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Algebras of finite representation type arising from maximal rigid objects ☆

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ABSTRACT

We give a complete classification of all algebras appearing as endomorphism algebras of maximal rigid objects in standard 2-Calabi–Yau categories of finite type. Such categories are equivalent to certain orbit categories of derived categories of Dynkin algebras. It turns out that with one exception, all the algebras that occur are 2-Calabi–Yau-tilted, and therefore appear in an earlier classification by Bertani-Økland and Oppermann. We explain this phenomenon by investigating the subcategories generated by rigid objects in standard 2-Calabi–Yau categories of finite type.

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Introduction

Motivated by trying to categorify the essential ingredients in the definition of cluster algebras by Fomin and Zelevinsky, the authors of [11] introduced the cluster category $\mathcal{C}_Q$ associated with a finite acyclic quiver Q. The notion was later generalised by Amiot [3, Section 3], dealing with quivers which are not necessarily acyclic. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Cluster categories are special cases of Hom-finite, triangulated 2-Calabi–Yau $\mathbb{K}$-categories (2-CY categories). In such categories, the cluster tilting objects, or more generally, maximal rigid objects, play a special role for the categorification of cluster algebras. For cluster categories in the acyclic case these two classes coincide, but in general maximal rigid objects in 2-CY categories are not necessarily cluster tilting.

The cluster-tilted algebras are the finite dimensional algebras obtained as endomorphism algebras of cluster tilting objects in cluster categories. These, and the more general class of 2-Calabi–Yau-tilted algebras, are of independent interest, and have been studied by many authors, see [19,24,25]. As a natural generalisation, one also considers the endomorphism algebras of maximal rigid objects in 2-CY categories, here called 2-*endorigid algebras*.

When $\Gamma = \mathrm{End}_{\mathcal{C}}(T)$ is a 2-Calabi–Yau-tilted algebra, it is not known if the category $\mathcal{C}$ is determined by Γ, but this is known to be true in the case of acyclic cluster categories [21, Section 2.1]. However, if we consider 2-endorigid algebras, then one frequently obtains the same algebras starting with different 2-CY categories. In this paper we investigate this phenomenon. We restrict to the case where the 2-CY categories in question only have a finite number of isomorphism classes of indecomposable objects. Also in this case, it is known that the 2-endorigid algebras are of finite representation type (this follows from the proof of [17, Corollary 6.5]). In [3] and [30], the structure of triangulated categories with finitely many indecomposables was studied. Such categories have Serre functors, and hence there is an associated AR-quiver. Here orbit categories of the form $D^b(\mathrm{mod}\,\mathbb{K}Q)/\varphi$ play a special role, where Q is a Dynkin quiver, $D^b(\mathrm{mod}\,\mathbb{K}Q)$ is the bounded derived category of the path algebra $\mathbb{K}Q$ and φ is an automorphism. These are triangulated categories which are *standard*, i.e. they can be identified with the mesh category of their AR-quiver, with a finite number of indecomposable objects. In [13, Appendix A], such orbit categories with the 2-CY property were classified. And as an application of that classification, the 2-CY-tilted algebras of finite representation type, coming from orbit categories, were classified in [7, Theorem 5.7] (one case was missed, as was noticed in [22, Section 1.4], see Section 2.1 for details). These classifications are crucial for our investigations. Our main result is a complete classification of the 2-endorigid algebras associated to standard 2-CY categories of finite type. In fact, we show that all such algebras, with one single exception, already appear in the classification of [7]. In order to prove this we show that the following holds in almost all cases: If we fix a 2-CY (orbit) category $\mathcal{C}$ of finite type, then there is an associated 2-CY category $\mathcal{C}'$ with cluster tilting objects, such that the full additive subcategories generated by the rigid objects in $\mathcal{C}$ and $\mathcal{C}'$ are equivalent.

It is known that in the case of standard 2-CY categories, the 2-CY-tilted algebras of finite representation type are Jacobian [7, Theorem 5.7] (when the algebraically closed field $\mathbb{K}$ is of characteristic zero): There is a potential (i.e. a sum of cycles), such that the algebra is the Jacobian of its Gabriel quiver with respect to this potential. Moreover, all finite dimensional Jacobian algebras are 2-CY-tilted, by the work of Amiot [3, Corollary 3.6]. However, as indicated, we point out that there is a 2-endorigid algebra which is not 2-CY-tilted, and therefore also not Jacobian.

In Section 1, we give some background material on maximal rigid and cluster tilting objects. In Section 2 we give our version of the classification of 2-CY orbit categories, and in particular we describe the rigid objects in these categories. Then, in Section 3, we define functors identifying the subcategories of rigid objects in the relevant cases. In Section 4, we give the example of a 2-endorigid algebra of finite type which is not 2-CY-tilted.

Notation. Unless stated otherwise, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. We write Σ for the shift functor in any orbit category, and [1] for the shift in any derived category. We will use the following notation:

$$\mathcal{A}_{n,t} = D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{t(n+1)-1}[1],$$
$$\mathcal{D}_{n,t} = D^{\mathrm{b}}\left(\mathbb{K}\mathrm{D}_{2t(n+1)}\right)/\tau^{n+1}\varphi^{n},$$

where φ is induced by an automorphism of order 2 of $\mathrm{D}_{2t(n+1)}$. The orbit categories that we consider are triangulated, by a theorem of Keller, see [18, Theorem 1].

1. Background

In this section, we give some background material on cluster tilting and maximal rigid objects in Hom-finite triangulated 2-Calabi–Yau categories over an algebraically closed field $\mathbb{K}$ (which is not required to be of characteristic 0 in this section).

Let d be a non-negative integer. A Hom-finite, triangulated $\mathbb{K}$-category $\mathcal{C}$ is called *d-Calabi–Yau* or d-CY for short, if we have a natural isomorphism

$$D\,\mathrm{Hom}(X,Y) \simeq \mathrm{Hom}(Y, X[d])$$

for objects X, Y in $\mathcal{C}$, where $D = \mathrm{Hom}_{\mathbb{K}}(\ ,\mathbb{K})$ is the ordinary $\mathbb{K}$-duality.

A main example here is the cluster category $\mathcal{C}_Q$ associated with a finite (connected) acyclic quiver Q [11, Section 1]. Here $\mathcal{C}_Q$ is the orbit category $D^b(\operatorname{mod}\mathbb{K}Q)/\tau^{-1}[1]$, where τ is the AR-translation on the bounded derived category $D^b(\operatorname{mod}\mathbb{K}Q)$. The cluster categories have been shown to be triangulated [18, Theorem 1]. Another main example is the stable category $\underline{\mathrm{mod}}\,\Lambda$, where Λ is a preprojective algebra of Dynkin type, investigated in [16].

An object M in a triangulated category is called *rigid* if $\mathrm{Ext}^1(M, M) = 0$, and *maximal rigid* if it is maximal with respect to this property. Let $\mathrm{add}\, M$ denote the additive closure of M. If also $\mathrm{Ext}^1(M, X) = 0$ implies $X \in \mathrm{add}\, M$, then M is said to be *cluster tilting*. For the cluster categories $\mathcal{C}_Q$ and the stable module categories $\underline{\mathrm{mod}}\,\Lambda$ of preprojective algebras of Dynkin type, the maximal rigid objects are also cluster tilting [11, Proposition 2.3], [16, Theorem 2.2], but this is not the case in general.

An object $\overline{T}$ is called an *almost complete cluster tilting* object in $\mathcal{C}_Q$, if there is an indecomposable object X, not in $\mathrm{add}\, \overline{T}$, such that $\overline{T} \amalg X$ is a cluster tilting object. It was shown in [11, Theorem 5.1] that if $\overline{T}$ is an almost complete cluster tilting object in $\mathcal{C}_Q$, then there is a unique indecomposable object $Y \not\simeq X$, such that $T^* = \overline{T} \amalg Y$ is a cluster tilting object.

There is an interesting property for cluster tilting objects which does not hold for maximal rigid objects. For T a cluster tilting object in a 2-CY category $\mathcal{C}$, there is an equivalence of categories $\mathcal{C}/\mathrm{add}\, T \to \mathrm{mod}\, \mathrm{End}(T)$, by [10, Theorem A], [20, Proposition 2.1 (c)].

For a connected 2-Calabi–Yau category, then either all maximal rigid objects are cluster tilting, or none of them are [32, Theorem 2.6]. And if for a maximal rigid object M there are no loops or 2-cycles in the quiver of $\mathrm{End}(M)$, then M is cluster tilting [9, Conjecture II.1.9], [29, Theorem 3.6’].

The main sources of examples for having maximal rigid objects which are not cluster tilting are 1-dimensional hypersurface singularities [13, Proposition 2.8] and cluster tubes [12, Section 2], see also [6,28,31].

