The Broué-Malle-Rouquier freeness conjecture

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- which makes H appear as a quotient of the group algebra of B.

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The ring of definition R of H is a ring of Laurent polynomials with integer coefficients. Its number of variables is related to

- the order of the (pseudo-)reflections
- the number of conjugacy classes of (distinguished) reflections (no more than 3 for irreducible W).

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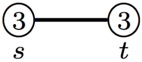
- representation theory of finite-dimensional algebras
- deterministic computational methods.

Laurent phenomenon : an example

G_4



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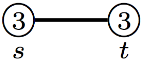

B

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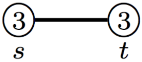

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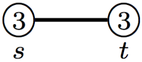

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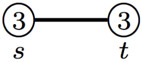

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Here, H could be defined over $\mathbb{Z}[a, b, c]$.

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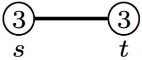

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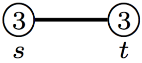

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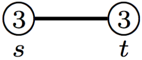

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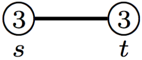

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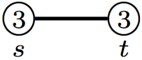

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In order to prove this, we can specialize to $a = b = 0$, and define H over $\mathbb{Z}[c]$.

Laurent phenomenon : torsion

$$
\begin{aligned}
c(s_1^2 s_2^2)^6 &= c s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= s_1 c s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= s_1 s_2^3 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= s_1 s_2^2 (s_2 s_1 s_2) s_2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= s_1 s_2^2 s_1 (s_2 s_1 s_2) s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= s_1 s_2^2 s_1 s_1 s_2 (s_1 s_1^2) s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c s_1 s_2^2 s_1 s_1 (s_2 s_2^2) s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^2 s_1 s_2^2 s_1 (s_1 s_1^2) s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^3 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^3 s_1 s_2 (s_2 s_1 s_2) s_2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^3 s_1 s_2 s_1 (s_2 s_1 s_2) s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^3 s_1 s_2 s_1 s_1 s_2 (s_1 s_1^2) s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^4 s_1 s_2 s_1 s_1 (s_2 s_2^2) s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^5 s_1 s_2 s_1 (s_1 s_1^2) s_2^2 s_1^2 s_2^2 \\
&= c^6 (s_1 s_2 s_1) s_2^2 s_1^2 s_2^2 \\
&= c^6 s_2 s_1 (s_2 s_2^2) s_1^2 s_2^2 \\
&= c^7 s_2 (s_1 s_1^2) s_2^2 \\
&= c^8 s_2 s_2^2 \\
&= c^9
\end{aligned}
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&= s_1cs_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
&= s_1s_2^3s_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
&= s_1s_2^2(s_2s_1s_2)s_2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
&= s_1s_2^2s_1(s_2s_1s_2)s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
&= s_1s_2^2s_1s_1s_2(s_1s_1^2)s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
&= cs_1s_2^2s_1s_1(s_2s_2^2)s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
&= c^2s_1s_2^2s_1(s_1s_1^2)s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
&= c^3s_1s_2^2s_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
&= c^3s_1s_2(s_2s_1s_2)s_2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
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&= c^4s_1s_2s_1s_1(s_2s_2^2)s_1^2s_2^2s_1^2s_2^2 \\
&= c^5s_1s_2s_1(s_1s_1^2)s_2^2s_1^2s_2^2 \\
&= c^6(s_1s_2s_1)s_2^2s_1^2s_2^2 \\
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\end{aligned}
$$

