PARABOLIC SUBGROUPS OF COMPLEX BRAID GROUPS I

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Abstract. In this paper we introduce a class of 'parabolic' subgroups for the generalized braid group associated to an arbitrary irreducible complex reflection group, which maps onto the collection of parabolic subgroups of the reflection group. Except for one case, we prove that this collection forms a lattice, so that intersections of parabolic subgroups are parabolic subgroups. In particular, every element admits a parabolic closure, which is the smallest parabolic subgroup containing it. We furthermore prove that it provides a simplicial complex which generalizes the curve complex of the usual braid group. In the case of real reflection groups, this complex generalizes the one previously introduced by Cumplido, Gebhardt, González-Meneses and Wiest for Artin groups of spherical type. We show that it shares similar properties, and similarly conjecture its hyperbolicity.[1]

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1. Introduction

Let W be a complex reflection group, and B the generalized braid group associated to it, as in [8]. In the case W is a real reflection group, B can be described as an Artin group, and it was shown in [14] how to generalize the curve complex of the usual braid group in this setting: curves are replaced with irreducible parabolic subgroups, which are defined algebraically as conjugates of the irreducible standard parabolic subgroups of the Artin group. Actually, these standard parabolic subgroups are the standard parabolic subgroups of the corresponding Garside structure, as defined in [30]. A disadvantage of this construction is however that it heavily depends on the presentation chosen for B, or equivalently on its construction as the group of fractions of some specific monoid.

Here we provide a generalization of this work to the general setting of a complex reflection group. For this we first provide, in Section 2, a purely topological definition of a parabolic subgroup of B, which is a subgroup isomorphic to the braid group of some parabolic subgroup of W. This topological definition as some kind of local fundamental group implies 'on the nose' two main properties. First of all it implies that, when two reflection groups are 'isodiscriminantal' (see the end of subsection 2.3), then the collections of the parabolic subgroups for their braid groups are the same. It also enables us to identify, when we have two equally likeable Garside monoids with group of fractions B, the conjugates of standard parabolic subgroups of the corresponding Garside structures, by showing that the corresponding conjugacy classes of subgroups are the parabolic subgroups we define topologically.

In this paper we prove the following 3 main theorems for all irreducible complex reflection groups but the exceptional group G_{31}. For this group, these statements are still conjectural, and will be addressed in a forthcoming paper. Let W be an irreducible complex reflection group distinct from G_{31}, and let B be its braid group.

Theorem 1.1. *For every element $x \in B$, there exists a unique minimal parabolic subgroup* $\mathrm{PC}(x)$ *of B containing it, and we have* $\mathrm{PC}(x^m) = \mathrm{PC}(x)$ *for every $m \neq 0$.*

This unique minimal parabolic subgroup of B containing x is called the parabolic closure of x.

Theorem 1.2. *If B_1, B_2 are two parabolic subgroups of B, then $B_1 \cap B_2$ is a parabolic subgroup. More generally, every intersection of a family of parabolic subgroups is a parabolic subgroup.*

Defining the join of two parabolic subgroups as the intersection of all the parabolic subgroups containing them then defines, as in [14], a lattice structure on the collection of parabolic subgroups.

The proof goes as follows. In Section 3 we prove that the main Garside monoids used for dealing with the complex braid groups are well adapted to our study of parabolic subgroups. More precisely, we prove in these cases that every parabolic subgroup is a conjugate of some subgroup generated by certain specific subsets of the generators of the given presentation, and that the Garside-theoretical notion and our topological notion of parabolic subgroups coincide. When there are no available Garside monoids, but B is a normal subgroup of finite index of another complex braid group endowed with a Garside structure, we show how to determine the collection of parabolic subgroups of the former from the one of the latter. This covers all cases of complex braid groups except G_{31}.

In Section 4 we develop the needed Garside machinery. Under suitable assumptions on the Garside monoid, we prove (see Theorem 4.31) that parabolic closures exist for arbitrary elements. In Section 5 we prove that these conditions are satisfied for the Garside monoids which appeared in Section 3. In so doing, we determine the standard parabolic subgroups for the Garside structure, and find the same 'standard' subgroups determined in Section 3, so that the Garside-theoretic notion of a parabolic subgroup and the topological notion coincide in each case. This proves that every element admits a parabolic closure. We then prove in Section 6, using additional properties of our Garside structures, that the intersection of two parabolic subgroups are parabolic subgroups. Using additional methods, we show in the same section how to conclude the proofs for the other groups.

From this construction we build an analogue of the curve complex for the usual braid groups, generalizing the ideas of [14]. Its vertices are the irreducible parabolic subgroups of B as defined above, and two vertices B_1, B_2 are connected if and only if either $B_1 \subset B_2$, or $B_2 \subset B_1$, or $B_1 \cap B_2 = [B_1, B_2] = 1$. This forms the curve graph Γ for B. The associated simplicial complex is then constructed as the clique complex of Γ, namely the flag complex made of all the simplices whose edges belong to the curve graph. It is easy to prove (see Proposition 2.10) that the group $B/Z(B)$ acts faithfully on it.

As in [14] for the case of real reflection groups, we prove that the graph admits a simpler description. For this, recall from [8] and [3] (see also [19]) that the center of any irreducible complex braid group B admits a canonical positive generator z_B. Here positive means that its image under the natural map $B \to \mathbb{Z}$ is positive. From this, a well-defined element $z_{B_0} \in B$ is associated to every irreducible parabolic subgroup B_0 of B, that is every parabolic subgroup whose associated (parabolic) reflection subgroup is irreducible.

Theorem 1.3. *If B_0 is an irreducible parabolic subgroup of B we have $B_0 = \mathrm{PC}(z_{B_0})$. Moreover, if B_1 and B_2 are two irreducible parabolic subgroups of the irreducible complex braid group B, an element $g \in B$ satisfies $(B_1)^g = B_2$ if and only if $(z_{B_1})^g = z_{B_2}$, and B_1, B_2 are adjacent if and only if $z_{B_1} z_{B_2} = z_{B_2} z_{B_1}$.*

Of course this theorem admits the following immediate corollary.

Corollary 1.4. *If B_0 is an irreducible parabolic subgroup of B, then the normalizer of B_0 is equal to the centralizer of z_{B_0}.*

The above results allow us to redefine the curve graph associated to B as the graph whose vertices correspond to the elements z_{B_0}, for B_0 an irreducible parabolic subgroup of B, and where two vertices are adjacent if and only if their corresponding elements commute. The curve complex is defined from the graph in the same way as above.

Since the simplicial complex constructed here generalizes the usual curve complex, it is natural to conjecture that the associated simplicial complex is hyperbolic in the sense of Gromov. We also mention, as pointed to us by L. Paris, that a generalization of [14] in a different direction has been done in [15], where the authors consider fundamental groups of simplicial hyperplane complement.

We end this introduction by a few comments on the Garside-theoretic aspects. The machinery used to prove the main theorems is established in the setting of a general Garside group satisfying two main properties, and the parabolic subgroups being defined in this case are the ones introduced by Godelle [29]. We introduce a new kind of conjugation (called 'swap') and a new kind of elements ('recurrent elements'), which turn out to be very useful tools to study conjugacy in Garside groups, simplifying the usual techniques. We have a notion of 'support' of an element (using recurrent elements) which generalizes the one in [14], and we provide in several cases much simpler proofs of the results that were given in [14] for Artin groups of spherical type. Notably, the proof that the intersection of two parabolic subgroups is a parabolic subgroup is significantly shorter, and is valid for the more general case of the Garside monoids involved in the study of complex braid groups.

The extension of our results to the single remaining case of G_{31} requires specific work, including a deeper understanding of the parabolic subgroups of centralizers of regular elements, in terms of divided (Garside) categories. This will be the theme of the forthcoming paper [25].

2. Parabolic subgroups

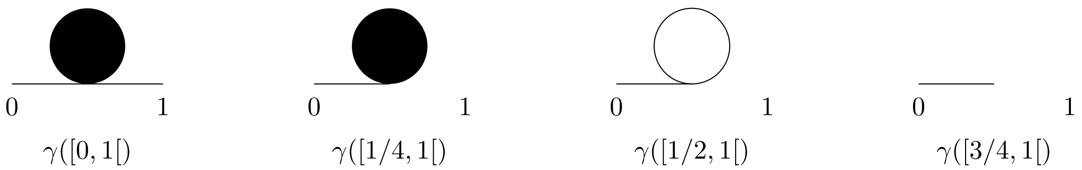
In this section we define a purely topological concept of a local fundamental group which is suitable for our purposes, and then define a parabolic subgroup of a complex braid group as such a local fundamental group with respect to a topological pair $(X/W, V/W)$. We then establish its basic properties.

2.1. **Normal rays.** Let Y be a topological space and $X \subset Y$ an open subset. A closed normal ray in the topological pair (X, Y) is a (continuous) path $\gamma : [0,1] \to Y$ such that

(1) $\eta([0,1[) \subset X$
(2) $\eta(1) \notin X$
(3) there exists $\alpha_0 \in]0,1[$ such that $\eta([1-\alpha, 1[)$ is simply connected for every $\alpha \in]0, \alpha_0]$.

Such a closed normal ray is said to terminate at $\eta(1)$ and to be based at $\eta(0) \in X$. An open normal ray is defined similarly to be a map $\eta :]0,1] \to Y$ with $\eta(]0,1[) \subset X$, fulfilling the same two conditions (2) and (3). Of course, if η is a closed normal ray, then its restriction $\check{\eta} = \eta_{|]0,1]}$ is an open normal ray and, if η is an open normal ray, one can build a closed normal ray via $\hat{\eta} : u \mapsto \eta(\frac{1+u}{2})$.

Probably every person reading this definition first wonders why it is not sufficient to check condition (3) for $\alpha = \alpha_0$, therefore we provide an example. Consider the case $Y = \mathbb{C}$, $X = \mathbb{C} \setminus \{0\}$ with basepoint 1, and consider the following path $\gamma : [0,1] \to \mathbb{C}$. From 0 to 1/4, γ is a straight path from 1 to $1/2 \in \mathbb{C}$, from 3/4 to 1 it is a straight path from 1/2 to 0, from 1/2 to 3/4 it is a circle tangent at 1/2, and from 1/4 to 1/2 a Peano curve filling the corresponding disc. Then $\gamma([0,1[)$ is simply connected but $\gamma([1/2,1[)$ is not. Therefore γ is not a normal ray.



Recall that, if $A \subset X$ is path-connected and simply-connected, then $\pi_1(X, A)$ is well-defined: either as the groups $\pi_1(X, a)$ for $a \in A$ canonically identified with each other via an arbitrary path joining the base points, or as the set of classes of paths from some point of A to some other one, up to a homotopy leaving the endpoints inside A.

Because of the defining conditions of a normal ray, the fundamental groups $\pi_1(X, \eta([1-\alpha, 1[))$ for $\alpha \in]0, \alpha_0]$ can be canonically identified under the natural maps $\pi_1(X, \eta([1-\alpha_2, 1[)) \to \pi_1(X, \eta([1-\alpha_1, 1[))$ for $\alpha_0 \geq \alpha_1 > \alpha_2 > 0$. We denote it $\pi_1(X, \eta)$. Of course, if $x \in \eta(]1-\alpha_0, 1[)$, then the natural morphism $\pi_1(X, x) \to \pi_1(X, \eta)$ is an isomorphism.

From this definition it is readily checked that, if η is closed, then we have a natural isomorphism $\pi_1(X, \check{\eta}) \to \pi_1(X, \eta)$ and, if η is open, we have similarly $\pi_1(X, \hat{\eta}) \xrightarrow{\simeq} \pi_1(X, \eta)$.

If η is a closed normal ray, an isomorphism $\pi_1(X, \eta(0)) \to \pi_1(X, \eta)$ can be defined via

$$\pi_1(X, \eta) \simeq \pi_1(X, \eta([1-\alpha_0, 1[) \simeq \pi_1(X, \eta(1-\alpha_0)) \to \pi_1(X, \eta(0)),$$

the map $\pi_1(X, \eta(1-\alpha_0)) \to \pi_1(X, \eta(0))$ being $[\gamma] \mapsto [\eta^{-1}_{|[0,1-\alpha_0]} * \gamma * \eta_{|[0,1-\alpha_0]}]$, where the concatenated path $\alpha * \beta$ means β followed by α.

We extend in an obvious way the notation $\pi_1(X, \eta)$ to $\pi_1(U \cap X, \eta)$ where $U \subset Y$ is some open neighborhood of $\eta(1)$, and $\pi_1(U \cap X, \eta)$ means $\pi_1(U \cap X, \eta')$ where η' is the restriction of η to some $]1-\alpha, 1[$ with $\alpha < \alpha_0$ and $\eta(]1-\alpha, 1[) \subset U$. By the above observations, this does not depend on the choice of such an α.

2.2. **Local fundamental groups.** The topological pair (X, Y) is said to admit a local fundamental group at η if the image of the obvious map $\pi_1(X \cap U, \eta) \to \pi_1(X, \eta)$ for $U \subset Y$ an open neighborhood of $\eta(1)$ does not depend on U for U small enough. This means that there exists an open neighborhood U_0 such that, for any other $U \subset U_0$, the composite of the maps $\pi_1(X \cap U, \eta) \to \pi_1(X \cap U_0, \eta) \to \pi_1(X, \eta)$ has the same image as $\pi_1(X \cap U_0, \eta) \to \pi_1(X, \eta)$. It is obviously equivalent to saying that this image is the same for U belonging to some fundamental system of neighborhoods of $\eta(1)$. If it is the case, we denote $\pi_1^{loc}(X, \eta)$ this image.

Notice that, if η is closed, then (X, Y) admits a local fundamental fundamental group at η if and only if it admits one at the open normal ray $\check{\eta}$, and conversely it admits a local fundamental group at the open normal ray η if and only if it admits one at the closed normal ray $\hat{\eta}$.

Proposition 2.1. *If $W < \mathrm{GL}(V)$ is a complex reflection group and X its hyperplane complement, then the topological pair $(X/W, V/W)$ admits local fundamental groups at every normal ray.*

Proof. Let η be such a normal ray. By the above remark we can assume that it is closed. Let $x_0 \in X$ such that $W.x_0 = \eta(0)$. By definition $\eta_{|[0,1[}$ is a path inside X/W, so since $X \to X/W$ is a covering map it can be lifted to a path $\tilde{\eta} : [0, 1[\to X$, and since W is finite $\tilde{\eta}$ can be extended to $[0, 1] \to V$ with $\tilde{\eta}(1) = v_0 \in V \setminus X$. Let W_0 the parabolic subgroup of W fixing v_0, L_0 the intersection of its reflecting hyperplanes, and $X_0 \supset X$ the complement of the reflecting hyperplanes of W_0. We denote $\eta_0 : [0, 1[\to X_0/W_0$ the composite of $\tilde{\eta}$ with the natural projection map $X_0 \to X_0/W_0$. Since $X_0 \to X_0/W_0$ is a finite covering map η_0 is also a (closed) normal ray. Choosing some W-invariant norm on V, for some $\varepsilon_0 > 0$ the open balls Ω_ε of radius $\varepsilon > 0$ and center v_0 do not cross any other reflecting hyperplanes than the ones of W_0 for $0 < \varepsilon \leq \varepsilon_0$. This provides morphisms $\pi_1(\Omega_\varepsilon \cap X/W_0, \eta_0) \to \pi_1(X/W, \eta)$, which do not depend on $\varepsilon \in]0, \varepsilon_0[$ as, for $0 < \varepsilon' < \varepsilon < \varepsilon_0$, the composite $\pi_1(\Omega_{\varepsilon'} \cap X/W_0, \eta_0) \to \pi_1(\Omega_\varepsilon \cap X/W_0, \eta_0) \to \pi_1(X_0/W_0, \eta_0)$ is known to be an isomorphism (see [8] Proposition 2.29), whence $\pi_1(\Omega_{\varepsilon'} \cap X/W_0, \eta_0) \stackrel{\simeq}{\to} \pi_1(\Omega_\varepsilon \cap X/W_0, \eta_0)$. Since the $\Omega_\varepsilon \cap X/W_0$ for $\varepsilon < \varepsilon_0$ are mapped homeomorphically into V/W with images forming a fundamental system of open neighborhoods of $\eta(1)$, this proves that $\pi_1^{loc}(X/W, \eta)$ is well-defined. □

Actually, this proof shows also that $\pi_1^{loc}(X/W, \eta) \simeq \pi_1(X_0/W_0, \eta_0)$ where the parabolic subgroup W_0 has been defined as the stabilizer of $\eta(1)$ and X_0 is its hyperplane complement. From these arguments, the proof of the following proposition is straightforward.

Proposition 2.2. *Let $\tilde{\eta}$ be a normal ray inside $(X, \mathbb{C}^n)$ where X is the hyperplane complement of the complex reflection group $W < \mathrm{GL}_n(\mathbb{C})$. Then the composite η of $\tilde{\eta}$ with $X \to X/W$ is a normal ray. Moreover*

1. *If $\tilde{\eta}(1)$ belongs to the intersection of all the reflecting hyperplanes of W, then $\pi_1^{loc}(X/W, \eta) \simeq \pi_1(X/W, \eta)$.*
2. *If W_0 is the stabilizer of $\tilde{\eta}(1)$ and $X_0 \supset X$ is its hyperplane complement, then $\pi_1^{loc}(X/W, \eta) \simeq \pi_1^{loc}(X_0/W_0, \eta_0)$, where η_0 is the composite of $\tilde{\eta}$ with $X_0 \to X_0/W_0$.*

2.3. **Parabolic subgroups of complex braid groups.** From the results above we can define the following concept.

Definition 2.3. Let $W < \mathrm{GL}(V)$ be a complex reflection group and X be its hyperplane complement. Let $v_0 \in X/W$ and $B = \pi_1(X/W, v_0)$ the braid group of W. A parabolic subgroup of B is the image of $\pi_1^{loc}(X/W, \eta)$ for some closed normal ray η based at v_0 under the maps $\pi_1^{loc}(X/W, \eta) \to \pi_1(X/W, \eta) \simeq \pi_1(X/W, v_0) = B$.

Choosing a preimage $\tilde{v}_0$ of v_0 inside X, the covering map $X \to X/W$ defines a projection map $B \to W$ with kernel $P = \pi_1(X, \tilde{v}_0)$. It is readily checked from e.g. [8] that the image of the parabolic subgroup attached to η under the projection map $\pi : B = \pi_1(X/W, v_0) \to W$ is a parabolic subgroup of W. More precisely, it is the parabolic subgroup fixing the collection of reflecting hyperplanes containing $\tilde{\eta}(1)$, where $\tilde{\eta}$ is the unique lift of η with $\tilde{\eta}(0) = \tilde{v}_0$. In this context, we shall need the following lemma.

