Old and New on the Broué-Malle-Rouquier conjecture

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Part 1 : Preliminaries, and the conjecture.

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Complex reflection groups, Braid groups, Hecke algebras

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$$\langle s_1, \dots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$$

or equivalently

$$\langle s_1, \dots, s_n \mid s_i^2 = 1, \underbrace{s_i s_j s_i \dots}_{m_{ij}} = \underbrace{s_j s_i s_j \dots}_{m_{ij}} \rangle$$

then $W < \mathrm{GL}_n(\mathbb{R})$ as a reflection group.

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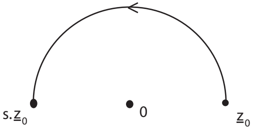

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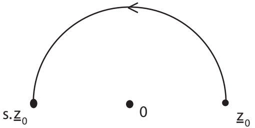

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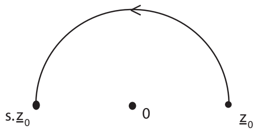

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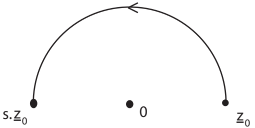

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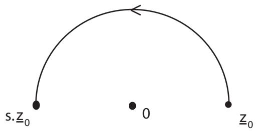

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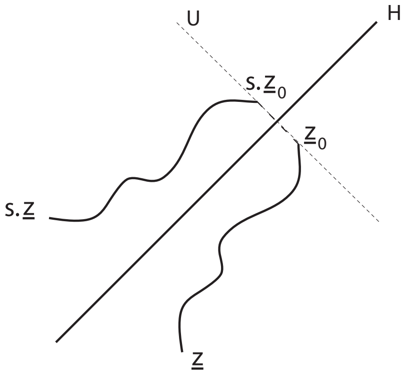
We get a homotopy class in $\pi_1(X/W, \underline{z}) = B$, called a braided reflection.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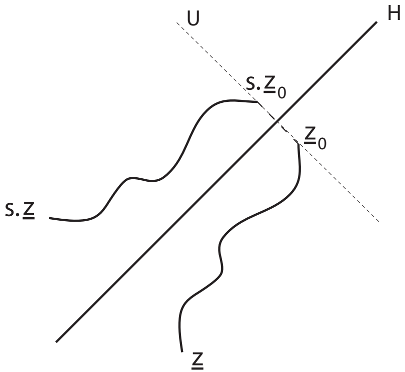

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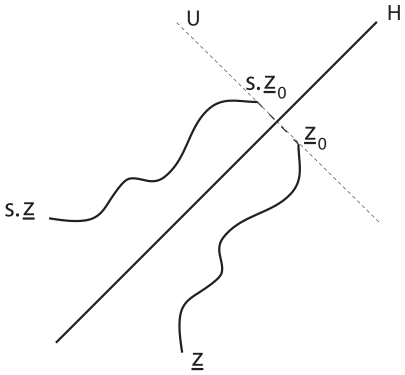

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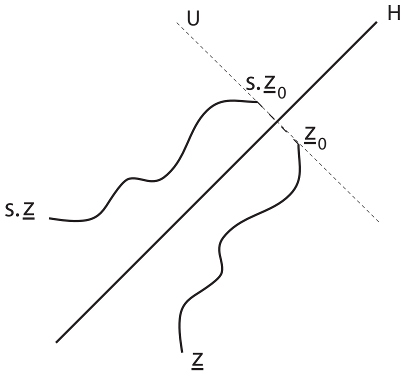

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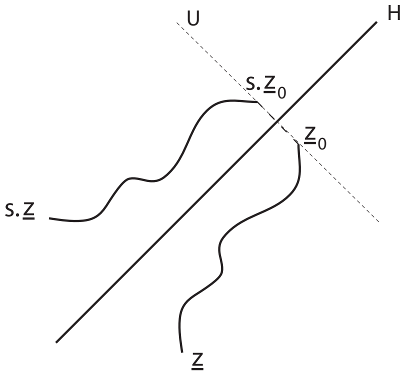

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Fact : every braided reflection is conjugated to one of them.

