FIBRATION OF HYPERPLANE ARRANGEMENT

Dear Ivan, I sum up what we calculated and write down the detail on the fibration of type D_n and F_4.

1. Type D_n

Let

$$Y = \{(y_1, ..., y_n) \in \mathbb{C}^n : y_i \pm y_j \neq 0 \text{ for } i \neq j\},$$

$$Z = \{(z_1, ...z_{n-1}) \in \mathbb{C}^{n-1} : z_i \neq 0, z_i \neq z_j \text{ for } i \neq j\}$$

and the map $Y \to Z$ defined by $z_i = y_n^2 - y_i^2$. Our goal is to show that π is a fibration with a cross section. We can imbed Y into $\mathbb{P}^n \times Z$ given by

$$(y_1, y_2, \dots, y_n) \mapsto (y_1, y_2, \dots, y_n, y_n^2 - y_1^2, \dots, y_n^2 - y_{n-1}^2).$$

In the following discussion, we always think of Y as such a subset of $\mathbb{P}^n \times Z$. Then the map $Y \to Z$ is simply the restriction of the natural projection $\pi : \mathbb{P}^n \times Z \to Z$. (You pointed out that this is just the graph of the map $Y \to Z$.)

1.1. **Cross section.** Let $y_n = \sqrt{|z_1| + \cdots |z_{n-1}|}$, then the real part of $y_n^2 - z_i$ $(1 \leq i \leq n-1)$ is always positive, therefore we can define $y_i = \sqrt{y_n^2 - z_i}$ continuously for $1 \leq i \leq n-1$ (choose a branch for the square root). Thus we have a cross section for $Y \to Z$.

1.2. **Transversality.** Now consider the $n-1$ hypersurfaces S_i $(1 \leq i \leq n-1)$ in $\mathbb{C}^n \times Z$, defined by

$$S_i : y_n^2 - y_i^2 - z_i = 0.$$

To include the points at infinity, it is better to consider the closure of the above hypersurfaces in $\mathbb{P}^n \times Z$, which is defined by

$$\overline{S}_i : y_n^2 - y_i^2 - z_i y_0 = 0.$$

Fixing $z = (z_1, z_2, \dots, z_{n-1})$, let $(\overline{S}_i)_z = \overline{S}_i \cap (\mathbb{P}^n \times \{z\})$. We can show that these $(\overline{S}_i)_z$ intersect transversally, by calculating the Jacobian:

$$\begin{pmatrix} z_1 & -2y_1 & 0 & ... & 0 & 2y_n \\ z_2 & 0 & -2y_2 & ... & 0 & 2y_n \\ ... & ... & ... & ... & ... & ... \\ z_{n-1} & 0 & 0 & ... & -2y_{n-1} & 2y_n \end{pmatrix}$$

It is easy to see this matrix has full rank for points in $\overline{C}_z = \cap(\overline{S}_i)_z$. Therefore, $\overline{C}_z$ is a smooth curve in $\mathbb{P}^n$.

1.3. **Connectness.** By Lefschetz hyperplane theorem, we know that if there is a smooth projective manifold M and a smooth hypersurface N determined by an ample line bundle on M, then

$$H^i(M) \to H^i(N)$$

is an isomorphism for $i < \dim N$ and an injective for $i = \dim N$. In particular, when $\dim N > 0$, we always have $H^0(M) \cong H^0(N) \cong \mathbb{Z}$, therefore N is connected.

Successively applying Lefschetz hyperplane theorem, we know that $\overline{C}_z = \cap(\overline{S}_i)_z$ is connected.

1.4. **Stratification.** Define the hyperplane at infinity $H = \mathbb{P}^n \setminus \mathbb{C}^n$, which is defined by $y_0 = 0$. Similar to 1.2, the hypersurfaces $\overline{S}_i$ and the infinity hyperplane $H \times Z$ intersect transversally in $\mathbb{P}^n \times Z$, because of the following Jacobian is of full-rank.

