E}(IT,U) \to \mathcal{E}(T,U)\big)$$

instead of $U \mapsto \mathcal{E}(X,U)$ and using lemma 4 for $\widetilde{B}$.

By applying lemma 4 to $\widetilde{B} \otimes kA_4$, we see that the vertical cokernel of diagram $(*)$ is constructible as a $\widetilde{B} \otimes kA_4$-module. Now the claim follows because the terms of the cokernel are B-modules and B is also the stable endomorphism algebra of $T \rightarrowtail IT$ in the Frobenius category of inflations of $\mathcal{E}$. $\square$

2.5. **Generalized cluster categories have constructible cones.** Let (Q,W) be a Jacobi-finite quiver with potential W in kQ (cf. section 3.3 of [Ami09]), and let Γ be the Ginzburg dg algebra associated with (Q,W) (cf. section 4.2 of [Gin]). The perfect derived category $\operatorname{per}\Gamma$ is the thick subcategory of the derived category $\mathcal{D}\Gamma$ generated by Γ. The finite dimensional derived category $\mathcal{D}_{\mathrm{fd}}\Gamma$ is the full subcategory of $\mathcal{D}\Gamma$ whose objects are the dg modules whose homology is of finite total dimension. It is easy to check that an object M belongs to $\mathcal{D}_{\mathrm{fd}}\Gamma$ if and only if $\operatorname{Hom}_{\mathcal{D}\Gamma}(P,M)$ is finite dimensional for each object P of $\operatorname{per}\Gamma$.

Lemma 12 (Appendix of [KY])**.** a) *The category $\mathcal{D}_{fd}\Gamma$ is contained in* $\operatorname{per}\Gamma$.
b) *An object of $\mathcal{D}\Gamma$ belongs to $\mathcal{D}_{fd}\Gamma$ if and only if it is quasi-isomorphic to a dg Γ-module of finite total dimension.*
c) *The category $\mathcal{D}_{fd}\Gamma$ is equivalent to the localization of the homotopy category $\mathcal{H}_{fd}\Gamma$ of right dg Γ-modules of finite total dimension with respect to its subcategory of acyclic dg modules.*

Note that we stated the previous lemma under some restrictions which do not appear in the appendix of [KY]. Recall that the generalized cluster category associated with (Q,W), defined in [Ami09], is the localization of the category $\operatorname{per}\Gamma$ by the full subcategory $\mathcal{D}_{\mathrm{fd}}\Gamma$.

It is proved in [Ami09] that the canonical t-structure on $\mathcal{D}\Gamma$ restricts to a t-structure on $\operatorname{per}\Gamma$. We will denote this t-structure by $(\operatorname{per}^{\leq 0}, \operatorname{per}^{\geq 0})$.

Denote by $\mathcal{F}$ the full subcategory of $\operatorname{per}\Gamma$ defined by:

$$\mathcal{F} = \operatorname{per}^{\leq 0} \cap {}^{\perp}(\operatorname{per}^{\leq -2}).$$

Recall from [Ami09] that the canonical functor from $\operatorname{per}\Gamma$ to $\mathcal{C}_\Gamma$ induces a k-linear equivalence from $\mathcal{F}$ to $\mathcal{C}_\Gamma$ and that the functor $\tau_{\leq -1}$ induces an equivalence from $\mathcal{F}$ to $\Sigma\mathcal{F}$.

Fix an object T in $\mathcal{C}_\Gamma$. Without loss of generality, assume that T belongs to $\mathcal{F}$. Note that the canonical cluster tilting object $\Gamma \in \mathcal{C}_\Gamma$ does belong to $\mathcal{F}$.

Lemma 13. *Let X be an object of* $\operatorname{per}\Gamma$. *If X is left orthogonal to* $\operatorname{per}^{\leq -3}$, *which happens for instance when X is in $\mathcal{F}$ or in $\Sigma\mathcal{F}$, then there is a functorial isomorphism*

$$\operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq -1}T, X) \xrightarrow{\simeq} \mathcal{C}_\Gamma(T, X).$$

Proof. Let $X \in \operatorname{per}\Gamma$ be left orthogonal to $\operatorname{per}^{\leq -3}$. By [Ami09, Proposition 2.8], we have $\mathcal{C}_\Gamma(T, X) = \varinjlim \operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq n}T, \tau_{\leq n}X)$. Moreover, for any n, we have

$$\operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq n}T, \tau_{\leq n}X) = \operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq n}T, X).$$

Let $n < -1$. The object $\tau_{[n+1,-1]}T$ belongs to $\mathcal{D}_{\mathrm{fd}}(\Gamma)$ and X belongs to $\operatorname{per}\Gamma$, so that the 3-Calabi–Yau property (see [Kel08]) implies that the morphism space $\operatorname{Hom}_{\operatorname{per}\Gamma}(\Sigma^{-1}\tau_{[n+1,-1]}T, X)$ is isomorphic to the dual of $\operatorname{Hom}_{\operatorname{per}\Gamma}(X, \Sigma^2\tau_{[n+1,-1]}T)$. This latter vanishes since X belongs to ${}^\perp(\operatorname{per}^{\leq -3})$. The same argument shows that the space $\operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{[n+1,-1]}T, X)$ also vanishes. Therefore applying the functor $\operatorname{Hom}_{\operatorname{per}\Gamma}(?, X)$ to the triangle

$$\Sigma^{-1}\tau_{[n+1,-1]}T \longrightarrow \tau_{\leq n}T \longrightarrow \tau_{\leq -1}T \longrightarrow \tau_{[n+1,-1]}T,$$

yields an isomorphism $\operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq n}T, X) \xrightarrow{\simeq} \operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq -1}T, X)$. □

Lemma 14. *Let $X, Y \in \operatorname{per}\Gamma$ and assume that X belongs to ${}^\perp(\operatorname{per}^{\leq -3})$. Then the functor $\tau_{\geq -2}$ induces a bijection* $\operatorname{Hom}_{\operatorname{per}\Gamma}(X, Y) \simeq \operatorname{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{\geq -2}X, \tau_{\geq -2}Y)$.

Proof. By assumption, X is left orthogonal to the subcategory $\operatorname{per}^{\leq -3}$. Therefore, the space $\operatorname{Hom}_{\operatorname{per}\Gamma}(X, Y)$ is isomorphic to $\operatorname{Hom}_{\operatorname{per}\Gamma}(X, \tau_{\geq -2}Y)$, and thus to $\operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\geq -2}X, \tau_{\geq -2}Y)$. Since X and Y are perfect over Γ, their images under $\tau_{\geq -2}$ are quasi-isomorphic to dg modules of finite total dimension. □

Proposition 15. *Let Γ be the Ginzburg dg algebra associated with a Jacobi-finite quiver. Then the category $\mathcal{D}_{fd}(\Gamma)$ has constructible cones.*