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It is a deformation of RW, meaning that, under $\varphi : a \mapsto 0, b \mapsto 1$, $H \otimes_\varphi \mathbb{Z} = \mathbb{Z}W$.

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As consequence, if we denote for avoiding confusion T_{s_i} the 's_i' of H, the element $T_{s_{i_1}} \dots T_{s_{i_r}}$ depends only on w, and can be denoted T_w.

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This proves that the T_w's generate H, and this is enough (see below).

Claim : the T_w form a R-basis for H.

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Example : $W = \mathfrak{S}_3 = \langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2, s_i^2 = 1 \rangle$.
$T_{s_1}.T_{s_2 s_1} = T_{s_1 s_2 s_1}$ $T_{s_2}.T_{s_2 s_1} = T_{s_2}^2 T_{s_1} = aT_{s_2 s_1} + bT_{s_1}$, etc.

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Conjecture

(BMR, 1998) H is a free R-module of rank $|W|$.

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Part 2 : History of the problem.

Classification of complex reflection groups

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- a finite set of 34 exceptions, denoted $G_4, \dots, G_{37}$.

The BMR conjecture is known for the infinite series by Ariki and Ariki-Koike (1993), so we only need to deal with the exceptional groups.

First results on exceptional groups : Broué-Malle

Among these 34 exceptional groups, there are 6 exceptional Coxeter groups ($H_3, H_4, F_4, E_6, E_7, E_8$), for which the conjecture is known to hold.

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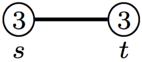

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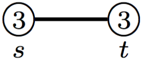

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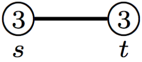

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$$H = \langle s, t, u \mid sts = tst, tut = utu, su = us, s^3 = as^2 + bs + c \rangle$$

First results on exceptional groups : Broué-Malle

Proof for $A_2^{(3)} = G_4$:

In $\mathcal{H}(A_2^{(3)}, \mathbf{u})$ bildet $B :=$

$$\{T_1^i,\ T_2T_1^i,\ T_2^2T_1^i,\ T_1T_2T_1^i,\ T_1^2T_2T_1^i,\ T_1T_2^2T_1^i,\ T_1^2T_2^2T_1^i,\ T_2T_1^2T_2T_1^i \mid 0 \le i \le 2\}$$

eine Basis. Denn man überzeugt sich leicht anhand der Relationen, daß die Elemente von B unter Linksmultiplikation mit T_1 und T_2 jeweils in Linearkombinationen aus B übergehen. So ist etwa im schwierigsten Fall

$$\begin{aligned} T_2{\cdot}T_1^2T_2^2T_1^i &= T_1^{-1}T_1T_2T_1^2T_2^2T_1^i = T_1^{-1}T_2T_1T_2T_1T_2^2T_1^i = T_1^{-1}T_2T_1^3T_2T_1^{i+1} \\ &= \alpha_1T_1^{-1}T_2^2T_1^{i+1} + \alpha_2T_1^{-1}T_2T_1T_2T_1^{i+1} + \alpha_3T_1^{-1}T_2T_1^2T_2T_1^{i+1} \\ &= \alpha_4T_2^2T_1^{i+1} + \alpha_5T_1T_2^2T_1^{i+1} + \alpha_6T_1^2T_2^2T_1^{i+1} + \alpha_2T_2T_1^{i+2} + \alpha_3T_2T_1^2T_2T_1^i \end{aligned}$$

für $0 \le i \le 2$ mit gewissen $\alpha_j \in K$. Die irreduziblen Matrixdarstellungen von $\mathcal{H}(A_2^{(3)}, \mathbf{u})$ werden in 5B konstruiert. Damit folgt auch hier die Behauptung.