$$\begin{pmatrix} z_1 & -2y_1 & 0 & \dots & 0 & 2y_n & -y_0 & 0 & \dots & 0 \\ z_2 & 0 & -2y_2 & \dots & 0 & 2y_n & 0 & -y_0 & \dots & 0 \\ \dots & & & \dots & & \dots & & & \dots & \\ z_{n-1} & 0 & 0 & \dots & -2y_{n-1} & 2y_n & 0 & 0 & \dots & -y_0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Now we can stratify $\mathbb{P}^n \times Z$ by these n hypersurfaces. In another word, define the closed strata to be the intersections of any collection of the above hypersurfaces:

$$H \times Z, \overline{S}_1, (H \times Z) \cap \overline{S}_1, \overline{S}_2, \overline{S}_1 \cap \overline{S}_2, \cdots$$

Notice that the above n hypersurfaces and a fiber $\mathbb{P}^n \times \{z\}$ intersect transversally. It is this transversality which guarantees the map from any of the strata to Z is a submersion.

Now use

THEOREM 1 (Thom's First Isotopy Lemma, proved in Mather's paper, also can be found in Goresky and MacPherson's *Stratified Morse Theory* pg 41). *Let $f : X \to Y$ be a C^2 mapping, let A be a closed subset of X which admits a C^2 Whitney prestratification $\mathcal{P}$. Suppose $f|_A : A \to Y$ is proper nd that for each stratum U of $\mathcal{P}$, $f|_U : U \to Y$ is a submersion. Then $f|_A : A \to Y$ is a locally trivial fibration. Moreover, $f|U : U \to Y$ is a fibration.*

(the last sentence is mentioned before(8.2) in Mather's paper: "Note that the local trivialization which this theorem provides preserves the strata")

Consider the "curve" $C = \cap S_i$, $B = \overline{C} \cap (H \times Z)$ which is the boundary of C, and their complement $\mathbb{P}^n \times Z \setminus \overline{C}$. They give another much simpler stratification of $\mathbb{P}^n \times Z$, each maps to Z as a submersion. Apply Thom's First Isotopy Lemma to our case, the map f is just $\pi : \mathbb{P}^n \times Z \to Z$, A is the "curve" $\overline{C}$, the prestratification is given by $\mathcal{P} = \{B, C\}$. This prestratification satisfies Whitney condition trivially.

So $\pi|_C : C \to Z$ is a fibration, which is what we want to prove!

2. TYPE F_4

Let

$$Y = \{(y_1, y_2, y_3, y_4) \in \mathbb{C}^4 : y_i \neq 0, y_i \pm y_j \neq 0 \text{ for } i \neq j, y_1 \pm y_2 \pm y_3 \pm y_4 \neq 0\},$$

$$Z = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_i \neq 0, z_i \neq z_j, \text{ for } i \neq j\}.$$

and define $Y \to Z$ by $z_i = y_1 y_2 y_3 y_4 (y_4^2 - y_i^2)$.

As we did before, we think of Y as a subset embedded in $\mathbb{P}^4 \times Z$ by the "graph" map $Y \hookrightarrow \mathbb{P}^4 \times Z$ defined as

$$(y_1, y_2, y_3, y_4) \mapsto \big(y_1, y_2, y_3, y_4, y_1y_2y_3y_4(y_4^2 - y_1^2), y_1y_2y_3y_4(y_4^2 - y_2^2), y_1y_2y_3y_4(y_4^2 - y_3^2)\big)$$

The map $Y \to Z$ is then the restriction of the projection $\pi : \mathbb{P}^4 \times Z \to Z$.

Let S_i be the hypersurface in $\mathbb{C}^4 \times Z$ defined by $z_i = y_1 y_2 y_3 y_4 (y_4^2 - y_i^2)$.

Let $C = S_1 \cap S_2 \cap S_3$. Let $C^\circ = C \setminus \{y_1 \pm y_2 \pm y_3 \pm y_4 \neq 0\}$. Then whether the map $Y \to Z$ is a fibration, is equivalent to whether the map $C^\circ \to Z$ is a fibration.