$$s_1s_2s_1 = s_2s_1s_2,\ s_1^3 = s_2^3 = c$$

$$c\left((s_1^2s_2^2)^6 - c^8\right) = 0$$

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&= s_1 s_2^2 (s_2 s_1 s_2) s_2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
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&= c s_1 s_2^2 s_1 s_1 (s_2 s_2^2) s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^2 s_1 s_2^2 s_1 (s_1 s_1^2) s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
&= c^3 s_1 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 \\
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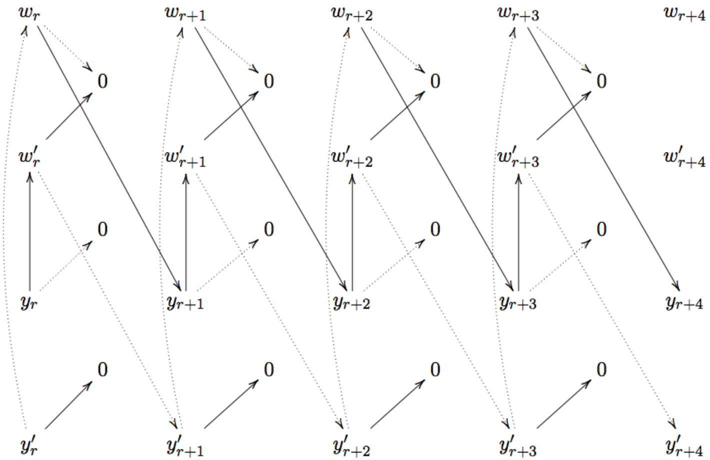
&= c^5 s_1 s_2 s_1 (s_1 s_1^2) s_2^2 s_1^2 s_2^2 \\
&= c^6 (s_1 s_2 s_1) s_2^2 s_1^2 s_2^2 \\
&= c^6 s_2 s_1 (s_2 s_2^2) s_1^2 s_2^2 \\
&= c^7 s_2 (s_1 s_1^2) s_2^2 \\
&= c^8 s_2 s_2^2 \\
&= c^9
\end{aligned}
$$

$$s_1 s_2 s_1 = s_2 s_1 s_2,\ s_1^3 = s_2^3 = c$$

$$c\left((s_1^2 s_2^2)^6 - c^8\right) = 0 \text{ but } (s_1^2 s_2^2)^6 \neq c^8.$$

Laurent phenomenon : infinite dimension



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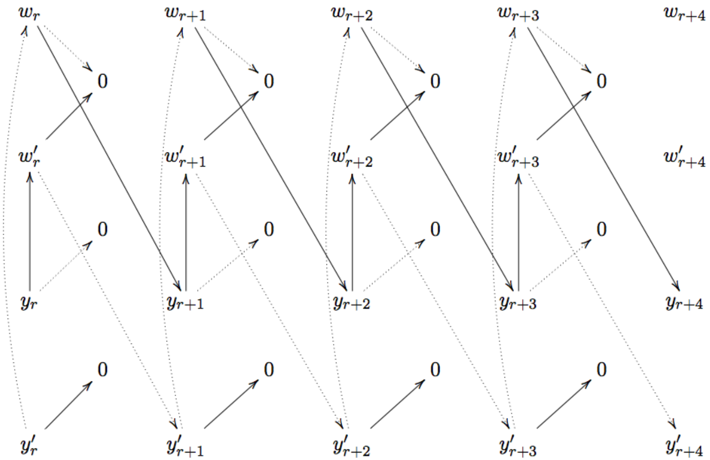

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If H is known to be finitely generated, then every specialization of H to complex numbers has dimension $|W|$.

Applications of the conjecture

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- Ariki-Koike algebras : in case of the general series, these algebras are related to the affine Hecke algebra of type $\tilde{A}$
- Exceptional 'Shephard' groups are specialization of the group algebra of Artin-Tits groups (including the usual braid group), and therefore their Hecke algebra provides a grab on their representation theory, and also on the existence of Markov traces.

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In these 3 cases, there exists a tower of parabolic subalgebras $H_1 \subset \cdots \subset H_n = H$ such that H_n is a free H_{n-1}-module of the expected rank. (M. 2012-2013)

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The plan for solving the conjecture follows this line.

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Recall : if $G \subset \mathrm{GL}_2(\mathbb{C})$ is an exceptional reflexion group of rank 2, then $G/Z(G) \simeq W_0 \subset W$ for W some finite Coxeter group of rank 3.

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With some more work, one shows that : the action of z on H is annihilated by some monic polynomial with coefficients in R (of very large degree), and therefore H is finitely generated over R.

But this approach does not give any control on the number of elements which are necessary in order to generate H.

