Open Boundary Conditions for Fluid Dynamic Problems

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1 Introduction

The overall context of this work is the design of efficient open boundary conditions (OBCs) for limited area models of the ocean circulation. In these limited areas, complex systems of equations are solved using a high resolution grid. On the artificial boundaries, boundary conditions with some available external information, must be prescribed. Such external information is generally available from previous simulations of large scale low resolution models. Ideally, the OBC must evacuate the outgoing information reaching the boundary, and must take into account the incoming part of the external information.

Several OBCs for different equations have been proposed in the past. The most recent ones rely on Absorbing Boundary Conditions (see Blayo and Debreu [2005] for a review of methods in oceanographic context). In this paper, we intend to analyse such conditions from a mathematical point of view on a model problem, and to propose improvements.

For f in $C^2([0,1])$, we are interested in computing the finite difference approximation of the solution of :

$$\begin{cases} \mathcal{L}u = f \text{ on } (0,1), \\ u(0) = 0, \\ u(1) = 0, \end{cases} \tag{1}$$

where $\mathcal{L} := -\frac{d^2}{dx^2} + \alpha$ is a diffusion operator with α a positive scalar. Note that other boundary conditions can be considered without changing the main results of the paper.

The interval $[0,1]$ is discretized with the uniform mesh $(x_i)_{0\leq i\leq 2N}$ such as $x_i - x_{i-1} = h = \frac{1}{2N}$, $x_0 = 0$ and $x_{2N} = 1$. We also consider the coarse mesh :

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$(x_{2i})_{0\leq i\leq N}$ and we denote by I the interpolation operator which is defined from the coarse grid to the fine grid.

We denote by L_H (resp. L_h) the classical three points discrete operator corresponding to (1) on the coarse (resp. fine) grid. These two operators are consistent with $\mathcal{L}$ and are second order accurate. We assume that a coarse grid solution has already been computed, that will be denoted $U^{ext} = (U_0^{ext}, \cdots, U_{2i}^{ext}, \cdots, U_{2N}^{ext}) \in \mathbb{R}^{N+1}$ in the following. Thus U^{ext} is solution of

$$\begin{cases} L_H U^{ext} = f_H, \\ U_0^{ext} = 0, \\ U_{2N}^{ext} = 0. \end{cases}$$

Let assume that we are interested in obtaining a more accurate solution, denoted U^{loc}, on a local domain $\Omega_{loc} = (0, x_{2i_0})$. We introduce the boundary operators B_h and B_H which come from the discretization of the same differential operator but B_h is defined on the fine grid, whereas B_H is defined on the coarse one. We propose to solve the following problem which uses the informations provided by the exterior problem :

$$\begin{cases} L_h U^{loc} = f_h, \\ U_0^{loc} = 0, \\ B_h U^{loc} = B_H U^{ext} + g, \end{cases} \tag{2}$$

with $U^{loc} = (U_0^{loc}, U_1^{loc}, \cdots, U_{2i_0-1}^{loc}, U_{2i_0}^{loc}) \in \mathbb{R}^{2i_0+1}$, the local solution.

The boundary condition is said “exact“ if U^{loc} coincides with the reference solution $U^{ref} = (U_0^{ref}, U_1^{ref}, \cdots, U_{2N}^{ref}) \in \mathbb{R}^{2N+1}$ defined by

$$\begin{cases} L_h U^{ref} = f_h, \\ U_0^{ref} = 0, \\ U_{2N}^{ref} = 0. \end{cases}$$

A first choice for B_h and B_H corresponds to a Dirichlet condition :

$$B_h W = W_{2i_0}, \quad B_H W = W_{2i_0}, \;\; g = 0. \tag{3}$$

A more efficient choice appears in Flather [1976] : he proposes to use a Sommerfeld condition, which is an absorbing boundary condition for the wave equation. In a more general context, it can be interpreted as :

$$B_h = \mathcal{A}_h, \; B_H = \mathcal{A}_H, \; g = 0, \tag{4}$$

where $\mathcal{A}_h$ (resp. $\mathcal{A}_H$) denotes an absorbing boundary operator discretized on the fine (resp. coarse) grid at x_{2i_0}.

