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S-FUNCTIONS AND SYMMETRY IN PHYSICS (*)

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Résumé. — Nous avons développé une chaîne complète de programmes pour le calcul extensif des caractères dans la théorie des groupes. En utilisant l'algèbre des fonctions S, nous pouvons calculer les produits de Kronecker ou les règles de branchement à l'intérieur des groupes continus compact U_n, O_n, R_n, Sp_n et G_2. On discute une méthode systématique simple de calcul de pléthysmes externes et leur emploi dans les calculs des propriétés des groupes continus. L'opération de pléthysme interne peut être employée pour calculer les propriétés similaires de groupes finis. On discute aussi l'évaluation systématique de cette opération par une méthode qui n'exige pas l'usage des tables de caractères.

Abstract. — We have developed a comprehensive set of computer programmes for the extensive calculation of the character theory of groups. Using the algebra of S-functions we may calculate Kronecker products or branching rules within the compact continuous groups U_n, O_n, R_n, Sp_n and G_2. This paper discusses a straightforward systematic method of calculating outer plethysms and its use in calculating continuous group properties. The operation of inner plethysm may be used to calculate similar properties of finite groups. This paper also discusses the systematic evaluation of this operation by a method that does not require the use of character tables.

1. **Introduction.** — There has been much interest in recent years in the properties of both finite and continuous groups. Physicists have approached the continuous groups from the standpoint of infinitesimal operators [1], and indeed there have been suggestions [2] to approach the algebra of finite groups from this direction also. Lately, however, interest is being revived [3-6] in the character theory of continuous groups owing to the fact that many results of group theory may be calculated very simply using such ideas. Selection rules and the labelling schemes for atoms and nuclei follow from considerations of character theory, without the need to construct explicit operator bases.

Littlewood's approach [7] to the theory of group characters emphasizes the correspondences between unitary and symmetric groups but does not require the use of character tables of the symmetric group. Instead he uses S-functions, which are functions on the characteristic roots of the representation matrices.

S-functions may be multiplied together either by taking an algebraic product of two functions of different degrees on the same set of variables, or by taking a product of two functions of the same degree on different sets of variables. In both cases the product, termed the outer (or ordinary) and inner product respectively, may be resolved as a direct sum S-functions.

$$\{ \lambda \} \{ \mu \} = \sum_{\nu} \Gamma_{\mu\lambda\nu} \{ \nu \} \tag{1a}$$

$$\{ \lambda \} \circ \{ \mu \} = \sum_{\mu} g_{\lambda\mu\nu} \{ \nu \} . \tag{1b}$$

The outer or inner square may be resolved into symmetric and antisymmetric parts, and indeed any n-th order product may be resolved into terms occurring according to the symmetries related to representations of the symmetric group S_n.

$$\{ \lambda \} \{ \lambda \} \cdots \{ \lambda \} \; n \text{ times} = \sum f_{\mu} \{ \lambda \} \otimes \{ \mu \} \tag{2a}$$

$$\{ \lambda \} \circ \{ \lambda \} \circ \cdots \circ \{ \lambda \} \; n \text{ times} = = \sum f_{\mu} \{ \lambda \} \odot \{ \mu \} \tag{2b}$$

where the sum is over all partitions (μ) of n and the coefficient f_{μ} is the degree of the representation $\{ \mu \}$ of S_n [8].

The sets of terms $\{ \lambda \} \otimes \{ \mu \}$ and $\{ \lambda \} \odot \{ \mu \}$ are termed the outer and inner plethysms. Littlewood introduced the outer plethysm as an immanant of an immanant from matrix theory [9]. When calculating branching rules [3, 4] it is useful to observe [10, 11] that outer plethysm is equal to the substitution of the terms of the S-function $\{ \lambda \}$ into the function $\{ \mu \}$ as the basic variables.

It is the purpose of this paper to give systematic methods for resolving the two operations into direct sums of S-functions.

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2. **Properties of plethysms.** — From the definitions of plethysms it follows that the operation is distributed on the right

$$\{\lambda\} \otimes (A \pm B) = \{\lambda\} \otimes A \pm \{\lambda\} \otimes B \quad (3a)$$

$$\{\lambda\} \odot (A \pm B) = \{\lambda\} \odot A \pm \{\lambda\} \odot B \quad (3b)$$

$$\{\lambda\} \otimes (AB) = (\{\lambda\} \otimes A)(\{\lambda\} \otimes B) \quad (4a)$$

$$\{\lambda\} \odot (AB) = (\{\lambda\} \odot A) \circ (\{\lambda\} \odot B) \quad (4b)$$