Lemma 2.4. *Let $H_1, \dots, H_r$ be a collection of hyperplanes such that $v_0 \notin W.H_i$, $i = 1, \dots, r$. Let B_0 be a parabolic subgroup of $B = \pi_1(X/W, v_0)$, obtained as the image of $\pi_1^{loc}(X/W, \eta)$ for some η, and let L_0 the intersection of the reflecting hyperplanes containing $\tilde{\eta}(1)$, for $\tilde{\eta} : [0, 1] \to V$ lifting η. Assume that $\forall i$ $L_0 \not\subset H_i$. Then B_0 can be obtained as the image of $\pi_1^{loc}(X/W, \eta')$ with η' such that $\forall i$ $\eta'([0, 1]) \cap W.H_i = \varnothing$.*

Proof. Let W_0 be the pointwise stabilizer of L_0, which is the parabolic subgroup of W obtained as the image of B_0 under $\pi : B \twoheadrightarrow W$. From the assumptions we get that the complement of the

H_i's is dense inside L_0. From this and the construction as the image of $\pi_1(\Omega_\varepsilon \cap X/W_0, \eta)$ for some $\varepsilon > 0$ in the proof of Proposition 2.1 it is then clear that η can be slightly modified near $\eta(1)$ in such a way that it remains the same on some $[0, 1-\beta]$, does not cross any of the H_i's on $[1-\beta, 1]$, and still provide the same image inside B_0. But now $\eta_{|_{[0,1-\beta]}}$ is a path inside some open set of $\mathbb{C}^n$ joining two elements not belonging to $F = H_1 \cup \cdots \cup H_r$. Since F is a finite union of (real) codimension 2 subspaces, $\eta_{[0,1-\beta]}$ is homotopically equivalent to a path which avoids F, so we can modify η accordingly on $[0, 1-\beta]$ and get the same image inside $B = \pi_1(X/W, v_0)$. ☐

We then have the following proposition, which summarizes the basic properties of parabolic subgroups of braid groups. Informally, (1) says that the concept of a parabolic is stable after reducing to essential reflection arrangements; (2) says that conjugates of parabolic subgroups are parabolic subgroups ; (3) says that a parabolic subgroup of B is naturally isomorphic to the braid group of a parabolic subgroup of W ; (4) says that, under this identification, parabolic subgroups of parabolic subgroups are parabolic subgroups.

Proposition 2.5. *Let $\mathcal{A}$ denote the hyperplane arrangement of W, and $B = \pi_1(X/W, v_0)$. Then the following hold.*

(1) Let V_+ be a linear subspace of V contained in all the reflecting hyperplanes. Let $\bar{V} = V/V_+$ and let $\bar{v}_0$, $\bar{X}$ be the images of v_0 and X inside $\bar{V}/W$ and $\bar{V}$, respectively. Then the natural isomorphism $\pi_1(X/W, v_0) \to \pi_1(\bar{X}/W, \bar{v}_0)$ induces a bijection between their sets of parabolic subgroups.

(2) If $B_0 < B$ is a parabolic subgroup, then gB_0g^{-1} is a parabolic subgroup for every $g \in B$

(3) If $B_0 < B$ is a parabolic subgroup, then $B_0 \simeq \pi_1(X_0/W_0, v_0')$ where v_0' is a preimage of v_0 under $X/W_0 \to X/W$, and X_0 is the hyperplane complement for the image W_0 of B_0 under $B \to W$.

(4) If $B_0 < B$ is a parabolic subgroup of B and v_0', W_0, X_0 as above, then every parabolic subgroup of $\pi_1(X_0/W_0, v_0')$ is identified with a parabolic subgroup of B via $\pi_1(X_0/W_0, v_0') \simeq B_0 < B$.

Proof. We first prove (1). The isomorphism $\pi_1(X/W, v_0) \to \pi_1(\bar{X}/W, \bar{v}_0)$ induces a bijection between their sets of subgroups. The fact that parabolic subgroups of $\pi_1(X/W, v_0)$ are mapped to parabolic subgroups of $\pi_1(\bar{X}/W, \bar{v}_0)$ is an immediate consequence of the commutativity of the diagram

$$\begin{array}{ccc} \pi_1^{loc}(X/W, \eta) & \longrightarrow & \pi_1^{loc}(\bar{X}/W, \bar{\eta}) \\ \downarrow & & \downarrow \\ \pi_1(X/W, \eta) & \xrightarrow{\simeq} & \pi_1(\bar{X}/W, \bar{\eta}) \end{array}$$

In order to prove that, conversely, every parabolic subgroup of $\pi_1(\bar{X}/W, \bar{v}_0)$ is obtained from a parabolic subgroup of $\pi_1(X/W, v_0)$, it then remains to show that every normal ray inside $\bar{X}/W$ can be lifted to some normal ray inside X/W, and this is clear from the existence of W-equivariant homeomorphisms $V \simeq \bar{V} \times V_+$ given by choosing some orthogonal complement of V_+ inside V for some W-invariant orthogonal form on V. This proves (1).

If B_0 is the image of $\pi_1^{loc}(X/W, \eta) \to \pi_1(X/W, \eta) \simeq \pi_1(X/W, v_0) = B$, and $g = [\gamma]$ for some $\gamma : [0, 1] \to X/W$ with $\gamma(0) = \gamma(1) = v_0$, then gB_0g^{-1} is the image of $\pi_1^{loc}(X/W, \eta * \gamma^{-1})$, as $\eta * \gamma^{-1}$ is easily checked to be another normal ray based at v_0. This proves (2). Part (3) was already noticed after Proposition 2.1 as a consequence of its proof.

We prove (4). Let us consider a parabolic subgroup given as the image of $\pi_1^{loc}(X_0/W_0, \tau) \to \pi_1(X_0/W_0, v_0')$ for some closed normal ray τ based at v_0', and let $\tilde{\tau} : [0, 1] \to V$ be a lift of τ. Let W_{00} be the stabilizer of $\tau(1)$ inside W_0, and L_{00} its fixed point set. If H is a reflecting hyperplane for W but not for W_0, it does not contain L_{00}, for otherwise it would contain the fixed point set of W_0 and would be a reflecting hyperplane for W_0. Therefore we can apply Lemma 2.4 and assume that τ does not cross any of these (orbits of) hyperplanes. It follows that it defines a normal ray inside X/W_0, and its image inside X/W defines a normal ray. Therefore the subgroup we are interested in is the image of the map

$$\pi_1\left(\frac{X_0 \cap \Omega_\varepsilon}{W_0}, \tau\right) = \pi_1\left(\frac{X \cap \Omega_\varepsilon}{W_0}, \tau\right) \to \pi_1\left(\frac{X}{W}, v_0\right)$$

which is a parabolic subgroup of B by the proof of Proposition 2.1.

□

We now relate conjugacy classes of parabolic subgroups with their images inside W.

Proposition 2.6. *Let B_1, B_2 be two parabolic subgroups of $B = \pi_1(X/W, v_0)$, and W_1, W_2 their images inside W. We set $P = \mathrm{Ker}(B \twoheadrightarrow W)$.*

(1) $W_1 = W_2$ if and only if B_1 and B_2 are P-conjugates.
(2) W_1 and W_2 are W-conjugates if and only if B_1 and B_2 are B-conjugates.

Proof. The 'if' parts are clear, so we shall prove only the other direction. Moreover, assuming that (1) is true, if W_1 and $W_2 = wW_1w^{-1}$ are W-conjugates we can find $b \in B$ with image w, so that bB_1b^{-1} is still a parabolic subgroup by Proposition 2.5 (2), with image W_2, so that it is a P-conjugate of B_2, which proves (2). Therefore it only remains to prove that if $W_1 = W_2$, then B_1, B_2 are P-conjugates.

In order to do this, we set $W_0 = W_1 = W_2$, denote $L_0 = V^{W_0}$ its fixed point set, and choose some lift $\tilde{v}_0 \in X$ of v_0. We fix a W-invariant hermitian scalar product on the ambient space. The parabolic subgroups B_i are given as the image of $\pi_1^{loc}(X/W_i, \eta_i)$ inside $\pi_1(X/W, v_0)$, for some closed normal rays η_i with $\eta_i(0) = v_0$.

We set $\tilde{\eta}_i : [0,1] \to V$ the lift of η_i such that $\tilde{\eta}_i(0) = \tilde{v}_0$, so that $\tilde{\eta}_i(1) = x_i^0 \in L_0$. Then, $P_i = P \cap B_i$ is the image of $\pi_1^{loc}(X, \tilde{\eta}_i)$ inside $\pi_1(X, \tilde{v}_0)$, and we need to prove in particular that P_1, P_2 are conjugates inside $P = \pi_1(X, \tilde{v}_0)$.

Since the complement L_0' inside L_0 of the union of the set $\mathcal{A} \setminus \mathcal{A}_0$ of the reflecting hyperplanes of W not containing L_0 is open and path connected, one can choose a C^∞ simple curve $\eta_0 : [0,1] \to L_0'$ with $\eta_0(\varepsilon_0) = x_1^0$, $\eta_0(1-\varepsilon_0) = x_2^0$, for some $\varepsilon_0 > 0$ close to 0, and we can assume $\eta_0'(t) \neq 0$ for every t. Then, $\eta_0(]0,1[)$ is a 1-dimensional submanifold of L_0', and is contained in some normal tube $T_0 = \{\eta_0(t) + D(t,r); t \in]0,1[\}$, where $D(t,r)$ is the disc of radius r centered at 0 inside the subspace orthogonal to $\gamma'(t)$ inside L_0. Moreover, r can be chosen small enough to be a tubular neighborhood, that is T_0 is homeomorphic to $]0,1[\times D$ for some $(\dim_{\mathbb{R}} L_0 - 1)$-disc D via $(t,v) \mapsto \eta_0(t) + h(t,v)$ with $h(t,v) \in D(t,r)$. Then we set $T = T_0 + F$, where $F = \{y \in L_0^{\perp}, \|y\| \leq r\}$. Up to possibly reducing r, one can assume that T does not cross any hyperplane of $\mathcal{A} \setminus \mathcal{A}_0$. We then have a homeomorphism $\Psi :]0,1[\times D \times F \to T$, $(t,v,f) \mapsto \eta_0(t) + h(t,v) + f$, with the property that the inverse image of $X \cap T$ is $]0,1[\times D \times F'$ with F' the complement inside F of the union of the hyperplanes of $\mathcal{A}_0$.

Then, notice that, for $0 < \varepsilon < \varepsilon_0$, $T^1(\varepsilon) = \{\eta_0(t) + h(t,v) + y;\ t \in] - \varepsilon + \varepsilon_0, \varepsilon_0 + \varepsilon[, v \in D, y \in L_0^{\perp}, \|y\| \leq r\}$ is an open neighborhood of x_1^0, and similarly $T^2(\varepsilon) = \{\eta_0(t) + h(t,v) + y; t \in] - \varepsilon + 1 - \varepsilon_0, 1 - \varepsilon_0 + \varepsilon[, v \in D, y \in L_0^{\perp}, \|y\| \leq r\}$ is an open neighborhood of x_2^0. Therefore, one can choose ε and $x_i = \eta_i(\alpha_i) \in T^i(\varepsilon)$ with $\alpha_i < 1$ so that $\pi_1^{loc}(X, \eta_i) \simeq \pi_1(X \cap T^i(\varepsilon), x_i)$ for $i = 1, 2$. Then, the map $\pi_1(X \cap T^i(\varepsilon), x_i) \to \pi_1(X, \tilde{v}_0)$, whose image is P_i, factors as $\pi_1(X \cap T^i(\varepsilon), x_i) \to \pi_1(X \cap T, x_i) \to \pi_1(X, \tilde{v}_0)$. Now, the inclusion map $N_i : X \cap T^i(\varepsilon) \to X \cap T$ is a homotopy equivalence. Indeed, if $\Psi^{(i)}$ denotes the restriction of Ψ to $H_i \times D \times F \to T$, with $H_1 =]\varepsilon_0 - \varepsilon, \varepsilon_0 + \varepsilon[$ and $H_2 =]1 - \varepsilon_0 - \varepsilon, 1 - \varepsilon_0 + \varepsilon[$, then $\Psi^{-1} \circ N_i \circ \Psi^{(i)}$ is the inclusion map $H_i \times D \times F \to T \subset]0,1[\times D \times F$, which is a homotopy equivalence. It follows that each $\pi_1(X \cap T^i(\varepsilon), x_i) \to \pi_1(X \cap T, x_i)$ is an isomorphism, and therefore each P_i is the image inside of $\pi_1(X, \tilde{v}_0)$ of $\pi_1(X \cap T, x_i)$. From this, and noticing that each $T_i(\varepsilon)$ and T are W_0-invariant, one gets that the natural morphism $\pi_1(X \cap T_i(\varepsilon)/W_0, x_i) \to \pi_1(X \cap T/W_0, x_i)$ is also an isomorphism, and therefore B_i is the image inside $\hat{B} = \pi_1(X/W_0, \tilde{v}_0)$ of $\pi_1(X \cap T/W_0, x_i)$. But then, letting $\hat{\eta}_i$ denote the restriction of η_i to $[0, \alpha_i]$, and h some path $x_1 \rightsquigarrow x_2$ inside $X \cap T$, we check immediately that the loop $\hat{\eta}_2^{-1} * h * \hat{\eta}_1$ based at $\tilde{v}_0$ conjugates the two images. Identifying $\hat{B}$ with the subgroup $\pi^{-1}(W_0)$ of B, this concludes the proof of the proposition. □

Our definition of a parabolic subgroup as depending only on the topological pair $(X/W, V/W)$ has the following consequence. Assume that we are given a choice of (homogeneous) basic invariants $f_1, \dots, f_n \in \mathbb{C}[V]$ for W, that is, W-invariant homogeneous polynomials such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_n]$, and denote $f = (f_1, \dots, f_n) : V \to \mathbb{C}^n$ the corresponding polynomial map. The union of the reflecting hyperplanes is mapped under f onto some hypersurface $\mathcal{H}$. Then f induces a map $\hat{f} : V/W \to \mathbb{C}^n$ which maps X/W to the hypersurface complement $\mathbb{C}^n \setminus \mathcal{H}$. This map is known to be (continuous and) bijective (see e.g. [34] Proposition 9.3), and easily seen to be proper, because $\mathbb{C}[V]$ is an integral extension of $\mathbb{C}[V]^W$ (see e.g. [34], Lemma 3.11). Therefore it is a homeomorphism, and the topological pair $(X/W, V/W)$ is homeomorphic to the topological pair $(\mathbb{C}^n \setminus \mathcal{H}, \mathbb{C}^n)$, and the concept of a parabolic subgroup depends only on the hypersurface $\mathcal{H}$.

Proposition 2.7. *Assume that $W, W' < \mathrm{GL}(V)$ are two complex reflection groups with two collections of basic invariants $f_1, \dots, f_n$ and $f'_1, \dots, f'_n$ for which the associated discriminant hypersurfaces $\mathcal{H} \subset \mathbb{C}^n$ are the same. Then the parabolic subgroups for W and W' define the same collection of parabolic subgroups of their common braid group $B = \pi_1(\mathbb{C}^n \setminus \mathcal{H}, z)$.*

The term isodiscriminantal was coined by Bessis in order to describe such pair of complex reflection groups. This applies in particular to the so-called Shephard groups. Indeed, it is immediately checked in rank 2 and for the infinite series, and it was shown by Orlik and Solomon (see [43]) that the exceptional Shephard groups of higher rank, namely G_{25}, G_{26} and G_{32}, have the same discriminant as the Coxeter groups of type A_3, B_3 and A_4, respectively.

2.4. **Subgroups of parabolic subgroups and products.** In this section we shall use systematically the construction of parabolic subgroups from the choice of convenient balls inside the hyperplane complement, due to [8], as in the proof of Proposition 2.1. We have the following companion to part (4) of Proposition 2.5.

Proposition 2.8. *Under the previous notations, if B_0, B_1 are two parabolic subgroups of $B = \pi_1(X/W, W.\tilde{v}_0)$, and $B_1 < B_0$, then B_1 is a parabolic subgroup of $B_0 \simeq \pi_1(X_0/W_0, W_0.\tilde{v}_0)$.*

Proof. In order to lighten notations in this proof, as no ambiguity should occur we allow ourselves to write $\pi_1(E/G, x)$ for $x \in E$ instead of $\pi_1(E/G, G.x)$ whenever E is a G-space. Since $B_1 < B_0$ we have $W_1 = \pi(B_1) < \pi(B_0) = W_0$; since W_1 is a parabolic subgroup of W, if follows that W_1 is a parabolic subgroup of W_0. Now let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}$ denote the hyperplane arrangements of W_0, W_1 and W, respectively. We have $\mathcal{A}_1 \subset \mathcal{A}_0$. As a $\mathbb{C}W_0$-module, V can be canonically decomposed as $V = V_0 \oplus U$, with $U = V^W$ being the isotypic component of the trivial representation. Then, every reflecting hyperplane inside $\mathcal{A}_0$ can be written as $H \oplus U$ for H a hyperplane of V_0. We denote $\mathcal{B}_0$ the collection of such hyperplanes, and $\mathcal{B}_1$ the subset of those which originate from $\mathcal{A}_1$.

Then, setting $Y_0 = V_0 \setminus \bigcup \mathcal{B}_0$, we have $X_0 = Y_0 \times U$ and Y_0 can be identified as $\overline{X}_0$ defined as in (1) of Proposition 2.5. Writing $\tilde{v}_0 = \tilde{v}'_0 + u_0 \in V_0 \oplus U$, this provides an identification of $B_0 \simeq \pi_1(X_0/W_0, \tilde{v}_0)$ as a parabolic subgroup of B with the braid group $\pi_1(Y_0/W_0, \tilde{v}'_0)$ which preserves the collection of parabolic subgroups.

We endow V with a W_0-invariant norm of the form $\|(y_1, y_2)\| = \max(\|y_1\|, \|y_2\|)$ for $(y_1, y_2) \in V_0 \times U = V$, for some arbitrary W_0-invariant norm on V_0 and some arbitrary norm on U. We choose it non-Euclidean in order to get a convenient Cartesian decomposition of the balls.

Since the statement is invariant under conjugation, we can assume that B_0 is the image of

$$j_0 : \pi_1\left(\frac{\Omega_0 \cap X}{W_0}, x_0\right) \to B = \pi_1\left(\frac{X}{W}, x_0\right)$$

induced by the natural inclusion and quotient by W, for some $x_0 \in \Omega_0 \setminus \bigcup \mathcal{A}$ and Ω_0 some open ball. Since B_1 is a parabolic subgroup of B admitting no other hyperplanes than the ones of $\mathcal{A}_1 \subset \mathcal{A}_0$, there is a W_1-invariant ball Ω_1 containing some $x_1 \in \Omega_1 \setminus \bigcup \mathcal{A}$ and meeting no other hyperplane than the ones of $\mathcal{A}_0$.

Then, a path $\gamma : x_0 \rightsquigarrow x_1$ inside X such that $b \mapsto (\pi(b).\gamma^{-1}) * b * \gamma$ defines an embedding

$$J_\gamma : \pi_1\left(\frac{\Omega_1 \cap X}{W_1}, x_1\right) \to \pi_1\left(\frac{X}{W}, x_0\right)$$

whose image is B_1. Because of the specific properties of the chosen norm, under $V = V_0 \oplus U$ we have a decomposition $\Omega_1 = \Omega_1^0 \times \Omega_1'$ with Ω_1^0 a W_1-invariant ball inside V_0. Then

$$\Omega_1 \cap X = \Omega_1 \cap X_0 = (\Omega_1^0 \times \Omega_1') \cap (Y_0 \times U) = (\Omega_1^0 \setminus \bigcup \mathcal{B}_0) \times U \subset Y_0 \times U$$

is naturally homotopically equivalent to $\Omega_1^0 \setminus \bigcup \mathcal{B}_0 = \Omega_1^0 \cap Y_0$. Therefore, the same formula $b \mapsto (\pi(b).\gamma^{-1}) * b * \gamma$ defines an embedding

$$J_\gamma^0 : \pi_1\left(\frac{\Omega_1 \cap X}{W_1}, x_1\right) = \pi_1\left(\frac{\Omega_1^0 \setminus \bigcup \mathcal{B}_0}{W_1} \times U, x_1\right) \to \pi_1\left(\frac{Y_0}{W_0} \times U, x_0\right)$$

The natural inclusions together with the morphisms J_γ, J_γ^0 induce the following commutative diagram, which proves that B_1 is indeed the image under j_0 of a parabolic subgroup of $B_0 = \pi_1(X_0/W_0, x_0)$ attached to $W_1 < W_0 < \mathrm{GL}(V)$.

$$\begin{array}{ccccccc}
\pi_1\left(\frac{X}{W}, x_0\right) & \xleftarrow{j_0} & \pi_1\left(\frac{\Omega \cap X}{W_0}, x_0\right) & \xrightarrow{\simeq} & \pi_1\left(\frac{X_0}{W_0}, x_0\right) & = & \pi_1\left(\frac{Y_0}{W_0} \times U, x_0\right) \\
 & \nwarrow^{J_\gamma} & \vdots\uparrow & & & & \uparrow J_\gamma^0 \\
 & & \pi_1\left(\frac{\Omega_1 \cap X}{W_1}, x_1\right) & & = & & \pi_1\left(\frac{\Omega_1^0 \cap Y_0}{W_1} \times \Omega_1', x_1\right)
\end{array}$$

□

An immediate consequence of the proposition and its proof is the following.

