ON THE RESIDUAL NILPOTENCE OF PURE ARTIN GROUPS

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Abstract. In this very short note we show that the residual nilpotence of pure Artin groups of spherical type is easily deduced from the faithfulness of the Krammer-Digne representations.

Let W be a finite Coxeter group, B the associated (spherical) Artin group of spherical type, and P the corresponding pure group, defined as the kernel of the projection $\pi : B \twoheadrightarrow W$. It is conjectured that the group P is residually nilpotent, meaning that the intersection of the terms $C^r P$ for $r \geq 1$ of its descending central series, defined by $C^1 P$ equals the commutator group (P, P) and $C^{r+1} P = (P, C^r P)$, is trivial. This fact is obvious for the dihedral types, since it reduces in this case to the study of the free groups, which is known for a long time. It was proved by Falk and Randell for types A_n, B_n (and $I_2(m)$) in [6] as an easy consequence of their previous work [5], and by Markushevich in [8, 9] for type D_n.

There is a short way to prove this for every Artin group of spherical type using the generalization of the Krammer representation by F. Digne (these representations where also obtained in type ADE by A.M. Cohen and D.B. Wales in [2]) — in the same spirit that the linearity of these groups imply their residual finiteness by Malcev theorem. It seems that this very simple application had been unnoticed yet.

Theorem. *The pure Artin groups of spherical type are residually nilpotent.*

Proof. First we may assume that W is reduced. Moreover let's assume first that W is of crystallographic type. Let N be the number of reflexions of W and $V \simeq \mathbb{R}^N$ the $\mathbb{R}$-vector space formed by linear formal combination of reflexions of W. We let $L = \mathbb{R}[q, q^{-1}, t]$, $A = \mathbb{R}[[h]]$ where q, t, h are indeterminates. There are embeddings $\iota : L \hookrightarrow A$ given, for example, by $q \mapsto e^h$ and $t \mapsto e^{\sqrt{2}h}$. We let $V_{q,t} = V \otimes_{\mathbb{R}} L$, and $V_h = V \otimes_{\mathbb{R}} A = V_{q,t} \otimes_\iota A$. There is a reduction morphism $\varphi : V_h \mapsto V$. F. Digne defined in [4] a faithful representation $R_W : B \to GL(V_{q,t}) \subset GL(V_h)$, and it is easily checked from his formulas that the following diagram is commutative

$$\begin{tikzcd} B \arrow[r] \arrow[d] & GL(V_h) \arrow[d, "\varphi"'] \\ W \arrow[r, "R_W^0"'] & GL(V) \end{tikzcd}$$

where the map $R_W^0 : W \to GL(V)$ is given by the natural conjugation action of W on its set of reflexions. It follows, using the identifications $V = \mathbb{R}^N$, $\mathrm{End}(V_h) = M_N(A)$, that $R_W(P) \subset 1 + hM_N(A)$ hence $R_W(C^r P) \subset 1 + h^{r+1} M_N(A)$ by an easy induction on r. In particular, if x belongs to the intersection of the terms $C^r P$, $R_W(x)$ belongs to the intersection of the terms $1 + h^{r+1} M_N(A)$, which is trivial. Hence $R_W(x) = 1$ and $x = 1$ by faithfulness of the Krammer-Digne representation.

If W is of type H_3, we let W' be of type D_6, and denote by B', P' the Artin group and pure Artin group associated to it. There exists an injective morphism

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$j : B \to B'$ which sends P to P' and induces an injective morphism $\check{j} : W \to W'$ such that the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d] & B' \arrow[d] \\ W \arrow[r, "\check{j}"'] & W' \end{tikzcd}$$

where the vertical arrows are the natural projections. If $\sigma_1, \sigma_2, \sigma_3$ denote the Artin generators of B such that $< \sigma_1, \sigma_2 >$ is of type $I_2(5)$ and $< \sigma_2, \sigma_3 >$ has type $I_2(3)$, and the Artin generators $\sigma'_1, \dots, \sigma'_6$ of B' are numbered according to Bourbaki (see [1]), j is defined by $\sigma_1 \mapsto \sigma'_3\sigma'_5$, $\sigma_2 \mapsto \sigma'_2\sigma'_4$, $\sigma_3 \mapsto \sigma'_1\sigma'_6$. This is one of the "folding morphisms" defined by J. Crisp in [3]. If W is of type H_4, let W' be of type E_8, let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be the Artin generators of H_4 such that $< \sigma_1, \sigma_2, \sigma_3 >$ is of type H_3 and number the Artin generators $\sigma'_1, \dots, \sigma'_8$ of B' according to Bourbaki. There is also a folding morphism defined by $\sigma_1 \mapsto \sigma'_3\sigma'_5$, $\sigma_2 \mapsto \sigma'_4\sigma'_6$, $\sigma_3 \mapsto \sigma'_3\sigma'_7$ and $\sigma_4 \mapsto \sigma'_1\sigma'_8$. Then j restricts to a monomorphism $P \to P'$, and $R_{W'} \circ j$ is faithful. Then $R_{W'} \circ j(P) \subset 1 + hM_N(A)$, where N is the number of reflexions of W'. It follows that P is residually nilpotent by the same argument. Since there are also folding morphisms from the groups of type $I_2(m)$ to any Artin group of Coxeter number m, the conclusion also holds for type $I_2(m)$ (alternatively one may use the faithfulness of the Iwahori-Hecke representation proved in [7]).

$\square$

Notice that we could have restrict ourselves to the representations constructed by A.M. Cohen and D.B. Wales (see [2]) in type ADE since all Artin groups of spherical type embed in one of these by a folding morphism.

References

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