Braid representations and arithmetic characters

Ivan Marin

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Preliminaries
- Profinite groups
- Geometric Galois actions
- The Grothendieck-Teichmüller story

Using representations of braid groups
- Profinite rigidity and Kummer characters
- Pro-unipotent rigidity and Soulé characters

Definitions

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Example : for $G = \mathbb{Z}$, we have $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$.

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hence $f(x, y)$ is well-defined in G for a pro-word $f \in \widehat{F}$.

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Fundamental groups

The algebraic fundamental group of $X_{\overline{\mathbb{Q}}}$ is the profinite completion of the topological fundamental group $\pi_1(X(\mathbb{C}))$.

The fundamental exact sequence

The mysterious group $\pi_1^{alg}(X)$ fits into the following short exact sequence

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which have no reasons to be injective in general !

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- It is the cyclotomic character

$$\sigma(\zeta) = \zeta^{\chi(\sigma)}$$

for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ and $\zeta \in \mu_\infty$.

(Kronecker-Weber theorem)

The kernel of χ is the commutator subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, and can be identified with $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}(\mu_\infty))$.

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where $B = \pi_1(Y)$.
In particular, if W is a Coxeter group, then B is an Artin-Tits group.

Belyi theorem and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

Belyi theorem

A smooth algebraic curve is defined over $\overline{\mathbb{Q}}$ if and only if it admits a covering over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

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Corollary

The map

$$\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to \mathrm{Out}\left(\pi_1(\widehat{\mathbb{C} \setminus \{0,1\}})\right)$$

is injective.

Grothendieck, Ihara, Drinfeld

The fundamental group of $\mathbb{C} \setminus \{0, 1\}$ is the free group F on two generators x, y.
The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ acts on $\widehat{F}$ by

$$\sigma.x = x^{\chi(\sigma)}, \quad \sigma.y = f_\sigma y^{\chi(\sigma)} f_\sigma^{-1}$$

hence every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ is determined by a couple

$$(\chi(\sigma), f_\sigma) \in \widehat{\mathbb{Z}}^\times \times \widehat{F}$$

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Theorem (Harbater-Schneps+Boggi-Lochak+Lochak)

$\widehat{GT} \simeq \mathrm{Out}^*(\widehat{B_n})$ for $n \geq 5$.

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Sissy question

Do usual representations of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ extend to $\widehat{GT}$, beyond the cyclotomic character ?

Action on B_3

Let $B_3 =< \sigma_1, \sigma_2 \; |\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 >$. The couple $(\lambda, f) \in \widehat{GT}$ acts on $\widehat{B_3}$:

$$(\lambda, f).\sigma_1 = \sigma_1^{\lambda}, \qquad (\lambda, f).\sigma_2 = f(\sigma_1^2, \sigma_2^2)\sigma_2^{\lambda} f(\sigma_1^2, \sigma_2^2)^{-1}$$

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On the arithmetic side,

$$\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}(\mu_\infty)) \hookrightarrow \widehat{GT}_1 \hookrightarrow \mathrm{Aut}(\widehat{B}_3)$$

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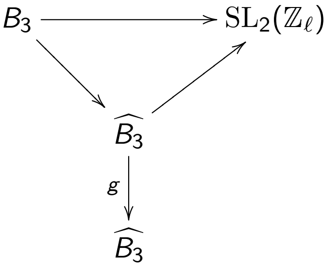

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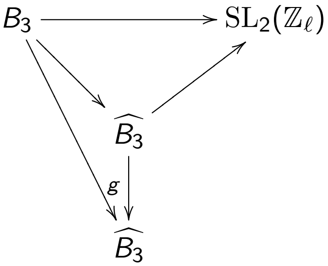

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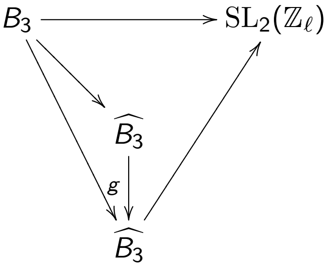
Hence $\mathrm{Hom}(B_3, \mathrm{SL}_2(\mathbb{Z}_\ell)) = \mathrm{Hom}(\widehat{B_3}, \mathrm{SL}_2(\mathbb{Z}_\ell))$, and $g \in \widehat{GT}$ sends $R \in \mathrm{Hom}(B_3, \mathrm{SL}_2(\mathbb{Z}_\ell))$ to $R \circ g$:

R :



The natural map $B_3 \to \mathrm{SL}_2(\mathbb{Z})$

We have a well-known map

$$B_3 \to \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{Z}_\ell)$$

given by

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

B_3 and $\mathrm{SL}_2(\mathbb{Z}_\ell)$

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Question : what is the corresponding character

$$\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}(\mu_\infty)) \to \mathbb{Z}_\ell ?$$

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$$\gamma(\sqrt[n]{2}) = \zeta_n^{\rho_2(\sigma)} \sqrt[n]{2}$$

From $B_3 \to \mathrm{SL}_2$ to the Burau

The morphism

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

is a degeneration at $q = -1$ of the (reduced) Burau representation

$$\sigma_1 \mapsto \begin{pmatrix} -q & 1 \\ 0 & 1 \end{pmatrix} \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ q & -q \end{pmatrix}$$

What can we get from this ?

The pro-$\mathbb{Q}_\ell$-unipotent Grothendieck-Teichmüller group

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Similarly,

- it is possible to associate a pro-algebraic group over $\mathbb{Q}_\ell$ to B_n, denoted $B_n(\mathbb{Q}_\ell)$.

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- The group $GT(\mathbb{Q}_\ell)$ acts on $B_n(\mathbb{Q}_\ell)$.
- The Burau representation extends to

$$R : B_n(\mathbb{Q}_\ell) \to \mathrm{GL}_{n-1}(K)$$

where $K = \mathbb{Q}_\ell((h))$, with $q = e^h \in K$.

Rigidity characters

Lemma

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with $\chi_d : GT_1(\mathbb{Q}_\ell) \to K^\times$.

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and κ_m are the so-called Soulé characters.

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where $[a^{m-1}]$ denotes the (non-negative) euclidean remainder of the division of a^{m-1} by ℓ^n. These elements are totally real and totally positive. The Soulé character κ_m is defined by

$$\sigma(\sqrt[\ell^n]{\varepsilon_{m,n}}) = \zeta_n^{\kappa_m(\sigma)} \sqrt[\ell^n]{\sigma(\varepsilon_{m,n})}.$$

Open questions

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Broader perspectives :

- Do the geometric Galois actions on the other Artin groups factorize through $\widehat{GT}$? (work in progress with P. Lochak)
- If yes, what give the corresponding generalized Burau representations ?