Corollary 2.9. *If B_0, B_1 are two parabolic subgroups of B with $B_1 < B_0$, then B_1 is equal to B_0 if and only if their images inside W are the same, if and only if they have the same rank.*

Assume now that $V = V_1 \oplus V_2$ and $W = W_1 \times W_2$ as reflection groups, that is $W_i < \mathrm{GL}(V)$ acting trivially on V_j for $\{i, j\} = \{1, 2\}$. The hyperplane arrangement associated to W is $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_i$ denotes the hyperplane arrangement attached to W_i. Of course $V_j \subset H$ for all $H \in \mathcal{A}_i$, $\{i, j\} = \{1, 2\}$. Let us choose $y = (y_1, y_2) \in V \setminus \bigcup \mathcal{A}$ as basepoint,

$$B = \pi_1\left(\frac{V \setminus \bigcup \mathcal{A}}{W}, (y_1, y_2)\right), \quad B_1 = \pi_1\left(\frac{V_1 \setminus \bigcup \mathcal{A}_1}{W_1}, y_1\right), \quad B_2 = \pi_1\left(\frac{V_2 \setminus \bigcup \mathcal{A}_2}{W_2}, y_2\right)$$

There is a natural embedding $B_1 \hookrightarrow B$ given by $[\gamma] \mapsto [t \mapsto (\gamma(t), y_2)]$ and a natural projection $B \twoheadrightarrow B_1$, and similarly for B_2. These maps provide a natural isomorphism $B \simeq B_1 \times B_2$. For $i = 1, 2$, let us choose W_i-invariant norms on V_i, and choose the norm on $V = V_1 \oplus V_2$ defined by $\|(z_1, z_2)\| = \max(\|z_1\|, \|z_2\|)$. Then, a ball Ω centered at (y_1^0, y_2^0) is the cartesian product $\Omega_1 \times \Omega_2$ of the balls of the same radius centered at $y_1^0 \in V_1$ and $y_2^0 \in V_2$. Let $W_i' < W_i$ the stabilizer of y_i^0. The stabilizer of $(y_1^0, y_2^0) \in V$ is $W_1^0 \times W_2^0 \subset W_1 \times W_2 = W$, and we have $\Omega \setminus \bigcup \mathcal{A} = (\Omega_1 \setminus \bigcup \mathcal{A}_1) \times (\Omega_2 \setminus \bigcup \mathcal{A}_2)$. If $\check{B}$, $\check{B}_1$ and $\check{B}_2$ denote the parabolic subgroups of B, B_1 and B_2, respectively, attached to these data, we get from this that $\check{B} = \check{B}_1 \times \check{B}_2$, where $\check{B}_i$ is identified with its image under the natural embedding $B_i \hookrightarrow B$. The case of an arbitrary parabolic subgroup, which is conjugated to one attached to such a ball, is immediately deduced from it, as

$$(\check{B}_1 \times \check{B}_2)^{([\gamma_1],[\gamma_2])} = \check{B}_1^{[\gamma_1]} \times \check{B}_2^{[\gamma_2]}$$

2.5. **The curve graph and the curve complex.** As in the introduction, we define the (nonoriented) curve graph Γ of B with vertices the irreducible parabolic subgroups of B, and edges the pairs $\{B_1, B_2\}$ with $B_1 \neq B_2$ such that, either $B_1 \subset B_2$, or $B_2 \subset B_1$, or $B_1 \cap B_2 = [B_1, B_2] = \{1\}$. We call rank of a given vertex the rank of the corresponding irreducible subgroup.

Similarly, we recall that, for B_0 an irreducible parabolic subgroup, $z_{B_0} \in B_0$ is defined as the unique positive generator of $Z(B_0) \simeq \mathbb{Z}$, where positive means the following. Consider the standard map $X \to \mathbb{C}^*$ obtained by taking the product of a W-invariant collection of defining linear forms for the hyperplane arrangement $\mathcal{A}$. The induced morphism $B = \pi_1(X/W) \to \pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$ identifies $Z(B_0)$ with a subgroup of $\mathbb{Z}$, and such a subgroup admits a unique positive generator. By definition this generator is z_{B_0}.

The group B obviously acts by conjugation on Γ, and this action factorizes through $B/Z(B)$. We have the following general fact.

Proposition 2.10. *The action of $B/Z(B)$ on Γ is faithful. It is actually already faithful on the set of vertices of rank* 1.

Proof. Let $b \in B$ acting trivially on the vertices of rank 1. For any braided reflection σ, the group $\langle\sigma\rangle$ is an irreducible parabolic subgroup of rank 1, so that $\langle\sigma^b\rangle = \langle\sigma\rangle \simeq \mathbb{Z}$. But since σ and σ^b have the same image under the homomorphism $B \to \mathbb{Z}$ defined above, this proves $\sigma^b = \sigma$. Since B is generated by its braided reflections this implies $b \in Z(B)$ and this proves the claim. □

The curve complex $\mathcal{K}$ associated to B can then be defined as the collection of finite sets $\{B_1, \dots, B_r\}$ of irreducible parabolic subgroups such that each B_i is adjacent to each B_j in Γ. By the above proposition, the natural conjugacy action of $B/Z(B)$ on $\mathcal{K}$ is faithful, too.

The statement of Theorem 1.3 says that the adjacency relation between two irreducible parabolic subgroups B_1 and B_2 is detected by the relation $[z_{B_1}, z_{B_2}] = 1$, that is $z_{B_1}z_{B_2} = z_{B_2}z_{B_1}$. In order to prove this, we shall later need the following result, which we can prove without using the classification of complex reflection groups.

Theorem 2.11. *Let W_1, W_2 be two irreducible parabolic subgroups of the 2-reflection group W, and B_1, B_2 two (irreducible) parabolic subgroups of its braid group B with image W_1, W_2. If $[z_{B_1}, z_{B_2}] = 1$, then either $W_1 \subset W_2$, or $W_2 \subset W_1$, or $W_1 \cap W_2 = [W_1, W_2] = \{1\}$*

This theorem is an immediate consequence of the forthcoming auxiliary results. The general idea of the proof is that we look at the Taylor expansion of degree 1 of the images of the z_{B_i}'s in the Hecke algebra representation of B – which is defined in the general case by a monodromy construction. This yields the following lemma.

Lemma 2.12. *Under the assumptions of Theorem 2.11, let $u_i \in \mathbb{Z}W$ denote the sum of the reflections of $W_i < W$ inside the group algebra of W. If $[z_{B_1}, z_{B_2}] = 1$, then $u_1u_2 = u_2u_1$.*

Proof. Let N_i denote the order of $Z(W_i)$. Then $z_{B_i}^{N_i}$ belongs to the pure braid group of W, and we have $[z_{B_1}^{N_1}, z_{B_2}^{N_2}] = 1$. Let X be the hyperplane complement for W and $\mathfrak{g}_X$ its holonomy Lie algebra, as in [31, 32]. We refer to [44] and to [39] §2 for basic results on it and for its use in the construction of linear representations of the braid group. It is a graded Lie algebra generated by elements $t_H, H \in \mathcal{A}$ of degree 1, where $\mathcal{A}$ is the hyperplane arrangement of W. Let ω_H be the logarithmic 1-form associated to H, that is $\omega_H = (1/2\pi \mathrm{i})\mathrm{dlog}\alpha_H$ where α_H is some linear form with kernel H. The monodromy of the 1-form $\omega = \sum_H t_H\omega_H$ provides a morphism from the pure braid group $P = \pi_1(X; x_0)$ to the invertible elements of the completed Hopf algebra $\widehat{\mathsf{U}\mathfrak{g}_X}$. This morphism can be described via Chen's iterated integrals, as in [10, 11]. In particular, the image of $[\gamma]$ for γ a loop based at x_0 is equal to $1 + \int_\gamma \omega$ plus terms of higher degree. Now, $z_{B_i}^{N_i}$ is the homotopy class of a simple loop γ_i based at x_0 around the reflecting hyperplanes of W_i, and therefore it is mapped to

$$1 + \int_\gamma \omega + \dots = 1 + \sum_H t_H \int_\gamma \omega_H + \dots = 1 + \sum_{H \in \mathcal{A}_i} t_H + \dots$$

where $\mathcal{A}_i \subset \mathcal{A}$ is the reflection arrangement for $W_i < W$, and the dots represent terms of higher degree inside $\widehat{\mathsf{U}\mathfrak{g}_X}$. It follows that $[z_{B_1}^{N_1}, z_{B_2}^{N_2}] = 1$ implies $t_1t_2 = t_2t_1$ where $t_i = \sum_{H\in\mathcal{A}_i} t_H$.

We then endow the group algebra $\mathbb{Z}W$ of W with the Lie bracket $[a, b] = ab-ba$. There is a Lie algebra morphism $\mathfrak{g}_X \to \mathbb{Z}W$, which has been investigated together with its image in [36, 37, 38], mapping each t_H to the reflection around H. Each t_i is obviously mapped to the sum u_i of the reflections of W_i, and the conclusion follows immediately from $[t_1, t_2] = 0$. □

Then, we translate this equation holding inside the infinitesimal Hecke algebra of [36, 38] into a commutation relations between some of the reflections involved here.

Lemma 2.13. *Let W_1, W_2 be two parabolic subgroups of the 2-reflection group W, and let $u_i \in \mathbb{Z}W$ denote the sum of the set $\mathcal{R}_i$ of the reflections of W_i. Then $u_1u_2 = u_2u_1$ if and only if, for every $s_1 \in \mathcal{R}_1 \setminus \mathcal{R}_2$ and $s_2 \in \mathcal{R}_2 \setminus \mathcal{R}_1$, we have $s_1s_2 = s_2s_1$.*

Proof. For $w \in W$, we set $K(w) = \mathrm{Ker}(w-1)$. Then $E_i = \bigcap_{s \in \mathcal{R}_i} K(s)$ is the fixed point set of W_i and, for $w \in W$, $w \in W_i \Leftrightarrow w_{|E_i} = \mathrm{Id}_{E_i}$. We endow again the algebra $\mathbb{Z}W$ with the Lie bracket $[a, b] = ab - ba$, so by asumption we have $[u_1, u_2] = 0$.

Let $\mathcal{A}$ be the hyperplane arrangement of W. We denote $\varepsilon : W \to \{\pm 1\}$ the determinant map mapping each reflection to -1, and $W^+ = \mathrm{Ker}\,\varepsilon$ its rotation subgroup. More generally, for $G < W$, we set $G^+ = G \cap W^+ = \mathrm{Ker}\,\varepsilon_{|G}$. Let $\mathcal{Z}$ denote the collection of codimension 2 subspaces of the form $H \cap H'$ for $H, H' \in \mathcal{A}$, $H \neq H'$ and, for $F \subset E$, let W_F be the pointwise stabilizer of F. If $Z \in \mathcal{Z}$ and $w \in W_Z^+ \setminus \{1\}$, we have $K(w) = Z$, as $K(w)$ necessarily has codimension 1 or 2, and if it had codimension 1 it would be a reflection, contradicting $w \in W^+$. From this one gets that, if $Z_1, Z_2 \in \mathcal{Z}$ with $Z_1 \neq Z_2$, then $W_{Z_1}^+ \cap W_{Z_2}^+ = \{1\}$.

For $s, u \in \mathcal{R}$, if $[s, u] \neq 0$ we have in particular $s \neq u$ whence $K(su) = K(s) \cap K(u) \in \mathcal{Z}$, and $su \in W_{K(su)}^+$. From this, it is easily checked that $[u_1, u_2]$ belongs to the linear span of $\{W_Z^+,\ Z \in \mathcal{Z}\}$, hence to $\bigoplus_{Z \in \mathcal{Z}} \mathbb{Z}W_Z^+$.

Let $s_1 \in \mathcal{R}_1 \setminus \mathcal{R}_2$, and $s_2 \in \mathcal{R}_2 \setminus \mathcal{R}_1$. We first consider the projection of $[u_1, u_2]$ onto $\mathbb{Z}W_Z^+$ for $Z = K(s_1 s_2)$. Given $s_1' \in \mathcal{R}_1$ and $s_2' \in \mathcal{R}_2$, suppose that the projection of $[s_1', s_2'] \neq 0$. This is only possible if $K(s_1' s_2') = Z = K(s_1 s_2)$. If $s_1' \in \mathcal{R}_1 \cap \mathcal{R}_2$, then $K(s_1')$ contains $E_1 + E_2$ and therefore $K(s_1 s_2) = K(s_1' s_2')$ contains E_2, whence $K(s_1) \supset E_2$ contradicting $s_1 \notin \mathcal{R}_2$. Therefore $s_1' \in \mathcal{R}_1 \setminus \mathcal{R}_2$ and similarly one proves $s_2' \in \mathcal{R}_2 \setminus \mathcal{R}_1$.

Now, we have $K(s_1) \supset K(s_1 s_2) + E_1$, and this inclusion is strict only if $E_1 \subset K(s_1 s_2)$, which implies $E_1 \subset K(s_2)$ and $s_2 \in \mathcal{R}_1$, a contradiction. Therefore $K(s_1) = K(s_1 s_2) + E_1$ and similarly $K(s_2) = K(s_1 s_2) + E_2$. Since (s_1', s_2') satisfies the same properties as (s_1, s_2), one gets $K(s_1') = K(s_1' s_2') + E_1 = K(s_1 s_2) + E_1 = K(s_1)$ and similarly $K(s_2') = K(s_2)$, so this proves $s_1' = s_1$ and $s_2' = s_2$. This proves that the projection onto $\mathbb{Z}W_Z^+$ of $[u_1, u_2]$ is equal to $[s_1, s_2]$.

Therefore, if $[u_1, u_2] = 0$, this implies that $s_1 s_2 = s_2 s_1$ for every $s_1 \in \mathcal{R}_1 \setminus \mathcal{R}_2$ and $s_2 \in \mathcal{R}_2 \setminus \mathcal{R}_1$.

Conversely, suppose that $s_1 s_2 = s_2 s_1$ for every $s_1 \in \mathcal{R}_1 \setminus \mathcal{R}_2$ and $s_2 \in \mathcal{R}_2 \setminus \mathcal{R}_1$. Set $\mathcal{R}_0 = \mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathcal{R}_i' = \mathcal{R}_i \setminus \mathcal{R}_0$. For short, let $\sum_S$ denote $\sum_{s \in S} s$. Since $\sum_{\mathcal{R}_i}$ belongs to the center of $\mathbb{Z}W_i$, we have $[s, \sum_{\mathcal{R}_i}] = 0$ for every $s \in \mathcal{R}_0$. Therefore, since we know by hypothesis that $[\sum_{\mathcal{R}_1'}, \sum_{\mathcal{R}_2'}] = 0$, we get that $[\sum_{\mathcal{R}_1}, \sum_{\mathcal{R}_2}]$ is equal to

$$[\sum_{\mathcal{R}_1'} + \sum_{\mathcal{R}_0}, \sum_{\mathcal{R}_2}] = [\sum_{\mathcal{R}_1'}, \sum_{\mathcal{R}_2}] = [\sum_{\mathcal{R}_1'}, \sum_{\mathcal{R}_0}] + [\sum_{\mathcal{R}_1'}, \sum_{\mathcal{R}_2'}] = [\sum_{\mathcal{R}_1} - \sum_{\mathcal{R}_0}, \sum_{\mathcal{R}_0}] + [\sum_{\mathcal{R}_1'}, \sum_{\mathcal{R}_2'}] = [\sum_{\mathcal{R}_1'}, \sum_{\mathcal{R}_2'}] = 0$$

and this proves the converse statement. $\square$

Finally, in order to use the irreducibility assumptions we have, we use the following group-theoretic result.

Proposition 2.14. *Let $W < \mathrm{GL}(E)$ be an irreducible complex 2-reflection group with set of reflections $\mathcal{R}$, and $W_0 \subsetneq W$ a nontrivial parabolic subgroup. Then W is generated by $\mathcal{R} \setminus \mathcal{R}_0$, for $\mathcal{R}_0 = W_0 \cap \mathcal{R}$.*

Proof. We assume $W_0 \neq \{1\}$, for otherwise the statement is trivial. Let us set $\mathcal{R}' = \mathcal{R} \setminus \mathcal{R}_0$ and $W' = \langle \mathcal{R}' \rangle$. We first prove that $\bigcap_{s \in \mathcal{R}'} K(s) = \{0\}$, where we denote $K(s) = \mathrm{Ker}(s-1)$. For this, set $F = \bigcap_{s \in \mathcal{R}'} K(s)$. Then $E = F \oplus F^{\perp}$ and, since $wK(s) = K(wsw^{-1})$ we get that F is W_0-stable. Since $\mathcal{R}'$ acts trivially on F and W is generated by $\mathcal{R} \subset W_0 \cup \mathcal{R}'$ it follows that F is W-stable. Since W is irreducible this proves $F = \{0\}$.

Now consider $s_0 \in \mathcal{R}_0$. We want to prove $s_0 \in W'$. We have $\{0\} \neq K(s_0)^{\perp} \not\subset \bigcap_{s \in \mathcal{R}'} K(s) = \{0\}$ so there exists $s \in \mathcal{R}'$ such that $K(s_0)^{\perp} \not\subset K(s)$. Let $Z = K(s_0) \cap K(s)$ and W_Z the parabolic subgroup of W fixing Z. It has rank 2. Assume there exists $s_0' \in \mathcal{R}_0 \cap W_Z$ with $K(s_0') \neq K(s_0)$. Then $Z = K(s_0) \cap K(s_0')$ and $W_Z \subset W_0$ whence $s \in W_0$, a contradiction. Therefore $\mathcal{R}_0 \cap W_Z = \{s_0\}$. Now, $K(s_0)^{\perp} \not\subset K(s)$ implies $ss_0 \neq s_0 s$, hence $ss_0 s^{-1} \in (\mathcal{R} \cap W_Z) \setminus \{s_0\} \subset \mathcal{R}'$. But then $s_0 = s^{-1}.ss_0s^{-1}.s$ is a product of elements of $\mathcal{R}'$ hence belongs to $\langle \mathcal{R}' \rangle = W'$. It follows that $\mathcal{R}_0 \subset W'$ hence $\mathcal{R} \subset W'$ and $W' = \langle \mathcal{R} \rangle = W$. This concludes the proof. $\square$

The above statement with the same proof holds more generally for an irreducible complex reflection group W having reflections of arbitrary order, and the set of reflections can even be replaced by the set of distinguished ones, but we shall not need this here.