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Ideally, we would like to have a polynomial of degree $|Z(G)|$, that is 2 for G_4 and 60 for G_{19}.
But the conjecture predicts that the minimal polynomial of z is of degree $|\mathrm{Irr}(G)| > |Z(G)|$, that is 7 for G_4 and 270 for G_{19}.
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Conjecture in rank 2 : E. Chavli's thesis (in progress)

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The tetrahedral series

	ST	$\|ZW\|$	orders	$\#gens$	$\#R$
$\surd$	4	2	3	2	2
$\surd$	5	6	3	2	4
$\surd$	6	4	3	2	3
$\surd$	7	12	3	3	5

$$G_4 = \langle s, t | sts = tst, s^3 = t^3 = 1 \rangle$$

The octahedral series

	ST	$\|ZW\|$	orders	$\#gens$	$\#R$
√	8	4	4	2	3
√	9	8	4	2	4
√	10	12	4	2	5
	11	24	4	3	6
√	12	2	2	3	1
√	13	4	2	3	2
√	14	6	3	2	3
√	15	12	3	3	4

$$G_8 = \langle s, t | sts = tst, s^4 = t^4 = 1 \rangle$$

The icosahedral series

	ST	$\|ZW\|$	orders	#gens	#R
$\surd$	16	10	5	2	4
	17	20	5	2	5
	18	30	5	2	6
	19	60	5	3	7
	20	6	3	2	2
	21	12	3	2	3
$\surd$	22	4	2	3	1

$$G_{16} = \langle s, t | sts = tst, s^5 = t^5 = 1 \rangle$$

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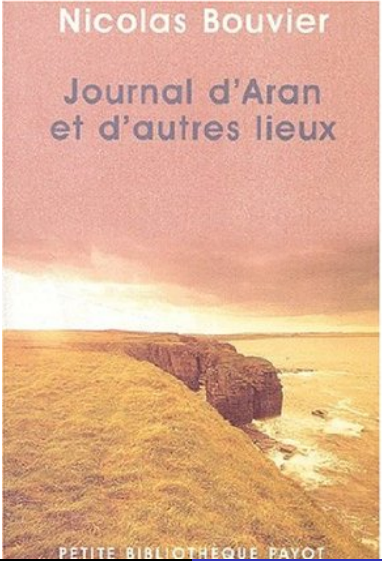

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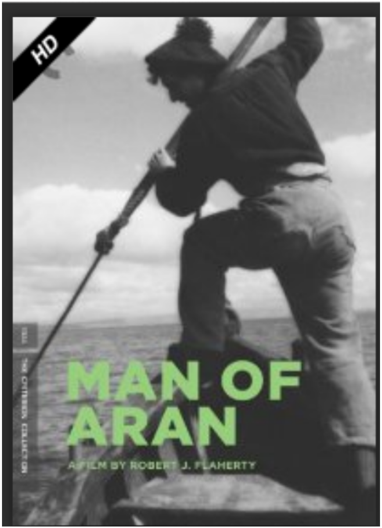

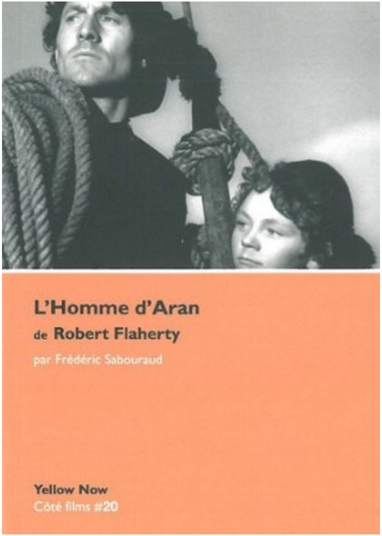

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3 months later the dream came true.

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Once landed, I realized that maybe I should do or at least propose something to deserve that invitation...

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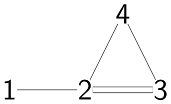

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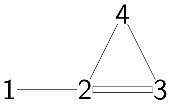

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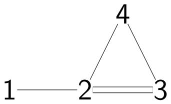

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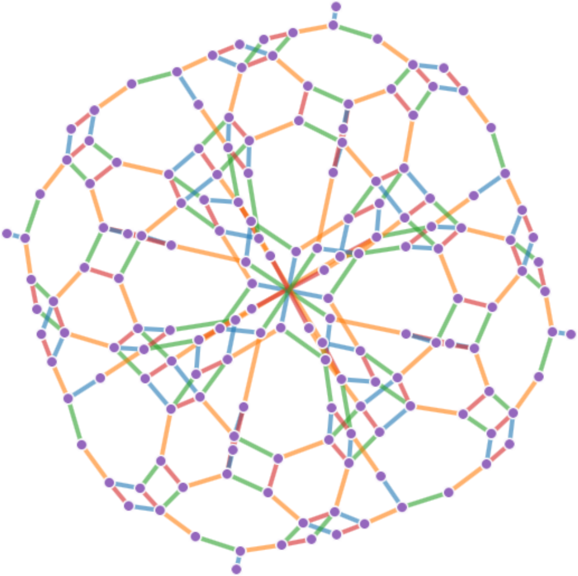

