Krammer representations for complex braid groups

Ivan Marin

June 2008

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Fact : every reflection group is a direct product of irreducible ones.

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(Broué-Malle-Rouquier) B_0 embeds in B

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(Steinberg) $W_0 < \mathrm{GL}(V)$ is a complex reflection group, $\mathcal{R}_0 \subset \mathcal{R}$

(Broué-Malle-Rouquier) B_0 embeds in B

(uniquely up to P-conjugation)

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Compare with : MCG are generated by Dehn twists,
and have special subgroups fixing curve systems.

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Remark. When $W = \mathfrak{S}_n$, $\mathcal{T}$ is the Lie algebra of (horizontal) chord diagrams.

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This was the only contruction known so far which worked for arbitrary complex reflection groups.

A new integrable 1-form

Let $N = \#\mathcal{R}$, and $\check{\rho} : W \to \mathrm{GL}_N(\mathbb{C})$ the natural permutation representation on $\mathcal{R}$.
Basis of $V = \mathbb{C}^N$: $v_s, s \in \mathcal{R}$, with $w.v_s = v_{wsw^{-1}}$.
Let $m \in \mathbb{C}$.

Theorem

The formulas

$$\left\{ \begin{array}{rcl} t_s.v_s & = & m v_s \\ t_s.v_u & = & v_{sus} - \alpha(s,u) v_s \text{ if } s \neq u \end{array} \right.$$

define an equivariant representation of $\mathcal{T}$, *where*

$$\alpha(s,u) = \#\{y \in \mathcal{R} \mid yuy = s\}$$

Irreducible components

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- For generic values of m, $\rho_c(\mathcal{T}) = \mathfrak{gl}(V_c)$

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- This theorem is true when W is Coxeter (I.M.).
- Among exceptional groups, only G_{13} has $\#\mathcal{R}/W > 1$, and its braid group is isomorphic to the one of Coxeter type $I_2(6)$.

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Proposition

(I.M.) If W is a Coxeter group, or of type G_{25}, G_{26}, G_{32}, then P is residually torsion-free nilpotent.

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so under the conjecture this settles the case of $\#\mathcal{R}/W = 1$ for W a reflection group.

First miracle

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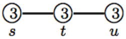
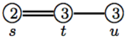
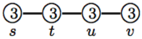
G_{25} 

G_{26} 

G_{32} 

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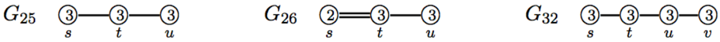

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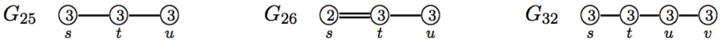

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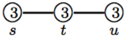

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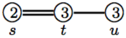

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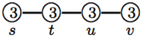

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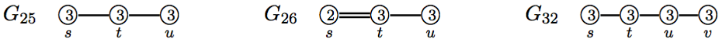

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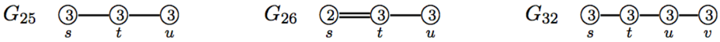

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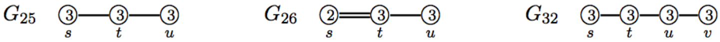

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Then (Miracle !) $R(P) \subset \mathrm{GL}_N^0(A)$.

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$$\begin{cases} \sigma_k x_{k,k+1} = tq^2 x_{k,k+1} & \\ \sigma_k x_{i,k} = (1-q)x_{i,k} + q x_{i,k+1} & i < k \\ \sigma_k x_{i,k+1} = x_{i,k} + tq^{k-i+1}(q-1)x_{k,k+1} & i < k \\ \sigma_k x_{k,j} = tq(q-1)x_{k,k+1} + q x_{k+1,j} & k+1 < j \\ \sigma_k x_{k+1,j} = x_{k,j} + (1-q)x_{k+1,j} & k+1 < j \\ \sigma_k x_{i,j} = x_{i,j} & i < j < k \text{ or } k+1 < i < j \\ \sigma_k x_{i,j} = x_{i,j} + tq^{k-i}(q-1)^2 x_{k,k+1} & i < k < k+1 < j \end{cases}$$

where t and q denote algebraically independent parameters.
Embed $\mathbb{Q}(q,t)$ into $K = \mathbb{C}((h))$ by $q \mapsto -je^h$ and $t \mapsto e^{\sqrt{2}h}$
Then (Miracle !) $R(P) \subset \mathrm{GL}_N^0(A)$.
Hence P is residually torsion-free nilpotent.

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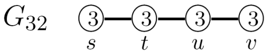

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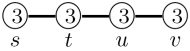

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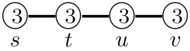

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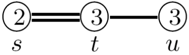

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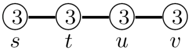

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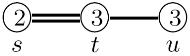

$$< s, t, u \mid stst = tsts, su = us, tut = utu, s^2 = t^3 = u^3 = 1 >$$

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These two morphisms are defined by $(s, t, u) \mapsto ((tu)^3, s, t)$.

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This is the right one !

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(Recall that residual torsion-free nilpotent groups are bi-orderable and residually p for all p.)