On exceptional cyclotomic algebras

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The Table below reports on the current status of the computations concerning the cyclotomic algebras associated to the exceptional finite complex reflection groups. The status as far as ranks are concerned is unsatisfactory; at least the remaining dimension 2 cases should be doable. As far as symmetrising forms are concerned the status is poor indeed; here it seems to be necessary to improve the used polynomial arithmetic first, where I used the one in GAP; and secondly one could try something more clever than the brute force attack I have tried.

Anyway, here are the results; the notation used is as follows:

The reflection groups G_i, for $i \in \{4, \ldots, 37\}$, are denoted as in [1]. The number n denotes the degree of the reflection representation and $|G_i|$ denotes the order of the reflection group G_i. In the last column we indicate which of the reflection groups are real and their corresponding types; the exclamation marks are explained below.

Let (W, S) be finite complex reflection group given by a Coxeter-type presentation with generating set S, as given in [5]. For $s \in S$ let $e_s \in \mathbb{N}$ denote the order of $s \in W$. Let $\mathbf{u} := \{u_{s,j}; s \in S, j \in \{0, \ldots, e_s - 1\}\}$ be a set of independent indeterminates over $\mathbb{Z}$ and let $A := \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$. Then the cyclotomic algebra associated to W is defined as

$$\mathcal{H}_A(\mathbf{u}) := \langle T_s; s \in S | \text{braid relations}, \prod_{j=0}^{e_s-1} (T_s - u_{s,j}) = 0 \rangle,$$

where the latter relations are called *generalized order relations*; see [4], and the comments below for G_{24} and G_{27}.

For computational purposes the cyclotomic algebras are slightly modified as follows. Let $\mathbf{u}' := \{u_{s,j}; s \in S, j \in \{1, \ldots, e_s - 1\}\}$ be a set of independent indeterminates over $\mathbb{Z}$, let $A' := \mathbb{Z}[\mathbf{u}', \mathbf{u}'^{-1}]$ and let

$$\mathcal{H}_{A'}(\mathbf{u}') := \langle T_s; s \in S | \text{braid relations}, \ (T_s - 1) \cdot \prod_{j=1}^{e_s-1} (T_s - u_{s,j}) = 0 \rangle.$$

This amounts to a rescaling of the generators $\{T_s; s \in S\}$.

Furthermore, for $s \in S$ and $j \in \{1, \ldots, e_s - 1\}$ let $v_{s,j} \in A'$ be the j-th elementary symmetric polynomial in $\{u_{s,j}; j \in \{1, \ldots, e_s - 1\}\}$, where $\deg(v_{s,j}) = j$.

Let $\mathbf{v} := \{v_{s,j}; s \in S, j \in \{1, \ldots, e_s - 1\}\}$; this again is a set of independent indeterminates over $\mathbb{Z}$. Let $A'' := \mathbb{Z}[\mathbf{v}, v_{s,e_s-1}^{-1}; s \in S]$, and $K'' := \mathrm{Quot}(A'')$; note that we thus only allow the inverses of $\{v_{s,e_s-1}^{-1}; s \in S\}$. As $\mathcal{H}_{A'}(\mathbf{u}')$ is already defined over A'', we let

$$\mathcal{H}_{A''}(\mathbf{v}) := \langle T_s; s \in S | \text{braid relations, generalized order relations in terms of } \mathbf{v} \rangle.$$

The notation in the 'rank' column is a follows:
'++' indicates that $\mathcal{H}_{A''}(\mathbf{v})$ is proven to be A''-free of rank $|W|$,
'+' indicates that $\mathcal{H}_{A''}(\mathbf{v})$ is only proven to have K''-dimension $|W|$.
The brackets indicate the results which are known from the theory of finite real reflection groups anyway.

For G_{24} and G_{27} the presentations given in [5] do not work; indeed deforming these Coxeter-type group presentations leads to a 'collapsing' presentation. Using braid relations of length 7 instead of 6 yields suitable presentations; this has been found by computer search, and has later been confirmed in [3, 2].