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Proof for $A_3^{(3)} = G_{25}$:

In $\mathcal{H}(A_3^{(3)}, \mathbf{u})$ sei $\mathcal{H}'$ die von T_1 und T_2 erzeugte Teilalgebra. Weiter sei

$$\begin{aligned} B := \{&1,\ T_3,\ T_2T_3,\ T_2^2T_3,\ T_1T_2T_3,\ T_1^2T_2T_3,\ T_1T_2^2T_3,\ T_1^2T_2^2T_3,\ T_3T_2^2T_3, \\ &T_1T_3T_2^2T_3,\ T_1^2T_3T_2^2T_3,\ T_2T_1^2T_2T_3,\ T_3T_2T_1^2T_2T_3,\ T_2T_1^2T_3T_2^2T_3, \\ &T_2^2T_1T_3T_2^2T_3,\ T_1T_2^2T_1T_3T_2^2T_3,\ T_2T_1T_3T_2^2T_3,\ T_3^2,\ T_2T_3^2,\ T_2^2T_3^2,\ T_1T_2T_3^2, \\ &T_1^2T_2T_3^2,\ T_1T_2^2T_3^2,\ T_1^2T_2^2T_3^2,\ T_2T_1^2T_2T_3^2,\ T_3T_2T_1^2T_2T_3^2,\ T_3^2T_2T_1^2T_2T_3^2\}. \end{aligned}$$

First results on exceptional groups : Broué-Malle

Proof for $A_3^{(3)} = G_{25}$:

Dann stellt man anhand der definierenden Relationen fest, daß

$$\sum_{S \in B} RS\mathcal{H}'$$

invariant unter Linksmultiplikation mit T_1, T_2 und T_3 bleibt, und daher schon gleich $\mathcal{H}$ sein muß. Das Nachprüfen dieser Aussage für die 81 Produkte sei dem Leser überlassen. Etwas vereinfacht wird die Rechnung durch konsequente Benutzung der Formel $T_iT_jT_i^2T_j = T_jT_i^2T_jT_i$ für $1 \leq i, j \leq 3$, welche unmittelbar aus den definierenden Relationen folgt. Da die Erzeuger T_1, T_2 von $\mathcal{H}'$ die Relationen von $\mathcal{H}(A_2^{(3)}, \mathbf{u})$ erfüllen, können wir ein Erzeugendensystem für $\mathcal{H}'$ wie oben wählen. Damit haben wir insgesamt $|B|\,|W(A_2^{(3)})| = 27 \cdot 24 = |W(A_3^{(3)})|$ Erzeuger für $\mathcal{H}$. Die irreduziblen Matrixdarstellungen werden später in 5F konstruiert, woraus die Aussage schließlich folgt.

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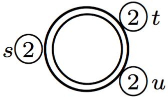
Computer approach 1 : Knuth-Bendix and Malle

For G_{12} and G_5, it is already difficult to do it by hand.

G_{12}



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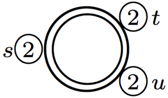

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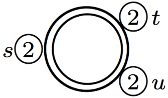

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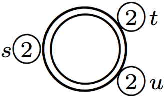

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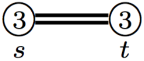
G_5



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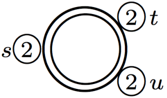

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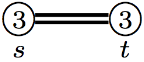

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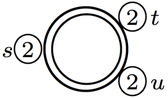

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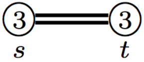

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Use of Knuth-Bendix algorithm, as follows.

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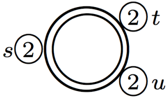

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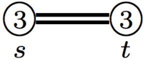

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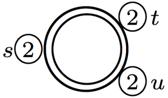

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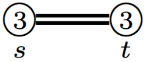

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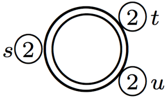

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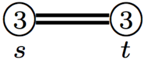

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- let $\mathcal{R}$ be the set of relations such that $M/\mathcal{R} = W$
- choose a partial ordering on M, compatible with multiplication, and write $x \to y$ if $\{x, y\} \in \mathcal{R}$ and $x > y$ (e.g. the length in the generators)

- Apply Knuth-Bendix in order to find a confluent set of relations, $\mathcal{R}_{con}$ i.e. such that $w_1 \to w_2$ and $w_1 \to w_3$ implies the existence of w_4 such that $w_2 \to w_4$ and $w_3 \to w_4$.