The purpose of this paper is to identify the "exact" boundary condition, that means we want to find B_h, B_H and g such that $U^{loc} = U^{ref}_{|\Omega_{loc}} = (U_0^{ref}, U_1^{ref}, \cdots, U_{2i_0-1}^{ref}, U_{2i_0}^{ref})$.

In Section 2 we build such an exact condition, by relying on ABCs for an equation with non zero right hand side. The performance of the method is illustrated by numerical results. In section 3 we propose a variant of the previous OBC which is theoretically simpler but is more demanding in terms of numerical computations. Numerical results are given for the 1-D Shallow Water equations.

2 New Open Boundary Conditions

The subsection 2.1 provides a result which will be useful to the building of the exact OBC.

2.1 Transparent boundary condition for an equation with right hand side

The most common boundary conditions introduced to solve such an open problem are the absorbing boundary conditions (ABC) (see e.g. Engquist and Majda [1977]). However, these conditions are generally computed when the support of the source term is strictly included in Ω_{loc}, which is not the case here. We thus need to revisit the ABC for an equation with a non zero right hand side.

Theorem 1. *Let w_0 be a real number and $\mathcal{F}$ be in $L^2(x_{2i_0}, 1)$. If w is the solution of the following equation :*

$$\begin{cases} \mathcal{L}w = \mathcal{F} \text{ on } (x_{2i_0}, 1), \\ w(x_{2i_0}) = w_0, \\ w(1) = 0, \end{cases} \tag{5}$$

then w satisfies the following boundary condition :

$$w'(x_{2i_0}) + \lambda w(x_{2i_0}) = p, \tag{6}$$

where $\lambda = -\frac{z'(x_{2i_0})}{z(x_{2i_0})}$, $p = \frac{\int_{x_{2i_0}}^1 \mathcal{F}(\sigma) z(\sigma) d\sigma}{z(x_{2i_0})}$ and $z(x) = e^{\sqrt{\alpha}(1-x)} - e^{-\sqrt{\alpha}(1-x)}$.

2.2 Exact Open Boundary Condition

In order to find the exact OBC for problem (2), we consider the equation for the error :

$$\begin{cases} L_h(U^{ref}_{|\Omega^{loc}} - U^{loc}) = 0, \\ U_0^{ref} - U_0^{loc} = 0, \\ B_h(U^{ref} - U^{loc}) = B_h U^{ref} - B_H U^{ext} - g, \end{cases} \tag{7}$$

so that if we can find B_h, B_H and g such that :

$$B_h U^{ref} - B_H U^{ext} = g, \tag{8}$$

then by linearity and by uniqueness of the solution, we have $U^{loc} = U^{ref}_{|\Omega^{loc}}$.

In order to find such B_h, B_H and g, we will interpret the Theorem 1 at the discrete level : $U^{ref} - IU^{ext}$ and $F = L_h(U^{ref} - IU^{ext})$ will play the role of w and $\mathcal{F}$.

More precisely, the operators B_h and B_H will be an approximation of the absorbing boundary operator $\frac{d}{dx} + \lambda$. We propose to use a finite volume scheme to discretize the derivative : if u is solution of $\mathcal{L}u = f$, then u' can be approximated at point x_i by $D_h^-(U)_i = (\frac{\alpha h}{2} + \frac{1}{h})U_i - \frac{1}{h}U_{i-1} - \frac{h}{2}f(x_i)$ or by $D_h^c(U)_i = \frac{1}{2h}(U_{i+1} - U_{i-1})$.

We propose then to use the following operators :

$$B_h U^{loc} = D_h^-(U^{loc})_{2i_0} + \lambda U^{loc}_{2i_0}, \;\; B_H U^{ext} = D_h^c(IU^{ext})_{2i_0} + \lambda U^{ext}_{2i_0}. \tag{9}$$

In order to find g, we need to know $F = L_h(U^{ref} - IU^{ext})$. That is the purpose of the next theorem.