Proof. We write $\mathfrak{n}$ for the ideal of Γ generated by the arrows of the Ginzburg quiver, and $\mathbf{p}$ for the left adjoint to the canonical functor $\mathcal{H}(\Gamma) \to \mathcal{D}(\Gamma)$. Let L, M and T be dg modules of finite total dimension. Since $\operatorname{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(L, \Sigma M)$ is finite dimensional, there exists a quasi-isomorphism $M \xrightarrow{w} M'$, where M' is of finite total dimension and such that any morphism $L \to \Sigma M$ may be represented by a fraction:

$$L \searrow \Sigma M' \xleftarrow{\Sigma w} \Sigma M.$$

We thus obtain a surjection $\operatorname{Ext}^1_{\mathcal{H}_{\mathrm{fd}}(\Gamma)}(L, M') \twoheadrightarrow \operatorname{Ext}^1_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(L, M)$. Fix a k-linear section s of this surjection. Choose m such that $M'\mathfrak{n}^m$ and $L\mathfrak{n}^m$ vanish. Then for the cone Y of any morphism from $\Sigma^{-1}M'$ to L, we have $Y\mathfrak{n}^m = 0$. For X being any one of L, M', Y we thus have isomorphisms

$$\mathcal{C}_\Gamma(T, X) \simeq \operatorname{Hom}_{\mathcal{H}(\Gamma)}(\mathbf{p}T, X) \simeq \operatorname{Hom}_{\mathcal{H}_{\mathrm{fd}}(\Gamma)}(T', X)$$

where T' denotes the finite dimensional quotient of $\mathbf{p}T$ by $(\mathbf{p}T)\mathfrak{n}^m$. The category $\mathcal{H}_{\mathrm{fd}}(\Gamma)$ is the stable category of a Hom-finite Frobenius category. By section 2.4, the category $\mathcal{H}_{\mathrm{fd}}(\Gamma)$ has constructible cones: There exists a constructible map $\varphi_{L,M'}$ (associated with T') as in section 1.3. By composing this map with the section s, we obtain a map $\varphi_{L,M}$ as required. □

Proposition 16. *Let Γ be the Ginzburg dg algebra associated with a Jacobi-finite quiver. Then the generalized cluster category $\mathcal{C}_\Gamma$ has constructible cones.*

Proof. Let L and M be in $\mathcal{C}_\Gamma$. Up to replacing them by isomorphic objects in $\mathcal{C}_\Gamma$, we may assume that L belongs to $\Sigma\mathcal{F}$ and M to $\mathcal{F}$. The projection then induces an isomorphism $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(L,\Sigma M) \xrightarrow{\simeq} \mathcal{C}_\Gamma(L,\Sigma M)$. Let ε be in $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(L,\Sigma M)$, and let $M \to Y \to L \xrightarrow{\varepsilon} \Sigma M$ be a triangle in $\mathrm{per}\,\Gamma$. Let us denote the sets of morphisms $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(\ ,\ )$ by $(\ ,\ )$. There is a commutative diagram

$$\begin{array}{ccccccccc}
(\tau_{\leq -1}T,\Sigma^{-1}L) & \to & (\tau_{\leq -1}T,M) & \to & (\tau_{\leq -1}T,Y) & \to & (\tau_{\leq -1}T,L) & \to & (\tau_{\leq -1}T,\Sigma M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}_\Gamma(T,\Sigma^{-1}L) & \to & \mathcal{C}_\Gamma(T,M) & \to & \mathcal{C}_\Gamma(T,Y) & \to & \mathcal{C}_\Gamma(T,L) & \to & \mathcal{C}_\Gamma(T,\Sigma M),
\end{array}$$

where the morphisms in the first two and in the last two columns are isomorphisms by lemma 13, and the middle one by the five lemma. Note that $\tau_{\leq -1}T$ belongs to $\Sigma\mathcal{F}$, so that, by lemma 14, we have isomorphisms:

$$\mathrm{Hom}_{\mathrm{per}\,\Gamma}(L,\Sigma M) \simeq \mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{\geq -2}L,\tau_{\geq -2}\Sigma M)$$

and

$$\mathcal{C}_\Gamma(T,X) \simeq \mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{[-2,-1]}T,\tau_{\geq -2}X)$$

for $X \in \left\{\Sigma^{-1}L, M, L, \Sigma M\right\}$ and thus also for X being the middle term of any triangle in $\mathrm{Ext}^1_{\mathrm{per}\,\Gamma}(L,M)$. Let $\varepsilon \in \mathcal{C}_\Gamma(L,\Sigma M)$ and let $M \to Y \to L \xrightarrow{\varepsilon} \Sigma M$ be a triangle in $\mathcal{C}_\Gamma$. Let $\overline{\varepsilon}$ be the morphism in $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{\geq -2}L,\tau_{\geq -2}\Sigma M)$ corresponding to ε and let $\tau_{\geq -2}M \to Z \to \tau_{\geq -2}L \xrightarrow{\overline{\varepsilon}} \tau_{\geq -2}\Sigma M$ be a triangle in $\mathcal{D}_{\mathrm{fd}}(\Gamma)$. Then the sequence obtained from $\Sigma^{-1}L \to M \to Y \to L \to \Sigma M$ by applying the functor $\mathcal{C}_\Gamma(T,?)$ is isomorphic to the one obtained from $\Sigma^{-1}\tau_{\geq -2}L \to \tau_{\geq -2}M \to Z \to \tau_{\geq -2}L \to \tau_{\geq -2}\Sigma M$ by applying the functor $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{[-2,-1]}T,?)$. By proposition 15, the cylinders of the morphisms $L \to \Sigma M$ are constructible with respect to T. □

3. Proof of theorem 1

Let T be a cluster tilting object of $\mathcal{C}$. Let L and M be two objects in $\mathcal{C}$, such that the cylinders of the morphisms $L \to \Sigma M$ and $M \to \Sigma L$ are constructible with respect to T. Let ε be a morphism in $\mathcal{C}(L,\Sigma M)_{\langle Y\rangle}$ for some $Y \in \mathcal{C}$, and let $M \xrightarrow{i} Y' \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ be a triangle in $\mathcal{C}$. The image of ε under $\varphi_{L,M}$ lifts the orbit of the matrix representation of $\overrightarrow{A_4}$ in $\mathrm{mod}\,B$ given by $FM \xrightarrow{Fi} FY' \xrightarrow{Fp} FL \xrightarrow{F\varepsilon} F\Sigma M$. In all of this section, we will take the liberty of denoting by Fi, Fp and FY' the image $\varphi_{L,M}(\varepsilon)$. Denote by Δ the dimension vector $\underline{\dim}FL + \underline{\dim}FM$. For any object Y in $\mathcal{C}$ and any non-negative e, f and g in $K_0(\mathrm{mod}\,B)$, let $W^Y_{LM}(e,f,g)$ be the subset of

$$\mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle Y\rangle} \times \coprod_{d\leq\Delta}\prod_{i=1}^{n}\mathrm{Gr}_{g_i}(k^{d_i})$$

formed by the pairs $([\varepsilon],E)$ such that E is a submodule of FY' of dimension vector g, $\underline{\dim}(Fp)E = e$ and $\underline{\dim}(Fi)^{-1}E = f$, where FY',Fi and Fp are given by $\varphi_{L,M}(\varepsilon)$. We let

- $W^Y_{LM}(g)$ denote the union of all $W^Y_{LM}(e,f,g)$ with $e \leq \underline{\dim}FL$ and $f \leq \underline{\dim}FM$ and
- $W^Y_{LM}(e,f)$ denote the union of all $W^Y_{LM}(e,f,g)$ with $g \leq \underline{\dim}FL+\underline{\dim}FM$.