The 2-Calabi–Yau-tilted algebras Γ satisfy some nice homological properties: They are Gorenstein of dimension ≤ 1, and $\mathrm{Sub}\, \Gamma$ is a Frobenius category whose stable category $\underline{\mathrm{Sub}}\, \Gamma$ is 3-Calabi–Yau [20]. Here $\mathrm{Sub}\, \Gamma$ denotes the full additive subcategory of $\mathrm{mod}\, \Gamma$ generated by the submodules of objects in $\mathrm{add}\, \Gamma$, and $\underline{\mathrm{Sub}}\, \Gamma$ denotes the corresponding stable category, that is: the category with the same objects, but with Hom-spaces given as the Hom-spaces in $\mathrm{mod}\, \Gamma$ modulo maps factoring through projective objects. By [32, Theorem 4.4], also the 2-endorigid algebras are Gorenstein of dimension ≤ 1.

2. Rigid objects in triangulated orbit categories of finite type

2.1. The classification

In [3, Theorem 7.2], Amiot classified all standard algebraic triangulated categories with finitely many indecomposable objects. By using geometric descriptions in type A [14] and in type D [26], and direct computations in type E, Burban–Iyama–Keller–Reiten extracted from Amiot’s list all 2-Calabi–Yau triangulated categories with cluster tilting objects, and with non-zero maximal rigid objects (see the appendix of [13]). In this section, we give a restatement of the results in the appendix of [13]. We note two changes from their lists:

(L1) The orbit category $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^4$ has cluster tilting objects (this case was first noticed by Ladkani in [22, Section 1.4]);

(L2) The orbit category $D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_4)/\tau^2\varphi$, where φ is induced by an automorphism of D_4 of order 2, has non-zero maximal rigid objects which are not cluster tilting.

Proposition 2.1 *(Amiot; Burban–Iyama–Keller–Reiten). The standard, 2-Calabi–Yau, triangulated categories with finitely many indecomposable objects and with cluster tilting objects are exactly the cluster categories of Dynkin types A, D or E and the orbit categories:*

- (Type A) $D^{\mathrm{b}}(\mathbb{K}\mathrm{A}_{3n})/\tau^n[1]$, *where* $n \geq 1$*;*
- (Type D) $D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_{kn})/(\tau\varphi)^n$, *where* $n \geq 1$, $k > 1$, $kn \geq 4$ *and* φ *is induced by an automorphism of* D_{kn} *of order* 2*;*
- (Type E) $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^4$ *and* $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^8$*.*

Proof. These categories are described in a table of the appendix of [13], and our description is based on that. We explain why and how our description in case of types A and D_4 differ from that of [13].

Apart from the cluster category, the orbit categories of $D^{\mathrm{b}}(\mathbb{K}\mathrm{A}_m)$ appearing in the table of [13] are given by the automorphisms:

$(\tau^{\frac{m}{2}}[1])^{\frac{m+3}{3}}$, if 3 divides m and m is even;
$\tau^{\frac{m+3}{6}+\frac{m+1}{2}}[1]$, if 3 divides m and m is odd.

We simplify this description by using the fact (folklore, see e.g. [18, Example 8.3 (2)]) that in the triangulated category $D^{\mathrm{b}}(\mathbb{K}\mathrm{A}_m)$, we have

$$\tau^{-(m+1)} = [2]. \tag{1}$$

Note that this is sometimes referred to as a *fractional Calabi–Yau property.*

Let $m = 3n$. Assume first that n is even. We then have:

$$\begin{aligned}(\tau^{\frac{m}{2}}[1])^{\frac{m+3}{3}} &= (\tau^{3\frac{n}{2}}[1])^{n+1} = (\tau^{3n+1})^{\frac{n}{2}}\tau^n[n+1] = (\tau^{3n+1})^{\frac{n}{2}}\tau^n[2]^{\frac{n}{2}}[1] \\ &= (\tau^{3n+1}[2])^{\frac{n}{2}}(\tau^n[1]) = \tau^n[1]\end{aligned}$$

where the last equality follows from (1).

Assume now that n is odd. Then, we have

$$\tau^{\frac{m+3}{6}+\frac{m+1}{2}}[1] = \tau^{2n+1}[1] = (\tau^n[1])^{-1}$$

where (1) is used for the last equation.

In both cases, the orbit category is $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{3n}\right)/\tau^{n}[1]$.

The orbit categories of $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{D}_{4}\right)$ appearing in the table of [13, Appendix A] are given by the automorphisms: $\tau^{k}\sigma$, where k divides 4, where σ is induced by an automorphism of D_4 satisfying $\sigma^{\frac{4}{k}} = 1$ and where $(k,\sigma) \neq (1,1)$. We thus have:

if $k = 1$, then σ is of order 2;
if $k = 2$, then σ is either the identity or of order 2;
if $k = 4$, then σ is the identity and the orbit category is the cluster category of type D_4.

We claim that if $k = 2$ and σ is of order 2, then the corresponding orbit category has non-zero maximal rigid objects, but does not have cluster tilting objects. Let thus σ be of order 2. By computing the Hom-hammocks in the Auslander–Reiten quiver:

$$\begin{tikzcd}
 & d \arrow[dr] & & \Sigma d \arrow[dr] & & d \\
a \arrow[ur] \arrow[r] \arrow[dr] & c \arrow[r] & \Sigma a \arrow[ur] \arrow[r] \arrow[dr] & \Sigma b \arrow[r] & a \arrow[ur] \arrow[r] \arrow[dr] & b \\
 & b \arrow[ur] & & \Sigma c \arrow[ur] & & c,
\end{tikzcd}$$

one finds that d and Σd are the only non-zero rigid objects and that there are no non-zero morphisms from d to b or c. This shows that d, and therefore also Σd, are maximal rigid objects which are not cluster tilting. This explains (L2). □

Proposition 2.2 *(Amiot; Burban–Iyama–Keller–Reiten). The standard, 2-Calabi–Yau, triangulated categories with finitely many indecomposable objects and with non-zero maximal rigid objects which are not cluster tilting are exactly the orbit categories:*

– (Type A) $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{t(n+1)-1}[1]$, *where* $n \geq 1$ *and* $t > 1$*;*
– (Type D) $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{D}_{2t(n+1)}\right)/\tau^{n+1}\varphi^{n}$, *where* $n, t \geq 1$, *and where* φ *is induced by an automorphism of* $\mathrm{D}_{2t(n+1)}$ *of order* 2*;*
– (Type E) $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{E}_{7}\right)/\tau^{2}$ *and* $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{E}_{7}\right)/\tau^{5}$*.*

Proof. Type A deserves a few comments. The tables in the appendix of [13] list all orbit categories of $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{m}\right)$ with non-zero maximal rigid objects which are not cluster tilting. They are given by the following automorphisms:

I. $(\tau^{\frac{m}{2}}[1])^{k}$, where m is even; k divides $m+3$; $k \neq 1$; $k \neq m+3$ and if 3 divides m, then $k \neq \frac{m+3}{3}$;
II. $\tau^{k+\frac{m+1}{2}}[1]$, where m is odd; k divides $\frac{m+3}{2}$; $\frac{m+3}{2k}$ is odd; $k \neq \frac{m+3}{2}$ and if 3 divides m, then $k \neq \frac{m+3}{6}$.

As in the proof of Proposition 2.1, we use the property given by equation (1), in order to give a uniform description of all the cases above. Note first that if $k = \frac{m+3}{3}$ or if $k = \frac{m+3}{6}$, then 3 divides m. Therefore, the condition "if 3 divides m" above is redundant.

Assume first we are in case I above, so m is even and we can write $m+3 = uk$, where u and k are greater than 1 and $u \neq 3$. We then have:

$$\begin{aligned}(\tau^{\frac{m}{2}}[1])^k = (\tau^{\frac{uk-3}{2}}[1])^k = \tau^{k\frac{uk-3}{2}}[1][k-1] = \tau^{k\frac{uk-3}{2}}[1][2]^{\frac{k-1}{2}} \\ = \tau^{k\frac{uk-3}{2}}[1](\tau^{-uk+2})^{\frac{k-1}{2}} = \tau^{\frac{u-1}{2}k-1}[1].\end{aligned}$$

Replacing u by $2t+1$ and k by $n+1$ gives

$$m = uk - 3 = (2t+1)(n+1) - 3 \text{ and } \frac{u-1}{2}k - 1 = t(n+1) - 1.$$

Hence, we obtain the orbit categories $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{t(n+1)-1}[1]$ (where $t > 1$ and $n \geq 1$).

Assume now we are in case II, so that m is odd and we can write $m+3 = 2uk$, where u is odd and greater than 3. We then have:

$$\tau^{k+\frac{m+1}{2}}[1] = \tau^{k+uk-1}[1] = \tau^{\frac{u+1}{2}2k-1}[1] = (\tau^{\frac{u-1}{2}2k-1}[1])^{-1}$$

where the last equation follows from equation (1).