Theorem 2. *Let f be in $C^2([0,1])$. Then, the exterior error projected on the fine grid, $U^{ref} - IU^{ext}$, satisfies :*

$$L_h(U^{ref} - IU^{ext}) = F^h + \mathcal{O}(h^4),$$

with $F^h = F^1 + h^2 F^2 - \frac{h^2}{2} F^3$. The even and odd components of F^i are defined by :

$$F^1 = \begin{pmatrix} \cdots \\ u''(x_{2i}) \\ -u''(x_{2i+1}) \\ \cdots \end{pmatrix}, F^2 = \begin{pmatrix} \cdots \\ cu^{(4)}(x_{2i}) \\ -cu^{(4)}(x_{2i+1}) \\ \cdots \end{pmatrix}$$

and $F^3 = \begin{pmatrix} \cdots \\ 0 \\ u^{(4)}(x_{2i+1}) + f''(x_{2i+1}) \\ \cdots \end{pmatrix}$, *with c a constant.*

We are now in position to propose a right hand side :

$$g = \frac{T_h(\tilde{F}^h)}{z(x_{2i_0})}, \tag{10}$$

where $(\tilde{F}^h_{2i})_{i_0 \leq i \leq N} = (F^h_{2i} z(x_{2i}))_{i_0 \leq i \leq N}$ and $T_h(Z)$ is the trapezoidal rule to integrate the vector Z : $T_h(Z) = \frac{h}{2}(Z_{2i_0} + Z_{2N}) + h \sum_{i=2i_0+1}^{2N-1} Z_i$. The term g will be handled in the next subsection.

The OBC defined by (9)-(10) will be named $(\text{ABC})_g$ whereas the one defined by (9) and $g = 0$ (corresponding to (4)) will be named (ABC). This last condition is the one which is usually used in oceanographic context.

2.3 Approximation of g

This section deals with the approximation of g given in (10). Vectors F^1 and F^2 have similar patterns : Lemma 1 explains how to integrate them, whereas F^3 is integrated thanks to Lemma 2.

Lemma 1. *Let w be a $C^4([a,b])$ function and $(x_i)_{0 \leq i \leq 2N}$ be a mesh of $[a,b]$ with $x_{i+1} - x_i = h$. Integrating $W = (w(x_0), -w(x_1), w(x_2), \cdots, -w(x_{2N-1}), w(x_{2N}))$ by a trapezoidal rule gives $\frac{h^2}{4}(w'(x_{2N}) - w'(x_0)) + \mathcal{O}(h^3)$.*

Lemma 2. *Let w be a $C^4([a,b])$ function and $(x_i)_{0 \leq i \leq 2N}$ be a mesh of $[a,b]$ with $x_{i+1} - x_i = h$. Integrating $W = (0, w(x_1), 0, \cdots, w(x_{2N-1}), 0)$ by a trapezoidal rule gives $\frac{1}{2}\int_a^b w(t)dt + \mathcal{O}(h)$.*

We are now in position to compute g given in (10). We have $T_h(\tilde{F}^h) = T_h(\tilde{F}^1) + \frac{h^2}{2} T_h(\tilde{F}^2) - \frac{h^2}{2} T_h(\tilde{F}^3)$ with

$$\begin{aligned}
T_h(\tilde{F}_1) &= \frac{h^2}{4}((u''z)'(x_{2N}) - (u''z)'(x_0)) + \mathcal{O}(h^3), \\
T_h(\tilde{F}_2) &= \mathcal{O}(h^2), \\
T_h(\tilde{F}_3) &= \frac{1}{2}\int_{x_{2i_0}}^1 (u^{(4)}z + f''z)(\sigma) d\sigma + \mathcal{O}(h).
\end{aligned}$$