Lemma 17. *The sets $W^Y_{LM}(e,f,g)$ are constructible.*

Proof. Denote by Δ the dimension vector $\underline{\dim} FL + \underline{\dim} FM$, and fix a dimension vector g. Consider the map induced by $\varphi_{L,M}$ which sends a pair (ε, E) in $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle} \times \coprod_{d \leq \Delta} \prod_{i \in Q_0} \mathrm{Gr}_{g_i}(k^{d_i})$ to (Fi, Fp, FY', E). By our assumption, this map (exists and) is constructible. Therefore, the subset of

$$\mathcal{C}(L, \Sigma M)_{\langle Y \rangle} \times \coprod_{d \leq \Delta} \prod_{i \in Q_0} \mathrm{Gr}_{g_i}(k^{d_i})$$

formed by the pairs (ε, E) such that E is a submodule of FY' is a constructible subset. We denote by $V_{LM}^Y(g)$ this constructible subset. We thus have a constructible function $V_{LM}^Y(g) \longrightarrow \mathbb{Z}^{2n}$ sending the pair (ε, E) to $(\underline{\dim}(Fi)^{-1}E, \underline{\dim}(Fp)E)$. This function induces a constructible function $\delta : W_{LM}^Y(g) \longrightarrow \mathbb{Z}^{2n}$, and the set $W_{LM}^Y(e, f, g)$ is the fiber of δ above (e, f). □

The fiber above the class $[\varepsilon]$ of the first projection $W_{LM}^Y(g) \to \mathbb{P}\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$ is $\{[\varepsilon]\} \times \mathrm{Gr}_g FY'$ and thus all fibers have Euler characteristics equal to that of $\mathrm{Gr}_g FY$. Therefore we have:

$$(**) \qquad \chi\big(W_{LM}^Y(g)\big) = \chi\big(\mathbb{P}\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}\big) \chi(\mathrm{Gr}_g FY).$$

Define $L(e, f)$ to be the variety $\mathbb{P}\mathcal{C}(L, \Sigma M) \times \mathrm{Gr}_e FL \times \mathrm{Gr}_f FM$. Consider the following map:

$$\begin{array}{ccc} \coprod_{Y \in \mathcal{Y}} W_{LM}^Y(e, f) & \stackrel{\psi}{\longrightarrow} & L(e, f) \\ ([\varepsilon], E) & \longmapsto & \big([\varepsilon], (Fp)E, (Fi)^{-1}E\big). \end{array}$$

By our assumption, the map ψ is constructible.

Let $L_1(e, f)$ be the subvariety of $L(e, f)$ formed by the points in the image of ψ, and let $L_2(e, f)$ be the complement of $L_1(e, f)$ in $L(e, f)$.

We want to compute

$$\begin{aligned} \dim \mathcal{C}(L, \Sigma M) X_L X_M &= \underline{x}^{-\operatorname{coind}(L \oplus M)} \sum_{e,f} \chi(L(e, f)) \prod_{i=1}^n x_i^{\langle S_i, e+f \rangle_a} \\ &= \sum_{e,f} \chi(L_1(e, f)) \underline{x}^{-\operatorname{coind}(L \oplus M)} \prod_{i=1}^n x_i^{\langle S_i, e+f \rangle_a} \\ &\quad + \sum_{e,f} \chi(L_2(e, f)) \underline{x}^{-\operatorname{coind}(L \oplus M)} \prod_{i=1}^n x_i^{\langle S_i, e+f \rangle_a}. \end{aligned}$$

Denote by s_1 (resp. s_2) the first term (resp. second term) in the right hand side of the last equality above.

As shown in [CC06], the fibers of ψ over $L_1(e, f)$ are affine spaces. For the convenience of the reader, we sketch a proof. Let $([\varepsilon], U, V)$ be in $L_1(e, f)$. Denote by Y the middle term of ε and by $\mathrm{Gr}_{U,V}$ the projection of the fiber $\psi^{-1}([\varepsilon], U, V)$ on the second factor $\mathrm{Gr}\, FY$. Let W be a cokernel of the injection of U in FM.

$$\begin{array}{ccccccc} W & & & & & & \\ \uparrow{\scriptstyle \pi} & & & & & & \\ FM & \xrightarrow{i} & FY & \xrightarrow{p} & FL & \longrightarrow & F\Sigma M \\ {\scriptstyle i_U}\uparrow & & \uparrow & & \uparrow{\scriptstyle i_V} & & \\ U & \longrightarrow & E & \twoheadrightarrow & V & & \end{array}$$

Lemma 18. (Caldero–Chapoton) *There is a bijection* $\mathrm{Hom}_B(V, W) \longrightarrow \mathrm{Gr}_{U,V}$.

Proof. Define a free transitive action of $\operatorname{Hom}_B(V,W)$ on $\operatorname{Gr}_{U,V}$ in the following way: For any E in $\operatorname{Gr}_{U,V}$ and any g in $\operatorname{Hom}_B(V,W)$, define E_g to be the submodule of FY of elements of the form $i(m)+x$ where m belongs to FM, x belongs to E and $gpx=\pi m$. Note that E_g belongs to $\operatorname{Gr}_{U,V}$ (since the kernel of i is included in U), that $E_0=E$ and that $(E_g)_h=E_{g+h}$. This action is free: An element $i(m)+x$ is in E if and only if m is in U. This is equivalent to the vanishing of πm, which in turn is equivalent to px belonging to the kernel of g. This action is transitive: Let E and E' be in $\operatorname{Gr}_{U,V}$. For any v in V, let $g(v)$ be $\pi(x'-x)$ where $x\in E$, $x'\in E'$ and $px=px'=v$. This defines a map $g:V\longrightarrow W$ such that $E_g=E'$. □

By lemma 18, we obtain the following equality between the Euler characteristics:

$$\sum_{\langle Y\rangle}\chi(W_{LM}^Y(e,f))=\chi(L_1(e,f)),$$

which implies the equality

$$s_1=\sum_{e,f,\langle Y\rangle}\chi\big(W_{LM}^Y(e,f)\big)\underline{x}^{-\operatorname{coind}(L\oplus M)}\prod_{i=1}^n x_i^{\langle S_i,e+f\rangle_a}.$$

If the pair $([\varepsilon],E)$ belongs to $W_{LM}^Y(e,f,g)$, then by [Pal08, lemma 5.1], we have

$$\sum_{i=1}^n\langle S_i,e+f\rangle_a[P_i]-\operatorname{coind}(L\oplus M)=\sum_{i=1}^n\langle S_i,g\rangle_a[P_i]-\operatorname{coind}(\operatorname{mt}(\varepsilon))$$

and $\operatorname{coind}(\operatorname{mt}(\varepsilon))=\operatorname{coind}Y$ since the morphism ε is in $\mathcal{C}(L,\Sigma M)_{\langle Y\rangle}$. Therefore,

$$\begin{aligned}
s_1 &= \sum_{e,f,g,\langle Y\rangle}\chi\big(W_{LM}^Y(e,f,g)\big)\underline{x}^{-\operatorname{coind}(L\oplus M)}\prod_{i=1}^n x_i^{\langle S_i,e+f\rangle_a}\\
&= \sum_{e,f,g,\langle Y\rangle}\chi\big(W_{LM}^Y(e,f,g)\big)\underline{x}^{-\operatorname{coind}Y}\prod_{i=1}^n x_i^{\langle S_i,g\rangle_a}\\
&= \sum_{g,\langle Y\rangle}\chi\big(W_{LM}^Y(g)\big)\underline{x}^{-\operatorname{coind}Y}\prod_{i=1}^n x_i^{\langle S_i,g\rangle_a}\\
&= \sum_{\langle Y\rangle}\sum_{g}\chi\big(\mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle Y\rangle}\big)\chi(\operatorname{Gr}_g FY)\underline{x}^{-\operatorname{coind}Y}\prod_{i=1}^n x_i^{\langle S_i,g\rangle_a}\quad\text{by }(**)\\
&= \sum_{\langle Y\rangle}\chi\big(\mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle Y\rangle}\big)X_Y.
\end{aligned}$$