The last equation brings out a difficulty with the notation. We have written the operation as a product, although it is more akin to a substitution. The inner product on the right of eq. (4b) appears since both sides are partitions of m, if $\{\lambda\}$ is a partition of m. Similarly, although outer plethysm may be carried out in any order :

$$(\{\lambda\} \otimes \{\mu\}) \otimes \{\nu\} = = \{\lambda\} \otimes (\{\mu\} \otimes \{\nu\}) \quad (5a)$$

inner plethysm may not :

$$(\{\lambda\} \odot \{\mu\}) \odot \{\nu\} = = \{\lambda\} \odot (\{\mu\} \otimes \{\nu\}) \quad (5b)$$

For operations on the left, cross terms appear in the products of eq. (2), and we obtain

$$(A+B) \otimes \{\lambda\} = \sum A \otimes (\{\lambda\}/\{\mu\})\, B \otimes \{\mu\} \quad (6a)$$

$$(A+B) \odot \{\lambda\} = \sum A \odot (\{\lambda\}/\{\mu\})\, B \odot \{\mu\} \quad (6b)$$

$$(A-B) \otimes \{\lambda\} = \sum (-1)^p A \otimes (\{\lambda\}/\{\mu\})\, B \otimes \{\tilde{\mu}\} \quad (7a)$$

$$(A-B) \odot \{\lambda\} = \sum (-1)^p A \odot (\{\lambda\}/\{\mu\}) \circ B \odot \{\tilde{\mu}\} \quad (7b)$$

$$(AB) \otimes \{\lambda\} = \sum A \otimes (\{\lambda\} \circ \{\mu\})\, B \otimes \{\mu\} \quad (8a)$$

$$(A \circ B) \odot \{\lambda\} = \sum A \odot (\{\lambda\} \circ \{\mu\}) \circ B \odot \{\mu\} \quad (8b)$$

where the sums are over all $\{\mu\}$ a partition of all p, except in eq. (8) where the sum is restricted to partitions of n only. We use also the operation of division of S-functions defined in a previous paper [4] as the inverse to outer multiplication. If we have eq. (1a) then we write

$$\{\lambda\}/\{\mu\} = \sum \Gamma_{\mu\nu\lambda} \{\nu\} \quad (9)$$

Conjugate relations for plethysm may be derived ;

$$\widetilde{\{\lambda\} \otimes \{\mu\}} = \{\tilde{\lambda}\} \otimes \{\mu\} \qquad n \text{ even}$$
$$= \{\tilde{\lambda}\} \otimes \{\tilde{\mu}\} \qquad n \text{ odd} \quad (10a)$$

$$\widetilde{\{\lambda\} \odot \{\mu\}} = \{\tilde{\lambda}\} \odot \{\mu\} \qquad n \text{ even}$$
$$= \{\tilde{\lambda}\} \odot \{\tilde{\mu}\} \qquad n \text{ odd}. \quad (10b)$$

3. **Dimensional relations.** — The operations may be checked dimensionally by the relations [8]

$$f(\{\lambda\} \otimes \{\mu\}) = \left(\frac{f\{\lambda\}}{m!}\right)^n \frac{f\{\mu\}}{n!} (mn)! \quad (11a)$$

$$f(\{\lambda\} \odot \{\mu\}) = \frac{f\{\mu\}}{n!} G^{(\mu)}(f\{\lambda\}) \quad (11b)$$

where (λ) and (μ) are partitions of m and n respectively, $f\{\lambda\}$ is the degree of representation $\{\lambda\}$ of S_m, and

$$G^{(\mu)}(d) = \prod_{i=1}^{r} \prod_{j=1}^{\mu_i} (d + j - i) \quad (12)$$

(μ) being a partition of r parts.

4. **The calculation of outer plethysms.** — Much work in the 1950's went into finding a systematic method of calculating plethysms. Ibrahim [2] has calculated tables for $mn \leqslant 18$ by a recursive method using the theorem

$$(\{\lambda\} \otimes \{\mu\})/\{1\} = = (\{\lambda\} \otimes (\{\mu\}/\{1\}))(\{\lambda\}/\{1\}) \quad (13)$$

The right hand side of eq. (13) is calculated first and various terms in the product tried until the one and only consistent set is found.

The method uses the general results

$$\{\lambda\} \otimes \{0\} = \{0\}, \{\lambda\} \text{ of one part only}$$
$$= 0 \text{ otherwise}.$$

However two theorems due to Murnaghan [13, 14] lead to a simpler recursive method where the choices in the method above do not occur. We denote by

$$[\{\lambda\}]_k$$

the S-function obtained by subtracting 1 from each part of the S-function $\{\lambda\}$, if $\{\lambda\}$ is of k parts, or zero if $\{\lambda\}$ is not of k parts.

e. g. $[\{6\} + \{51\} + \{42\} + \{411\}]_2 = = \{4\} + \{31\}$

Murnaghan's theorems state :

$$[\{m\} \otimes \{n\}]_k = = \sum_{r=1}^{n-k} (-1)^{r-1} \{m\} \otimes \{1^r\} [\{m\} \otimes \{n-r\}]_k + + (-1)^{n-k} (\{m-1\} \otimes \{1^n\}) \{n-k\} \quad (14a)$$

$$[\{m\} \otimes \{1^n\}]_k = = \sum_{r=1}^{n-k} (-1)^{r-1} \{m\} \otimes \{r\} [\{m\} \otimes \{1^{n-r}\}]_k + + (-1)^{n-k} (\{m-1\} \otimes \{n\}) \{n-k\} \quad (14b)$$

omitting all terms in the products of more than k parts. We use roman letters to indicate S-functions of one part.