This statement then proves that our provisional result $u_1u_2 = u_2u_1$ actually means the following, and this completes the proof of Theorem 2.11.

Corollary 2.15. *Let W_1, W_2 be two irreducible parabolic subgroups of the 2-reflection group W, with sets of reflections $\mathcal{R}_i, i = 1, 2$. Then the condition that, for every $s_1 \in \mathcal{R}_1 \setminus \mathcal{R}_2$ and $s_2 \in \mathcal{R}_2 \setminus \mathcal{R}_1$, we have $s_1s_2 = s_2s_1$, is equivalent to saying that, either $W_1 \subset W_2$, or $W_2 \subset W_1$, or $W_1 \cap W_2 = [W_1, W_2] = \{1\}$.*

Proof. One implication being obvious, we prove the other one. Assume $s_1s_2 = s_2s_1$ for every $s_1 \in \mathcal{R}_1 \setminus \mathcal{R}_2$ and $s_2 \in \mathcal{R}_2 \setminus \mathcal{R}_1$. If $\mathcal{R}_1 \subset \mathcal{R}_2$ we have $W_1 \subset W_2$ and similarly $\mathcal{R}_2 \subset \mathcal{R}_1 \Rightarrow W_2 \subset W_1$. So we can assume that $W_0 = W_1 \cap W_2$ is a proper parabolic subgroup of both W_1 and W_2. Let $\mathcal{R}_0 = \mathcal{R} \cap W_0$. We have $\mathcal{R}_1 \setminus \mathcal{R}_2 = \mathcal{R}_1 \setminus \mathcal{R}_0$, so by Proposition 2.14 we know that W_1 is generated by $\mathcal{R}_1 \setminus \mathcal{R}_2$. It follows that W_1 commutes with $\mathcal{R}_2 \setminus \mathcal{R}_1$. But since $\mathcal{R}_2 \setminus \mathcal{R}_1 = \mathcal{R}_2 \setminus \mathcal{R}_0$ generates W_2 it follows that $[W_1, W_2] = \{1\}$. Now, $W_0 \subset W_2$ commutes with all W_1 ; since it is a parabolic subgroup of W_1 this contradicts the irreducibility of W_1 unless $W_0 = \{1\}$, and this concludes the proof. □

3. Description of the parabolic subgroups

The goal of this section is to describe the parabolic subgroups of the braid groups of every irreducible complex reflection group up to conjugacy. The subsections below cover all of them, except for the group G_{31}.

3.1. **Real reflection groups and Shephard groups.** By Proposition 2.7, Shephard groups have the same collection of parabolic subgroups as their associated real reflection group, so we can concentrate on the case where W is a real reflection group. It admits a Coxeter system (W, S), and its braid group B admits a presentation as an Artin group, with set of generators Σ, called the Artin-Brieskorn generators of B, which are in natural 1-1 correspondence with S. More precisely, picking a Weyl chamber $C \subset X$, and choosing some base-point $x \in C$, the element of Σ in $B = \pi_1(X/W, \bar{x})$ corresponding to $s \in S$ is obtained by joining x to its image under s, viewed as an element of W, by a straight line, only modified at the intersection with the wall $\mathrm{Ker}(s-1)$ so that it makes a positive half turn in the normal complex direction (see [7]).

If $S_0 \subset S$, there is a standard parabolic subgroup $W_0 < W$ generated by S_0, and the subgroup of B generated by the corresponding copy $\Sigma_0 \subset \Sigma$ is a parabolic subgroup of B. Indeed, C is included inside a unique Weyl chamber for W_0, denoted C_0. Let L_0 denote the intersection of the reflecting hyperplanes for W_0, and L'_0 the complement of all the other reflecting hyperplanes of W inside L_0. Let us choose as normal ray a straight path η from x to some point in $L'_0 \cap \overline{C}$. It is then immediately checked that $\pi_1^{loc}(X/W, \eta)$ can be identified with the Artin subgroup associated to S_0, the Artin-Brieskorn generators of B corresponding to the walls of C_0 being mapped to the Artin-Brieskorn generators of the braid group B_0 of W_0. This proves that every such subgroup $\langle \Sigma_0 \rangle$ is a parabolic subgroup of B. We call it a standard parabolic subgroup of B.

Proposition 3.1. *For a braid group B associated to a real reflection group or a Shephard group, the parabolic subgroups are the conjugates of the standard parabolic subgroups of B, viewed as an Artin group by the above construction.*

Proof. We can focus on the case of W being a real reflection group. We already saw that conjugates of standard parabolic subgroups are parabolic subgroups. Conversely, let B_0 be a parabolic subgroup of B, and let $W_0 < W$ denote its image under the natural projection $B \to W$. Then W_0 is a parabolic subgroup of W, hence there exists $g \in W$ such that gW_0g^{-1} is a standard parabolic subgroup of the chosen Coxeter system (W, S). Let B'_0 be the standard parabolic subgroup associated to gW_0g^{-1}. By Proposition 2.6, B_0 and B'_0 are conjugate in B, as their images under the natural projection are conjugate in W. Hence B_0 is conjugate to a standard parabolic subgroup. □

3.2. **The groups $G(de,e,n)$, $d>1$.** Recall that $G(de,e,n)$ is the group of $n\times n$ monomial matrices with coefficients inside $\mu_{de}(\mathbb{C})$ such that the product of their nonzero entries belongs to $\mu_d(\mathbb{C})$, where $\mu_k(\mathbb{C})$ denotes the group of complex k-th roots of 1. In this section we set $X_n(r)=\{(z_1,\dots,z_n)\in\mathbb{C}^n \mid z_i\neq 0, z_i/z_j\notin\mu_r(\mathbb{C})\}$. It is the hyperplane complement for all the subgroups $W=G(de,e,n)$ of the group $\hat{W}=G(r,1,n)$ with $r=de$ and $d>1$. We choose some basepoint $x\in X_n(r)$ in order to define the braid groups B and $\hat{B}$ of W and $\hat{W}$, respectively. Since $\hat{W}$ is a Shephard group, the parabolic subgroups of $\hat{B}$ are already known up to conjugacy. Here we show how to deduce the parabolic subgroups of B from the ones of $\hat{B}$.

Let $\bar{\varphi}: G(r,1,n)\twoheadrightarrow\mu_r(\mathbb{C})/\mu_d(\mathbb{C})$ denote the composite of the morphism $G(r,1,n)\twoheadrightarrow\mu_r(\mathbb{C})$ given by the product of the (nonzero) monomial entries with the natural projection morphism. Composed with the projection map $\hat{B}\twoheadrightarrow\hat{W}=G(r,1,n)$ it provides a surjective morphism $\varphi:\hat{B}\twoheadrightarrow\mu_{de}(\mathbb{C})/\mu_d(\mathbb{C})\simeq\mathbb{Z}/e\mathbb{Z}$ whose kernel is naturally identified with B.

Proposition 3.2. *The parabolic subgroups of B are exactly the kernels of the restriction of φ to the parabolic subgroups of $\hat{B}$. Moreover, if B_0 is an irreducible parabolic subgroup of B associated to the parabolic subgroup W_0, then $B_0=\hat{B}_0\cap B$ with $\hat{B}_0$ an irreducible parabolic subgroup of $\hat{B}$ associated to the irreducible parabolic subgroup $\hat{W}_0$ of $\hat{W}$, and we have $z_{B_0}=z_{\hat{B}_0}^{|Z(\hat{W}_0)|/|Z(W_0)|}$.*

Proof. Since the hyperplane arrangements of W and $\hat{W}$ are the same, their parabolic subgroups are in natural 1-1 correspondance with the elements of the intersection lattice of this hyperplane arrangement. Let E_0 be some element of this intersection lattice, and W_0, $\hat{W}_0$ their pointwise stabilizer in W and $\hat{W}$, respectively. It is sufficient to prove that, for $\eta: x\rightsquigarrow y_0$ some normal ray inside $(X_n(r),\mathbb{C}^n)$ and a ball Ω of center $y_0\in E_0$ and radius $\epsilon>0$ with respect to some $\hat{W}$-invariant norm, then the corresponding parabolic subgroups $B_0,\hat{B}_0$ of B and $\hat{B}$ satisfy $B_0=\hat{B}_0\cap B$.

We set $\Omega^*=\Omega\cap X_n(r)$. By definition, B_0 and $\hat{B}_0$ are then the images of $\pi_1(\Omega^*/W_0,W_0.\eta)\to\pi_1(\Omega^*/W,W.\eta)$ and $\pi_1(\Omega^*/\hat{W}_0,\hat{W}_0.\eta)\to\pi_1(\Omega^*/\hat{W},\hat{W}.\eta)$, respectively. Because of the covering map properties, the natural maps between the topological spaces involved induce the following commutative diagrams of groups, with the additional property that all commutative squares are cartesian.

$$\begin{array}{ccccccc}
\pi_1(\frac{\Omega^*}{W_0},W_0.\eta) & \hookrightarrow & \pi_1(\frac{X_n(r)}{W_0},W_0.\eta) & \hookrightarrow & \pi_1(\frac{X_n(r)}{W},W.\eta) & \xrightarrow{\simeq} & \pi_1(\frac{X_n(r)}{W},W.x) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_1(\frac{\Omega^*}{\hat{W}_0},\hat{W}_0.\eta) & \hookrightarrow & \pi_1(\frac{X_n(r)}{\hat{W}_0},\hat{W}_0.\eta) & \longrightarrow & \pi_1(\frac{X_n(r)}{\hat{W}},\hat{W}.\eta) & \xrightarrow{\simeq} & \pi_1(\frac{X_n(r)}{\hat{W}},\hat{W}.x)
\end{array}$$

This implies that B_0 is identified inside $\hat{B}$ with $\hat{B}_0\cap B=\hat{B}_0\cap\operatorname{Ker}\varphi$, which proves the first claim. Finally, if W_0 is irreducible then $\hat{W}_0$ is also irreducible, as it acts on the same space and contains W_0. We choose some $y_0+v\in\Omega^*\cap\eta(]1-\alpha,1[)$ for α small enough, and naturally identify B_0 and $\hat{B}_0$ with fundamental groups based at $W_0.(y_0+v)$ and $\hat{W}_0.(y_0+v)$, respectively. Then, the paths $\gamma: t\mapsto y_0+v\exp(2\pi\mathrm{i}t/|Z(W_0)|)$ and $\hat{\gamma}: t\mapsto y_0+v\exp(2\pi\mathrm{i}t/|Z(\hat{W}_0)|)$ provide loops $t\mapsto W_0.\gamma(t)$ and $t\mapsto\hat{W}_0.\gamma(t)$ whose homotopy classes are z_{B_0} and $z_{\hat{B}_0}$, respectively. It is then immediately checked that $t\mapsto\hat{W}_0.\gamma(t)$ has for homotopy class $z_{\hat{B}_0}^{|Z(\hat{W}_0)|/|Z(W_0)|}$, which proves the last claim. □

3.3. **Interval monoids.** The presentations we are going to use are strongly connected with monoids with strong properties. These turn out to be interval monoids, so we define this concept first.

One first considers a finite group, which in our case will always be the reflection group W, and some generating set S for W.

The length with respect to S of $w\in W$ is defined as the minimal $\ell_S(w)=r\geq 0$ so that one can write $w=s_1\dots s_r$ where $s_i\in S$, with $\ell_S(1)=0$. An expression $(s_1,\dots,s_r)\in S^r$ such that $w=s_1s_2\dots s_r$ with $r=\ell_S(w)$ is called a reduced expression and the set of such reduced expressions is denoted $Red(w)$.

From this one can define two partial orderings on W, setting $a \prec b$ if $\ell_S(a) + \ell_S(a^{-1}b) = \ell(b)$ and $b \succ a$ if $\ell_S(ba^{-1}) + \ell_S(a) = \ell_S(b)$. An element $c \in W$ is then said to be balanced if $[1, c] = \{a \in W; 1 \prec a \prec c\} = \{a \in W; c \succ a \succ 1\}$, that is if $\{u \in W; \ell_S(u) + \ell_S(u^{-1}c) = \ell_S(c)\} = \{u \in W; \ell_S(cu^{-1}) + \ell_S(u) = \ell_S(c)\}$. The interval monoid M attached to the data (W, S, c) is defined by taking for generators a copy of $[1, c]$, denoting $\mathbf{u}$ the generator corresponding to $u \in [1, c]$, and for relations

$$\mathbf{w} = \mathbf{uv} \text{ when } w = uv, \ell_S(w) = \ell_S(u) + \ell_S(v)$$

Such monoids with their main properties have been basically introduced in [41] and we refer to [16] for a modern treatment of the subject. A general important result is that, if the partial orderings $\prec$ and $\succ$ restricted to the set $[1, c]$ are lattices, then M together with $\mathbf{c}$ provides a Garside structure on the group G defined by the same presentation as M, where the concept of a Garside structure will be recalled in detail in section 4 below.

Another important property is that such an interval monoid is homogeneous, and more precisely there exists a monoid morphism $\ell : M \to \mathbb{N} = \mathbb{Z}_{\geq 0}$ such that M is generated by its elements of degree 1 – these are the atoms of the monoid M, and they are exactly the $\mathbf{s}$ for $s \in S \cap [1, c]$. Then, another general property is that, for $w \in [1, c]$, one has $\ell(\mathbf{w}) = \ell_S(w)$.

The archetypical example of interval monoids providing Garside structures is given by the Artin monoids associated to Artin groups: there, the set S is given by the simple reflections of the Coxeter presentation of the real reflection group. The complex braid groups for $G(e, e, n)$ and for many exceptional groups can also be described as the group of fractions of interval monoids.

In our case, the set S will contain only elements of order 2. This has the following consequence, which will be of crucial importance in the proof of our main theorems (see section 6).

Proposition 3.3. *If all the elements of S have order 2 then, For every $u \in [1, c]$, $\mathbf{u} \in M$ is squarefree, that is no expression as a product of atoms may contain* **aa** *for some $a \in S$.*

Proof. This is the consequence of the fact that, for $u \in [1, c]$, if $\mathbf{u} = \mathbf{a_1 a_2} \dots \mathbf{a_r}$ with $a_i \in S$ and $r = \ell(\mathbf{u}) = \ell_S(u)$, then $a_1 a_2 \dots a_r$ has to be a reduced decomposition, which excludes the possibility $a_i a_{i+1} = aa = 1$ for some i, as every element of S has order 2. □

3.4. **Groups $G(e, e, n)$.** Let $W = G(e, e, n)$, with $e \geq 1$. The group B admits the following ('standard') presentation with generating set $S = \{t_0, t_1, \dots, t_{e-1}, s_3, s_4, \dots, s_n\}$ and relations

(1) $t_i t_{i+1} = t_j t_{j+1}$, with the convention $t_e = t_0$,
(2) $s_3 t_i s_3 = t_i s_3 t_i$
(3) $s_k t_i = t_i s_k$ for $k \geq 4$
(4) $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$ for $k \geq 3$
(5) $s_k s_l = s_l s_k$ when $|l - k| \geq 2$.

We refer to [12, 9, 42] for general results on this presentation and the corresponding monoid, which was first introduced by Corran and Picantin in [12]. We call it the standard monoid for $G(e, e, n)$. When $e = 1$ (resp. $e = 2$) one recovers the Artin monoid of type A_{n-1} (resp. D_n). For products of reflection groups of these types, we take for presentation of B the obvious direct product presentations. This covers in particular the case of the reflection subgroups of W of the form $G(e, e, n_1) \times G(1, 1, n_2) \times \cdots \times G(1, 1, n_k) \subset W$ with $n_1 + n_2 + \cdots + n_k = n$, which are actually parabolic subgroups of W.

We recall that the reflecting hyperplanes for W are given by the equations $z_i = \zeta z_j$ for $i \neq j$ and $\zeta \in \mu_e(\mathbb{C})$. We also recall from [12, 9, 42] that, under the natural morphism $B \twoheadrightarrow W$, the action on $\mathbb{C}^n$ of the above generators of B is given by the permutation matrices $(k-1, k)$ for s_k and, for t_k, by

$$\begin{pmatrix} 0 & \zeta_e^k \\ \zeta_e^{-k} & 0 \end{pmatrix} \oplus \mathrm{Id}_{n-2} \quad \text{where } \zeta_e = \exp(2\mathrm{i}\pi/e).$$

A nontrivial important result from [42] is the following

Theorem 3.4. *(Neaime) The standard monoid for $G(e, e, n)$ is an interval monoid with respect to the image of S under $B \twoheadrightarrow W$.*

The basepoint corresponding to this presentation is $\underline{b} = (b_1, \dots, b_n) \in \mathbb{R}^n \subset \mathbb{C}^n$ with $0 < b_1 < b_2 < \cdots < b_n$, when needed with the condition that b_{i+1}/b_i is big enough. Notice that the set of such basepoints is convex and therefore simply connected and also that, when $e \in \{1, 2\}$, it lies inside a natural Weyl chamber.

It was proven in [9] Proposition 6.2 that, with respect to such a basepoint, the generators we have correspond to local fundamental groups as follows. For $s \in S$, consider the straight line segment $[\underline{b}, s.\underline{b}]$, where $s.\underline{b}$ is given by the action of B on $\mathbb{C}^n$ described above. It crosses only one reflecting hyperplane, at its middle point $\beta(s) = (\underline{b} + s.\underline{b})/2$. Then s is a generator of the local fundamental group for X/W w.r.t. the normal ray $[\underline{b}, \beta(s)]$. The same property holds, with the same proof, if W is replaced by one of its reflection subgroups of the form $G(e, e, k) \times G(1, 1, n-k)$ for $0 \le k < n$. Actually these subgroups are maximal parabolic subgroups of $G(e, e, n)$. More generally, as a consequence of [47] Theorem 3.11, every maximal parabolic subgroup of W is a conjugate of the stabilizer of either $\underline{b}_0(\zeta) = (\zeta, 1, \dots, 1)$ for $\zeta \in \mu_e(\mathbb{C})$ or of

$$\underline{a}_k = (0, \dots, 0, \underbrace{1, \dots, 1}_{n-k}) \text{ for } 1 \le k \le n-1.$$

The former are isomorphic to $\mathfrak{S}_n = G(1, 1, n)$ while the latter are isomorphic to $G(e, e, k) \times G(1, 1, n-k)$ for $1 \le k < n$. Notice that, since $b_{k+1} > 0$, the stabilizer of $\underline{a}_k$ is equal to the stabilizer of $b_{k+1}\underline{a}_k = (0, \dots, 0, b_{k+1}, \dots, b_{k+1})$.

From this we deduce the following[2]

Proposition 3.5. *Let $W = G(e, e, n)$, and $S_0 \subsetneq S = \{s_k, t_i, 3 \le k \le n, 0 \le i < e\}$ be such that $\{i; t_i \in S_0\}$ has cardinality 1 or e, and which is maximal for this property. Then $\langle S_0 \rangle \subset B$ is a maximal parabolic subgroup, and every maximal parabolic subgroup of B is a conjugate of such a subgroup.*

Proof. By the description of the maximal parabolic subgroups given above and Proposition 2.6 it is sufficient to prove that, for every $\underline{b}_0$ equal to $\underline{b}_0(\zeta)$ or $b_{k+1}\underline{a}_k$ as above, there exists a normal ray $\eta : \underline{b} \rightsquigarrow \underline{b}_0$ such that the parabolic subgroup W_0 stabilizing $\underline{b}_0$ is generated by the subset $S_0 \subset S$ given by $\{s \in S; s.\underline{b}_0 = \underline{b}_0\}$. Indeed, it is readily checked that these collections S_0 are the ones of the statement.