Replacing u by $2t+1$, and $2k$ by $n+1$ gives

$$m = 2uk - 3 = (2t+1)(n+1) - 3 \text{ and } \frac{u-1}{2}2k - 1 = t(n+1) - 1$$

and also in this case we obtain the orbit categories $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{t(n+1)-1}[1]$ (where $t > 1$, $n \geq 1$). □

Remark 2.3. For a given value of n, the orbit categories

$$D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{t(n+1)-1}[1]$$

share some similarities, and are compared in Section 3. Note that when $t = 1$, we have

$$D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{t(n+1)-1}[1] = D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{3n}\right)/\tau^{n}[1].$$

Hence, by Proposition 2.1 this orbit category has cluster tilting objects. On the other hand, if $t > 1$ it has non-zero maximal rigid objects which are not cluster tilting. This family can be expanded by including the cluster tubes, thought of as a limit obtained when t goes to infinity. This point of view will be corroborated in Sections 3 and 4, where the endomorphism algebras of the maximal rigid objects in these categories are shown to be independent of the specific value of t.

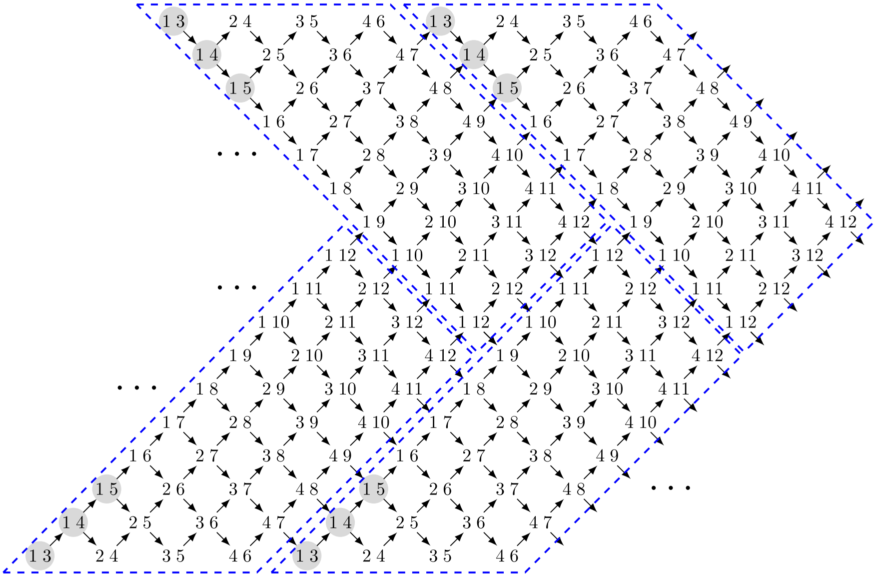

Fig. 1. A bijection between $\frac{2\pi}{5}$-periodic collections of arcs of the heptakaidecagon and isomorphism classes of basic objects in $\mathcal{A}_{3,2}$. The maximal rigid object of Corollary 2.7 is highlighted in grey.

2.2. The rigid objects

We will now describe indecomposable rigid objects in the orbit categories listed in Subsection 2.1, and then consider the additive subcategories generated by the set of rigid objects.

2.2.1. Type A

In order to compute the rigid objects in the orbit categories $\mathcal{A}_{n,t} = D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{t(n+1)-1}[1]$, we use the geometric description [14] of the cluster category of type A. The following lemma was implicitly used in the appendix of [13].

Lemma 2.4.

(1) *There is a bijection between isomorphism classes of basic objects in $\mathcal{A}_{n,t}$ and collections of arcs of the $(2t+1)(n+1)$-gon which are stable under rotation by $\frac{2\pi}{2t+1}$. Such a bijection is given in Fig. 1 for $t = 2, n = 3$ and is sketched in Fig. 2 for the general case.*

(2) *Under the bijection above, rigid objects correspond to non-crossing collections of arcs. In particular:*
   - (a) *The isomorphism classes of indecomposable rigid objects in $\mathcal{A}_{n,t}$ are parametrised by the arcs $[i\ (i+2)], \ldots, [i\ (i+n+1)]$ for $i = 1, \ldots, n+1$.*
   - (b) *The maximal non-crossing collections correspond to (isoclasses of) basic maximal rigid objects and such an object is cluster tilting if and only if the collection of arcs is a triangulation (if and only if $t = 1$).*

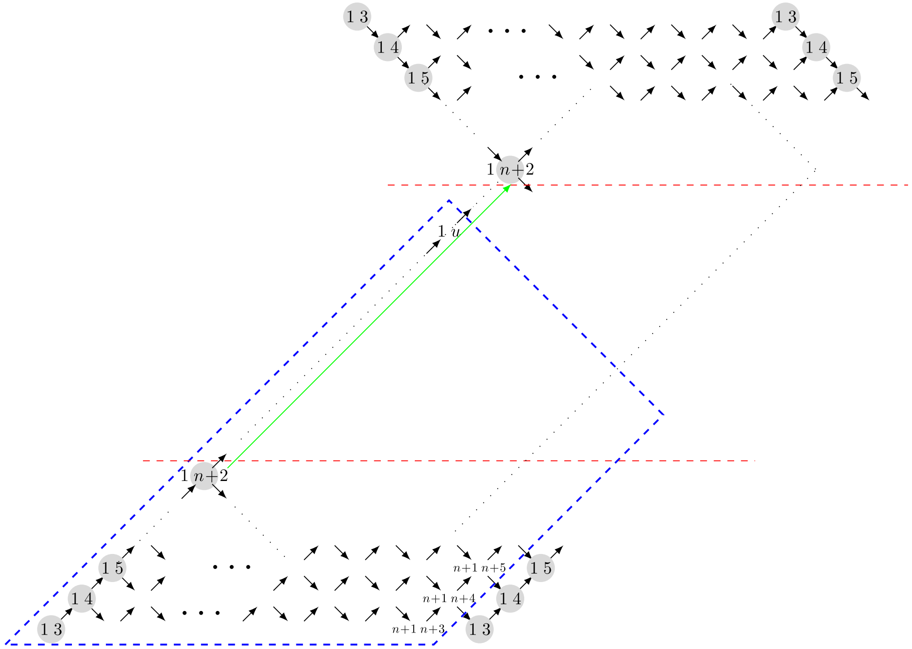

Fig. 2. A fundamental domain for $\mathcal{A}_{n,t}$ inside the derived category is encircled by a dashed (blue in the web version) line. Below the bottom (and above the top) horizontal, dashed (red in the web version) line lie all rigid indecomposable objects. The Hom-hammock of $(1\, n+2)$ is emphasised by a dotted rectangle. The longest (green in the web version) arrow gives rise to a loop in the quiver $\mathcal{Q}_{\mathcal{R}_c}$. Here u equals $(t+1)(n+1)$.

Proof. Let $n, t \geq 1$, let $N = (2t+1)(n+1)$, let $\mathcal{A}_{n,t}$ be the triangulated orbit category $D^{\mathrm{b}}(\mathbb{K}\mathrm{A}_{N-3})/\tau^{t(n+1)-1}[1]$ and let $\mathcal{C}_{\mathrm{A}_{N-3}}$ be the cluster category of type A_{N-3}. Using that $\mathcal{A}_{n,t}$ is 2-CY, the universal property of orbit categories yields a functor $\mathcal{C}_{\mathrm{A}_{N-3}} \xrightarrow{F} \mathcal{A}_{n,t}$. Note that this covering functor commutes with shift functors since the latter are induced by the shift in the orbit category $D^{\mathrm{b}}(\mathbb{K}\mathrm{A}_{N-3})$.