On $(x_{2i_0}, 1)$, we only know U^{ext}, so u will be approximated in this formula by IU^{ext}. The last term which is an integral, will be approximated by a trapezoidal rule. We also use the equation $\alpha u - u'' = f$ to make the fourth order derivative term to disappear. We finally obtain :

$$\begin{aligned}
g = \frac{h^2}{4z(x_{2i_0})} \Big(& (\alpha D_h^-(IU^{ext})_{2N} - f'(x_{2N})) z(x_{2N}) + (\alpha U^{ext}_{2N} - f(x_{2N})) z'(x_{2N}) \\
& - (\alpha D_h^c(IU^{ext})_{2i_0} - f'(x_{2i_0})) z(x_{2i_0}) - (\alpha U^{ext}_{2i_0} - f(x_{2i_0})) z'(x_{2i_0}) \\
& - \alpha^2 T_h((U^{ext}_{2i} z(x_{2i}))_{i_0 \leq i \leq N}) + \alpha T_h((f(x_{2i}) z(x_{2i}))_{i_0 \leq i \leq N}) \Big).
\end{aligned} \tag{11}$$

2.4 Numerical Results

In this section α is equal to 10 and f is chosen such that $u(x) = \sin(2\pi x)$ is the exact solution of (1). Figure 1 shows the difference between the exact solution and the numerical one on the coarse and fine grids (exterior error and reference error). For several locations of the interface we can also see the solution obtained with a Dirichlet (3), (ABC) (4) or $(\text{ABC})_g$ (9)-(11) conditions. We first observe that without correction, the (ABC) solution in

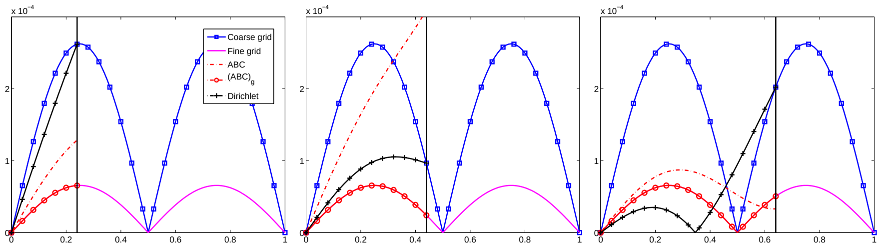

Fig. 1 Error between the exact solution and the numerical one for the interface at $x_{2i_0} = 0.24$, $x_{2i_0} = 0.44$ and $x_{2i_0} = 0.64$.

not systematically better than the Dirichlet solution. Then we can see that the corrected condition (9) provides a solution which is as good as the reference one.

3 Variant for the Open Boundary Condition

3.1 Principle

The method presented in the previous section gives very good results. However for more complex equations, the right hand side (11) can not be easily computed. That is why we propose now another way to get this perturbed term. We first remark that the exact condition (8) is not unique : for every choice of B_h and B_H, we can build a corresponding g such that the condition is exact. Choosing B_h and B_H to be absorbing boundary operators leads however to a g that does not depend on U^{ref}. If we choose B_h equal to the identity, i.e. if we consider a Dirichlet condition, then g will depend on U^{ref} : $g = U^{ref}_{2i_0} - U^{ext}_{2i_0}$ and we propose to approximate this term by a Richardson type procedure. Because the numerical scheme is second order accurate, we have :

$$U_i^{ref} = u(x_i) + h^2 c(x_i) + \mathcal{O}(h^4),$$
$$U_i^{ext} = u(x_i) + 4h^2 c(x_i) + \mathcal{O}(h^4),$$

which yields $g = U^{ref}_{2i_0} - U^{ext}_{2i_0} = -3h^2 c(x_{2i_0}) + \mathcal{O}(h^4)$. If we now solve the equation on even a coarser grid with a $4h$ space step, let U^{EXT} denotes this solution, then we have :

$$U^{ext}_{2i_0} - U^{EXT}_{2i_0} = -12h^2 c(x_{2i_0}) + \mathcal{O}(h^4).$$

We propose then to approximate g by $(U^{ext}_{2i_0} - U^{EXT}_{2i_0})/4$.

Numerical tests have been implemented for the Laplacian model problem and show that this perturbed Dirichlet condition gives the same very good results as the $(\mathrm{ABC})_g$ condition. In the next section, we implement this strategy for the linear 1-D Shallow Water equations.