Recall that since $\mathcal{C}$ is 2-Calabi–Yau, there is an isomorphism

$$\phi_{L,M}:\mathcal{C}(\Sigma^{-1}L,M)\longrightarrow D\mathcal{C}(M,\Sigma L).$$

We denote by ϕ the induced duality pairing:

$$\begin{aligned}
\phi:\mathcal{C}(\Sigma^{-1}L,M)\times\mathcal{C}(M,\Sigma L) &\longrightarrow k\\
(a,b) &\longmapsto \phi_{L,M}(a)b.
\end{aligned}$$

Let $C_{e,f}(Y,g)$ consist of all pairs $\big(([\varepsilon],U,V),([\eta],E)\big)$ in $L_2(e,f)\times W_{ML}^Y(g)$ such that $\phi(\Sigma^{-1}\varepsilon,\eta)\neq 0$, $(Fi)^{-1}E=V$ and $(Fp)E=U$, where Fi, Fp are given by $\varphi_{M,L}(\eta)$. The set $C_{e,f}(Y,g)$ is constructible, by our assumption. Let $C_{e,f}$ be the union of all $C_{e,f}(Y,g)$, where Y runs through the set of representatives $\mathcal{Y}$, and g

through $K_0(\operatorname{mod} B)$. We then consider the following two projections

$$\begin{array}{ccc} C_{e,f} & \text{and} & C_{e,f}(Y,g) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ L_2(e,f) & & W^Y_{ML}(f,e,g). \end{array}$$

The aim of the next proposition is to show that the projections p_1 and p_2 are surjective, and to describe their fibers.

Let U be in $\operatorname{Gr}_e FL$, and V be in $\operatorname{Gr}_f FM$. Let $U \xrightarrow{i_U} L$ and $V \xrightarrow{i_V} M$ lift these two inclusions to the triangulated category $\mathcal{C}$. As in section 4 of [Pal08], let us consider the following two morphisms: α from $\mathcal{C}(\Sigma^{-1}L, U) \oplus \mathcal{C}(\Sigma^{-1}L, M)$ to $\mathcal{C}/(T)\big(\Sigma^{-1}V, U\big) \oplus (\Sigma^{-1}V, M) \oplus \mathcal{C}/(\Sigma T)\big(\Sigma^{-1}L, M\big)$ and

$$\alpha' : (\Sigma T)(U, \Sigma V) \oplus \mathcal{C}(M, \Sigma V) \oplus (\Sigma^2 T)(M, \Sigma L) \longrightarrow \mathcal{C}(U, \Sigma L) \oplus \mathcal{C}(M, \Sigma L)$$

defined by:

$$\alpha(a,b) = (a\Sigma^{-1}i_V, i_U a \Sigma^{-1} i_V - b\Sigma^{-1}i_V, i_U a - b)$$

and

$$\alpha'(a,b,c) = \Big((\Sigma i_V)a + c\, i_U + (\Sigma i_V) b\, i_U, -c - (\Sigma i_V) b\Big).$$

Remark that the maps α and α' are dual to each other via the pairing ϕ. In the following lemma, orthogonal means orthogonal with respect to this pairing.

Proposition 19. [CK08, proposition 3] *With the same notations as above, the following assertions are equivalent:*

(i) *The triple $([\varepsilon], U, V)$ belongs to $L_2(e,f)$.*
(ii) *The morphism $\Sigma^{-1}\varepsilon$ is not orthogonal to $\mathcal{C}(M, \Sigma L) \cap \operatorname{Im} \alpha'$.*
(iii) *There is an $\eta \in \mathcal{C}(M, \Sigma L)$ such that $\phi(\Sigma^{-1}\varepsilon, \eta) \neq 0$ and such that if*

$$L \xrightarrow{i} N \xrightarrow{p} M \xrightarrow{\eta} \Sigma L$$

is a triangle in $\mathcal{C}$, then there exists $E \in \operatorname{Gr} FN$ with $(Fi)^{-1}E = V$ and $(Fp)E = U$.

Proof. Let us start with the equivalence of (i) and (ii). The same proof as that in [CK08, proposition 3] applies in this setup: Denote by p the canonical projection of $\mathcal{C}(\Sigma^{-1}L, U) \oplus \mathcal{C}(\Sigma^{-1}L, M)$ onto $\mathcal{C}(\Sigma^{-1}L, M)$. Then, by [Pal08, lemma 4.2], assertion (i) is equivalent to $\Sigma^{-1}\varepsilon$ not belonging to $p(\operatorname{Ker}\alpha)$. That is, the morphism $\Sigma^{-1}\varepsilon$ is not in the image of the composition:

$$q : \operatorname{Ker}\alpha \longrightarrow \mathcal{C}(\Sigma^{-1}L, U) \oplus \mathcal{C}(\Sigma^{-1}L, M) \longrightarrow \mathcal{C}(\Sigma^{-1}L, M).$$

So (i) holds if and only if $\Sigma^{-1}\varepsilon$ is not in the orthogonal of the orthogonal of the image of q. The orthogonal of the image of q is the kernel of its dual, which is given by the composition:

$$\mathcal{C}(M, \Sigma L) \longrightarrow \mathcal{C}(U, \Sigma L) \oplus \mathcal{C}(M, \Sigma L) \longrightarrow \operatorname{Coker}\alpha'.$$

Therefore assertion (i) is equivalent to the morphism $\Sigma^{-1}\varepsilon$ not being in the orthogonal of $\mathcal{C}(M, \Sigma L) \cap \operatorname{Im}\alpha'$ which proves that (i) and (ii) are equivalent.

By [Pal08, lemma 4.2], a morphism in $\mathcal{C}(M, \Sigma L)$ is in the image of α' if and only if it satisfies the second condition in (iii). Therefore (ii) and (iii) are equivalent. □

A variety X is called an *extension of affine spaces* in [CK08] if there is a vector space V and a surjective morphism $X \longrightarrow V$ whose fibers are affine spaces of constant dimension. Note that extensions of affine spaces have Euler characteristics equal to 1.

