Artin groups as Zariski-dense subgroups of GL_N

Ivan Marin

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Using the same ideas, Long proved that the Frattini subgroup $\Phi(G)$ (= intersection of the proper maximal subgroups) is trivial.

Generalizations ?

The proof of Long relies upon the Nielsen-Thurston classification, and the result also holds for mapping class groups.

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Only one consequence had been generalized to Artin groups :

Theorem (L.Paris 2004)

If $B \simeq X \times Y$ then $X \subset Z(B)$ or $Y \subset Z(B)$.

L.Paris used combinatorical methods. These methods do not extend to subgroups.

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Theorem implies that B is strongly linear.

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Theorem implies that "many" subgroups of B are strongly linear.

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B linear finitely generated implies that $\Phi(B)$ is nilpotent (Platonov theorem). Since $\Phi(B) \lhd B$ it follows that $\Phi(B) \subset Z(B)$. Through the length morphism $\ell : B \to \mathbb{Z}$ we get maximal subgroups $\ell^{-1}(p\mathbb{Z})$ for all p. Hence

$$\Phi(B) \subset Z(B) \cap \mathrm{Ker}\ell = \{e\}$$

The Krammer representations

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Theorem (Krammer & Bigelow + Digne & Cohen-Wales)

If B is of type ADE, there exists $R_K : B \hookrightarrow \mathrm{GL}_N(\mathcal{L})$ with $N = \#\mathcal{R}$ such that $(R_K)_{q=t=1}$:

$$\begin{array}{ccc} B & \longrightarrow & \mathrm{GL}_N(\mathbb{Q}) \\ & \searrow \quad \nearrow_{\text{permutation on } \mathcal{R}} & \\ & W & \end{array}$$

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And it is known (e.g. by Crisp 2000) that for all B there exists B' of type ADE such that

$$\begin{array}{ccc} B & \hookrightarrow & B' \\ \downarrow & & \downarrow \\ W & \hookrightarrow & W' \end{array}$$

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Interlude

This proves the residual torsion-free nilpotence hence the biorderability of P, because $R_K(P)$ is a subgroup of

$$\mathrm{GL}_N^{o}(\mathcal{M}) = \{X \in \mathrm{GL}_N(\mathcal{M}) \mid X \equiv 1 \ mod \ h\}$$

which has this property.

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Main steps

We have $R(\sigma_i^2) = 1 + h \dots$, so we can define $u_i = \log R(\sigma_i^2)$ modulo h, in $\mathfrak{gl}_N(\mathbb{C})$.

Lemma

$$\mathrm{Lie}\overline{R(H)} \supset < u_1, \dots, u_n >_{\mathrm{Lie}} \otimes_{\mathbb{C}} K$$

Proposition

$$< u_1, \dots, u_n >_{\mathrm{Lie}} = \mathfrak{gl}_N(\mathbb{C})$$

This implies the theorem.

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By direct computation this proves the theorem for small and exceptional types.

Types ADE

Let $E =< v_s; s \in \mathcal{R} >$ as a $\mathbb{Q}$-vector space, and $m \in \mathbb{Q}$.
We define $t_s \in \mathrm{End}(E)$ by

$$\begin{cases} t_s.v_s & = & mv_s & \\ t_s.v_u & = & v_u & \text{if } su = us, s \neq u \\ t_s.v_u & = & v_{sus} - v_s & \text{if } su \neq us \end{cases}$$

Let $s_i \in W$ be the image of $\sigma_i \in B$.

Lemma

$u_i = t_{s_i}$

Lemma

$< t_{s_i} | 1 \leq i \leq n >_{Lie} = < t_s \mid s \in \mathcal{R} >_{Lie}$

Prop

$< t_s \mid s \in \mathcal{R} >_{Lie} = \mathfrak{gl}(E)$

By induction on the rank of W.

Backwards

These formulas are enough to define the (generalized) Krammer representation in types ADE.
Recall $P = \pi_1(X)$, $B = \pi_1(X/W)$.
Let $\alpha_H \in (\mathbb{C}^n)^*$ such that $H = \mathrm{Ker}\alpha_H$, for $H \in \mathcal{A}$.
Then $\omega_H = \frac{d\alpha_H}{\alpha_H} \in \Omega_1(X)$
Letting $t_H = t_s$, then

$$\omega = \frac{1}{2\mathrm{i}\pi} h \sum_{H \in \mathcal{A}} t_H \omega_H \in \Omega^1(X) \otimes \mathfrak{gl}_N(\mathcal{M})$$

is integrable : $d\omega + \omega \wedge \omega = 0$.
By monodromy we get $R : B \to \mathrm{GL}_N(\mathcal{M}) \subset \mathrm{GL}_N(K)$. with $t_{s_i} = \log R(\sigma_i^2)$ modulo h.

Theorem

If $m \notin \mathbb{Q}$ then $R \simeq R_K$.