Artin groups as Zariski-dense subgroups of $\mathrm{GL}_{\it N}$

Ivan Marin



Definitions
Long's theorem in type *A*

The density theorem

Statement Consequences

Sketch of proof

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Using the same ideas, Long proved that the Frattini subgroup $\Phi(G)$ (= intersection of the proper maximal subgroups) is trivial.



Generalizations?

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Only one consequence had been generalized to Artin groups :

Theorem (L.Paris 2004)

If
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 then $X \subset Z(B)$ or $Y \subset Z(B)$.

L.Paris used combinatorical methods. These methods do not extend to subgroups.

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Theorem implies that *B* is strongly linear.



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Theorem implies that "many" subgroups of B are strongly linear.



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Proof

B linear finitely generated implies that $\Phi(B)$ is nilpotent (Platonov theorem). Since $\Phi(B) \lhd B$ it follows that $\Phi(B) \subset Z(B)$. Through the length morphism $\ell: B \to \mathbb{Z}$ we get maximal subgroups $\ell^{-1}(p\mathbb{Z})$ for all p. Hence

$$\Phi(B) \subset Z(B) \cap \mathrm{Ker} \ell = \{e\}$$



The Krammer representations

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$$\mathcal{L} = \mathbb{Q}[q, q^{-1}, t, t^{-1}].$$

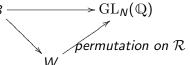
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Theorem (Krammer & Bigelow + Digne & Cohen-Wales)

If B is of type ADE, there exists $R_K: B \hookrightarrow \operatorname{GL}_N(\mathcal{L})$ with $N = \#\mathcal{R}$

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And it is known (e.g. by Crisp 2000) that for all B there exists B' of type ADE such that $B \longleftrightarrow B'$ commutes.



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Interlude

This proves the residual torsion-free nilpotence hence the biorderability of P, because $R_K(P)$ is a subgroup of

$$\operatorname{GL}_N^o(\mathcal{M}) = \{ X \in \operatorname{GL}_N(\mathcal{M}) \mid X \equiv 1 \mod h \}$$

which has this property.



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Now we start from $R: B \hookrightarrow \operatorname{GL}_N(\mathcal{M})$ for some N. If $B = \langle \sigma_1, \dots, \sigma_n \rangle$, let $H = \langle \sigma_1^2, \dots, \sigma_n^2 \rangle$. It is sufficient to show that $\overline{R(H)} = \operatorname{GL}_N(K)$.

Main steps

We have $R(\sigma_i^2) = 1 + h \dots$, so we can define $u_i = \log R(\sigma_i^2)$ modulo h, in $\mathfrak{gl}_N(\mathbb{C})$.

Lemma

$$\mathrm{Lie}\overline{R(H)} \supset < u_1, \dots, u_n >_{\mathrm{Lie}} \otimes_{\mathbb{C}} K$$

Proposition

$$< u_1, \ldots, u_n >_{\operatorname{Lie}} = \mathfrak{gl}_{\mathcal{N}}(\mathbb{C})$$

This implies the theorem.



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In fact $u_i \in \mathfrak{gl}_N(\mathbb{Z}[m])$. Then the proposition is true as soon as $\dim_{\mathbb{C}} < u_1, \ldots, u_n >_{\mathrm{Lie}} \geq N^2$ for one $m \in \mathbb{C}$, as soon as $\dim_{\mathbb{Q}} < u_1, \ldots, u_n >_{\mathrm{Lie}} \geq N^2$ for one $m \in \mathbb{Q}$, as soon as $\dim_{\mathbb{F}_p} < \overline{u_1}, \ldots, \overline{u_n} >_{\mathrm{Lie}} \geq N^2$ for one $m \in \mathbb{Z}$ and one prime p.

By direct computation this proves the theorem for small and exceptional types.

Types ADE

Let $E = \langle v_s; s \in \mathcal{R} \rangle$ as a \mathbb{Q} -vector space, and $m \in \mathbb{Q}$. We define $t_s \in \operatorname{End}(E)$ by

$$\begin{cases} t_s.v_s &= mv_s \\ t_s.v_u &= v_u & \text{if } su = us, s \neq u \\ t_s.v_u &= v_{sus} - v_s & \text{if } su \neq us \end{cases}$$

Let $s_i \in W$ be the image of $\sigma_i \in B$.

Lemma

$$u_i = t_{s_i}$$

Lemma

$$< t_{s_i} | 1 \le i \le n >_{Lie} = < t_s | s \in \mathcal{R} >_{Lie}$$

Prop

$$< t_s \mid s \in \mathcal{R} >_{Lie} = \mathfrak{gl}(E)$$

By induction on the rank of W.



Backwards

These formulas are enough to define the (generalized) Krammer representation in types ADE.

Recall
$$P = \pi_1(X)$$
, $B = \pi_1(X/W)$.

Let $\alpha_H \in (\mathbb{C}^n)^*$ such that $H = \operatorname{Ker} \alpha_H$, for $H \in \mathcal{A}$.

Then
$$\omega_H = \frac{d\alpha_H}{\alpha_H} \in \Omega_1(X)$$

Letting $t_H = t_s$, then

$$\omega = \frac{1}{2\mathrm{i}\pi} h \sum_{H \in \mathcal{A}} t_H \omega_H \in \Omega^1(X) \otimes \mathfrak{gl}_N(\mathcal{M})$$

is integrable : $d\omega + \omega \wedge \omega = 0$.

By monodromy we get $R: B \to GL_N(\mathcal{M}) \subset GL_N(K)$. with $t_{s_i} = \log R(\sigma_i^2)$ modulo h.

Theorem

If $m \notin \mathbb{Q}$ then $R \simeq R_K$.