Under our assumptions, up to possibly increasing the value of the ratios b_{i+1}/b_i, we can assume that the straight line segment $[\underline{b}, \underline{b}_0]$ crosses the hyperplane arrangement only at $\underline{b}_0$, and we take $\eta = [\underline{b}, \underline{b}_0]$ for normal ray.

We prove that $\langle S_0 \rangle$ is the corresponding parabolic subgroup by completing the following commutative diagram, whose plain arrows are the natural ones and X_0 is the hyperplane complement of W_0.

$$\begin{array}{ccccc}
\pi_1(\frac{X}{W}, W.\eta) & \longleftarrow & \pi_1^{loc}(\frac{X}{W}, W.\eta) & & \\
 & & \uparrow & \searrow^{\simeq} & \\
 & & \langle S_0 \rangle & \overset{\simeq}{\dashrightarrow} & \pi_1(\frac{X_0}{W_0}, W_0.\eta)
\end{array}$$

We have a natural embedding

$$\langle S_0 \rangle \subset B = \pi_1(X/W, W.\underline{b}) = \pi_1(X/W, W.\eta(0)) \simeq \pi_1(X/W, W.\eta),$$

which provides our first 'dashed' arrow $\langle S_0 \rangle \hookrightarrow \pi_1^{loc}(\frac{X}{W}, W.\eta)$. The second one is obtained as its composite with the natural map $\pi_1^{loc}(\frac{X}{W}, W.\eta) \to \pi_1(\frac{X_0}{W_0}, W_0.\eta)$. It remains to prove that this composite map is surjective.

When $\underline{b}_0 = b_{k+1}\underline{a}_k$ or when $\underline{b}_0 = \underline{b}_0(1)$ this is immediate because in these two cases the elements of S_0 are mapped to the generators of the corresponding braid group of $\pi_1(\frac{X_0}{W_0}, W_0.\eta)$. In case $\underline{b}_0 = \underline{b}_0(\zeta)$ with $\zeta \neq 1$, we consider the map $p : (z_1, z_2, \dots, z_n) \mapsto (\zeta^{-1}z_1, z_2, \dots, z_n)$. It maps X_0

[2] Part of this statement was already claimed in [9] Prop. 6.3, but the proof was incomplete, because of the erroneous statement given there that the stabilizer of $(\zeta, 1, 1, \dots, 1)$ is conjugate inside W to the stabilizer of $(1, 1, \dots, 1)$.

to the hyperplane complement X_0' associated to $W_0' = \mathfrak{S}_n$, and we have $p(\underline{b}_0) = \underline{1} = (1, \dots, 1)$. Without loss of generality we can replace the path $\eta = [\underline{b}, \underline{b}_0]$ by some basepoint inside $[\underline{b}, \underline{b}_0]$ close enough to $\underline{b}_0$, and thus the path $p \circ \eta$ by

$$\underline{c}_0(\varepsilon) = p(\underline{b}_0 + \varepsilon(\underline{b} - \underline{b}_0)) = \underline{1} + \varepsilon(\zeta^{-1}b_1 - 1, b_2 - 1, \dots, b_n - 1) = (1-\varepsilon)\underline{1} + \varepsilon(\zeta^{-1}b_1, b_2, \dots, b_n)$$

for $\varepsilon > 0$ small enough. We want to prove that $\pi_1(X_0'/W_0', W_0'.\underline{c}_0(\varepsilon))$ is generated by the images of S_0. For fixed $k \in \{1, \dots, n-1\}$, we introduce $\underline{d}_0(\varepsilon) = (1-\varepsilon)\underline{1} + \varepsilon\underline{b}$, $\underline{c}_0(\varepsilon, u) = (1-u)\underline{c}_0(\varepsilon) + u\underline{d}_0(\varepsilon)$ and, for $\alpha > 0$ small enough,

$$A(t,u) = (1-t)\underline{c}_0(\varepsilon,u) + t\sigma_k.\underline{c}_0(\varepsilon,u) + \alpha i t(1-t)(\sigma_k - 1).\underline{c}_0(\varepsilon,u)$$

where $\sigma_k = (k, k+1) \in \mathfrak{S}_n$. The images of S_0 are the homotopy classes of the maps $t \mapsto A(t,0)$ for $t \in [0,1]$. The maps $t \mapsto A(t,1)$ provide homotopy classes inside $\pi_1(X_0'/W_0', W_0'.\underline{d}_0(\varepsilon))$ which are the Artin generators of the braid group of $W_0' \simeq \mathfrak{S}_n$; indeed, it is readily checked that, provided $b_1 > 1$, $\underline{d}_0(\varepsilon)$ belongs to the Weyl chamber associated to the Coxeter system $\{\sigma_1, \dots, \sigma_{n-1}\}$. In order to get that $\pi_1(X_0'/W_0', W_0'.\underline{c}_0(\varepsilon))$ is generated by the images of S_0 it is then sufficient to check that the map A is a homotopy map with values in X_0'.

We have that $A(0, \bullet)$ describes the line segment $[\underline{c}_0(\varepsilon), \underline{d}_0(\varepsilon)]$, and $A(1,u) = \sigma_k.A(0,u)$, so we only need to prove that $A(t,u) \in X_0'$ for every $(t,u) \in [0,1]^2$. Writing $A(t,u) = (1-\varepsilon)\underline{1} + \varepsilon\check{A}(t,u)$, it is equivalent to proving that $\check{A}(t,u) \in X_0'$. Now we compute that

$$\underline{c}_0(\varepsilon,u) = (1-\varepsilon)\underline{1} + \varepsilon\check{c}_0(\varepsilon,u), \quad \underline{\check{c}}_0(\varepsilon,u) = \big(((1-u)\zeta^{-1} + u)b_1, b_2, \dots, b_n\big)$$

and $\check{A}(t,u) = (1-t)\underline{\check{c}}_0(\varepsilon,u) + t\sigma_k.\underline{\check{c}}_0(\varepsilon,u) + \alpha i t(1-t)(\sigma_k - 1).\underline{\check{c}}_0(\varepsilon,u)$. Let β_i be the linear form $\underline{z} \mapsto z_{i+1} - z_i$. If $i \geq 2$, then the real part of $\beta_i(\check{A}(t,u))$ is positive, except when $i = k$; in this case, it is still nonzero, except for $t = 1/2$, but then its imaginary part is nonzero. So it only remains to consider the case $i = 1$. For $k \geq 2$, the real part of $\beta_1(\check{A}(t,u))$ is positive, so we can assume $k = 1$. We have

$$\beta_1(\check{A}(t,u)) = ((1-2t) - 2\alpha i t(1-t))\,\beta_1(\underline{\check{c}}_0(\varepsilon,u))$$

and we have that $\beta_1(\underline{c}_0(\varepsilon,u))$ has positive real part for every $u \in [0,1]$ so that $\beta_1(\check{A}(t,u)) \neq 0$ for every t, u, which concludes the proof. ☐

Using the compatibility of parabolic subgroups with products established in subsection 2.4, this statement is immediately extended to the case of an arbitrary parabolic subgroup of $G(e,e,n)$. From this one gets a complete description of the parabolics as follows.

Corollary 3.6. *Let $W = G(e,e,n)$, and $S_0 \subsetneq S = \{s_k, t_i, 3 \leq k \leq n, 0 \leq i < e\}$ be such that $\{i; t_i \in S_0\}$ has cardinality 0, 1 or e. Then $\langle S_0 \rangle \subset B$ is a parabolic subgroup, and every parabolic subgroup of B is a conjugate of such a subgroup.*

Proof. We prove that $\langle S_0 \rangle \subset B$ is a parabolic subgroup by descending induction on $|S_0|$. If S_0 is maximal for this property, we have the conclusion by Proposition 3.5. Otherwise, we have $S_0 \subsetneq S_1 \subsetneq S$ with $\{i; t_i \in S_1\}$ of cardinality 0, 1 or e, and we can assume that S_1 is minimal for this property. Then, by the induction assumption, $\langle S_1 \rangle \subset B$ is a parabolic subgroup B_1, attached to the parabolic subgroup W_1 of W. On the other hand, S_1 can be identified with the generators of the standard presentation for the corresponding braid group of B_1, and $S_0 \subsetneq S_1$ is maximal for the property that $\{i; t_i \in S_0\}$ has cardinality 0, 1 or e. From Proposition 3.5 it then follows that $\langle S_0 \rangle \subset B_1 \subset B$ is a parabolic subgroup of B_1, and therefore of B by Proposition 2.5 (4).

Conversely, assume that B_0 is a parabolic subgroup of B mapped to the parabolic subgroup W_0 of W. Then it is included inside a maximal parabolic subgroup B_1, with corresponding maximal parabolic subgroup W_1. By Proposition 3.5 we can assume that $B_1 = \langle S_1 \rangle$ with $S_1 \subsetneq S$ maximal such that $\{i; t_i \in S_1\}$ has cardinality 0, 1 or e. Then B_0 is a parabolic subgroup of B_1 by Proposition 2.8, and by induction on the rank of W one gets that B_0 is a conjugate inside B_1 (hence inside B) of $\langle S_0 \rangle$, where $\{i; t_i \in S_0\}$ has cardinality 0, 1 or e. ☐

3.5. **Well-generated 2-reflection groups.** Our main references are [3] and [46] – see also [20]. Let $W < \mathrm{GL}_n(\mathbb{C})$ be a well-generated complex reflection group, that is a group that can be generated by at most n reflections. All the exceptional groups of rank at least 3 fall into this category except G_{31}. We denote $\mathcal{R}$ the set of all reflections, and assume that all of them have order 2. This is also the case for all well-generated irreducible complex reflection groups, except for some of them which are Shephard groups, so that we already know their parabolic subgroups.

By definition the Coxeter number $h = h(W)$ of W is its largest reflection degree. We have $h = 2|\mathcal{R}|/n$. We construct an interval monoid by considering the generating set $\mathcal{R}$. When W is irreducible, a Coxeter element for W is a regular element for the eigenvalue $\exp(2\pi \mathrm{i}/h)$, in the sense of Springer. When it is not, this notion is also defined, by taking the direct sum of Coxeter elements for the irreducible constituents of W. We refer to [3], Definition 7.1, for more details. Let c be such a Coxeter element for W. Such an element is balanced, and then the 'dual braid monoid' $M(c) = M_W(c)$ in the sense of [3] is defined as the interval monoid attached to $(W, \mathcal{R}, c)$. We denote $G(c)$ the group with the same presentation.

The following two propositions are proven in [3] and in the references there.

Proposition 3.7. *Let W be well-generated reflection group of rank n, and c a Coxeter element.*

(1) If W is essential (that is $\bigcap_{w\in W} \mathrm{Ker}(w - \mathrm{Id}) = \{0\}$), then $\ell(c) = n$.
(2) The poset $[1, c]$ is a lattice.

Proposition 3.8.

(1) $M(c)$ is a Garside monoid, with fundamental element $\mathbf{c}$, and $u \mapsto \mathbf{u}$ is an isomorphism of lattices from $[1, c]$ to the set of divisors of $\mathbf{c}$, for the partial ordering induced by left divisibility.
(2) Let $\mathcal{R}_c = \mathcal{R} \cap [1, c]$. Then $M(c)$ has a presentation with generators $\mathbf{u}, u \in \mathcal{R}_c$ and relations $\mathbf{uv} = \mathbf{vw}$ whenever $u, v, w = v^{-1}uv \in \mathcal{R}_c$. These generators are the atoms of $M(c)$.
(3) For $u \in [1, c]$, $\ell(u)$ is equal to the length of $\mathbf{u}$ with respect to the atoms of $M(c)$.
(4) The Garside group $G(c)$ associated to $M(c)$ is isomorphic to the braid group B of W, in such a way that each atom $\mathbf{u}, u \in \mathcal{R}_c$ is sent to a braided reflection associated to u.

From this we get immediately the following. The first two items are classical and the last one is immediate.

Proposition 3.9.

(1) The length with the respect to the atoms of $M(c)$ defines a monoid morphism $\ell : M(c) \to (\mathbb{N}, +)$, that is $\ell(m_1m_2) = \ell(m_1) + \ell(m_2)$ for every $m_1, m_2 \in M(c)$.
(2) For every $u \in [1, c]$, $\mathbf{u} \in M(c)$ is balanced, that is its set of left and right divisors are the same.
(3) For every $u \in [1, c]$, $\mathbf{u} \in M(c)$ is squarefree, that is no expression as a product of atoms may contain $\mathbf{aa}$ for some $a \in \mathcal{R}_c$.

Proof. (1) is immediate for instance from the homogeneity of the presentation. For (2), if $a \prec u$ then $ab = u$ and with $\ell(a) + \ell(b) = \ell(u)$. But since $\mathcal{R}$ is a union of conjugacy classes we have $\ell(b) = \ell(aba^{-1})$, hence $(aba^{-1})a = u$ with $\ell(aba^{-1}) + \ell(a) = \ell(u)$, so a divides u on the right too, and $\mathbf{u}$ is balanced. (3) is a special case of Proposition 3.3. □

In the sequel we shall need the following classical lemma. We denote by $\pi : M(c) \to W$ the natural monoid morphism mapping each $\mathbf{u}$ to $u \in [1, c]$, and we denote by $j : [1, c] \to M(c)$ the map sending u to $\mathbf{u}$.

Lemma 3.10. *For $m \in M(c)$ with $\pi(m) \in [1, c]$, m divides $\mathbf{c}$ if and only if $\ell(m) = \ell(\pi(m))$, where $\ell(m)$ is the length of m with respect to the atoms of $M(c)$.*

Proof. By the previous proposition we have $\ell(m) = \ell(\pi(m))$ when m divides $\mathbf{c}$, so we only need to prove the converse. Notice that one always has $\ell(m) \geq \ell(\pi(m))$, and assume $m \in M(c)$ satisfies $\ell(m) = \ell(\pi(m))$. Write m as m_1m_2 with $m_1, m_2 \in M(c)$, and notice that

$$\ell(m_1) + \ell(m_2) = \ell(m) = \ell(\pi(m)) = \ell(\pi(m_1)\pi(m_2)) \leq \ell(\pi(m_1)) + \ell(\pi(m_2)) \leq \ell(m_1) + \ell(m_2)$$

so that we have $\ell(m_i) = \ell(\pi(m_i))$ for $i = 1, 2$. Then by induction on $\ell(m)$ one can assume $m = m_1 m_2$ with m_1 diving $\mathbf{c}$ and m_2 an atom, that is $m_2 = \mathbf{s}$ for $s = \pi(m_2) \in [1, c]$ a reflection. But then $\ell(\pi(m)) = \ell(m) = \ell(m_1) + 1 = \ell(\pi(m_1)) + \ell(s)$. But from the very definition of $M(c)$ this implies that $j(\pi(m)) = m_1\mathbf{s}$, that is $j(\pi(m)) = m$. This proves that m divides $\mathbf{c}$ by Proposition 3.8 (1).

□

In addition, we will need the following proposition, which combines results from [3] and [46].

Proposition 3.11. *Let W be a well-generated complex reflection group, and c a Coxeter element.*

(1) Every parabolic subgroup of W is well-generated.

(2) Let W_0 be a parabolic subgroup of W, and c_0 a Coxeter element of W_0. Then W_0 is the pointwise stabilizer of $\mathrm{Ker}(c_0 - \mathrm{Id})$, and there exists $g \in W$ such that $gc_0g^{-1} \in [1, c]$. Moreover, for $w \in W_0$, its length with respect to $\mathcal{R}$ coincides with its length with respect to $\mathcal{R}_0 = \mathcal{R} \cap W_0$.

(3) Let $c_0 \in [1, c]$. Then c_0 is a Coxeter element for the pointwise stabilizer W_0 of $\mathrm{Ker}(c_0 - \mathrm{Id})$. Moreover, if $(s_1, \dots, s_k) \in Red(c_0)$, then $\langle s_1, \dots, s_k \rangle = W_0$.

Proof. For (1), this is a direct check on the tables, for instances the ones of Appendix C in [44]. For (2) and (3), see Proposition 1.36 of [46] and its proof. □

An immediate consequence of this proposition is that the inclusion $[1, c_0] \subset [1, c]$ is an injective morphism of posets. Moreover, we have a well-defined homomorphism $\varphi : M(c_0) \to M(c)$ extending the inclusion map $[1, c_0] \subset [1, c]$, for any $c_0 \in [1, c]$ and W_0 the corresponding parabolic subgroup.

For the sequel, we shall need the following additional property.

$$(3.1) \qquad \forall c_0 \in [1, c] \quad W_0 \cap [1, c] = [1, c_0]$$

We were unable to find a reference for it – nor a proof avoiding the classification of complex reflection groups. For the groups we are interested in it is easily checked by computer – for this notice that, since all Coxeter elements are conjugates, it is sufficient to check this for one of them. Then, for each $c_0 \in [1, c]$, it is sufficient to get a decomposition of c_0 in order to determine W_0 as the group generated by the reflections it contains. However, it actually holds true in general, as shows the following proposition. The main idea of using [35] in order to prove the non-real case was communicated to us by Theodosios Douvropoulos. We give his proof here.

Proposition 3.12. *Property (3.1) holds true for every well-generated complex reflection group.*

Proof. We endow the ambient space V with an hermitian structure for which W acts by unitary transformations, that is $W \subset U(V)$. We use the Brady-Watt order defined on the unitary group by $a \prec_{BW} b$ if $\dim \mathrm{codim}\, \mathrm{Ker}(a-1) + \dim \mathrm{codim}\, \mathrm{Ker}(a^{-1}b-1) = \dim \mathrm{codim}\, \mathrm{Ker}(b-1)$, or equivalently $\mathrm{Im}(a-1) \oplus \mathrm{Im}(a^{-1}b-1) = \mathrm{Im}(b-1)$, defined in [5]. We denote $[a, b]_{BW}$ the set of all unitary transformations u such that $a \prec_{BW} u$ and $u \prec_{BW} b$. By Theorem 2 of [5] (see also Note 1 in [5]) we have that $[1, u]_{BW}$ is isomorphic as a poset to the set of subspaces of V containing $\mathrm{Ker}(u-1)$ ordered by reverse inclusion, the bijection being given by $u' \mapsto \mathrm{Ker}(u-1)$.

On the other hand, it is proved in [35] that $[1, c] = W \cap [1, c]_{BW}$ for every W – the real case being a classical result, the general case being proved using the classification, see Corollary 6.6 of [35].

Now, let $w_0 \in [1, c] \cap W_0$. Then $w_0 \in W_0$ implies that $\mathrm{Ker}(w_0 - 1)$ contains $\bigcap_{w \in W_0} \mathrm{Ker}(w-1)$, which is equal to $\mathrm{Ker}(c_0 - 1)$. Since $w_0, c_0 \in [1, c]_{BW}$, by the theorem of Brady-Watt this implies $w_0 \prec_{BW} c_0$. But then $w_0 \in W_0 \cap [1, c_0]_{W_0} = [1, c_0]$, and this proves the claim. □

Let $\mathcal{R}_0 = \mathcal{R} \cap W_0$. We have $\mathcal{R}_{c_0} = \mathcal{R}_0 \cap [1, c_0] = \mathcal{R}_0 \cap [1, c] = \mathcal{R} \cap W_0 \cap [1, c] = \mathcal{R} \cap [1, c_0]$. For $s, t \in \mathcal{R}_{c_0}$ the right lcm of $\mathbf{s}, \mathbf{t}$ is equal to $\mathbf{w}$ for w the join of s, t in the corresponding poset $[1, c_0]$. Since it is also equal to the join of s, t inside $[1, c]$ we get that $\mathrm{lcm}(\varphi(\mathbf{s}), \varphi(\mathbf{t})) = \varphi(\mathrm{lcm}(\mathbf{s}, \mathbf{t}))$.