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Note : a crucial tool for doing so is the efficient implementation in Chevie of the multiplication for the usual Hecke algebras. This should probably also work with PyCox. A crucial thing here is of course that $\#R = 1$, i.e. we can work with $R = \mathbb{Z}[q, q^{-1}]$.

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Theorem

The BMR freeness conjecture holds for higher rank 2-reflection groups, except possibly G_{34} (M.-Pfeiffer, 2014).

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$\Rightarrow$ an algebraic description of H is available for all these groups.

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Proof for higher rank 2-groups : general strategy

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If we manage to do so, this proves that H is generated by $|W|$ elements, and this proves the conjecture by general arguments.

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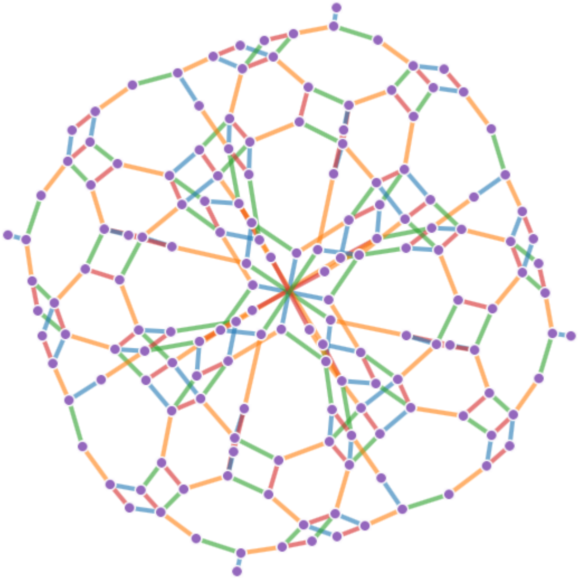
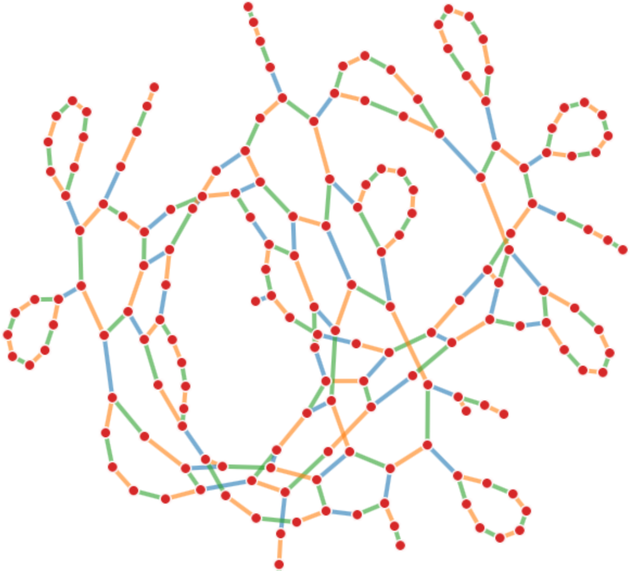


Coset graph : G_{27}



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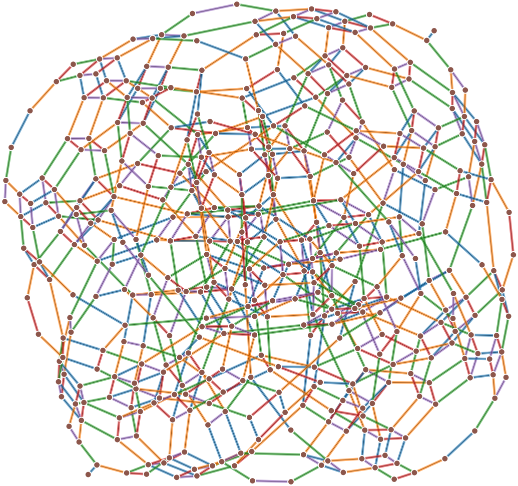

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To be continued. . .