Proposition 20. [CK08, proposition 4]

a) *The projection* $C_{e,f} \xrightarrow{p_1} L_2(e,f)$ *is surjective and its fibers are extensions of affine spaces.*
b) *The projection* $C_{e,f}(Y,g) \xrightarrow{p_2} W^Y_{ML}(f,e,g)$ *is surjective and its fibers are affine spaces.*
c) *If* $C_{e,f}(Y,g)$ *is not empty, then we have*

$$\sum_{i=1}^n \langle S_i, e+f\rangle_a [P_i] - \operatorname{coind}(L\oplus M) = \sum_{i=1}^n \langle S_i, g\rangle_a [P_i] - \operatorname{coind} Y.$$

Proof. Let us first prove assertion a). The projection p_1 is surjective by the equivalence of i) and iii) in proposition 19. Let X be the fiber of p_1 above some $([\varepsilon], U, V)$ in $L_2(e,f)$. Let V be the set of all classes $[\eta]$ in $\mathbb{P}\big(\mathcal{C}(M,\Sigma L)\cap \operatorname{Im}\alpha'\big)$ such that $\phi(\Sigma^{-1}\varepsilon, \eta)$ does not vanish. The set V is the projectivization of the complement in $\mathcal{C}(M,\Sigma L)\cap \operatorname{Im}\alpha'$ of the hyperplane $\operatorname{Ker}\phi(\Sigma^{-1}\varepsilon,\ )$. Hence V is a vector space. Let us consider the projection $\pi : X \longrightarrow V$. This projection is surjective by [Pal08, lemma 4.2]. Let η represent a class in V, and let Fi, Fp be given by $\varphi_{M,L}(\eta)$. Then the fiber of π above $[\eta]$ is given by the submodules E of FY such that $(Fi)^{-1}E = V$ and $(Fp)E = U$. Lemma 18 thus shows that the fibers of π are affine spaces of constant dimension.

Let us prove assertion b). Let $([\eta], E)$ be in $W^Y_{ML}(f,e,g)$. The fiber of p_2 above $([\eta], E)$ consists of the elements of the form $\big(([\varepsilon], U, V), ([\eta], E)\big)$ where U and V are fixed submodules given by $[\eta]$ and E, and $[\varepsilon] \in \mathbb{P}\mathcal{C}(L,\Sigma M)$ is such that $\phi(\Sigma^{-1}\varepsilon, \eta)$ does not vanish. Therefore the projection p_2 is surjective and its fibers are affine spaces.

To prove assertion c), apply [Pal08, lemma 5.1] and remark that if Y' belongs to $\langle Y\rangle$, then Y' and Y have the same coindex. □

As a consequence, we obtain the following equalities:

$$\chi(C_{e,f}) = \chi(L_2(e,f)) \text{ and } \chi\big(C_{e,f}(Y,g)\big) = \chi\big(W^Y_{ML}(f,e,g)\big).$$

We are now able to compute s_2 :

$$\begin{aligned}
s_2 &= \sum_{e,f} \chi(L_2(e,f))\underline{x}^{-\operatorname{coind}(L\oplus M)} \prod_{i=1}^n x_i^{\langle S_i, e+f\rangle_a} \\
&= \sum_{e,f} \chi(C_{e,f})\underline{x}^{-\operatorname{coind}(L\oplus M)} \prod_{i=1}^n x_i^{\langle S_i, e+f\rangle_a} \text{ by 20 a)} \\
&= \sum_{e,f,g,\langle Y\rangle} \chi\big(C_{e,f}(Y,g)\big)\underline{x}^{-\operatorname{coind}(L\oplus M)} \prod_{i=1}^n x_i^{\langle S_i, e+f\rangle_a} \\
&= \sum_{e,f,g,\langle Y\rangle} \chi\big(C_{e,f}(Y,g)\big)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^n x_i^{\langle S_i, g\rangle_a} \text{ by 20 c)} \\
&= \sum_{e,f,g,\langle Y\rangle} \chi\big(W^Y_{ML}(f,e,g)\big)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^n x_i^{\langle S_i, g\rangle_a} \text{ by 20 b)} \\
&= \sum_{g,\langle Y\rangle} \chi\big(W^Y_{ML}(g)\big)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^n x_i^{\langle S_i, g\rangle_a} \\
&= \sum_{g,\langle Y\rangle} \chi\big(\mathbb{P}\mathcal{C}(M,\Sigma L)_{\langle Y\rangle}\big)\chi(\operatorname{Gr}_g FY)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^n x_i^{\langle S_i, g\rangle_a} \text{ by } (**) \\
&= \sum_{\langle Y\rangle} \chi\big(\mathbb{P}\mathcal{C}(M,\Sigma L)_{\langle Y\rangle}\big) X_Y. \quad \square
\end{aligned}$$

4. Fu–Keller's cluster character

In this section, it is proven that the cluster character X' defined by C. Fu and B. Keller, in [FK10], satisfies a multiplication formula similar to that of [GLS07]. Note that the notations used here differ from those of [FK10].

Let k be the field of complex numbers and let $\mathcal{E}$ be a k-linear Hom-finite Frobenius category with split idempotents. Assume that its stable category $\mathcal{C}$ is 2-Calabi–Yau and that $\mathcal{E}$ contains a cluster tilting object T. Denote the endomorphism algebra $\operatorname{End}_{\mathcal{E}}(T)$ by A and recall that the algebra $\operatorname{End}_{\mathcal{C}}(T)$ is denoted by B. The object T is assumed to be basic with indecomposable summands $T_1, \ldots, T_n$ where the projective-injective ones are precisely $T_{r+1}, \ldots, T_n$. For $i = 1, \ldots, n$, we denote by S_i the simple top of the projective A-module $\mathcal{E}(T, T_i)$. The modules in $\operatorname{mod} B$ are identified with the modules in $\operatorname{mod} A$ without composition factors isomorphic to one of the S_i, $r < i \leq n$.

4.1. **Statement of the multiplication formula.** We first recall the definition of X' from [FK10]. For any two finitely generated A-modules L and M, put

$$\langle L\,,\, M\rangle_3 = \sum_{i=0}^{3} (-1)^i \dim_k \operatorname{Ext}^i_A(L, M).$$

Proposition [Fu-Keller]: *If $L, M \in \operatorname{mod} B$ have the same image in $K_0(\operatorname{mod} A)$, then we have*

$$\langle L\,,\, Y\rangle_3 = \langle M\,,\, Y\rangle_3$$

for all finitely generated A-module Y.

Therefore, the number $\langle L\,,\, S_i\rangle_3$ only depends on the dimension vector of L, for $L \in \operatorname{mod} B$. Put $\langle \underline{\dim} L\,,\, S_i\rangle_3 = \langle L\,,\, S_i\rangle_3$, for $i = 1, \ldots, n$.

Recall that F is the functor $\mathcal{C}(T, ?)$ from the category $\mathcal{C}$ to $\operatorname{mod} B$. Denote by G the functor $F\Sigma \simeq \operatorname{Ext}^1_{\mathcal{E}}(T, ?)$. For $a \in K_0(\operatorname{proj} A)$, the notation $\underline{x}^a$ stands for the product $\prod x_i^{a_i}$, where a_i is the multiplicity of $[P_i]$ in a.

Proposition [Fu-Keller] *For $M \in \mathcal{E}$, define the Laurent polynomial*

$$X'_M = \underline{x}^{\operatorname{ind} M} \sum_{e \in K_0(\operatorname{mod} A)} \chi(\operatorname{Gr}_e(GM)) \prod_{i=1}^{n} x_i^{-\langle e\,,\, S_i\rangle}.$$

Note that the A-module GM does not have composition factors isomorphic to one of the S_i, $r < i \leq n$, so that the sum might as well be taken over the Grothendieck group $K_0(\operatorname{mod} B)$.

Theorem [Fu-Keller] *The map $M \mapsto X'_M$ is a cluster character on $\mathcal{E}$. It sends T_i to x_i, for all $i = 1, \ldots, n$*

For a class $\varepsilon \in \operatorname{Ext}^1_{\mathcal{E}}(L, M)$, let $\operatorname{mt}(\varepsilon)$ be the middle term of a conflation which represents ε.