The notation in the 'basis' column is a follows:
A basis of $\mathcal{H}_{A''}(\mathbf{v})$ is called *monomial*, if its elements consist of products of the generators $\{T_s; s \in S\}$; the letter 'm' indicates the existence of a monomial basis.
A word $T_{s_1} \cdot \cdots \cdot T_{s_l} \in \mathcal{H}_{A''}(\mathbf{v})$, for $s_i \in S$, is called *reduced*, if the corresponding word $s_1 \cdot \cdots \cdot s_l \in W$ is reduced, i. e. of minimal length among all words in the generators S representing $s_1 \cdot \cdots \cdot s_l \in W$. A monomial basis of $\mathcal{H}_{A''}(\mathbf{v})$ is called *reduced*, if its elements are reduced; the letter 'r' indicates the existence of a reduced basis.
Again the brackets indicate the results which are known from the theory of finite real reflection groups anyway.

The notation in the 'form' column is a follows:
A basis $\mathcal{T}$ of $\mathcal{H}_{A''}(\mathbf{v})$ is called *quasi-symmetric*, if $1 \in \mathcal{T}$ and the A''-linear form $t_{\mathcal{T}} : \mathcal{H}_{A''}(\mathbf{v}) \to A''$, mapping $1 \mapsto 1$ and $h \mapsto 0$ for all $h \in \mathcal{T} \setminus \{1\}$, is a symmetrising form for $\mathcal{H}_{A''}(\mathbf{v})$. In particular this implies that the determinant of the Gram matrix of $t_{\mathcal{T}}$ up to sign is a product of powers of the $\{v_{s,e_s-1}^{-1}; s \in S\}$.
'+' indicates the existence of a quasi-symmetric reduced basis,
'++' indicates that all reduced bases are quasi-symmetric, and that the A''-linear form $t_{\mathcal{T}}$ does not depend on the choice of a reduced basis,
'+++' indicates that the Broué-Malle-Michel Condition [4, 2.1.1.c] on $t_{\mathcal{T}}(\pi)$ is fulfilled.
Again the brackets indicate the results which are known from the theory of finite real reflection groups anyway.

Note that the A''-linear form $t_{\mathcal{T}}$ specializes to the symmetrising form for the group ring of the underlying finite complex reflection group, see Condition [4, 2.1.1.b].

G_i	n	$\|G_i\|$	rank	basis	form	
4	2	24	++	r	+++	
5	2	72	++	r	+++	
6	2	48	++	r	+++	
7	2	144	++	r	+−+	
8	2	96	++	r	+++	
9	2	192	++	r	+++	
10	2	288	++	r	+−+	
11	2	576	++	m		
12	2	48	++	r	+++	
13	2	96	++	r	+−+	
14	2	144	++	r	+++	
15	2	288	++	r	−	
16	2	600	++	m		
17	2	1200	+	m		
18	2	1800				
19	2	3600				
20	2	360	++	m		
21	2	720	++	m		
22	2	240	++	m		
23	3	120	++	m(r)	(++)	H_3
24	3	336	++	m		!
25	3	648	++	m		
26	3	1296	++	m		
27	3	2160	++	m		!
28	4	1152	++	m(r)	(++)	F_4
29	4	7680	+	m		
30	4	14400	+(+)	m(r)	(++)	H_4
31	4	46080				
32	4	155520				
33	5	51840				
34	6	39191040				
35	6	51840	(++)	(r)	(++)	E_6
36	7	2903040	(++)	(r)	(++)	E_7
37	8	696729600	(++)	(r)	(++)	E_8

References

[1] G. Shephard, J. Todd: Finite unitary reflection groups, Canadian J. Math. 6, 1954, 274–304.

[2] D. Bessis: Zariski theorems and diagrams for braid groups, Invent. Math. 145 (3), 2001, 487–507.

[3] D. Bessis, J. Michel: Private communication.

[4] M. Broué, G. Malle, J. Michel: Towards spetses I, Transform.Groups 4 (2–3), 1999, 157–218.

[5] M. Broué, G. Malle, R. Rouquier: On complex reflection groups and their associated braid groups, *in*: Representations of groups, Banff, 1994, 1–13, CMS Conf. Proc. 16, Amer. Math. Soc., 1995.