In the cluster category $\mathcal{C}_{\mathrm{A}_{N-3}}$, we have $\tau^{t(n+1)-1}[1] = \tau^{t(n+1)}$. Moreover, in the derived category $D^{\mathrm{b}}(\mathbb{K}\mathrm{A}_{N-3})$, we have $\tau^{N-2} = [-2]$. Therefore, τ is of order N in $\mathcal{C}_{\mathrm{A}_{N-3}}$, and τ^{n+1} is of order $2t+1$. Since $\gcd(t, 2t+1) = 1$, then $\tau^{t(n+1)}$ is also of order $2t+1$ and generates the same group as τ^{n+1}. The functor F is thus a $(2t+1)$-covering functor, with $F(\tau^{n+1}X)$ isomorphic to FX for any object X. Since F commutes with shifts, we have, for any two objects X, Y in $\mathcal{A}_{n,t}$: $\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(X,Y) = \mathrm{Hom}_{\mathcal{A}_{n,t}}(X, \Sigma Y) \simeq \bigoplus_{FY' \simeq Y} \mathrm{Hom}_{\mathcal{C}_{A_{N-3}}}(X, \Sigma Y') = \bigoplus_{FY' \simeq Y} \mathrm{Ext}^1_{\mathcal{C}_{A_{N-3}}}(X, Y')$. We can thus use the description of the cluster category $\mathcal{C}_{A_{N-3}}$ in terms of diagonals of the N-gon [14] in order to compute the rigid indecomposable objects in $\mathcal{A}_{n,t}$: Isomorphism classes of indecomposable objects in $\mathcal{A}_{n,t}$ are in bijection with collections of $2t+1$ diagonals of the N-gon which are stable under the automorphism sending a diagonal $[i\ j]$ to $[(i+n+1)\ (j+n+1)]$. Moreover, such a collection corresponds to a rigid indecomposable object in $\mathcal{A}_{n,t}$ if and only if none of its diagonals cross. This shows that isomorphism classes of indecompos-

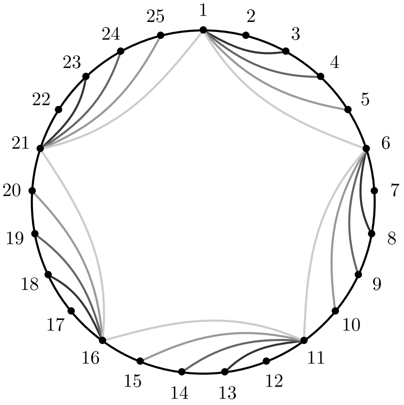

Fig. 3. A collection of arcs of the icosikaipentagon corresponding to a maximal rigid object in $\mathcal{A}_{4,2}$.

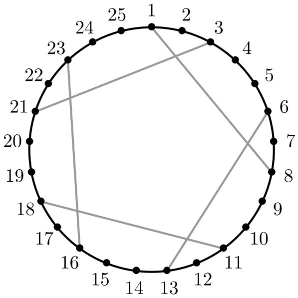

Fig. 4. A collection of arcs of the icosikaipentagon corresponding to a non-rigid indecomposable object of $\mathcal{A}_{4,2}$.

able rigid objects in $\mathcal{A}_{n,t}$ are parametrised by the arcs $[i\ (i+2)], \dots, [i\ (i+n+1)]$ for $i = 1, \dots, n+1$.

Consider a maximal collection $\mathfrak{A}$ of non-crossing arcs, stable under rotation by $\frac{2\pi}{2t+1}$, that is not a triangulation. Then there exists an arc γ which does not cross any arc in the collection (such an arc will correspond to a non-rigid indecomposable object). Necessarily, none of the rotations of γ by multiples of $\frac{2\pi}{2t+1}$ cross any arc in the collection. This implies that the maximal rigid object corresponding to $\mathfrak{A}$ is not cluster tilting. □

Remark 2.5. For an example of an arc corresponding to an indecomposable object which is not rigid, see Fig. 4.

Let $\mathcal{R}_{\mathcal{A}_{n,t}}$ be the full additive subcategory of $\mathcal{A}_{n,t}$ generated by the rigid objects. We will show in Section 3 that this category (up to equivalence) only depends on n. Here we provide a first step towards that result. Recall that for an additive Hom-finite Krull–Schmidt category $\mathcal{U}$, the quiver $\mathcal{Q}_{\mathcal{U}}$ of $\mathcal{U}$ has vertices corresponding to the isomorphism classes of indecomposable objects, and there are $\dim \operatorname{Irr}(X, Y)$ arrows from the vertex

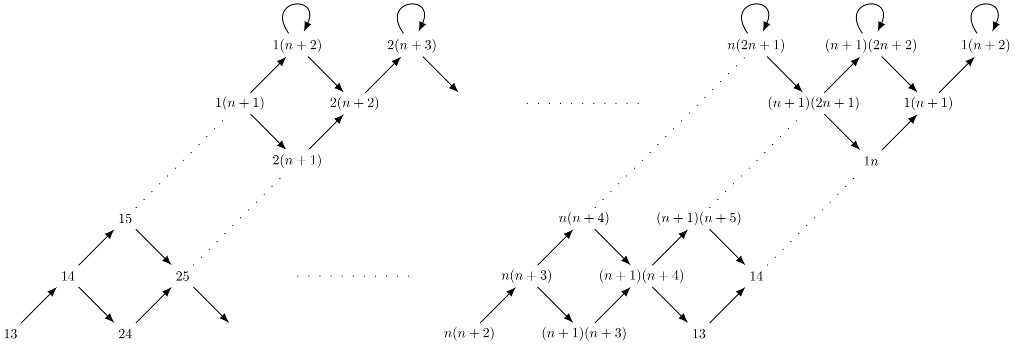

Fig. 5. The quiver $\mathcal{Q}_n$.

corresponding to X to the vertex corresponding to Y, where $\mathrm{Irr}(X, Y)$ is the space of irreducible maps from X to Y.

Proposition 2.6. *The quiver $\mathcal{Q}_{\mathcal{R}_{\mathcal{A}_{n,t}}}$ is isomorphic to the quiver $\mathcal{Q}_n$ depicted in Fig. 5.*

Proof. Consider the Auslander–Reiten quiver of $\mathcal{A}_{n,t}$ depicted in Fig. 2. Clearly, the irreducible maps in $\mathcal{A}_{n,t}$ with source and target in $\mathcal{R}_{\mathcal{A}_{n,t}}$, are also irreducible in $\mathcal{R}_{\mathcal{A}_{n,t}}$. It is also straightforward to verify, by using standard computations of Hom-hammocks in the Auslander–Reiten quiver of the derived category $D^b(\mathbb{K}A_{(2t+1)(n+1)-3})$ (see Fig. 2), that the map from $(1\ n+2)$ to $(1\ n+2)$ (and all shifts of this) is irreducible in $\mathcal{R}_{\mathcal{A}_{n,t}}$, and that there are no further irreducible maps in $\mathcal{R}_{\mathcal{A}_{n,t}}$. Hence the quiver $\mathcal{Q}_{\mathcal{R}_{\mathcal{A}_{n,t}}}$ is isomorphic to the quiver $\mathcal{Q}_n$ depicted in Fig. 5. □

As a special case of the computations necessary for the proof of Proposition 2.6 we also obtain the following. Note that the cluster tilting case $t = 1$ of this fact can also be found in [7, Theorem 5.7].

Corollary 2.7. *Let $n, t \in \mathbb{N}$ and let T be the maximal rigid object of the orbit category $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{t(n+1)-1}[1]$ corresponding to the collection of arcs generated by $[1\ 3], [1\ 4], \ldots, [1\ n+2]$ (see Lemma 2.4). Then the endomorphism algebra of T is given by the quiver*

$$1 \longrightarrow 2 \longrightarrow 3 \cdots\cdots n-1 \longrightarrow n \circlearrowright \alpha \ ,$$

with ideal of relations generated by α^2.

Remark 2.8. See Fig. 3 for the collection of arcs corresponding to the maximal rigid object in Corollary 2.7.