Theorem 21. *For all $L, M \in \mathcal{E}$, we have*

$$\chi(\mathbb{P}\operatorname{Ext}^1_{\mathcal{E}}(L, M)) X'_L X'_M = \int_{[\varepsilon] \in \mathbb{P}\operatorname{Ext}^1_{\mathcal{E}}(L,M)} X'_{\operatorname{mt}(\varepsilon)} + \int_{[\varepsilon] \in \mathbb{P}\operatorname{Ext}^1_{\mathcal{E}}(M,L)} X'_{\operatorname{mt}(\varepsilon)}.$$

The proof of this theorem is postponed to section 4.3. The next section is dedicated to proving that the map sending $[\varepsilon]$ to $X'_{\operatorname{mt}(\varepsilon)}$ is "integrable with respect to χ".

4.2. **Constructibility.** Let L, M be objects in $\mathcal{E}$. For $\varepsilon \in \mathcal{C}(L, \Sigma M)$, the middle term in $\mathcal{E}$ of the class in $\mathrm{Ext}^1_{\mathcal{E}}(L, M)$ corresponding to ε will be denoted by $\mathrm{mt}(\varepsilon)$. Note that this notation does not coincide with the one in section 1. Nevertheless, those two definitions yield objects which are isomorphic in $\mathcal{C}$.

Lemma 22. *The map*

$$\lambda : \mathcal{C}(L, \Sigma M) \longrightarrow K_0(\mathrm{proj}\, A)$$

which sends ε *to the index of* $\mathrm{mt}(\varepsilon)$ *is constructible.*

Proof. Let $\varepsilon \in \mathcal{C}(L, \Sigma M)$, and let $M \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} L$ be a conflation whose class in $\mathrm{Ext}^1_{\mathcal{E}}(L, M)$ corresponds to ε. Let d be the dimension vector of $\mathrm{Coker}\, \mathcal{E}(T, p)$. By the proof of [FK10, lemma 3.4], we have

$$\mathrm{ind}\, Y = \mathrm{ind}(L \oplus M) - \sum_{i=1}^{n} \langle d\, ,\, S_i \rangle_3 [P_i].$$

Thanks to lemma 10 and lemma 4, the formula above shows that the map λ is constructible. □

If an object Y occurs as $\mathrm{mt}(\varepsilon)$ for some $\varepsilon \in \mathrm{Ext}^1_{\mathcal{E}}(L, M)$, we let $\langle\langle Y \rangle\rangle$ denote the set of all isomorphism classes of objects $Y' \in \mathcal{E}$ such that:

- Y' is the middle term of some conflation in $\mathrm{Ext}^1_{\mathcal{E}}(L, M)$,
- $\mathrm{ind}\, Y' = \mathrm{ind}\, Y$ and
- for all e in $K_0(\mathrm{mod}\, B)$, we have $\chi(\mathrm{Gr}_e\, GY') = \chi(\mathrm{Gr}_e\, GY)$.

We denote the set of all $\varepsilon \in \mathcal{C}(L, \Sigma M)$ with $\mathrm{mt}(\varepsilon) \in \langle\langle Y \rangle\rangle$ by $\mathcal{C}(L, \Sigma M)_{\langle\langle Y \rangle\rangle}$. Note that the set $\mathcal{C}(L, \Sigma M)_{\langle\langle Y \rangle\rangle}$ is a (constructible) subset of $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$.

As for proposition 9, the following proposition follows easily from corollary 6 and the previous lemma.

Proposition 23. *The sets* $\mathcal{C}(L, \Sigma M)_{\langle\langle Y \rangle\rangle}$ *are constructible subsets of* $\mathcal{C}(L, \Sigma M)$. *Moreover, the set* $\mathcal{C}(L, \Sigma M)$ *is a finite disjoint union of such constructible subsets.*

This proposition shows that the right hand side in the multiplication formula of theorem 21 is well-defined.

4.3. **Proof of theorem 21.** The proof is essentially the same as that in section 3, where the use of [Pal08, lemma 5.1] is replaced by that of [FK10, lemma 3.4].

Let L and M be two objects in $\mathcal{E}$. Let ε be a morphism in $\mathcal{C}(L, \Sigma M)_{\langle\langle Y \rangle\rangle}$ for some $Y \in \mathcal{E}$, and let $M \xrightarrow{-i} Y' \xrightarrow{-p} L \xrightarrow{-\varepsilon} \Sigma M$ be a triangle in $\mathcal{C}$. Remark that, by section 2.4, the category $\mathcal{C}$ has constructible cones. The image of $\Sigma\varepsilon$ under $\varphi_{\Sigma L, \Sigma M}$ lifts the orbit of the matrix representation of $\overrightarrow{A_4}$ in $\mathrm{mod}\, B$ given by $GM \xrightarrow{Gi} GY' \xrightarrow{Gp} GL \xrightarrow{G\varepsilon} G\Sigma M$. In all of this section, we will take the liberty of denoting by Gi, Gp and GY' the image $\varphi_{\Sigma L, \Sigma M}(\Sigma\varepsilon)$. Denote by Δ the dimension vector $\underline{\dim} GL + \underline{\dim} GM$. For any non-negative e, f and g in $K_0(\mathrm{mod}\, B)$, let $\mathcal{W}^Y_{LM}(e, f, g)$ be the subset of

$$\mathbb{P}\mathcal{C}(L, \Sigma M)_{\langle\langle Y \rangle\rangle} \times \coprod_{d \leq \Delta} \prod_{i=1}^{n} \mathrm{Gr}_{g_i}(k^{d_i})$$

formed by the pairs $([\varepsilon], E)$ such that E is a submodule of GY' of dimension vector g, $\underline{\dim}(Gp)E = e$ and $\underline{\dim}(Gi)^{-1}E = f$. We let

- $\mathcal{W}^Y_{LM}(g)$ denote the union of all $\mathcal{W}^Y_{LM}(e, f, g)$ with $e \leq \underline{\dim} GL$ and $f \leq \underline{\dim} GM$ and
- $\mathcal{W}^Y_{LM}(e, f)$ denote the union of all $\mathcal{W}^Y_{LM}(e, f, g)$ with $g \leq \underline{\dim} GL + \underline{\dim} GM$.

Note that, by lemma 17, the sets $\mathcal{W}_{LM}^Y(e,f,g)$ are constructible.

The fiber above the class $[\varepsilon]$ of the first projection $\mathcal{W}_{LM}^Y(g) \to \mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle\langle Y\rangle\rangle}$ is $\{[\varepsilon]\} \times \mathrm{Gr}_g\, GY'$ and thus all fibers have Euler characteristics equal to that of $\mathrm{Gr}_g\, GY$. Therefore we have:

$$(**) \qquad \chi\big(\mathcal{W}_{LM}^Y(g)\big) = \chi\big(\mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle\langle Y\rangle\rangle}\big)\chi(\mathrm{Gr}_g\, GY).$$

Define $\mathcal{L}(e,f)$ to be the variety $\mathbb{P}\mathcal{C}(L,\Sigma M) \times \mathrm{Gr}_e\, GL \times \mathrm{Gr}_f\, GM$. Consider the following map:

$$\begin{array}{ccc} \coprod_{\langle\langle Y\rangle\rangle} \mathcal{W}_{LM}^Y(e,f) & \xrightarrow{\psi} & \mathcal{L}(e,f) \\ ([\varepsilon],E) & \longmapsto & \big([\varepsilon],(Gp)E,(Gi)^{-1}E\big). \end{array}$$

By lemma 10, the map ψ is constructible.