From this, general arguments (see [9] Lemmas 5.1, 5.2) imply that φ is injective, so that $M(c_0)$ is identified with a submonoid of $M(c)$. Moreover, if $m' \in M(c)$ divides (on the left) some

$m \in M(c_0)$, then $m' \in M(c_0)$; indeed, if $m = m'm''$, there should be a sequence of Hurwitz relations applied on some decompositions of m in atoms of $M(c_0)$ so that it is changed to some decomposition involving some atoms of $M(c)$ which do not belong to $M(c_0)$. But for Hurwitz relations $\mathbf{uv} = \mathbf{vw}$, if $u, v \in W_0$ then $v, w \in W_0$, so this is not possible. This property then implies by general arguments ([9] Lemma 5.3) the following.

Proposition 3.13. *Let $c_0 \in [1, c]$.*

1. *The map $\varphi : M(c_0) \to M(c)$ is injective and induces an injective group homomorphism $G(c_0) \to G(c)$, identifying $G(c_0)$ with a subgroup of $G(c)$.*
2. *For every $a, b \in M(c_0)$ $\mathrm{lcm}(\varphi(a), \varphi(b)) = \varphi(\mathrm{lcm}(a, b))$.*
3. *The image of $M(c_0)$ inside $M(c) \subset G(c)$ is equal to $G(c_0) \cap M(c)$.*

We then have the following.

Proposition 3.14. *For $c_0 \in [1, c]$ the image of the injective homomorphism $G(c_0) \to G(c) = B$ is a parabolic subgroup of B, that we call a standard parabolic subgroup. Moreover, every parabolic subgroup of B is conjugate to such a standard parabolic subgroup.*

Proof. The first part of the proposition is a reformulation of Proposition 20 and Proposition 39 of [20]. The second one then follows from the combination of Proposition 3.11 and Proposition 2.6. □

Of course, this concept of standard parabolic subgroup here depends on the choice of the Coxeter element c, although we know that the concept of parabolic subgroup does not.

3.6. **Exceptional groups of rank 2.** Most exceptional groups of rank 2 are Shephard groups, or isodiscriminantal to a group of the form $G(de, e, n)$. The only ones which are not are G_{12}, G_{13} and G_{22}.

If $W = G_{12}, G_{22}$, we choose the presentation of B given in [8], that is $\langle s, t, u \mid stus = tust = ustu\rangle$ and $\langle s, t, u \mid stust = tustu = ustus\rangle$. In both cases all reflections have order 2 and they form a conjugacy class. Since the generators s, t, u are braided reflections this proves that every proper parabolic subgroup of B is a conjugate, say, of $\langle s\rangle$.

For $W = G_{13}$, we use the presentation of B obtained in [1]. By the construction of [1], the generators are braided reflections, as they are meridians with respect to irreducible components of the discriminant, which can be written as $z_1(z_1^2 - z_2^3) = 0$. Since there are two such irreducible components, every braided reflection is conjugate to one of the generators. These generators are g_1, g_2, g_3 with relations $g_1g_2g_3g_1 = g_3g_1g_2g_3, g_3g_1g_2g_3g_2 = g_2g_3g_1g_2g_3$. Then, as noticed in [1], this group is isomorphic to the Artin group $\langle a, b \mid ababab = bababa\rangle$ of type $I_2(6)$, an isomorphism being given by $a = g_3g_1g_2g_3$, $ab = g_3g_1g_2$, that is $b = g_3^{-1}$, and its inverse by $g_3 = b^{-1}$, $g_1 = a^{-1}b^{-1}a$, $g_2 = \Delta a^{-2}$ with $\Delta = ababab$. Notice that Δ is central here. Applying the results of subsection 3.1 we get that every parabolic subgroup of B is a conjugate of either $\langle b^{-1}\rangle$ or $\langle \Delta a^{-2}\rangle$.

4. Parabolic closures in Garside groups

In this section we describe some basic properties of Garside groups, recall the definition of a parabolic subgroup in such a group, and show that, under certain conditions, an element in a Garside group admits a parabolic closure, that is, a unique minimal (by inclusion) parabolic subgroup containing it.

4.1. **Garside groups and normal forms.** A Garside group is a group G which admits a so-called Garside structure (G, G^+, Δ), where G^+ is a submonoid of G (the submonoid of positive elements) and $\Delta \in G^+$ is called the Garside element, satisfying some suitable properties [17]. We recall that such a Garside structure determines two partial orders in G: we say that $a \preccurlyeq b$ (a is a prefix of b) if $a^{-1}b \in G^+$, and we say that $b \succcurlyeq c$ (c is a suffix of b) if $bc^{-1} \in G^+$. Both are lattice orders in G, the former is invariant under left-multiplication and the latter is invariant under right-multiplication. We will denote $a \vee b$ and $a \wedge b$ the join and meet of a and b, respectively, with respect to $\preccurlyeq$, and we will denote $a \vee^{\Lsh} b$ and $a \wedge^{\Lsh} b$ the join and meet of a and b, respectively, with respect to $\succcurlyeq$. Notice that $\wedge$ and $\vee$ are invariant under left-multiplication: if $d = a \wedge b$ and

$m = a \vee b$ then $cd = ca \wedge cb$ and $cm = ca \vee cb$ for every $a, b, c \in G$. Similarly, $\wedge^{\Lsh}$ and $\vee^{\Lsh}$ are invariant under right-multiplication.

By definition, an element a is positive if and only if $1 \preccurlyeq a$ or, equivalently, $a \succcurlyeq 1$. We will say that an element x is negative if x^{-1} is positive. Then x is negative if and only if $x \preccurlyeq 1$ or, equivalently, $1 \succcurlyeq x$. We say that a positive element $\delta \in G^+$ is balanced if the set of its positive prefixes coincides with the set of its positive suffixes, and in this case we call it the set of divisors of δ:

$$\mathrm{Div}(\delta) = \{a \in G;\ 1 \preccurlyeq a \preccurlyeq \delta\} = \{a \in G;\ \delta \succcurlyeq a \succcurlyeq 1\}.$$

We remark that, given a balanced element δ and positive elements $a, b, c \in G^+$ such that $\delta = abc$, one has $b \in \mathrm{Div}(\delta)$. Indeed, we have $ab \preccurlyeq \delta$, hence $ab \in \mathrm{Div}(\delta)$ which implies that $\delta \succcurlyeq ab$, and then $\delta \succcurlyeq b$, that is, $b \in \mathrm{Div}(\delta)$.

In a Garside structure, the Garside element Δ is balanced, and its divisors are called simple elements. We denote by S the set of simple elements: $S = \mathrm{Div}(\Delta)$. We will assume that the set S is finite (this is usually part of the definition of a Garside group). The nontrivial simple elements which do not admit proper prefixes are called atoms. It is required that the set S of simple elements (and hence the set $\mathcal{A}$ of atoms) generates G.

For every element $x \in G$ one has $\Delta^p \preccurlyeq x \preccurlyeq \Delta^q$ for some integers $p \leq q$. The maximal integer p and the minimal integer q satisfying this property are called the *infimum* and the *supremum* of x, respectively. Every element $x \in G$ admits a unique decomposition $x = \Delta^p x_1 \cdots x_r$ (its *left normal form*), where $x_1, \ldots, x_r$ are proper simple elements (not trivial and not Δ) such that $x_i x_{i+1} \wedge \Delta = x_i$ for $i = 1, \ldots, r-1$. In this case $p = \inf(x)$ and $p + r = \sup(x)$.

If $x \in G^+$ then $\inf(x) = p \geq 0$, and we can write its left normal form as $x = \Delta\Delta \cdots \Delta x_1 \cdots x_r$, where the first p factors are equal to Δ. This is called the *left-weighted factorization* of x. It is the only way to decompose x as $x = s_1 \cdots s_q$, where each s_i is a nontrivial simple element and $s_i s_{i+1} \wedge \Delta = s_i$ for every i.

The analogous definitions can be done with respect to $\succcurlyeq$, so every element admits a right normal form, and every positive element admits a right-weighted factorization. It turns out that the infimum and the supremum of an element with respect to $\preccurlyeq$ coincide with its infimum and its supremum with respect to $\succcurlyeq$.

Let us now see a decomposition which is valid for every element in G, and which will be the most important for our purposes.

Definition 4.1. Given $x \in G$, we say that $x = a^{-1}b$ is the reduced left-fraction decomposition of x if $a, b \in G^+$ and $a \wedge b = 1$. We will say that a and b are the left-denominator and the left-numerator of x, respectively: $a = D_L(x)$, $b = N_L(x)$.

It is well-known that the reduced left-fraction decomposition of an element $x \in G$ exists and is unique. We will later use the following simple result.

Proposition 4.2. *Let $x \in G$ and suppose that $x = c^{-1}d$ for some $c, d \in G^+$. Let $\alpha = c \wedge d$ and write $c = \alpha a$ and $d = \alpha b$. Then $x = a^{-1}b$ is the reduced left-fraction decomposition of x.*

Proof. It is clear that a and b are positive (as $\alpha \preccurlyeq c$ and $\alpha \preccurlyeq d$). Also $x = c^{-1}d = a^{-1}\alpha^{-1}\alpha b = a^{-1}b$. On the other hand, $\alpha = c \wedge d = \alpha a \wedge \alpha b$. Left-multiplying by α^{-1} we get $a \wedge b = 1$, so $a^{-1}b$ is the reduced left-fraction decomposition of x. □

Let us point out that x is positive if and only if $D_L(x) = 1$, and that x is negative if and only if $N_L(x) = 1$. Also, if $x = a^{-1}b$ is the reduced left-fraction decomposition of x, then $x^{-1} = b^{-1}a$ is the reduced left-fraction decomposition of x^{-1}.

In the same way as one defines the reduced left-fraction decomposition of an element $x \in G$, there is also a reduced right-fraction decomposition $x = uv^{-1}$, where $u, v \in G^+$ and $u \wedge^{\Lsh} v = 1$. This decomposition is also unique, and can be obtained from a given decomposition $x = wy^{-1}$ with $w, y \in G^+$ by removing from w and y their greatest common suffix. We denote $u = N_R(x)$ and $v = D_R(x)$ the right-numerator and right-denominator of x, respectively.

Given a Garside structure (G, G^+, Δ), it is well known that (G, G^+, Δ^N) is also a Garside structure, for any $N \geq 1$. The positive elements and the prefix and suffix orders of both structures coincide, but the simple elements of the latter structure are those elements x such that $1 \preccurlyeq x \preccurlyeq \Delta^N$. It follows that, for every positive element $a \in G^+$, there is some N big enough so that a is a simple element with respect to (G, G^+, Δ^N).

We shall later need the following well known result:

Proposition 4.3. *Let G be a Garside group. For every $x \in G$ there exists a central element $z \in Z(G)$ such that $zx \in G^+$.*

Proof. It suffices to multiply the left normal form of x by a sufficiently large power of Δ to make it positive. Since conjugation by Δ permutes the atoms (which is a finite generating set), some power of Δ is central, so one can take a central power of Δ whose exponent is as big as needed. □

4.2. **Parabolic subgroups of a Garside group.** The main concept of this paper is that of parabolic subgroup. In the framework of Garside groups, parabolic subgroups have been defined by Godelle in [30].

Definition 4.4. Let (G, G^+, Δ) be a Garside structure. The support of a balanced element $\delta \in G^+$ is the set of atoms which are divisors of δ:

$$\mathrm{Supp}(\delta) = \mathrm{Div}(\delta) \cap \mathcal{A}.$$

The word support comes from the following property, which is clear from the previous arguments: An atom belongs to the support of δ if and only if it appears in some representative of δ as a product of atoms.

Definition 4.5. [30] Let (G, G^+, Δ) be a Garside structure.

(i) Let δ be a balanced element of $\mathrm{Div}(\Delta)$, let G_δ be the subgroup of G generated by $\mathrm{Supp}(\delta)$, and let $G_\delta^+ = G_\delta \cap G^+$. We say that G_δ is a standard parabolic subgroup of G if $\mathrm{Div}(\delta) = \mathrm{Div}(\Delta) \cap G_\delta^+$.

(ii) A parabolic subgroup of G is a subgroup of G which is conjugate to a standard parabolic subgroup.

Godelle shows in [30] that standard parabolic subgroups in a Garside group are also Garside groups, and that both Garside structures are closely related. We enumerate here the main properties:

Theorem 4.6. [30] *Let (G, G^+, Δ) be a Garside structure. Let $\delta \in \mathrm{Div}(\Delta)$ be a balanced element such that G_δ is a standard parabolic subgroup of G. Then $(G_\delta, G_\delta^+, \delta)$ is a Garside structure, where the lattice of G_δ is a sublattice of the lattice of G. Moreover, G_δ^+ is closed under positive prefixes and positive suffixes in G. This implies that the gcd (lcm) of two elements in G_δ is the same seen in G_δ and seen in G. And the reduced left-fraction (right-fraction) decomposition of an element of G_δ is the same seen in G_δ and seen in G.*

4.3. **LCM-Garside structures.** In this paper, the Garside structures we will be interested in will satisfy a very convenient property. For every subset $X = \{x_1, \ldots, x_r\} \subset \mathcal{A}$, denote $\Delta_X = x_1 \vee \cdots \vee x_r$, which is always a simple element. If $X = \varnothing$, we consider $\Delta_X = 1$. If Δ_X is balanced, we will denote $G_X = G_{\Delta_X}$ the subgroup generated by $\mathrm{Div}(\Delta_X)$, that is, the subgroup generated by $\mathrm{Supp}(\Delta_X)$. The property we require to a Garside structure is the following:

Definition 4.7. Let (G, G^+, Δ) be a Garside structure. We say that it is an LCM-Garside structure if:

(1) $\Delta = \Delta_{\mathcal{A}}$.
(2) For every $X \subset \mathcal{A}$, the element Δ_X is balanced.
(3) For every $X \subset \mathcal{A}$, the subgroup $G_X = G_{\Delta_X}$ is a standard parabolic subgroup.

Lemma 4.8. *If (G, G^+, Δ) is an LCM-Garside structure, then the standard parabolic subgroups of G are precisely the subgroups of the form G_X, for $X \subset \mathcal{A}$.*

Proof. If (G, G^+, Δ) is an LCM-Garside structure, all subgroups of the form G_X are standard parabolic subgroups. Conversely, let G_δ be a standard parabolic subgroup, for some balanced element $\delta \in \mathrm{Div}(\Delta)$. Consider $X = \mathrm{Supp}(\delta) = \{x_1, \dots, x_r\} \subset \mathcal{A}$. Since $x_i \preccurlyeq \delta$ for $i = 1, \dots, r$, it follows that $\Delta_X = x_1 \vee \cdots \vee x_r \preccurlyeq \delta$. This implies that $\mathrm{Div}(\Delta_X) \subset \mathrm{Div}(\delta)$. Hence

$$X \subset \mathrm{Supp}(\Delta_X) = \mathrm{Div}(\Delta_X) \cap \mathcal{A} \subset \mathrm{Div}(\delta) \cap \mathcal{A} = \mathrm{Supp}(\delta) = X,$$

which implies that $\mathrm{Supp}(\Delta_X) = X = \mathrm{Supp}(\delta)$, and then $G_X = G_{\Delta_X} = G_\delta$. □

We notice that, given $X \subset \mathcal{A}$, we do not necessarily have $\mathrm{Supp}(\Delta_X) = X$. Subsets of atoms satisfying this property will be important for us:

Definition 4.9. Let (G, G^+, Δ) be an LCM-Garside structure with set of atoms $\mathcal{A}$. For every nonempty subset $X = \{x_1, \dots, x_r\} \subset \mathcal{A}$, let $\Delta_X = x_1 \vee \cdots \vee x_r$. We define the *closure of* X as:

$$\overline{X} = \mathrm{Supp}(\Delta_X) = \mathrm{Div}(\Delta_X) \cap \mathcal{A}.$$

We say that X is saturated if $\overline{X} = X$.

With the above definitions and results, the next result follows immediately:

Proposition 4.10. *Let (G, G^+, Δ) be an LCM-Garside structure. A standard parabolic subgroup is a subgroup generated by a saturated set of atoms:*

$$G_X = \langle x_1, \dots, x_r \rangle, \qquad X = \{x_1, \dots, x_r\} = \overline{X}.$$

A parabolic subgroup is a subgroup P of G which is conjugate to a standard parabolic subgroup: $P = (G_X)^g = g^{-1} G_X g$ for some $g \in G$ and some saturated set of atoms X.

4.4. **Swaps and recurrent elements.** In Garside groups, the conjugacy problem is solved by using special kinds of conjugations (cyclings, decyclings, cyclic slidings) in order to compute suitable finite sets (super summit sets, ultra summit sets, sets of sliding circuits), see [21, 26, 27]. Their definitions and computations are sometimes technical, although they are quite efficient in practice.

In this paper we will show that there is a much simpler procedure to treat problems related to conjugacy, centralizers and parabolic subgroups in a Garside group. The algorithms are not faster than the ones mentioned above, but it is much better when one needs to show theoretical results.

Since we have a theoretical goal (to prove the existence of parabolic closures in some Garside groups), we will use this new approach. We will just use one kind of conjugation (that we call *swap*), and one finite set of elements (*recurrent elements* for swap).

Definition 4.11. Let G be a Garside group. We define the *left-swap* function (or just the swap function) to be the map ϕ which sends $x = a^{-1}b$ (written as a reduced left-fraction) to $\phi(x) = ba^{-1}$.

Notice that $\phi(x) = axa^{-1} = bxb^{-1}$, so $\phi(x)$ is conjugate to x.

Definition 4.12. Given $x \in G$, we say that x is recurrent for swap (or just recurrent) if $\phi^m(x) = x$ for some $m > 0$.

One could think of ϕ as a conjugation that simplifies elements, and think of recurrent elements as those which are as *simplified* as possible. It is important to see that every element can be conjugated to a recurrent element using swaps:

Proposition 4.13. *For every $x \in G$, there are integers $0 \leq m < n$ such that $\phi^m(x) = \phi^n(x)$. The elements in the set $\{\phi^m(x), \dots, \phi^{n-1}(x)\}$ are all recurrent, and the set will be called a circuit for swap.*

Proof. The second sentence in the statement is trivial, so we just need to show the first one.

Let $x = a^{-1}b$ be the reduced left-fraction decomposition of x. This reduced left-fraction decomposition of x depends only on G^+ (which determines the lattice order) and is independent of the Garside element we take. Hence, to simplify the arguments we will work with a Garside structure (G, G^+, Δ^N), with N big enough so that a and b are simple elements. To avoid confusion, we will denote $\mathbf{\Delta} = \Delta^N$.