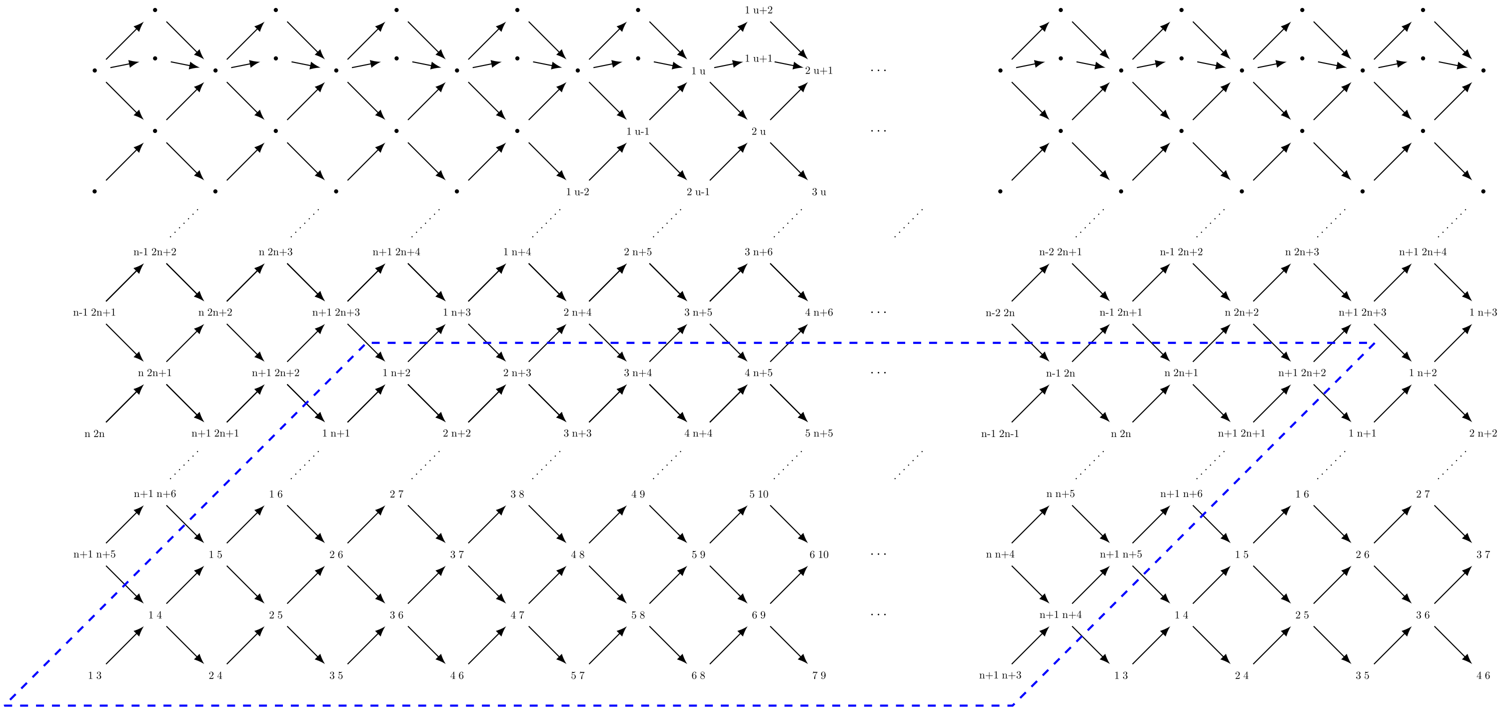

Fig. 6. The Auslander–Reiten quiver of $\mathcal{D}_{n,t}$. The objects in $\mathcal{R}_{\mathcal{D}_{n,t}}$ are in the area inside the dashed (blue in the web version) lines. Here $u = 2t(n+1)$.

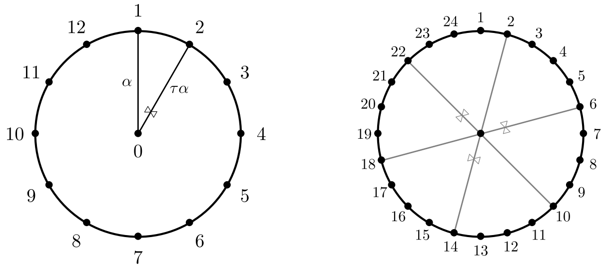

Fig. 7. Action of τ on a tagged arc (left) and a non-rigid indecomposable object of $\mathcal{D}_{3,3}$ (right).

2.2.2. Type D

Let $n, t \geq 1$ and let $P_{n,t}$ be a once-punctured $2t(n+1)$-gon. We denote by ρ the automorphism on the tagged arcs (see [15, Section 7] and [26, Section 3]) obtained by rotating by $\frac{\pi}{t}$ and switching tags, as in Fig. 7.

Recall that $\mathcal{D}_{n,t}$ is the orbit category $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{D}_{2t(n+1)}\right)/\tau^{n+1}\varphi^{n}$.

Lemma 2.9.

(1) *There is a bijection between isomorphism classes of basic objects in $\mathcal{D}_{n,t}$ and collections of arcs of $P_{n,t}$ which are stable under ρ. Such a bijection is illustrated in Fig. 6.*

(2) *Under the above bijection, rigid objects correspond to non-crossing collections of arcs. In particular:*

  (a) *The isomorphism classes of indecomposable rigid objects in $\mathcal{D}_{n,t}$ are parametrised by the arcs $[i\ (i+2)], \ldots, [i\ (i+n+1)]$ for $i = 1, \ldots, n+1$.*

  (b) *The maximal non-crossing collections which are stable under ρ correspond to (isoclasses of) basic maximal rigid objects.*

Proof. The proof is similar to that of Lemma 2.4. There is a $2t$-covering functor from the cluster category $\mathcal{C}_{\mathrm{D}_{2t(n+1)}}$ to the triangulated orbit category $\mathcal{D}_{t,n} = D^{\mathrm{b}}\left(\mathbb{K}\mathrm{D}_{2t(n+1)}\right)/\tau^{n+1}\varphi^{n}$. We note that φ acts on arcs by switching tags and that τ acts on arcs $[i\ 0]$ with an endpoint at the puncture 0 by sending it to $[i+1\ 0]$ and by switching tags. Therefore an arc with an endpoint at the puncture corresponds to a non-rigid indecomposable object in $\mathcal{D}_{n,t}$ and the rest of the proof is similar to that in type A above. □

Consider now the full additive subcategory $\mathcal{R}_{\mathcal{D}_{n,t}}$, generated by the rigid objects in $\mathcal{D}_{n,t}$. We will show, in Section 3, that $\mathcal{R}_{\mathcal{D}_{n,t}}$ is equivalent to $\mathcal{R}_{\mathcal{A}_{n,t}}$. For this, we will need the following.

Proposition 2.10. *The quiver $\mathcal{Q}_{\mathcal{R}_{\mathcal{D}_{n,t}}}$ is isomorphic to the quiver $\mathcal{Q}_n$ depicted in Fig. 5.*

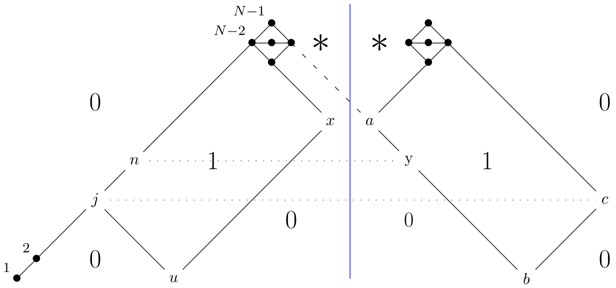

Fig. 8. Hom-hammocks in the derived category $D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_N)$, N even. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. Consider the Auslander–Reiten quiver of $\mathcal{D}_{n,t}$ depicted in Fig. 6. Clearly, the irreducible maps in $\mathcal{A}_{n,t}$ with source and target in $\mathcal{R}_{\mathcal{D}_{n,t}}$, are also irreducible in $\mathcal{R}_{\mathcal{D}_{n,t}}$. To proceed, we will need some basic facts about Hom-hammocks in the derived category $D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_N)$, for N even. First note that, in the derived category $D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_N)$, we have $\tau^{-N+1} = [1]$. Thus $\tau^{-N+2} = \tau[1]$ is a Serre functor in $D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_N)$: For any $X, Y \in D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_N)$, there are bi-natural isomorphisms $\mathrm{Hom}_{D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_N)}(X,Y) \simeq D\,\mathrm{Hom}_{D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_N)}(Y, \tau^{-N+2}X)$. In particular, the Hom-hammock of any object X ends in $\tau^{-N+2}X$ and is symmetric with respect to the vertical line (the blue line in Fig. 8) going through $\tau^{-\frac{N}{2}+1}X$. Without any computations, we thus obtain that the Hom-hammocks have the shape given in Fig. 8, where a part of the Hom-hammock of the indecomposable object denoted by j is described. The left-hand side of the figure is easily computed, since all meshes involved are commutative squares. The rectangle on the left-hand side indicates some indecomposable objects X such that $\dim \mathrm{Hom}(j, X) = 1$. Outside this rectangle, to its left and to its right, the zeros indicate that all morphisms from j to some of the indecomposable objects in these regions are zero morphisms. The star indicates a part of the Hom-hammock that we do not compute. The right-hand side of the figure is deduced from the left-hand side by symmetry. We have indicated some specific indecomposable objects in the figure. They are related by the following equalities: $u = \tau^{-j+1}(1)$, $x = \tau^{-j+1}(N-j-1)$, $a = \tau^{-1}(x) = \tau^{-j}(N-j-1)$, $b = \tau^{-N+2}(1)$, $c = \tau^{-N+2}(j)$ and $y = \tau^{-N+n+1}(n)$.

Using these Hom-hammocks, it is easy to verify that there is a non-zero map from n to y, which becomes an irreducible endomorphism in the category $\mathcal{R}_{\mathcal{D}_{n,t}}$. For this, note that there is a one-dimensional subspace of morphisms from n to $y = \left(\tau^{n+1}\right)^{1-2t}(n)$ (factoring through $N-1$) which do not factor through any indecomposable object in the τ-orbit of $1, \ldots, n-1$. The same will obviously hold for the shifts of this map.

We claim that there are no other irreducible maps in $\mathcal{R}_{\mathcal{D}_{n,t}}$. This can be checked, using the Hom-hammocks of Fig. 5. We leave the details to the reader, but point out the following useful fact.

Note that the only indecomposable objects in the rectangles of Fig. 8 that belong to the τ^{n+1} orbit of $1, 2, \ldots, n$ are $1, 2, \ldots, n$ and y. We claim that any morphism from some j, with $1 \leq j \leq n$, to y factors through n. This holds since $\dim \mathrm{Hom}_{D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_N)}(j, y) = 1$ and the composition $1 \to 2 \to \cdots \to n \to y$ is non-zero (as can be seen from the case $j = 1$ in Fig. 8).