As in section 3, let $\mathcal{L}_1(e,f)$ be the subvariety of $\mathcal{L}(e,f)$ formed by the points in the image of ψ, and let $\mathcal{L}_2(e,f)$ be the complement of $\mathcal{L}_1(e,f)$ in $\mathcal{L}(e,f)$.

Using the notations above, we have

$$\mathbb{P}\mathcal{C}(L,\Sigma M)X'_L X'_M = \underline{x}^{\mathrm{ind}(L\oplus M)} \sum_{e,f} \chi(\mathcal{L}(e,f)) \prod_{i=1}^n x_i^{-\langle e+f,S_i\rangle_3}$$

For $j=1,2$, denote by σ_j the term

$$\underline{x}^{\mathrm{ind}(L\oplus M)} \sum_{e,f} \chi(\mathcal{L}_j(e,f)) \prod_{i=1}^n x_i^{-\langle e+f,S_i\rangle_3}$$

so that $\mathbb{P}\mathcal{C}(L,\Sigma M)X'_L X'_M = \sigma_1 + \sigma_2$.

As shown in [CC06], the fibers of ψ over $\mathcal{L}_1(e,f)$ are affine spaces (see lemma 18). Therefore we have the following equality between Euler characteristics:

$$\sum_{\langle\langle Y\rangle\rangle} \chi(\mathcal{W}_{LM}^Y(e,f)) = \chi(\mathcal{L}_1(e,f)),$$

which implies the equality

$$\sigma_1 = \sum_{e,f,\langle\langle Y\rangle\rangle} \chi\big(\mathcal{W}_{LM}^Y(e,f)\big)\underline{x}^{\mathrm{ind}(L\oplus M)} \prod_{i=1}^n x_i^{-\langle e+f,S_i\rangle_3}.$$

If the pair $([\varepsilon],E)$ belongs to $\mathcal{W}_{LM}^Y(e,f,g)$, then by [FK10, lemma 3.4], we have

$$\mathrm{ind}(L\oplus M) - \sum_{i=1}^n \langle e+f,S_i\rangle_3[P_i] = \mathrm{ind}(\mathrm{mt}(\varepsilon)) - \sum_{i=1}^n \langle g,S_i\rangle_3[P_i]$$

and $\operatorname{ind}(\operatorname{mt}(\varepsilon)) = \operatorname{ind} Y$ since the morphism ε is in $\mathcal{C}(L, \Sigma M)_{\langle\langle Y \rangle\rangle}$. Therefore,

$$
\begin{aligned}
\sigma_1 &= \sum_{e,f,g,\langle\langle Y\rangle\rangle} \chi\big(\mathcal{W}_{LM}^{Y}(e,f,g)\big)\underline{x}^{\operatorname{ind}(L\oplus M)} \prod_{i=1}^{n} x_i^{-\langle e+f, S_i\rangle_3} \\
&= \sum_{e,f,g,\langle\langle Y\rangle\rangle} \chi\big(\mathcal{W}_{LM}^{Y}(e,f,g)\big)\underline{x}^{\operatorname{ind} Y} \prod_{i=1}^{n} x_i^{-\langle g, S_i\rangle_3} \\
&= \sum_{g,\langle\langle Y\rangle\rangle} \chi\big(\mathcal{W}_{LM}^{Y}(g)\big)\underline{x}^{\operatorname{ind} Y} \prod_{i=1}^{n} x_i^{-\langle g, S_i\rangle_3} \\
&= \sum_{\langle\langle Y\rangle\rangle} \sum_{g} \chi\big(\mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle\langle Y\rangle\rangle}\big)\chi(\operatorname{Gr}_g GY)\underline{x}^{\operatorname{ind} Y} \prod_{i=1}^{n} x_i^{-\langle g, S_i\rangle_3} \text{ by } (**) \\
&= \sum_{\langle\langle Y\rangle\rangle} \chi\big(\mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle\langle Y\rangle\rangle}\big) X'_Y.
\end{aligned}
$$

Recall that we denote by ϕ the duality pairing:

$$
\begin{array}{rcl}
\phi : \mathcal{C}(\Sigma^{-1}L, M) \times \mathcal{C}(M, \Sigma L) & \longrightarrow & k \\
(a,b) & \longmapsto & \phi_{L,M}(a)b
\end{array}
$$

induced by the 2-Calabi–Yau property of $\mathcal{C}$.

Let $\mathcal{C}_{e,f}(Y,g)$ consist of all pairs $\big(([\varepsilon],U,V),([\eta],E)\big)$ in $\mathcal{L}_2(e,f) \times \mathcal{W}_{ML}^{Y}(g)$ such that $\phi(\Sigma^{-1}\varepsilon,\eta) \neq 0$, $(G\iota)^{-1}E = V$ and $(G\pi)E = U$ (where $G\iota$, $G\pi$ are given by $\varphi_{\Sigma M,\Sigma L}(\Sigma\eta)$). The set $\mathcal{C}_{e,f}(Y,g)$ is constructible, by lemma 10. Let $\mathcal{C}_{e,f}$ be the union of all $\mathcal{C}_{e,f}(Y,g)$, where Y runs through a set of representatives for the classes $\langle\langle Y\rangle\rangle$, and g through $K_0(\operatorname{mod} B)$. We then consider the following two projections

$$
\begin{array}{ccc}
\mathcal{C}_{e,f} & \quad\text{and}\quad & \mathcal{C}_{e,f}(Y,g) \\
\Big\downarrow{\scriptstyle \rho_1} & & \Big\downarrow{\scriptstyle \rho_2} \\
\mathcal{L}_2(e,f) & & \mathcal{W}_{ML}^{Y}(f,e,g).
\end{array}
$$

As shown in section 3, the projections ρ_1 and ρ_2 are surjective. Moreover, by [FK10, lemma 3.4], if $\mathcal{C}_{e,f}(Y,g)$ is not empty, then we have

$$
\operatorname{ind}(L \oplus M) - \sum_{i=1}^{n} \langle e+f, S_i\rangle_3 [P_i] = \operatorname{ind} Y - \sum_{i=1}^{n} \langle g, S_i\rangle_3 [P_i].
$$

As a consequence, we obtain the following equalities:

$$
\chi(\mathcal{C}_{e,f}) = \chi(\mathcal{L}_2(e,f)) \text{ and } \chi\big(\mathcal{C}_{e,f}(Y,g)\big) = \chi\big(\mathcal{W}_{ML}^{Y}(f,e,g)\big).
$$