Using a table of $\{ m \} \otimes \{ n \}$ and $\{ m \} \otimes \{ 1^n \}$ generated by the above method it is a straightforward, if lengthy, process to build up any outer plethysm using eqs (3*a*-8*a*). The method of doing this is similar to the method of building up inner products described in a previous paper [4]. We first express $\{ \lambda \}$ as a sum of products of S-functions of one part by means of the determinant

$$\{ \lambda \} = | h_{\lambda_s - s + t} | \tag{15}$$

where $h_r = \{ r \}, r \geqslant 0, h_r = 0, r < 0$. This method has two distinct advantages over Ibrahim's method for machine calculations other than ease of computation. It only requires a small table of recursively computed results, and if plethysms are required on a restricted number of basis variables, and this is usually the case in physics, one may omit null terms when they first make their appearance. We have used this method for calculations on the University's IBM 360/44 Computer.

5. **The calculation of inner plethysms.** — Littlewood [15] gives the special result :

$$\{ m - 1, 1 \} \odot \{ 1^k \} = \{ m - k, 1^k \} \tag{16}$$

It is clear that a calculation of inner plethysms could proceed for all plethysms $\{ m - 1, 1 \} \odot \{ \mu \}$, by using the conjugate relation to eq. (15), and eq. (3*b*) and (4*b*). Also eq. (5*b*) leads to

$$\begin{aligned}\{ m - k, 1^k \} \odot \{ \mu \} &= (\{ m - 1, 1 \} \odot \{ 1^k \}) \odot \{ \mu \} \\ &= \{ m - 1, 1 \} \odot (\{ 1^k \} \otimes \{ \mu \})\end{aligned} \tag{17}$$

and thus we may evaluate such inner plethysms, using outer plethysms. For a more general S-function $\{ \lambda \}$ on the left we require to express it in terms of inner products of terms $\{ m - k, 1^k \}$. This may be done by the following procedure :

THEOREM : It is always possible to write an S-function $\{ \lambda \}$ in terms of *d*-fold inner products

$$\begin{aligned}\{ \lambda \} = \sum \pm \{ m - a, 1^a \} \circ \{ m - b, 1^b \} \circ \\ \circ \{ m - c, 1^c \} \circ \cdots \circ \{ m - d, 1^d \}\end{aligned} \tag{18}$$

To prove this, we define a new ordering among the S-functions $\{ \lambda \}$, $\{ \mu \}$, partitions of m, that differs from the usual ordering. We say $\{ \lambda \}$ preceeds $\{ \mu \}$ if,

(i) $\lambda_1 > \mu_1$

or if their first rows are equal, by the length of the first column :

(ii) $(\tilde{\lambda})_1 > (\tilde{\mu})_1$.

If both first row and first column are equal we order with respect to the smallness of their lowest rows $\lambda_r < \mu_r$ e. g.

$$\{ 421 \} > \{ 43 \} > \{ 3211 \} > \{ 331 \} > \\ > \{ 322 \} > \{ 2^2 1^3 \} > \{ 2^3 1 \} .$$

We may replace $\{ \lambda \}$, $\{ \lambda \}$ not of the form

$$\{ m - k, 1^k \} ,$$

by the set of S-functions given by

$$\{ \lambda'' \} \circ \{ \mu \} - \sum \{ \nu \} \tag{19}$$

where $\{ \lambda'' \} \circ \{ \mu \} = \{ \lambda \} + \sum \{ \nu \}$ and we are to prove that all the terms $\{ \lambda'' \}$, $\{ \mu \}$ and $\{ \nu \}$ preceed $\{ \lambda \}$ in the above ordering.

If

$$\{ \lambda \} = \{ \lambda_1, \lambda_2, \ldots, \lambda_r 1^n \}$$

and

$$\lambda_r = a + 1 \qquad a > 0$$

we take $\{ \lambda'' \} = \{ \lambda_1 + a, \lambda_2, \ldots, \lambda_{r-1}, 1^{n+1} \}$

$$\{ \mu \} = \{ m - a, a \}$$

It is clear that both $\{ \lambda'' \}$ and $\{ \mu \}$ preceed $\{ \lambda \}$ in the ordering.