Hence the quiver $\mathcal{Q}_{\mathcal{R}_{\mathcal{D}_{n,t}}}$ is isomorphic to the quiver $\mathcal{Q}_n$ depicted in Fig. 5. □

As for type A, we obtain the following as a special case of the computations necessary for the proof of Proposition 2.10.

Corollary 2.11. *Let $n, t \in \mathbb{N}$ and let T be the maximal rigid object of the orbit category $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{D}_{2t(n+1)}\right)/\tau^{n+1}\varphi^n$ corresponding to the collection of arcs generated by $[1\ 3], [1\ 4], \ldots, [1\ n]$ (see Lemma 2.9). Then the endomorphism algebra of T is given by the quiver*

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r, dotted, no head] & n-1 \arrow[r] & n \arrow[loop right, "\alpha"]
\end{tikzcd},$$

with ideal of relations generated by α^2.

Proof. The computation of the Gabriel quiver is essentially included in the proof of Proposition 2.10. It is easy to verify that the only relation is α^2. □

Let Λ_n denote the algebra appearing in Corollaries 2.7 and 2.11. We will need some properties of the module category $\operatorname{mod}\Lambda_n$. Recall that a module M is called τ-rigid if $\operatorname{Hom}(M, \tau M) = 0$, see [1, Definition 0.1]. Now let $\mathcal{R}_n$ denote the full additive subcategory generated by the indecomposable τ-rigid modules in $\operatorname{mod}\Lambda_n$. It follows from Proposition 2.1, with $t = 1$, that in particular Λ_n is a 2-CY-tilted algebra, and so by [1, Theorem 4.1], a module is τ-rigid if and only if it is of the form $\operatorname{Hom}_{\mathcal{C}}(T, X)$, where X is a rigid object in $\mathcal{C} = D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{3n}\right)/\tau^n[1]$.

It is easy to check that the quiver $\mathcal{Q}_{\mathcal{R}_n}$ can be obtained by deleting the vertices labelled by $(n+1)(n+3), \ldots, (n+1)(2n+2)$ in the quiver $\mathcal{Q}_n$ of Fig. 5.

2.2.3. Type E

In this section we investigate the rigid (and maximal rigid) objects in the orbit categories $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{E}_7\right)/\tau^2$ and $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{E}_7\right)/\tau^5$, appearing in Proposition 2.2. There is also a geometric machinery available in type E, see [23]. However, our description instead relies on simple brute force computations, and we leave out almost all details.

For type $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{E}_7\right)/\tau^5$, the Auslander–Reiten quiver is given in Fig. 9. There are 5 indecomposable rigid objects, all in the bottom τ-orbit in the figure. Let x be any of these five. Then $x \oplus \tau^2 x$ is maximal rigid, and all maximal rigid objects are obtained this way. In particular, they all have the same endomorphism ring.

Proposition 2.12. *The endomorphism algebra of any maximal rigid object in the orbit category $D^{\mathrm{b}}\left(\mathbb{K}\mathrm{E}_7\right)/\tau^5$ is isomorphic to the path algebra of the quiver:*

$$\begin{tikzcd}
\bullet \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & \bullet \arrow[loop right, "\gamma"]
\end{tikzcd},$$

with ideal of relations generated by $\beta\alpha - \gamma\beta$, α^2, γ^2.

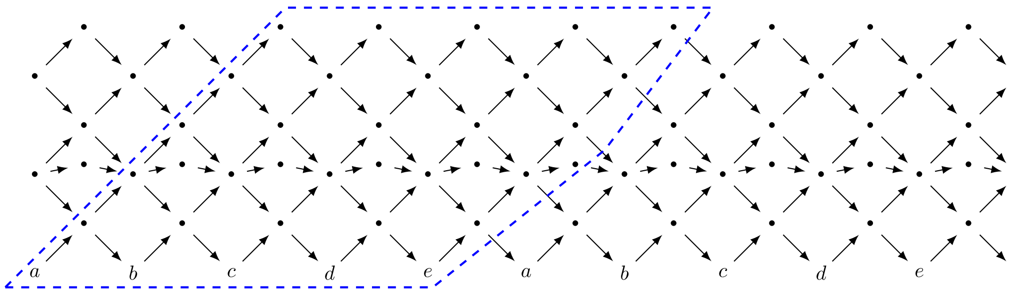

Fig. 9. The orbit category $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_7)/\tau^5$.

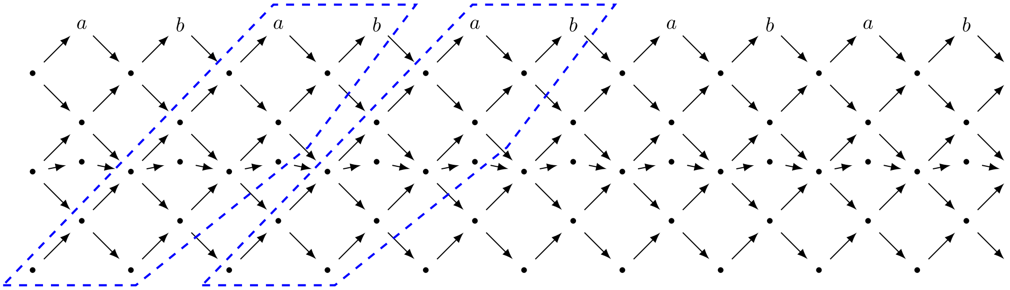

Fig. 10. The orbit category $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_7)/\tau^2$.

Remark 2.13. This latter 2-endorigid algebra is shown not to be 2-CY-tilted in Section 4.1.

Let us now consider $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_7)/\tau^2$. Its Auslander–Reiten quiver is given in Fig. 10. There are only two indecomposable rigid objects, both in the top τ-orbit in the figure.

Now the full subcategory $\mathcal{R}_{D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_7)/\tau^2}$ generated by the rigid objects only contains two indecomposable objects with no maps between them. In particular, we have the following.

Proposition 2.14. *Any maximal rigid object in the orbit category $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_7)/\tau^2$ is indecomposable and its endomorphism algebra is given by a loop α with relation α^3.*

We can compare this to the case $D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_4)/\tau\varphi$, which appears in Proposition 2.1. The AR-quiver of $D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_4)/\tau\varphi$ is given by

$$\begin{tikzcd}
& b \arrow[dr] & & c \arrow[dr] & & b \\
a \arrow[ur] \arrow[r] \arrow[dr] & c \arrow[r] & \Sigma a \arrow[r] \arrow[ur] \arrow[dr] & \Sigma b \arrow[r] & a \arrow[r] \arrow[ur] \arrow[dr] & c \\
& d \arrow[ur] & & d \arrow[ur] & & d
\end{tikzcd}$$

and the only indecomposable rigid objects are b and c. So, we also have that $\mathcal{R}_{D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_4)/\tau^2}$ contains exactly two indecomposable objects, with no maps between them. Moreover it is easily verified that each of these indecomposables are maximal rigid, and that the endomorphism rings are the same as in Proposition 2.14.

2.3. Tables

In Table 1, we summarise some known results on orbit categories with cluster tilting objects, which can be found in [2,13,7]. Table 2 summarises results from [2,13] and from the current section. For each orbit category, we give the number of isomorphism classes of indecomposable objects, the number of summands of any basic maximal rigid object (or equivalently, the rank of the Grothendieck group of its endomorphism algebra), the number of isomorphism classes of indecomposable rigid objects, and the quiver with relations of the endomorphism algebra of some maximal rigid object. Recall that φ denotes an automorphism of the derived category of type D induced by an automorphism of order two of a Dynkin diagram of type D.

Remark 2.15. In the second row of Table 1, the following conventions are used:

- If $n = 1$, then $a = 0$ and $b = 0$;
- if $k = 2$, then there is no loop α, and in the relations, α should be replaced by ab.

Remark 2.16. Let $\mathcal{C}$ be the orbit category appearing in the last row of Table 1. Because of the shape of the quiver in the last column, one might be tempted to think that $\mathcal{C}$ should categorify a cluster algebra of type F_4. However, $\mathcal{C}$ has 24 indecomposable rigid objects only, while there are 28 almost positive roots in type F_4.