We are now able to compute σ_2 :

$$
\begin{aligned}
\sigma_2 &= \sum_{e,f} \chi(\mathcal{L}_2(e,f))\underline{x}^{\operatorname{ind}(L\oplus M)} \prod_{i=1}^{n} x_i^{-\langle e+f,S_i\rangle_3} \\
&= \sum_{e,f} \chi(\mathcal{C}_{e,f})\underline{x}^{\operatorname{ind}(L\oplus M)} \prod_{i=1}^{n} x_i^{-\langle e+f,S_i\rangle_3} \\
&= \sum_{e,f,g,\langle\langle Y\rangle\rangle} \chi\big(\mathcal{C}_{e,f}(Y,g)\big)\underline{x}^{\operatorname{ind}(L\oplus M)} \prod_{i=1}^{n} x_i^{-\langle e+f,S_i\rangle_3} \\
&= \sum_{e,f,g,\langle\langle Y\rangle\rangle} \chi\big(\mathcal{C}_{e,f}(Y,g)\big)\underline{x}^{\operatorname{ind} Y} \prod_{i=1}^{n} x_i^{-\langle g,S_i\rangle_3} \\
&= \sum_{e,f,g,\langle\langle Y\rangle\rangle} \chi\big(\mathcal{W}_{ML}^{Y}(f,e,g)\big)\underline{x}^{\operatorname{ind} Y} \prod_{i=1}^{n} x_i^{-\langle g,S_i\rangle_3} \\
&= \sum_{g,\langle\langle Y\rangle\rangle} \chi\big(\mathcal{W}_{ML}^{Y}(g)\big)\underline{x}^{\operatorname{ind} Y} \prod_{i=1}^{n} x_i^{-\langle g,S_i\rangle_3} \\
&= \sum_{g,\langle\langle Y\rangle\rangle} \chi\big(\mathbb{P}\mathcal{C}(M,\Sigma L)_{\langle\langle Y\rangle\rangle}\big)\chi(\operatorname{Gr}_g GY)\underline{x}^{\operatorname{ind} Y} \prod_{i=1}^{n} x_i^{-\langle g,S_i\rangle_3} \\
&= \sum_{\langle\langle Y\rangle\rangle} \chi\big(\mathbb{P}\mathcal{C}(M,\Sigma L)_{\langle\langle Y\rangle\rangle}\big)X'_Y.
\end{aligned}
$$

□

References

[Ami09] Claire Amiot. Cluster categories for algebras of global dimension 2 and quivers with potential. *Ann. Inst. Fourier (Grenoble)*, 59(6):2525–2590, 2009.

[BIRS09] A. B. Buan, O. Iyama, I. Reiten, and J. Scott. Cluster structures for 2-Calabi-Yau categories and unipotent groups. *Compos. Math.*, 145(4):1035–1079, 2009.

[BM06] Aslak Bakke Buan and Robert Marsh. Cluster-tilting theory. In *Trends in representation theory of algebras and related topics*, volume 406 of *Contemp. Math.*, pages 1–30. Amer. Math. Soc., Providence, RI, 2006.

[CC06] Philippe Caldero and Frédéric Chapoton. Cluster algebras as Hall algebras of quiver representations. *Comment. Math. Helv.*, 81(3):595–616, 2006.

[CK06] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras. II. *Ann. Sci. École Norm. Sup. (4)*, 39(6):983–1009, 2006.

[CK08] Philippe Caldero and Bernhard Keller. From triangulated categories to cluster algebras. *Invent. Math.*, 172(1):169–211, 2008.

[Dim04] Alexandru Dimca. *Sheaves in topology.* Universitext. Springer-Verlag, Berlin, 2004.

[DWZ08] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. Quivers with potentials and their representations. I. Mutations. *Selecta Math. (N.S.)*, 14(1):59–119, 2008.

[DWZ10] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. Quivers with potentials and their representations II: Applications to cluster algebras. *J. Amer. Math. Soc.*, 23(3):749–790, 2010.

[FK10] Changjian Fu and Bernhard Keller. On cluster algebras with coefficients and 2-Calabi-Yau categories. *Trans. Amer. Math. Soc.*, 362(2):859–895, 2010.

[FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[Gin] Victor Ginzburg. Calabi–Yau algebras. *Preprint arXiv:math/0612139v3 [math.AG].*

[GLS] Christof Geiss, Bernard Leclerc, and Jan Schröer. Cluster algebra structures and semicanoncial bases for unipotent groups. *preprint arXiv:math/0703039v3 [math.RT].*

[GLS07] Christof Geiss, Bernard Leclerc, and Jan Schröer. Semicanonical bases and preprojective algebras. II. A multiplication formula. *Compos. Math.*, 143(5):1313–1334, 2007.

[GLS08a] Christof Geiss, Bernard Leclerc, and Jan Schröer. Partial flag varieties and preprojective algebras. *Ann. Inst. Fourier (Grenoble)*, 58(3):825–876, 2008.

[GLS08b] Christof Geiss, Bernard Leclerc, and Jan Schröer. Preprojective algebras and cluster algebras. In *Trends in representation theory of algebras and related topics*, EMS Ser. Congr. Rep., pages 253–283. Eur. Math. Soc., Zürich, 2008.

[Huba] Andrew Hubery. Acyclic cluster algebras via Ringel–Hall algebras. *Current work, available at Andrew Hubery's webpage: http://www.math.upb.de/~hubery/Cluster.pdf.*

[Hubb] Andrew Hubery. Hall polynomials for affine quivers. *Preprint arXiv:math/0703178v2 [math.RT].*

[Joy06] Dominic Joyce. Constructible functions on Artin stacks. *J. London Math. Soc. (2)*, 74(3):583–606, 2006.

[Kel] Bernhard Keller. Cluster algebras, quiver representations and triangulated categories. *Submitted to the proceedings of the Workshop on Triangulated Categories (Leeds, 2006), arXiv:0807.1960v6 [math.RT].*

[Kel08] Bernhard Keller. Calabi-Yau triangulated categories. In *Trends in representation theory of algebras and related topics*, EMS Ser. Congr. Rep., pages 467–489. Eur. Math. Soc., Zürich, 2008.

[KY] Bernhard Keller and Dong Yang. Derived equivalences from mutations of quivers with potential. *Preprint arXiv:0906.0761v3 [math.RT].*

[Lus97] G. Lusztig. Constructible functions on the Steinberg variety. *Adv. Math.*, 130(2):287–310, 1997.

[Pal08] Yann Palu. Cluster characters for 2-Calabi-Yau triangulated categories. *Ann. Inst. Fourier (Grenoble)*, 58(6):2221–2248, 2008.

[Rin90] Claus Michael Ringel. Hall polynomials for the representation-finite hereditary algebras. *Adv. Math.*, 84(2):137–178, 1990.

[Rin96] Claus Michael Ringel. Green's theorem on Hall algebras. In *Representation theory of algebras and related topics (Mexico City, 1994)*, volume 19 of *CMS Conf. Proc.*, pages 185–245. Amer. Math. Soc., Providence, RI, 1996.

[Sch08] Olivier Schiffmann. Lectures on Hall algebras. *Lecture notes, comptes-rendus de l'école doctorale d'été de Grenoble arXiv:math/0611617v1 [math.RT]*, 2008.

[Xu10] Fan Xu. On the cluster multiplication theorem for acyclic cluster algebras. *Trans. Amer. Math. Soc.*, 362(2):753–776, 2010.

[XX] Jie Xiao and Fan Xu. Green's formula with $\mathbb{C}^*$-action and Caldero-Keller's formula for cluster algebras. *To appear in series of Progess in Math. preprint arXiv: math.QA/0707.1175.*

Université Paris 7 - Denis Diderot, UMR 7586 du CNRS, case 7012, 2 place Jussieu, 75251 Paris Cedex 05, France.

E-mail address: `palu@math.jussieu.fr`