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- For $w \in W$, let $T_w \in H$ be given by a minimal element of M mapped to w.
- Check that $w_1 \to w_2$ implies that, inside H, $T_{w_1} = T_{w_2} + \sum \alpha_i U_i$, with $\alpha_i \in R$ and U_i the image of a term in M smaller than T_{w_1}.

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Die definierenden Relationen von $\mathcal{H}$ haben bereits die gewünschte Form. Für K_2 bestimmt man mit dem Computer eine konfluente Präsentation. Diese besteht aus 24 Relationen, welche wir hier nicht wiedergeben wollen. Man bestätigt, daß diese, modulo Termen kleinerer Länge, auch in der Algebra $\mathcal{H}(K_2)$ gelten, und erhält so ein Erzeugendensystem der Ordnung $|W(K_2)|$. Die irreduziblen Matrixdarstellungen von

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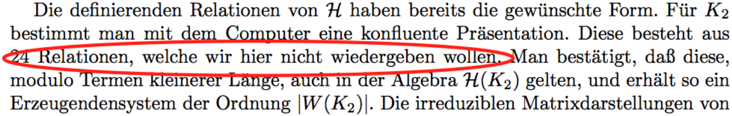

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Needless to say, more detailed accounts are needed ...

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Written account in rank 2, as 'semi-private communication'

G_i	n	$\lvert G_i \rvert$	rank
4	2	24	++
5	2	72	++
6	2	48	++
7	2	144	++
8	2	96	++
9	2	192	++
10	2	288	++
11	2	576	++
12	2	48	++
13	2	96	++
14	2	144	++
15	2	288	++
16	2	600	++
17	2	1200	+
18	2	1800	
19	2	3600	
20	2	360	++
21	2	720	++
22	2	240	++

Written account in higher rank

23	3	120	++
24	3	336	++
25	3	648	++
26	3	1296	++
27	3	2160	++
28	4	1152	++
29	4	7680	+
30	4	14400	+(+)
31	4	46080	
32	4	155520	
33	5	51840	
34	6	39191040	
35	6	51840	(++)
36	7	2903040	(++)
37	8	696729600	(++)

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(But there is no efficient control on the number of elements needed to generate H)

Part 3 : Recent work.

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What about $s_{i+1}s_i^{-1}s_{i+1}$, and
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Lemma

$\forall x \in u_i \ (s_{i+1}^{-1}s_is_{i+1}^{-1})x \in x(s_{i+1}^{-1}s_is_{i+1}^{-1}) + u_iu_{i+1}u_i$
$\forall x \in u_i \ (s_{i+1}s_i^{-1}s_{i+1})x \in x(s_{i+1}s_i^{-1}s_{i+1}) + u_iu_{i+1}u_i$
(commutation lemma)

The cubic Hecke algebras : case $n = 3$

$$
\begin{aligned}
(s_{i+1}^{-1} s_i s_{i+1}^{-1}) s_i^{-1} &= s_{i+1}^{-1} s_i s_{i+1}^{-1} s_i^{-1} \\
&= s_{i+1}^{-1} (c s_i^{-2} + b s_i^{-1} + a) s_{i+1}^{-1} s_i^{-1} \\
&= c s_{i+1}^{-1} s_i^{-2} s_{i+1}^{-1} s_i^{-1} + b s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1} s_i^{-1} + a s_{i+1}^{-1} s_{i+1}^{-1} s_i^{-1} \\
&= c s_{i+1}^{-1} s_i^{-2} s_{i+1}^{-1} s_i^{-1} + b s_i^{-1} s_{i+1}^{-1} s_i^{-1} s_i^{-1} + a s_{i+1}^{-1} s_{i+1}^{-1} s_i^{-1} \\
&\in c s_{i+1}^{-1} s_i^{-1} (s_i^{-1} s_{i+1}^{-1} s_i^{-1}) + u_i u_{i+1} u_i \\
&\in c (s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1}) s_i^{-1} s_{i+1}^{-1} + u_i u_{i+1} u_i \\
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&\in c s_i^{-1} s_{i+1}^{-1} (c^{-1} s_i - a c^{-1} - b c^{-1} s_i^{-1}) s_{i+1}^{-1} + u_i u_{i+1} u_i \\
&\in s_i^{-1} s_{i+1}^{-1} (s_i - a - b s_i^{-1}) s_{i+1}^{-1} + u_i u_{i+1} u_i \\
&\in s_i^{-1} s_{i+1}^{-1} s_i s_{i+1}^{-1} - a s_i^{-1} s_{i+1}^{-1} s_{i+1}^{-1} - b s_i^{-1} (s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1}) + u_i u_{i+1} u_i \\
&\in s_i^{-1} s_{i+1}^{-1} s_i s_{i+1}^{-1} - a s_i^{-1} s_{i+1}^{-1} s_{i+1}^{-1} - b s_i^{-1} s_i^{-1} s_{i+1}^{-1} s_i^{-1} + u_i u_{i+1} u_i \\
&\in s_i^{-1} (s_{i+1}^{-1} s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i
\end{aligned}
$$