3. Comparing subcategories generated by rigid objects

Our aim, in this section, is to compare the full subcategories of rigid objects of the triangulated categories listed in Table 2. In order to do so, we will follow a strategy we now describe: Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbb{K}$-linear, Krull–Schmidt, Hom-finite, 2-Calabi–Yau, triangulated categories. We assume that $T \in \mathcal{C}$ is a cluster tilting object and $U \in \mathcal{D}$ a maximal rigid object. Let $\mathcal{R}_{\mathcal{C}}$, resp. $\mathcal{R}_{\mathcal{D}}$, be the full subcategory of $\mathcal{C}$, resp. $\mathcal{D}$, generated by the rigid objects. Let $\mathcal{Q}_{\mathcal{R}_{\mathcal{C}}}$ be a quiver whose vertices are the (isoclasses of) indecomposable rigid objects of $\mathcal{C}$, and whose arrows form a basis for the irreducible morphisms in $\mathcal{R}_{\mathcal{C}}$. Define $\mathcal{Q}_{\mathcal{R}_{\mathcal{D}}}$ similarly. Finally, let $\mathcal{Q}_{\mathcal{C}}^{\tau-\mathrm{rig}}$ be the quiver similarly given by the irreducible morphisms of the image of $\mathcal{C}(T,-)|_{\mathcal{R}_{\mathcal{C}}}$ in $\operatorname{mod}\operatorname{End}_{\mathcal{C}}(T)$. Define $\mathcal{Q}_{\mathcal{D}}^{\tau-\mathrm{rig}}$ similarly.

Assume that the following hold:

(a) The indecomposable rigid objects of $\mathcal{C}$ are all shifts of indecomposable summands of T; and similarly for $\mathcal{D}$.

Table 1
Orbit categories with cluster tilting objects, which are not acyclic cluster categories.

Orbit category	Indecomposables	Rank	Indec. rigids	Quiver	Relations
$D^{\mathrm{b}}(\mathbb{K}\mathrm{A}_{3n})/\tau^n[1]$	$\frac{3n(n+1)}{2}$	n	$n(n+1)$	$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \cdots\cdots n-1 \arrow[r] & n \arrow[loop right, "\alpha"] \end{tikzcd}$	α^2
$D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_{kn})/\tau^n\varphi^n$, $kn \geq 4$, $k > 1$	kn^2	n	$n(n+1)$	$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \cdots\cdots n-1 \arrow[r, shift left, "a"] & n \arrow[l, shift left, "b"] \arrow[loop right, "\alpha"] \end{tikzcd}$	$\alpha^{k-1} - ab$, αa, $b\alpha$
$D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^4$	32	2	8	$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[loop right, "\alpha"] \end{tikzcd}$	α^3
$D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^8$	64	4	24	$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r, shift left, "a"] & 3 \arrow[l, shift left, "b"] \arrow[r] & 4 \end{tikzcd}$	aba, bab

Table 2
Orbit categories with non-cluster tilting, maximal rigid objects.

Orbit category	Indecomposables	Rank	Indec. rigids	Quiver	Relations
$D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{k(n+1)-1}[1]$, $t > 1$	$\frac{1}{2}[(2t+1)(n+1)-3](n+1)$	n	$n(n+1)$	$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \cdots\cdots n-1 \arrow[r] & n \arrow[loop right, "\alpha"] \end{tikzcd}$	α^2
$D^{\mathrm{b}}\left(\mathbb{K}\mathrm{D}_{2t(n+1)}\right)/\tau^{n+1}\varphi^n$	$2t(n+1)^2$	n	$n(n+1)$	$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \cdots\cdots n-1 \arrow[r] & n \arrow[loop right, "\alpha"] \end{tikzcd}$	α^2
$D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_7)/\tau^2$	14	1	2	$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$	α^3
$D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_7)/\tau^5$	35	2	5	$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[loop right, "\gamma"] \end{tikzcd}$	$\beta\alpha - \gamma\beta$, α^2, γ^2

(b) There is some isomorphism of quivers $\sigma : \mathcal{Q}_{\mathcal{R}_\mathcal{C}} \to \mathcal{Q}_{\mathcal{R}_\mathcal{D}}$ satisfying the following properties:
  (b1) The map σ commutes with shifts on objects and on irreducible morphisms;
  (b2) It sends T to U;
  (b3) It induces an isomorphism between $\operatorname{End}_\mathcal{C}(T)$ and $\operatorname{End}_\mathcal{D}(U)$.
(c) The finite dimensional algebra $\operatorname{End}_\mathcal{C}(T)$ is generalised standard, i.e. the morphisms in the module category are given by linear combinations of paths in its Auslander–Reiten quiver [27].
(d) The quiver $\mathcal{Q}_\mathcal{C}^{\tau-\mathrm{rig}}$ is isomorphic to the full subquiver of $\mathcal{Q}_{\mathcal{R}_\mathcal{C}}$ whose vertices are not in $\operatorname{add}\Sigma T$; and similarly for $\mathcal{D}$.

Lemma 3.1. *Under the assumptions listed above, any morphism in $\mathcal{R}_\mathcal{C}$ is a linear combination of paths in $\mathcal{Q}_{\mathcal{R}_\mathcal{C}}$ and σ induces an equivalence of categories $\mathcal{R}_\mathcal{C} \to \mathcal{R}_\mathcal{D}$.*

Proof. Assume that $T = T_1 \oplus \cdots \oplus T_n$ is basic, and T_i is indecomposable for each i. We prove the statement in three steps:

(1) Any morphism in $\mathcal{R}_\mathcal{C}$ (resp. $\mathcal{R}_\mathcal{D}$) is a linear combination of paths in $\mathcal{Q}_{\mathcal{R}_\mathcal{C}}$ (resp. $\mathcal{Q}_{\mathcal{R}_\mathcal{D}}$).
(2) The morphism σ induces a well-defined functor $\mathcal{R}_\mathcal{C} \to \mathcal{R}_\mathcal{D}$, which is faithful.
(3) The induced functor is dense and full.

(1) Let f be a morphism in $\mathcal{R}_\mathcal{C}$. By assumption (a), we may assume that it is of the form $\Sigma^a T_i \to \Sigma^b T_j$ for some $a, b \in \mathbb{Z}$, and $i, j \in \{1, \dots, n\}$. By assumption (c), the morphism $\mathcal{C}(T, \Sigma^{-a} f)$ is a linear combination of paths in $\mathcal{Q}_{\mathcal{R}_\mathcal{C}}^{\tau-\mathrm{rig}}$. Let $g \in \mathcal{R}_\mathcal{C}$ be the corresponding linear combination of paths in $\mathcal{Q}_{\mathcal{R}_\mathcal{C}}$. Such a morphism exists by assumption (d). We then have $\mathcal{C}(T, \Sigma^{-a} f - g) = 0$ so that $\Sigma^{-a} f - g$ belongs to the ideal (ΣT). Since the domain of $\Sigma^{-a} f$ lies in $\operatorname{add} T$, and T is rigid, we have $\Sigma^{-a} f = g$ and f is a linear combination of paths in $\mathcal{Q}_{\mathcal{R}_\mathcal{C}}$.

(2) Let f be a linear combination of paths in $\mathcal{Q}_{\mathcal{R}_\mathcal{C}}$. We claim that $f = 0$ in $\mathcal{R}_\mathcal{C}$ if and only if $\sigma f = 0$ in $\mathcal{R}_\mathcal{D}$. Indeed:

$$\begin{aligned} f = 0 \text{ in } \mathcal{R}_\mathcal{C} &\Leftrightarrow \Sigma^{-a} f = 0 \text{ in } \mathcal{R}_\mathcal{C} \\ &\Leftrightarrow \mathcal{C}(T, \Sigma^{-a} f) = 0 \text{ in } \operatorname{mod}\operatorname{End}_\mathcal{C}(T) \\ &\Leftrightarrow \mathcal{D}(\sigma T, \sigma\Sigma^{-a} f) = 0 \text{ in } \operatorname{mod}\operatorname{End}_\mathcal{D}(U) \\ &\Leftrightarrow \mathcal{D}(U, \Sigma^{-a}\sigma f) = 0 \text{ in } \operatorname{mod}\operatorname{End}_\mathcal{D}(U) \\ &\Leftrightarrow \Sigma^{-a}\sigma f = 0 \text{ in } \mathcal{R}_\mathcal{D} \\ &\Leftrightarrow \sigma f = 0 \text{ in } \mathcal{R}_\mathcal{D}. \end{aligned}$$

The second equivalence uses the fact that the domain of $\Sigma^{-a} f$ belongs to $\operatorname{add} T$, the fourth equivalence follows from assumptions (b1) and (b2). The third equivalence follows from assumption (d) as follows: This assumption implies that σ induces an isomorphism from $\mathcal{Q}_{\mathcal{R}_\mathcal{C}}^{\tau-\mathrm{rig}}$ to $\mathcal{Q}_{\mathcal{R}_\mathcal{D}}^{\tau-\mathrm{rig}}$ which commutes with the inclusions into $\mathcal{Q}_{\mathcal{R}_\mathcal{C}}$ and $\mathcal{Q}_{\mathcal{R}_\mathcal{D}}$.