We have $aa' = \mathbf{\Delta}$ and $bb' = \mathbf{\Delta}$ for some simple elements a' and b'. If we denote τ the conjugation by $\mathbf{\Delta}$, we have:

$$\phi(x) = ba^{-1} = ba'\mathbf{\Delta}^{-1} = \mathbf{\Delta}^{-1}\tau^{-1}(b)\tau^{-1}(a') = \left(\tau^{-1}(b')\right)^{-1}\tau^{-1}(a').$$

Since conjugation by $\mathbf{\Delta}$ preserves the Garside structure, it follows that $\tau^{-1}(b')$ and $\tau^{-1}(a')$ are simple elements. In other words, there are simple elements c and d such that $\phi(x) = c^{-1}d$. By Proposition 4.2, if $\alpha = c \wedge d$ and we write $c = \alpha c_0$ and $d = \alpha d_0$ for some simple elements c_0, d_0, then $\phi(x) = c_0^{-1}d_0$ is the reduced left-fraction decomposition of $\phi(x)$. Therefore the left-numerator and the left-denominator of $\phi(x)$ are simple elements. Since the number of simple elements is finite, the sequence $\{\phi^i(x)\}_{i\geq 0}$ must become periodic. ☐

It is worth noticing that if x is positive then it is recurrent, as its left-denominator is trivial and then $\phi(x) = x$. The same happens if x is negative, as in this case the left-numerator of x is trivial and then $\phi(x) = x$.

The following result is just a simple observation:

Proposition 4.14. *Let (G, G^+, Δ) be a Garside structure, and let G_δ be a standard parabolic subgroup of G. If $x \in G_\delta$, there is $\alpha \in G_\delta$ such that x^α is recurrent.*

Proof. Let $x = a^{-1}b$ be the reduced left-fraction decomposition of x in G. We know that $a^{-1}b$ is also the reduced left-fraction decomposition of x in G_δ, hence $a, b \in G_\delta^+$. But then $\phi(x) = axa^{-1} \in G_\delta$.

Applying the same reasoning to each iterated swap of x, it follows that all elements in the sequence $\{\phi^i(x)\}_{i\geq 0}$ belong to G_δ, and one can take a conjugating element in G_δ when conjugating $\phi^i(x)$ to $\phi^{i+1}(x)$ for every $i \geq 0$. By Proposition 4.13 some element in the sequence is recurrent, so the result follows. ☐

Notice that, given a standard parabolic subgroup $G_\delta \subset G$, the definition of swap of an element in G_δ does not depend on the Garside structure (either (G, G^+, Δ) or$(G_\delta, G_\delta^+, \delta)$) that one is considering, since the left-fraction decompositions in G_δ and in G coincide. Hence, an element of G_δ is recurrent with respect to (G, G^+, Δ) if and only if it is recurrent with respect to $(G_\delta, G_\delta^+, \delta)$. We can then just say that the element is recurrent, without mentioning any Garside structure.

We finish this section by relating recurrent elements with positive (or negative) elements in a conjugacy class.

Lemma 4.15. *Let $x \in G$. If x is conjugate to a positive element, then $\phi^m(x) \in G^+$ for some $m \geq 0$.*

Proof. For those familiar with Garside theory, we will parallel the usual proof showing that iterated cycling takes x to a positive element [21, 4].

Let $x = a^{-1}b$ be the reduced left-fraction decomposition of x. We will use the Garside structure (G, G^+, Δ^N) such that a and b are simple elements. Let us denote $\mathbf{\Delta} = \Delta^N$ to avoid confusion. Let a' be the simple element such that $a'a = \mathbf{\Delta}$. We have $x = \mathbf{\Delta}^{-1}a'b$. Recall that $a \wedge b = 1$, hence $\mathbf{\Delta} \wedge a'b = a'a \wedge a'b = a'(a \wedge b) = a'$. So $a'b$ is a left-weighted decomposition.

We are assuming that x is conjugate to a positive element, so there exists $c \in G$ such that $cxc^{-1} \in G^+$. By Proposition 4.3 we can assume that c is positive, multiplying it by a suitable central element if necessary. We then have $c, d \in G^+$ such that $cxc^{-1} = d$. Hence $c\mathbf{\Delta}^{-1}a'bc^{-1} = d$, and therefore $c\mathbf{\Delta}^{-1}a'b = dc \in G^+$. If we denote $\tilde{c} = \mathbf{\Delta}c\mathbf{\Delta}^{-1} \in G^+$, we finally obtain

$$\mathbf{\Delta}^{-1}\tilde{c}a'b \in G^+.$$

This means $1 \preccurlyeq \mathbf{\Delta}^{-1}\tilde{c}a'b$, hence $\mathbf{\Delta} \preccurlyeq \tilde{c}a'b$. Now let us see (a usual procedure in Garside theory) that we can remove b from the above expression, as $a'b$ is left weighted: We have that $\mathbf{\Delta} \preccurlyeq \tilde{c}\mathbf{\Delta}$ (as $\mathbf{\Delta}^{-1}\tilde{c}\mathbf{\Delta}$ is positive). Hence

$$\mathbf{\Delta} \preccurlyeq (\tilde{c}\mathbf{\Delta}) \wedge (\tilde{c}a'b) = \tilde{c}(\mathbf{\Delta} \wedge a'b) = \tilde{c}a',$$

the last equality holding as $a'b$ is left-weighted. Now recall that $\mathbf{\Delta}$ is a prefix of an element if and only if it is a suffix of the same element (both assertions mean that the infimum of the element is at least 1). Hence $\tilde{c}a' \succcurlyeq \mathbf{\Delta}$. That is,

$$\mathbf{\Delta}c = \tilde{c}\mathbf{\Delta} = \tilde{c}a'a \succcurlyeq \mathbf{\Delta}a.$$

Conjugating by $\mathbf{\Delta}$ (which preserves the Garside structure), we get $c\mathbf{\Delta} \succcurlyeq a\mathbf{\Delta}$, and finally $c \succcurlyeq a$.

Therefore, we have shown that if c is a positive element such that cxc^{-1} is positive, then $c \succcurlyeq a$. We can then decompose $c = c_1a$, and we obtain $cxc^{-1} = c_1a(a^{-1}b)a^{-1}c_1^{-1} = c_1(ba^{-1})c_1^{-1} = c_1\phi(x)c_1^{-1}$, where $1 \preccurlyeq c_1 \prec c$.

If $\phi(x)$ is positive, we are done. Otherwise, we repeat the process and we will find $1 \preccurlyeq c_2 \prec c_1 \prec c$ such that $c_2\phi^2(x)c_2^{-1}$ is positive, so we can repeat the process again with $\phi^2(x)$. This process must stop, as it cannot exist an infinite descending chain of positive elements in G, hence $\phi^m(x) \in G^+$ for some $m \geq 0$, as we wanted to show. □

Remark 4.16. It is interesting to point out a consequence of the above proof: If x is conjugate to a positive element, and $\phi^i(x) = a_i^{-1}b_i$ is the reduced left-fraction decomposition of $\phi^i(x)$ for each $i \geq 0$, then $c = a_{m-1}\cdots a_1a_0$ is the minimal positive element, with respect to $\succcurlyeq$ such that cxc^{-1} is positive (provided m is the smallest nonnegative integer such that $\phi^m(x)$ is positive). In other words, iterated swaps conjugate x to a positive element in the fastest possible way (conjugating by positive elements on the left). If one prefers to use $\preccurlyeq$ and to conjugate by positive elements on the right, one can parallel the same arguments just by using reduced right-fractions and right-swaps, which are defined in the symmetric way.

We can now characterize the sets of recurrent elements in a conjugacy class, in two particular cases. Given $x \in G$, let $\mathsf{R}(x)$ be the set of recurrent elements conjugate to x, which coincides with the set of circuits for swap in the conjugacy class of x. Let $C^+(x)$ be the set of positive elements conjugate to x and let $C^-(x)$ be the set of negative elements conjugate to x (any of the two latter sets could be empty).

Proposition 4.17. *Let (G, G^+, Δ) be a Garside structure and let $x \in G$. One has:*

(1) If x is conjugate to a positive element, then $\mathsf{R}(x) = C^+(x)$.

(2) If x is conjugate to a negative element, then $\mathsf{R}(x) = C^-(x) = \big(C^+(x^{-1})\big)^{-1}$.

Proof. (1) Suppose that x is conjugate to a positive element. If $y \in C^+(x)$ then y is conjugate to x and $\phi(y) = y$, hence $y \in \mathsf{R}(x)$. Conversely, let $y \in \mathsf{R}(x)$. Since y is conjugate to x, it is conjugate to a positive element. By Lemma 4.15, $\phi^m(y)$ is positive for some $m \geq 0$, and then $\phi^{m+k}(y) = \phi^m(y)$ for every $k \geq 0$. But y is recurrent, so $\phi^n(y) = y$ for some $n > 0$, and this implies that $y = \phi^{nm}(y) = \phi^m(y)$. Hence y is positive, and the orbit of y under ϕ consists of just one element, namely y.

(2) If x is conjugate to a negative element, x^{-1} is conjugate to a positive element, hence $\mathsf{R}(x^{-1}) = C^+(x^{-1})$ by the previous property. Now recall that for every $z \in G$ with reduced left-fraction decomposition $z = c^{-1}d$, the reduced left-fraction decomposition of z^{-1} is $d^{-1}c$. Hence $\phi(z^{-1}) = cd^{-1} = (dc^{-1})^{-1} = \phi(z)^{-1}$. Therefore, taking inverses commutes with ϕ and also preserves conjugations and conjugating elements. It follows that $\mathsf{R}(x^{-1}) = \mathsf{R}(x)^{-1}$, that $\big(C^-(x)\big)^{-1} = C^+(x^{-1})$, and then $\mathsf{R}(x) = \big(\mathsf{R}(x^{-1})\big)^{-1} = \big(C^+(x^{-1})\big)^{-1} = C^-(x)$.

□

4.5. **Transport for swap and convexity.** Let (G, G^+, Δ) be a Garside structure and let $x \in G$. In this section we will explain some important properties of $\mathsf{R}(x)$. Unfortunately, we do not know whether the set $\mathsf{R}(x)$ is always finite, even in the case in which it consists only of the positive conjugates of x. In this later case, the set is finite if G has homogenous relations, like in braid groups or, more generally, in Artin-Tits groups of spherical type. But in a general Garside group this set could a priori be infinite.

In any case, we will see that $\mathsf{R}(x)$ satisfies the same properties as similar sets, which are usually introduced to solve the conjugacy problem like super summit sets [21], ultra summit sets [26], or sets of sliding circuits [27, 28]. One of the basic properties is that their elements are connected through conjugations by simple elements. Although this can be shown by comparing the set $\mathsf{R}(x)$

with other sets like the ultra summit set of x (see [26]), we will proceed to show all details avoiding the use of Garside normal forms.

We start with a very basic property.

Lemma 4.18. *Let (G, G^+, Δ) be a Garside structure and let $x \in G$. If $y \in \mathsf{R}(x)$ then $y^\Delta \in \mathsf{R}(x)$.*

Proof. Since conjugation by Δ preserves the lattice structure of G, it follows that if the reduced left-fraction decomposition of some element z is $a^{-1}b$ then the reduced left-fraction decomposition of z^Δ is $(a^\Delta)^{-1}(b^\Delta)$. Hence, $\phi(z^\Delta) = (b^\Delta)(a^\Delta)^{-1} = (ba^{-1})^\Delta = \phi(z)^\Delta$. That is, applying a left swap commutes with conjugation by Δ.

Given $y \in \mathsf{R}(x)$, there is some $m > 0$ such that $\phi^m(y) = y$. By the argument in the previous paragraph, we have $\phi^m(y^\Delta) = (\phi^m(y))^\Delta = y^\Delta$. Therefore $y^\Delta \in \mathsf{R}(x)$. □

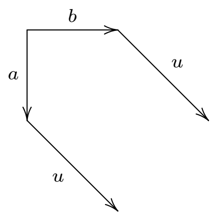
We will consider positive conjugates joining elements of G. One important concept, taken from [26], is the transport of a conjugating element.

Proposition 4.19. *Let (G, G^+, Δ) be a Garside structure and let $y, z \in G$ which are conjugate by a positive element $u \in G^+$, that is, $y^u = z$. Consider the reduced left-fraction decompositions $y = a^{-1}b$ and $z = c^{-1}d$. Then we define the transport of u at y as $u^{(1)} = auc^{-1}$, and the following conditions hold:*

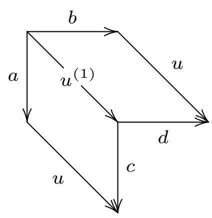
(1) $u^{(1)} = auc^{-1} = bud^{-1} = au \wedge bu$.
(2) $\phi(y)^{\left(u^{(1)}\right)} = \phi(z)$.

Proof. We know that $y^u = z$, hence $u^{-1}(a^{-1}b)u = c^{-1}d$, which is equivalent to $auc^{-1} = bud^{-1}$.

Now let us represent the positive elements au and bu:



Recall that z is precisely $(au)^{-1}(bu)$. From Proposition 4.2, cancelling the element $\alpha = au \wedge bu$ in the middle of the above expression, one obtains the reduced left-fraction decomposition of z, that is, $c^{-1}d$. In other words, $au = \alpha c$ and $bu = \alpha d$, which means that $au \wedge bu = \alpha = auc^{-1} = bud^{-1} = u^{(1)}$:



This shows the first condition.

The second condition is an easy observation. Since $u^{(1)} = auc^{-1} = bud^{-1}$, one has:

$$\phi(y)^{\left(u^{(1)}\right)} = (ba^{-1})^{\left(u^{(1)}\right)} = (bud^{-1})^{-1}(ba^{-1})(auc^{-1}) = dc^{-1} = \phi(z).$$

Hence, if u conjugates y to z, then $u^{(1)}$ conjugates $\phi(y)$ to $\phi(z)$. □

The transport satisfies very useful properties:

Proposition 4.20. *Let (G, G^+, Δ) be a Garside structure and let $y \in G$. Suppose that $u, v \in G^+$ are positive elements and let $u^{(1)}$, $v^{(1)}$, $(u \wedge v)^{(1)}$ and $(\Delta^m)^{(1)}$ be the transports of u, v, $u \wedge v$ and Δ^m at y, respectively, for $m \geq 0$. Then one has:*

(1) $(\Delta^m)^{(1)} = \Delta^m$ for every $m \geq 0$.
(2) $(u \wedge v)^{(1)} = u^{(1)} \wedge v^{(1)}$.
(3) If $u \preccurlyeq v$ then $u^{(1)} \preccurlyeq v^{(1)}$.

Proof. Let $a^{-1}b$ be the reduced left-fraction decomposition of y. We know from Proposition 4.19 that for every positive element $w \in G^+$, its transport at y is $w^{(1)} = aw \wedge bw$. Then the first condition is shown as follows:

$$(\Delta^m)^{(1)} = a\Delta^m \wedge b\Delta^m = \Delta^m a^{\Delta^m} \wedge \Delta^m b^{\Delta^m} = \Delta^m(a^{\Delta^m} \wedge b^{\Delta^m}) = \Delta^m(a \wedge b)^{\Delta^m} = \Delta^m,$$

where we have used that conjugation by Δ preserves the lattice structure, and that $a \wedge b = 1$.

The second condition also follows easily from the definition of transport:

$$(u \wedge v)^{(1)} = a(u \wedge v) \wedge b(u \wedge v) = au \wedge av \wedge bu \wedge bv = (au \wedge bu) \wedge (av \wedge bv) = u^{(1)} \wedge v^{(1)}.$$

Finally, $u \preccurlyeq v$ if and only if $u \wedge v = u$. If this is the case, $u^{(1)} \wedge v^{(1)} = (u \wedge v)^{(1)} = u^{(1)}$, which is equivalent to $u^{(1)} \preccurlyeq v^{(1)}$. This shows the third condition. □

Notice that one can iterate the transport map, provided each transport is performed successively at $y, \phi(y), \phi^2(y), \ldots$ As usual, one denotes $u^{(2)} = (u^{(1)})^{(1)}$ and, in general, $u^{(k)} = (u^{(k-1)})^{(1)}$ for $k \geq 2$. Let us see that the transport map behaves well when applied to conjugating elements between recurrent elements.

Lemma 4.21. *Let (G, G^+, Δ) be a Garside structure and let $x \in G$. Given two elements $y, z \in \mathsf{R}(x)$ and a positive element $u \in G^+$ such that $y^u = z$, then there exists $k > 0$ such that $\phi^k(y) = y$ and $u^{(k)} = u$.*

Proof. Let $m = \sup(u)$ and recall that we have $1 \preccurlyeq u \preccurlyeq \Delta^m$. By Proposition 4.20 we have $1 \preccurlyeq u^{(k)} \preccurlyeq \Delta^m$ for every $k > 0$. This implies that the set of iterated transports of u is finite.

Now let p, q be positive integers such that $\phi^p(y) = y$ and $\phi^q(z) = z$. These integers exist as y and z are left recurrent elements. Taking $n = \mathrm{lcm}(p, q)$ we have $\phi^n(y) = y$ and $\phi^n(z) = z$. Since the element $u^{(k)}$ conjugates $\phi^k(y)$ to $\phi^k(z)$ for every $z \geq 0$, it follows that $u^{(in)}$ conjugates y to z, for every $i \geq 0$. Since the set $\{u^{(in)}\}_{i \geq 0}$ is finite, there are some $0 \leq i_1 < i_2$ such that $u^{(i_1 n)} = u^{(i_2 n)}$. We take i_1 as small as possible.

Suppose that $i_1 > 0$ and notice that, by definition, the transport at a given element is an injective map. This implies, taking the preimages of $u^{(i_1 n)} = u^{(i_2 n)}$ under transport (based at the elements in the orbit of y) n times, that $u^{((i_1 - 1)n)} = u^{((i_2 - 1)n)}$. This contradicts the minimality of i_1. Hence $i_1 = 0$ and then $u = u^{(k)}$ where $k = i_2 n > 0$, as we wanted to show. □

We can then show a very important property, satisfied by all sets which are used in general to solve the conjugacy problem in Garside groups:

Proposition 4.22. *Let (G, G^+, Δ) be a Garside structure and let $x \in G$. Given $y \in \mathsf{R}(x)$, if $\alpha, \beta \in G$ are such that $y^\alpha, y^\beta \in \mathsf{R}(x)$, then $y^{\alpha \wedge \beta} \in \mathsf{R}(x)$.*

Proof. Suppose first that α and β are positive. By Lemma 4.21 we know that there are $k, m \geq 0$ such that $\phi^k(y) = y$ and $\alpha^{(k)} = \alpha$, and $\phi^m(y) = y$ and $\beta^{(m)} = \beta$. Taking N to be a multiple of k and m, it follows that $\alpha^{(N)} = \alpha$ and $\beta^{(N)} = \beta$, and both are conjugating elements starting at y. By Proposition 4.20, $(\alpha \wedge \beta)^{(N)} = \alpha^{(N)} \wedge \beta^{(N)} = \alpha \wedge \beta$. Since $(\alpha \wedge \beta)^{(N)}$ conjugates y to $\phi^N(y^{\alpha\wedge\beta})$ by definition of transport, and $\alpha \wedge \beta$ obviously conjugates y to $y^{\alpha\wedge\beta}$, it follows that $\phi^N(y^{\alpha\wedge\beta}) = y^{\alpha\wedge\beta}$, that is, $y^{\alpha\wedge\beta} \in \mathsf{R}(x)$, as we wanted to show.

Now, if α and β are arbitrary elements, let $M > 0$ such that Δ^M is central, and $\Delta^M\alpha$ and $\Delta^M\beta$ are positive. Then $y^{\Delta^M\alpha} = y^\alpha \in \mathsf{R}(x)$ and $y^{\Delta^M\beta} = y^\beta \in \mathsf{R}(x)$. Hence, by the above paragraph, $y^{\Delta^M\alpha\wedge\Delta^M\beta} \in \mathsf{R}(x)$. The proof finishes by noticing that $y^{\Delta^M\alpha\wedge\Delta^M\beta} = y^{\Delta^M(\alpha\wedge\beta)} = y^{\alpha\wedge\beta}$. □

As usual, the above property allows us to show that the elements of $\mathsf{R}(x)$ are connected by simple conjugating elements.