3.2 Numerical Results for the linear 1-D Shallow Water equations

In this section we consider the open boundary problem for the following equations :

$$\begin{cases} \dfrac{\partial u}{\partial t} + u_0 \dfrac{\partial u}{\partial x} + g\dfrac{\partial h}{\partial x} + ru = 0 & \text{on } (-L,L)\times(0,T), \\ \dfrac{\partial h}{\partial t} + u_0 \dfrac{\partial h}{\partial x} + h_0\dfrac{\partial u}{\partial x} + G_x u = 0 & \text{on } (-L,L)\times(0,T) \\ (\sqrt{\dfrac{h_0}{g}}u + h)(-20,\cdot) = 0 \text{ and } (\sqrt{\dfrac{h_0}{g}}u - h)(20,\cdot) = 0 & \text{on } (0,T), \\ u(\cdot,0) = 0 \text{ and } h(\cdot,0) = \dfrac{1}{4+2\cosh(\frac{x+5}{2})} & \text{on } (-L,L), \end{cases}$$

where u_0, G_x, r and g are constants. We make the assumptions $u_0 + c > 0$ and $u_0 - c < 0$, where $c = \sqrt{gh_0}$ so that there are two characteristics : $\frac{1}{2}(u\sqrt{h_0/g} + h)$ travels with the positive velocity $u_0 + c$ and $\frac{1}{2}(u\sqrt{h_0/g} - h)$ travels with the negative velocity $u_0 - c$.

The parameters for the numerical experiment are : $L = 20$, $h_0 = 25$ m, $c = 16$ m.s^{-1}, $u_0 = 0.1$ m.s^{-1}, $r = 3.10^{-3}$s^{-1}, $g = 10$ m.s^{-2} and $G_x = 0.095$. The interface is at $x_{2i_0} = -13$ and we can see on Figure 2-left that a wave enters the local domain $(-20,-13)$ during the experiment.

Figure 2-right shows the logarithm of the error on the local domain $(-20,-13)$ between a very fine solution and several numerical ones : exterior solution (coarse grid) and reference solution (fine grid). This figure also shows the error when the local solution is computed with (4) where B_h and B_H are the operators corresponding to the incoming characteristics and when this boundary condition is perturbed by a g obtained by a Richardson procedure (see Section 3.1). We can see again that this last method gives a result

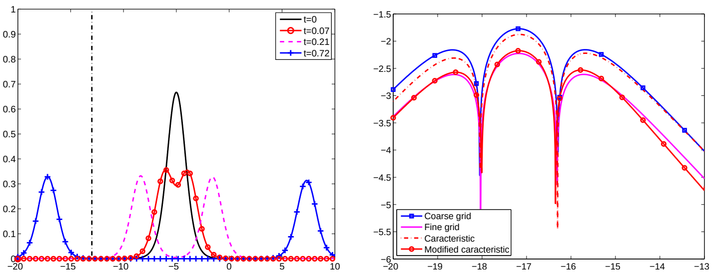

Fig. 2 Left panel : solutions at several instants. Right panel : logarithm of the error in the local domain at $t = 0.72$.

very close to the reference solution while the classical ABC (characteristics) improves just a little bit the coarse solution.

4 Conclusion

For the very simple case of the Laplace equation, we have properly designed an exact open boundary condition (8). More precisely, we have emphasized the necessity to correct the usually used absorbing boundary conditions. We have performed all the exact computations and we have obtained very good results. If one is ready to solve the exterior problem with a very coarse grid, one can easily compute numerically the corrected term. We have shown such encouraging numerical results for the linear Shallow Water equations.

References

E. Blayo and L. Debreu. Revisiting open boundary conditions from the point of view of characteristic variables. *Ocean Modelling*, 9(3):231–252, 2005.

B. Engquist and A. Majda. Absorbing boundary conditions for the numerical simulation of waves. *Math. Comp.*, 31(139):629–651, 1977.

R A Flather. A tidal model of the north-west european continental shelf. *Mmoires Socit Royale des Sciences de Lige*, 10(6):141–164, 1976.