2.1. **Cross section.** This is ok, as you pointed out. Let

$$u = \frac{y_4^2 - y_1^2}{z_1} = \frac{y_4^2 - y_2^2}{z_2} = \frac{y_4^2 - y_3^2}{z_3}. \tag{1}$$

For each (z_1, z_2, z_3), choose a continuous function $u = u(z_1, z_2, z_3)$ that $0 < u \ll 1$ and $|z_i u| \ll 1$. The solution for

$$z_1 = y_1 y_2 y_3 y_4 (y_4^2 - y_1^2)$$

or equivalently,

$$\sqrt{y_4^2 - z_1 u}\sqrt{y_4^2 - z_2 u}\sqrt{y_4^2 - z_3 u} \cdot y_4 u = 1 \tag{2}$$

has no multiple roots. (we can see this by square the equation, then consider the degree 8 polynomial of y_4.) So we can choose a branch of the solution $y_4 = y_4(u)$ as a continuous function when $z = (z_1, z_2, z_3)$ varies.

This is **almost** a cross section of $\pi : Y \to Z$, except that we haven't check the condition $y_1 \pm y_2 \pm y_3 \pm y_4 \neq 0$. But this difficulty can be easily got rid of, since even if the cross section does intersect the hypersurface $y_1 \pm y_2 \pm y_3 \pm y_4 = 0$, by a small perturbation we can get around this hypersurface since the hypersurface is of complex codimension 1.

So, the cross section exists.

2.2. **Transversality.** For any $z = (z_1, z_2, z_3)$, let

$$(S_i)_z := S_i \cap (\mathbb{C}^4 \times \{z\}),$$
$$C_z := (S_1)_z \cap (S_2)_z \cap (S_3)_z = C \cap (\mathbb{C}^4 \times \{z\}).$$

We have calculated that $(S_1)_z, (S_2)_z, (S_3)_z$ intersect transversally in $\mathbb{C}^4$. Therefore C_z is a smooth curve.

Notice that $\overline{S}_i$ do not intersect transversally at the infinity points in $\mathbb{P}^4$. So we have to change our method, i.e., we define $\overline{C}_z$ as the closure of C_z in $\mathbb{P}^4$ rather than the intersection of $\overline{S}_i$'s, then prove the smoothness of $\overline{C}_z$ by looking at each infinity point.

2.3. **Smoothness.** Firstly, we want to find out where $\overline{C}_z$ meets infinity. Consider the equation (2), there are five cases, totally 24 points, where the curve goes to infinity: y_1, y_2, y_3, y_4 or $u \to 0$.

Case $y_1 \to 0$. In this case, $y_2, y_3, y_4, u \to \infty$, hence $y_1/y_4 \to 0$. By (1),

$$\frac{y_4^2 - y_1^2}{y_4^2 - y_2^2} = \frac{z_1}{z_2} \quad \Rightarrow \quad \frac{1 - (y_1/y_4)^2}{1 - (y_2/y_4)^2} = \frac{z_1}{z_2} \quad \Rightarrow \quad y_2/y_4 \to \sqrt{\frac{z_1 - z_2}{z_1}}$$

Similarly

$$y_3/y_4 \to \sqrt{\frac{z_1 - z_3}{z_1}}$$

Thus in this case $\overline{C}_z$ meets infinity at 4 points

$$(y_0 : y_1 : y_2 : y_3 : y_4) = (0 : 0 : \sqrt{\frac{z_1 - z_2}{z_1}} : \sqrt{\frac{z_1 - z_3}{z_1}} : 1)$$

It is more symmetric if we add a dummy variable $z_4 = 0$, and rewrite the above as

$$(0 : 0 : \sqrt{z_1 - z_2} : \sqrt{z_1 - z_3} : \sqrt{z_1 - z_4})$$

Case $y_2 \to 0$, $\overline{C}_z$ meets infinity at 4 points

$$(0 : \sqrt{z_2 - z_1} : 0 : \sqrt{z_2 - z_3} : \sqrt{z_2 - z_4})$$

Case $y_3 \to 0$, $\overline{C}_z$ meets infinity at 4 points

$$(0 : \sqrt{z_3 - z_1} : \sqrt{z_3 - z_2} : 0 : \sqrt{z_3 - z_4})$$

Case $y_4 \to 0$, $\overline{C}_z$ meets infinity at 4 points

$$(0 : \sqrt{z_4 - z_1} : \sqrt{z_4 - z_2} : \sqrt{z_4 - z_3} : 0)$$

Case $u \to 0$, in this case $y_1, y_2, y_3, y_4 \to \infty$, and by (1) $y_i^2 - y_j^2 \to 0$, hence $y_i/y_j = \pm 1$. $\overline{C}_z$ meets infinity at 8 points:

$$(0 : 1 : \pm 1 : \pm 1 : \pm 1)$$

Next, we show that at each of these infinity point, $\overline{C}_z$ is smooth. This can be seen by changing the coordinates. For example, in the first case $y_1 \to 0$, we take the coordinate $x_i := y_i/y_4$, then locally around $(0 : 0 : \sqrt{\frac{z_1-z_2}{z_1}} : \sqrt{\frac{z_1-z_3}{z_1}} : 1)$ the curve $\overline{C}_z$ is defined by

$$z_i x_0^6 = x_1 x_2 x_3 (1 - x_i^2), \text{ for } i = 1, 2, 3 \tag{3}$$

By some argument using implicit function theorem, we can see that as the solution the equations (3), x_1, x_2 and x_3 are holomorphic functions of x_4, therefore $\overline{C}_z$ is smooth at $(0 : 0 : \sqrt{\frac{z_1-z_2}{z_1}} : \sqrt{\frac{z_1-z_3}{z_1}} : 1)$.

Other cases are similar.

It can also be shown that the intersection of the curve $\overline{C}_z$ with infinity hyperplane at each of the above 24 points is transversal. So the curve $\overline{C}_z$ is of degree 24.

2.4. **Connectness.** I have a down-to-earth argument saying that $\overline{C}_z$ is connected, by proving that each point is path-connected to one of the 8 points $(0 : 1 : \pm 1 : \pm 1 : \pm 1)$, and any two of the 8 points are path-connected.

2.5. **Stratification.** Define $B = \overline{C} \setminus C$ be the intersection of C with the infinity hypersurface in $\mathbb{P}^4 \times Z$. Then $\mathcal{P} := \{B, C\}$ gives a prestratification of $\overline{C}$. B restricts to each fiber $\mathbb{P}^4 \times \{z\}$ is just 24 points, which can be thought locally as 24 sections of the projection $\pi : \mathbb{P}^4 \times Z \to Z$. So the map $B \to Z$ is locally homeomorphic, therefore is a submersion. The map $C \to Z$ is also a submersion, which can be seen by the transversality of 4 hypersurfaces S_1,S_2,S_3 and $\mathbb{P}^4 \times \{z\}$ (compute the Jacobian again!).

Then by Thom's First Isotopy Lemma, $C \to Z$ is a fibration. Each fiber is a smooth curve with 24 puncture points. So far so good. HOWEVER, if we consider hypersurfaces $y_1 \pm y_2 \pm y_3 \pm y_4 = 0$, every good thing we expect fails! Now I explain this:

2.6. **Non-fibration!** Over a general $z \in Z$, the curve $\overline{C}_z$ intersects $\{y_1 + y_2 + y_3 + y_4 = 0\}$ at 24 points, with 6 different finite points and 3 points at infinity:

$$(0 : 1 : 1 : -1 : -1), (0 : 1 : -1 : -1 : 1), (0 : 1 : -1 : 1 : -1),$$

each with multiplicity 6. (The situation is similar in other cases $y_1 \pm y_2 \pm y_3 \pm y_4 = 0$. So $\overline{C}_z$ intersects the 8 hyperplanes $\{y_1 + y_2 + y_3 + y_4 = 0\}$ at totally $6 \times 8 = 48$ finite points)

But over some special points (e.g. $z = (1, 2, 3)$ or $z = (1, -3, -8)$), the curve $\overline{C}_z$ intersects $\{y_1 + y_2 + y_3 + y_4 = 0\}$ only at infinity.

Denote by $C_z = C \cap (\mathbb{P}^4 \times \{z\})$, $C_z^\circ = C^\circ \cap (\mathbb{P}^4 \times \{z\})$. The above argument shows that: for a general $z \in Z$, C_z° is the smooth curve $\overline{C}_z$ with $24 + 6 \times 8 = 72$ puncture points. But for some special points (e.g. $z = (1, -3, -8)$), C_z° is the smooth curve $\overline{C}_z$ with only 24 puncture points.

Therefore, each fiber might have different homotopy type, so the map $Y \to Z$ is not a fibration!

Best regards, Li Li