Moreover, we have

Lemma

$s_{i+1}^{-1} s_i s_{i+1}^{-1} \in c^{-1}(s_{i+1} s_i^{-1} s_{i+1}) s_i + u_i u_{i+1} u_i$

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The cubic Hecke algebras : case $n = 3$

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$$\begin{array}{rl} & (s_{i+1} s_i^{-1} s_{i+1}) s_i \\ = & s_{i+1}(s_i^{-1} s_{i+1} s_i) \\ = & s_{i+1}(s_{i+1} s_i s_{i+1}^{-1}) \\ = & s_{i+1}^2 s_i s_{i+1}^{-1} \\ = & a s_{i+1} s_i s_{i+1}^{-1} + b s_i s_{i+1}^{-1} + c s_{i+1}^{-1} s_i s_{i+1}^{-1} \\ \in & u_i u_{i+1} u_i + c s_{i+1}^{-1} s_i s_{i+1}^{-1} \end{array}$$

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Corollary

A_3 is a finitely generated R-module.

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And $s_2 s_1^2 s_2 \equiv s_2 s_1^{-1} s_2 \mod u_1 u_2 u_1$.

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c(s_1^2 s_2^2)^6 &= c s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= s_1 c s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= s_1 s_2^3 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= s_1 s_2^2 (s_2 s_1 s_2) s_2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= s_1 s_2^2 s_1 (s_2 s_1 s_2) s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
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&= c s_1 s_2^2 s_1 s_1 (s_2 s_2^2) s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^2 s_1 s_2^2 s_1 (s_1 s_1^2) s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^3 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
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&= c^5 s_1 s_2 s_1 (s_1 s_1^2) s_2^2 s_1^2 s_2^2 \\
&= c^6 (s_1 s_2 s_1) s_2^2 s_1^2 s_2^2 \\
&= c^6 s_2 s_1 (s_2 s_2^2) s_1^2 s_2^2 \\
&= c^7 s_2 (s_1 s_1^2) s_2^2 \\
&= c^8 s_2 s_2^2 \\
&= c^9
\end{aligned}
$$

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Theorem

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In this proof, we chose a specific maximal parabolic subgroup/subalgebra, the one generated by $\langle s_1, s_2 \rangle$, and we first decomposed the algebra as a module over its subalgebra, mimicing the (double) cosets decomposition.

The cubic Hecke algebras : case $n = 4$

Extra work leads a collection of 27 elements that generate A_4 as a A_3-module, whence $24 \times 27 = 648 = |G_{25}|$ elements generating A_4 as a R-module.

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What about other maximal parabolics ?

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Theorem

The BMR conjecture holds true for G_4, G_{25}, G_{32}.

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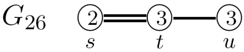

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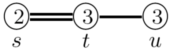

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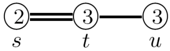
G_{26} 

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Decomposing further as a A-module proves the BMR conjecture for G_{26}.

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Questions :

- Is there an easy way to stay inside a monoid of positive elements ?
- Can we use the BKL-like 'dual' monoid for other (well-generated) reflection groups ?
- Any connection between Garside normal forms, simple elements, and nice bases for these Hecke algebras ?

Last slide

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Thank you !