(3) By construction, the functor $\mathcal{R}_\mathcal{C} \to \mathcal{R}_\mathcal{D}$ induced by σ is dense. For all $i = 1, \ldots, n$, let U_i be σT_i. Let g be a morphism in $\mathcal{R}_\mathcal{D}$. As above, we may assume that it is of the form $U_i \to \Sigma^k U_j$, for some $k \in \mathbb{Z}$ and some $i, j \in \{1, \ldots, n\}$. There is some $f \in \mathcal{C}(T_i, \Sigma^k T_j)$ whose image in $\operatorname{mod}\operatorname{End}_\mathcal{C}(T)$ is associated with $\mathcal{D}(U, g)$ in $\operatorname{mod}\operatorname{End}_\mathcal{D}(U)$. We thus have $\mathcal{D}(U, g - \sigma f) = 0$, which implies $\sigma f = g$. The functor induced by σ is full. □

Proposition 3.2. *For all $t \geq 1$, there are equivalences of additive categories:*

(1) $\mathcal{R}_{\mathcal{A}_{n,t}} \simeq \mathcal{R}_{\mathcal{A}_{n,1}}$;
(2) $\mathcal{R}_{\mathcal{D}_{n,t}} \simeq \mathcal{R}_{\mathcal{A}_{n,1}}$;
(3) $\mathcal{R}_{E_{7,2}} \simeq \mathcal{R}_{\mathcal{D}_{4,\tau\varphi}}$.

Proof. For each case, we need to check the assumptions of Lemma 3.1. This is done in Sections 2.2.1, 2.2.2 and 2.2.3, respectively. □

4. 2-endorigid algebras of finite type

4.1. A 2-endorigid algebra which is not 2-CY tilted

We refer to [4] or [5] for introductory notions on quivers and their representations. Consider the algebra $\Gamma = \mathbb{K}Q/I$, where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[loop right, "\gamma"] \end{tikzcd}$$

and the relations are $\beta\alpha - \gamma\beta$, α^2, γ^2.

The indecomposable projective objects in $\operatorname{mod}\Gamma$ are given by

$$P_1 = \left(\begin{array}{ccc} & 1 & \\ 1 & & 2 \\ & 2 & \end{array} \right) \text{ and } P_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

(where we write the simple modules appearing in their composition series from bottom to top) while the indecomposable injectives are

$$I_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } I_2 = P_1.$$

We have a minimal injective coresolution of $\Gamma = P_1 \amalg P_2$ given by

$$0 \to P_1 \amalg P_2 \to I_2 \amalg I_2 \to I_1 \to 0$$

and hence Γ has injective dimension 1, that is, Γ is Gorenstein of dimension 1. Then, see [20], we have that $\operatorname{Sub}\Gamma$ is a Frobenius category with projective (= injective) objects $\operatorname{add}\Gamma$, and $\underline{\operatorname{Sub}}\,\Gamma$ is a triangulated category, with suspension functor isomorphic to $\Omega^{-1}_{\operatorname{Sub}\Gamma}$.

We claim that Γ is not a 2-CY-tilted algebra. To see this, consider the simple S_2 and the module $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The exact sequence

$$0 \to S_2 \to P_2 \to S_2 \to 0$$

in $\operatorname{Sub}\Gamma$, shows that $\Omega^{-1}(S_2) \simeq S_2 \simeq \Omega^1(S_2)$. Hence also $\Omega^{-3}(S_2) \simeq S_2$. We then have that $\underline{\operatorname{Hom}}(S_2, X) \neq 0$, while clearly $\underline{\operatorname{Hom}}(X, S_2) = 0$. Therefore $\underline{\operatorname{Sub}}\,\Gamma$ is not 3-Calabi–Yau, and this implies that Γ is not a 2-CY-tilted algebra, by [20, Theorem 3.3]. The same argument shows that Γ is not d-CY-tilted for $d \geq 2$.

4.2. Standard 2-Calabi–Yau categories

Recall that our base field $\mathbb{K}$ is assumed to be algebraically closed and of characteristic 0. In that setup, it is known from [8] that all finite-dimensional algebras of finite representation type are standard: Their module categories are the path categories on their Auslander–Reiten quivers modulo all mesh relations. In this section, we address the following related question: Let $\mathcal{C}$ be a triangulated category of finite type. If $\mathcal{C}$ is 2-CY with cluster tilting objects, is it standard? We were not able to answer this question so far. However, we prove here that $\mathcal{C}$ is generalised standard [27] in the following sense.

Definition 4.1. A $\mathbb{K}$-linear, Krull–Schmidt, Hom-finite, triangulated category with a Serre functor is called *generalised standard* if all of its morphisms are given by linear combinations of paths in its Auslander–Reiten quiver.

Proposition 4.2. *Let $\mathcal{C}$ be a $\mathbb{K}$-linear, Krull–Schmidt, 2-Calabi–Yau, triangulated category. Assume that $T \in \mathcal{C}$ is a cluster tilting object whose endomorphism algebra is generalised standard. Then $\mathcal{C}$ is generalised standard.*

Proof. Let Γ be the Auslander–Reiten quiver of $\mathcal{C}$, and $\overline{\Gamma}$ be the one of $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$. By [10, Proposition 3.2] the AR-sequences in $\operatorname{mod}\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ are induced by the AR-triangles in $\mathcal{C}$. It follows that $\overline{\Gamma}$ is naturally a full subquiver of Γ and that we can pick a basis $(e_\alpha)_{\alpha\in\Gamma_1}$ of irreducible morphisms in $\mathcal{C}$ adapted to Γ (i.e. satisfying the mesh relations) such that $(\mathcal{C}(T, e_\alpha))_{\alpha\in\overline{\Gamma}_1}$ is a basis of irreducible morphisms in $\operatorname{mod}\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ adapted to $\overline{\Gamma}$. In what follows, we will use the following notation: if $p = \sum_i \lambda_i \alpha^i_{k_i} \cdots \alpha^i_1$ is a linear combination of paths in Γ, we write e_p for the morphism $\sum_i \lambda_i e_{\alpha^i_{k_i}} \circ \cdots \circ e_{\alpha^i_1}$. We note that the statement of the proposition is an immediate consequence of the two claims below.

Claim 1. *Any morphism f in $\mathcal{C}$ is of the form $f = e_p + g$, where p is a linear combination of paths in Γ and g belongs to the ideal (ΣT).*

Proof of Claim 1: Since $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ is generalised standard, $\mathcal{C}(T, f)$ is of the form $\mathcal{C}(T, e_p)$ where p is a linear combination of paths in $\bar{\Gamma}$, viewed as a subquiver of Γ. We thus have $f = e_p + g$ for some $g \in (\Sigma T)$.

Claim 2. *Any morphism $g \in (\Sigma T)$ is of the form e_p, for some linear combination p of paths in Γ.*

Proof of Claim 2: Let $X \xrightarrow{g} Y$ belong to (ΣT). Then there are some $U \in \operatorname{add} T$, $\Sigma U \xrightarrow{a} Y$ and $X \xrightarrow{b} \Sigma U$ such that $g = ab$. Applying Claim 1 to Σb gives a linear combination q of paths in Γ and a morphism h in (ΣT) such that $\Sigma b = e_q + h$. Since T is rigid and Σb has codomain in $\operatorname{add} \Sigma^2 T$, then h is zero. A similar argument shows that $\Sigma^{-1} a$ is of the form e_r. The claim follows. □

Corollary 4.3. *Let $\mathcal{C}$ be a $\mathbb{K}$-linear, Krull–Schmidt, 2-Calabi–Yau, triangulated category. Assume that $T \in \mathcal{C}$ is a cluster tilting object whose endomorphism algebra is of finite representation type. Then $\mathcal{C}$ is generalised standard.*

4.3. The standard 2-endorigid algebras of finite representation type

We call a finite dimensional $\mathbb{K}$-algebra *standard 2-endorigid* if it is isomorphic to the endomorphism algebra of a maximal rigid object in a standard, ($\mathbb{K}$-linear, Krull–Schmidt) 2-Calabi–Yau, triangulated category.

The standard 2-CY-tilted algebras of finite representation type were classified by Bertani–Oppermann in [7, Theorem 5.7], where a quiver with potential is given for each isomorphism class. Ladkani noticed, see [22, Section 2.6], that a 2-CY category with cluster tilting objects was missing in the list given in [13, Appendix]. For a comprehensive classification of all standard 2-CY-tilted algebras of finite representation type one thus has to take the algebra appearing in [22] into account.

Theorem 4.4. *The connected, standard 2-endorigid algebras of finite representation type are exactly the standard 2-CY-tilted algebras of finite representation type listed in [7, Theorem 5.7] (see also [22, Section 2.6]) and the non-Jacobian 2-endorigid algebra of Section 4.1.*

Proof. The theorem follows from the classification [2,13] of all standard 2-Calabi–Yau triangulated categories with maximal rigid objects (see Table 1 and Table 2) and from the equivalences of categories in Proposition 3.2. □

Remark 4.5. We note that the conclusion of Corollary 4.3 being weaker than one would like, we do not know if the list discussed above contains all 2-endorigid algebras of finite representation type.

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