We may write

$$\{ \mu \} = \{ m - a \} \{ a \} - \{ m - a + 1 \} \{ a - 1 \} \tag{20}$$

and by Littlewood's inner product theorem [4] :

$$\begin{aligned}\{ \lambda'' \} \circ \{ \mu \} = (\{ \lambda'' \}/\{ \xi_a \}) \{\xi_a \} - \\ - \{ \lambda'' \}/\{ \xi_{a-1} \}) \{\xi_{a-1} \}\end{aligned} \tag{21}$$

where $\{ \xi_a \}$ is summed over all partitions of a. All terms $\{ \lambda'' \}/\{ \xi_a \}$, $\{ \lambda'' \}/\{ \xi_{a-1} \}$ will have at least

$$\lambda_1 + 1$$

cells in the first row except one term of $\{ \lambda'' \}/\{ a \}$. This one term of $\{ \lambda'' \}/\{ a \}$ is

$$\{ \lambda_1, \lambda_2, \ldots, \lambda_{r-1}, 1^{n+1} \} .$$

All the $\{ \nu \}$ arising from the products of the other terms with their respective $\{ \xi \}$ must preceed $\{ \lambda \}$ in our ordering on account of condition 1. One term in

$$\{ \lambda_1, \lambda_2, \ldots, \lambda_{r-1}, 1^{n+1} \} \{ a \} = \sum \{ \nu \}$$

is $\{ \lambda \}$, all other terms will have $\nu_r < a + 1$, i. e. $\nu_r < \lambda_r$ (condition 3). This proves that by use of eq. (16) we may systematically replace the lowest $\{ \lambda \}$ in an inner product expression by terms « *higher* » in this ordering.

e. g. $\{ 321 \} \odot \{ 11 \} = (\{ 41^2 \} \circ \{ 51 \} -$
$- \{ 51 \} \circ \{ 51 \} - \{ 31^3 \} + \{ 6 \}) \odot \{ 11 \}$

by eq. (6*b*, 7*b*)

$$\begin{aligned}&= (\{ 41^2 \} \circ \{ 51 \}) \odot \{ 11 \}) + \\ &+ (\{ 51 \} \circ \{ 51 \}) \odot \{ 2 \} \\ &+ \{ 31^3 \} \odot \{ 2 \} + \{ 6 \} \odot \{ 11 \} \\ &- \{ 41^2 \} \circ \{ 51 \} \circ \{ 51 \} \circ \{ 51 \} \\ &+ \{ 411 \} \circ \{ 51 \} \circ \{ 31^3 \} + \{ 41^2 \} \circ \{ 51 \} \\ &+ \{ 51 \} \circ \{ 51 \} \circ \{ 31^3 \} - \\ &- \{ 51 \} \circ \{ 51 \} - \{ 31^3 \}\end{aligned}$$

but we have also by eq. (8*b*)

$$(\{ 41^2 \} \circ \{ 51 \}) \odot \{ 11 \} =$$
$$= \{ 41^2 \} \odot \{ 2 \} \circ \{ 51 \} \odot \{ 11 \} + + \{ 41^2 \} \odot \{ 11 \} \circ \{ 51 \} \odot \{ 2 \}$$

and

$$(\{ 51 \} \circ \{ 51 \} \odot \{ 2 \} =$$
$$= (\{ 51 \} \odot \{ 2 \})^2 + (\{ 51 \} \odot \{ 11 \})^2 .$$

Use of eq. (3*b*, 4*b*) and eq. (16) gives

$$\{ 51 \} \odot \{ 11 \} = \{ 41^2 \}$$
$$\{ 51 \} \odot \{ 2 \} = \{ 51 \} \circ \{ 51 \} - \{ 41^2 \} .$$

Using also eq. (15) we also obtain

$$\{ 31^3 \} \odot \{ 2 \} =$$
$$= \{ 41^2 \} \circ \{ 41^2 \} - \{ 51 \} \circ \{ 41^2 \} + \{ 51 \}$$

and

$$\{ 41^2 \} \odot \{ 11 \} = \{ 51 \} \circ \{ 31^3 \} - \{ 21^4 \}$$
$$\{ 41^2 \} \odot \{ 2 \} =$$
$$= \{ 41^2 \} \circ \{ 41^2 \} - \{ 51 \} \circ \{ 31^3 \} + \{ 21^4 \}$$

and finally using $\{ 6 \} \odot \{ 11 \} = 0$ leads to the result :

$$\{ 321 \} \odot \{ 11 \} = 3 \{ 41^2 \} + \{ 3^2 \} + 2 \{ 321 \} + + 3 \{ 31^3 \} + 2 \{ 2^2 1^2 \} + \{ 21^4 \}$$

verifying the result given by Robinson [8] by other means.

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