Corollary 4.23. *Let (G, G^+, Δ) be a Garside structure and let $x \in G$. For every pair of distinct elements $y, z \in \mathsf{R}(x)$, there is a sequence of simple elements $u_1, \ldots, u_m \in S$ such that $y^{u_1\cdots u_k} \in \mathsf{R}(x)$ for $k = 1, \ldots, m$, and $y^{u_1\cdots u_m} = z$.*

Proof. Since $y, z \in \mathsf{R}(x)$, they are conjugate. Let $u \in G^+$ be a nontrivial positive element such that $y^u = z$. By Lemma 4.18, $y^\Delta \in \mathsf{R}(x)$. Hence, by Proposition 4.22, $y^{u\wedge\Delta} \in \mathsf{R}(x)$.

Let us denote $u_1 = u \wedge \Delta$. Since u is nontrivial and the set of atoms generate G^+ as a monoid, there must exist an atom $a \in \mathcal{A}$ such that $a \preccurlyeq u$. Since every atom is a prefix of Δ, it follows that $a \preccurlyeq u_1$, hence u_1 is nontrivial. We can then write $u = u_1 v_1$ for some $v_1 \in G^+$. Then v_1 is a positive element which conjugates $y^{u_1} \in \mathsf{R}(x)$ to $z \in \mathsf{R}(x)$. If v_1 is nontrivial, we can apply the same reasoning to find a nontrivial simple element $u_2 \preccurlyeq v_1$ such that $y^{u_1 u_2} \in \mathsf{R}(x)$. Then $u = u_1 u_2 v_2$ for some $v_2 \in G_+$.

We can continue this process, but we can make at most $||u||$ steps before we obtain that v_k is trivial, as u cannot be decomposed in more than $||u||$ nontrivial positive elements. So there is some $m \leq ||u||$ such that $u = u_1 \cdots u_m$, where $y^{u_1\cdots u_k} \in \mathsf{R}(x)$ for $k = 1, \dots, m$ by construction, and $y^{u_1\cdots u_m} = y^u = z$. $\square$

The above result can be used to compute elements in $\mathsf{R}(x)$ starting with a single element $y \in \mathsf{R}(x)$. One just needs to conjugate y by all simple elements (a finite set), and keep the new conjugates of y which belong to $\mathsf{R}(x)$. Corollary 4.23 implies that every element in $\mathsf{R}(x)$ can be obtained in this way. The problem is that we are not sure whether $\mathsf{R}(x)$ is finite or not, so the process may never terminate.

To avoid the above problem, we can restrict our attention to finite subsets of $\mathsf{R}(x)$, which also satisfy the property analogous to Proposition 4.22, and therefore it will be possible to compute them completely using the above procedure.

Definition 4.24. Let (G, G^+, Δ) be a Garside structure and let $x \in G$. For every $m > 0$, let $\mathsf{R}^m(x)$ be the following subset of the conjugacy class:

$$\mathsf{R}^m(x) = \{y \in \mathsf{R}(x);\ sup(N_L(y)) \leq m \text{ and } \sup(D_L(y)) \leq m\}.$$

It is clear that $\mathsf{R}^m(x)$ is finite, and that it is nonempty for some $m > 0$. One can also show that $\mathsf{R}^m(x)$ satisfies the property analogous to Proposition 4.22, and then one can solve the conjugacy problem in G using this set. But this is not the goal of this paper, and the solution is not computationally better than the one using sets of sliding circuits [27, 28].

4.6. **Support of an element and parabolic closure.** In this section we will show that, if (G, G^+, Δ) is an LCM-Garside structure, and furthermore it is what we will call *support-preserving*, then every element $x \in G$ admits a parabolic closure, that is, a unique parabolic subgroup $\mathrm{PC}_G(x)$ which is minimal (with respect to inclusion) among all parabolic subgroups of G containing x. We will sometimes write $\mathrm{PC}(x)$ instead of $\mathrm{PC}_G(x)$ if the group G is clear by the context.

For this purpose we need to generalize the notion of *support* to the case of an arbitrary element of G, not necessarily a balanced, positive element.

Definition 4.25. Let (G, G^+, Δ) be an LCM-Garside structure. Let $x \in G$ whose reduced left-fraction decomposition is $x = a^{-1}b$. Write $a = x_1 \cdots x_r$ and $b = x_{r+1} \cdots x_m$ as products of (not necessarily distinct) atoms. If we denote $X = \{x_1, \dots, x_m\}$, we define the support of x as $\mathrm{Supp}(x) = \overline{X}$.

Notice that, if x is a positive balanced element, this definition coincides with the one given in Definition 4.4.

There are some Garside groups, for instance Artin groups of spherical type, in which all words representing a positive element involve the same set of atoms. In those cases, the support of $x = a^{-1}b$ (where this is its reduced left-fraction decomposition) is the set of atoms which appear in the word $v^{-1}w$, where v is any representative of a and w is any representative of b. But some other Garside groups, like the braid group of $G(e, e, n)$, do not satisfy the mentioned property, as we will later see. This forces us to show the following:

Proposition 4.26. *Let (G, G^+, Δ) be an LCM-Garside structure. The support of an element $x \in G$ is well defined.*

Proof. Consider first a positive element $a \in G^+$. We will show that if one can write $a = x_1 \cdots x_r = y_1 \cdots y_s$ with $X = \{x_1, \dots, x_r\} \subset \mathcal{A}$ and $Y = \{y_1, \dots, y_s\} \subset \mathcal{A}$, then $\overline{X} = \overline{Y}$.

Since (G, G^+, Δ) is an LCM-Garside structure, a is also a positive element in the standard parabolic subgroups $G_X = G_{\overline{X}}$ and $G_Y = G_{\overline{Y}}$. This means that $x_1 \cdots x_r$ is positive in $G_{\overline{Y}}$. Since $G_{\overline{Y}}^+$ is closed under suffixes and prefixes, it follows that $x_i \cdots x_r \in G_{\overline{Y}}^+$, and then $x_i \in G_{\overline{Y}}^+$, for $i = 1, \ldots, r$. Since the atoms in $G_{\overline{Y}}$ are precisely the elements in $\overline{Y}$, we have shown that $X \subset \overline{Y}$, and then $\overline{X} \subset \overline{Y}$. Exchanging the roles of X and Y we get $\overline{Y} \subset \overline{X}$, thus $\overline{X} = \overline{Y}$, as we wanted to show. So the support is well defined for positive elements.

Let now $x \in G$ be an arbitrary element whose reduced left-fraction decomposition is $x = a^{-1}b$. Write $a = x_1 \cdots x_r$ and $b = x_{r+1} \cdots x_m$ as product of atoms, and let $X_1 = \{x_1, \ldots, x_r\}$ and $X_2 = \{x_{r+1}, \ldots, x_m\}$. Suppose that we can also write $a = y_1 \cdots y_k$ and $b = y_{k+1} \cdots y_l$, and denote $Y_1 = \{y_1, \ldots, y_k\}$ and $Y_2 = \{y_{k+1}, \ldots, y_l\}$. We have already shown that $\overline{X_1} = \overline{Y_1}$, and that $\overline{X_2} = \overline{Y_2}$.

Notice that $\Delta_Z = \Delta_{\overline{Z}}$ for every set Z of atoms. Also, by definition, $\Delta_{Z_1 \cup Z_2} = \Delta_{Z_1} \vee \Delta_{Z_2}$ for all sets of atoms Z_1 and Z_2. Hence:

$$\Delta_{X_1 \cup X_2} = \Delta_{X_1} \vee \Delta_{X_2} = \Delta_{\overline{X_1}} \vee \Delta_{\overline{X_2}} = \Delta_{\overline{Y_1}} \vee \Delta_{\overline{Y_2}} = \Delta_{Y_1} \vee \Delta_{Y_2} = \Delta_{Y_1 \cup Y_2}$$

Therefore:

$$\overline{X_1 \cup X_2} = \mathrm{Div}(\Delta_{X_1 \cup X_2}) \cap \mathcal{A} = \mathrm{Div}(\Delta_{Y_1 \cup Y_2}) \cap \mathcal{A} = \overline{Y_1 \cup Y_2}.$$

So the support is well defined for every element in G. □

Now we introduce another property which we will require for a Garside group G to satisfy.

Definition 4.27. Let (G, G^+, Δ) be an LCM-Garside structure. We say that this structure is support-preserving if, for every pair of conjugate positive elements $y, z \in G^+$, and every $\alpha \in G$ such that $y^\alpha = z$, one has:

$$\left(G_{\mathrm{Supp}(y)}\right)^\alpha = G_{\mathrm{Supp}(z)}.$$

Proposition 4.28. *Let (G, G^+, Δ) be a support-preserving LCM-Garside structure, and let $x \in G$. For every $y, z \in \mathsf{R}(x)$ and every $\alpha \in G$ such that $y^\alpha = z$, one has:*

$$\left(G_{\mathrm{Supp}(y)}\right)^\alpha = G_{\mathrm{Supp}(z)}.$$

Proof. If x is conjugate to a positive element, then $y, z \in \mathsf{R}(x) = C^+(x)$ by Proposition 4.17. Hence the result holds as (G, G^+, Δ) is support-preserving.

If x is conjugate to a negative element, then $y^{-1}, z^{-1} \in \mathsf{R}(x)^{-1} = C^+(x^{-1})$ by Proposition 4.17. Hence $\left(G_{\mathrm{Supp}(y^{-1})}\right)^\alpha = G_{\mathrm{Supp}(z^{-1})}$. Now notice that $\mathrm{Supp}(\beta^{-1}) = \mathrm{Supp}(\beta)$ for every $\beta \in G$, hence $\left(G_{\mathrm{Supp}(y)}\right)^\alpha = G_{\mathrm{Supp}(z)}$.

Finally, suppose that x is conjugate to neither a positive nor a negative element. For every $i \geq 0$, let $\phi^i(y) = a_i^{-1}b_i$ be the reduced left-fraction decomposition of $\phi^i(y)$, and let $\phi^i(z) = c_i^{-1}d_i$ be the reduced left-fraction decomposition of $\phi^i(z)$.

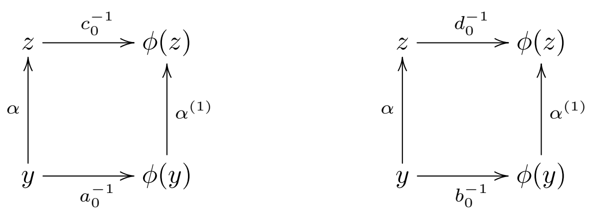
We have $y^\alpha = z$. By Proposition 4.3 we can assume that α is positive, multiplying it by a central element if necessary. We then have

$$(a_0\alpha)^{-1}(b_0\alpha) = \alpha^{-1}y\alpha = z = c_0^{-1}d_0.$$

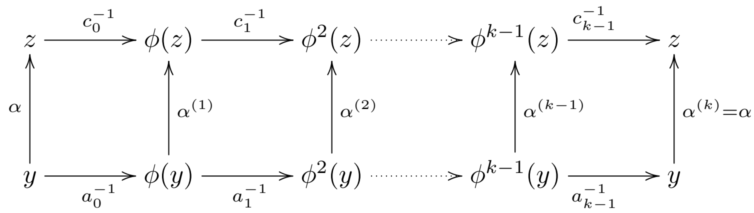
Consider the transport

$$\alpha^{(1)} = a_0\alpha \wedge b_0\alpha = a_0\alpha c_0^{-1} = b_0\alpha d_0^{-1}.$$

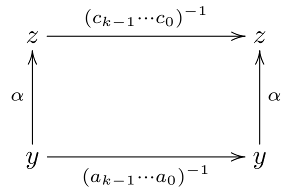
Notice that we have the commutative diagrams of conjugations:



Now recall that both y and z are recurrent, so there exists some $k > 0$ such that $\phi^k(y) = y$, $\phi^k(z) = z$ and $\alpha^{(k)} = \alpha$ by Lemma 4.21. We obtain the following commutative diagram of conjugations:



Simplifying the diagram, we have:



Let us denote $g_1 = a_{k-1}\cdots a_0$ and $h_1 = c_{k-1}\cdots c_0$. Both elements are positive, and we have $\alpha = g_1\alpha h_1^{-1}$.

Now notice that we also had $\alpha^{(1)} = b_0\alpha d_0^{-1}$. Repeating the above arguments, if we define $g_2 = b_{k-1}\cdots b_0$ and $h_2 = d_{k-1}\cdots d_0$, we have $\alpha = \alpha^{(k)} = g_2\alpha h_2^{-1}$. Therefore $\alpha = g_1 g_2 \alpha h_2^{-1} h_1^{-1}$. That is:

$$(g_1 g_2)^\alpha = h_1 h_2.$$

Since g_1g_2 and h_1h_2 are positive and (G, G^+, Δ) is support-preserving, it follows that

$$\left(G_{\mathrm{Supp}(g_1g_2)}\right)^\alpha = G_{\mathrm{Supp}(h_1h_2)}.$$

The proof will then finish by showing that $\mathrm{Supp}(g_1g_2) = \mathrm{Supp}(y)$ and that $\mathrm{Supp}(h_1h_2) = \mathrm{Supp}(z)$.

Recall that $g_1g_2 = a_{k-1}\cdots a_0 b_{k-1}\cdots b_0$ where all factors in this expression are positive elements. Hence $\mathrm{Supp}(g_1g_2) \supset \overline{\mathrm{Supp}(a_0) \cup \mathrm{Supp}(b_0)} = \mathrm{Supp}(y)$. On the other hand, since $y \in G_{\mathrm{Supp}(y)}$ and (G, G^+, Δ) is an LCM-Garside structure, it follows from Proposition 4.14 that all elements in $\{\phi^i(y)\}_{i\geq 0}$ belong to $G_{\mathrm{Supp}(y)}$. Hence all positive elements $a_{k-1}, \ldots, a_0, b_{k-1}, \ldots, b_0$ belong to $G_{\mathrm{Supp}(y)}$. Therefore $\mathrm{Supp}(g_1g_2) \subset \mathrm{Supp}(y)$, and hence $\mathrm{Supp}(g_1g_2) = \mathrm{Supp}(y)$.

The same argument shows that $\mathrm{Supp}(h_1h_2) = \mathrm{Supp}(z)$, and this finally implies that $\left(G_{\mathrm{Supp}(y)}\right)^\alpha = G_{\mathrm{Supp}(z)}$, as we wanted to show. □

We recall that the parabolic closure of an element is the unique minimal (for inclusion) parabolic subgroup containing it. We want to show that such a parabolic closure exists for every element in any of the groups we are interested in. Let us first show this for recurrent elements.

Theorem 4.29. *If (G, G^+, Δ) is a support-preserving LCM-Garside structure, then every recurrent element $x \in G$ admits a parabolic closure, namely* $\mathrm{PC}(x) = G_{\mathrm{Supp}(x)}$.

Proof. Let x be a recurrent element, and let $X = \mathrm{Supp}(x)$. We claim that G_X is the parabolic closure of x. It is clear that G_X is parabolic and that $x \in G_X$, hence we only need to show its minimality with respect to inclusion.

Let H be a parabolic subgroup such that $x \in H$. We can assume that $H \neq G$, otherwise it is clear that $G_X \subset H$. Since H is parabolic, it is conjugate by some element $\alpha \in G$ to a proper standard parabolic subgroup G_Y, where $\overline{Y} = Y$. It follows that $x^\alpha \in H^\alpha = G_Y$. Since (G, G^+, Δ) is an LCM-Garside structure, Proposition 4.14 tells us that $(x^\alpha)^\beta$ is recurrent for some $\beta \in G_Y$. Since $x^\alpha \in G_Y$ it follows that $x^{\alpha\beta} \in G_Y$. If its reduced left-fraction decomposition is $x^{\alpha\beta} = a^{-1}b$, we have that $a, b \in G_Y$, so one can write a and b using atoms from Y. Since Y is a saturated set of atoms, it follows that $\mathrm{Supp}(x^{\alpha\beta}) \subset Y$. Hence $G_{\mathrm{Supp}(x^{\alpha\beta})} \subset G_Y$.

Now recall that (G, G^+, Δ) is support-preserving, and that x and $x^{\alpha\beta}$ are recurrent. Hence, by Proposition 4.28:

$$(G_X)^{\alpha\beta} = (G_{\mathrm{Supp}(x)})^{\alpha\beta} = G_{\mathrm{Supp}(x^{\alpha\beta})} \subset G_Y.$$

Therefore:

$$G_X \subset (G_Y)^{\beta^{-1}\alpha^{-1}} = (G_Y)^{\alpha^{-1}} = H.$$

This shows the minimality of G_X, hence $G_X = G_{\mathrm{Supp}(x)}$ is the parabolic closure of x. □

Now the existence of the parabolic closure of an arbitrary element will be a consequence of the following result:

Lemma 4.30. *Let $x, c \in G$. If x admits a parabolic closure, so does its conjugate x^c. Namely,* $\mathrm{PC}(x^c) = \mathrm{PC}(x)^c$.

Proof. Let $\mathrm{PC}(x)$ be the parabolic closure of x. We need to show that $\mathrm{PC}(x)^c$ is the parabolic closure of x^c.

First notice that $\mathrm{PC}(x)^c$ is parabolic, as it is the conjugate of a parabolic subgroup. Notice also that $x^c \in \mathrm{PC}(x)^c$, as $x \in \mathrm{PC}(x)$. Finally, suppose that H is a parabolic subgroup containing x^c. Then $x^c \in H$ implies $x \in H^{c^{-1}}$, where $H^{c^{-1}}$ is a parabolic subgroup (being conjugate to H). Hence, by minimality of $\mathrm{PC}(x)$ we obtain $\mathrm{PC}(x) \subset H^{c^{-1}}$, and then $\mathrm{PC}(x)^c \subset H$. Therefore $\mathrm{PC}(x)^c$ is the parabolic closure of x^c, that is, $\mathrm{PC}(x^c) = \mathrm{PC}(x)^c$. □

Theorem 4.31. *If (G, G^+, Δ) is a support-preserving LCM-Garside structure, then every element of G admits a parabolic closure.*

Proof. Let $x \in G$. Applying iterated swaps, one can conjugate x to a recurrent element y. By Theorem 4.29 y admits a parabolic closure, and by Lemma 4.30 so does x. □

The hypotheses of Theorem 4.31 are satisfied by some well-known Garside groups, namely Artin groups of spherical type, but the existence of parabolic closures in these groups was already shown in [14].

We can also extend the results in [14] concerning parabolic closures of powers of elements:

Theorem 4.32. *If (G, G^+, Δ) is a support-preserving LCM-Garside structure, given $x \in G$ and m a nonzero integer, the parabolic closures of x and x^m coincide.*

Proof. The proof is basically the same as that of Theorem 8.2 in [14], replacing $RSSS_\infty(x)$ with $\mathsf{R}(x)$.

The idea is to assume, up to conjugation, that $x \in \mathsf{R}(x)$. Then consider the pair (x, x^m). The conjugating element which applies a swap to x^m is the inverse of its left-denominator, which is precisely $x^m \wedge 1$. Since both x^m and 1 conjugate x to itself, it follows by Proposition 4.22 that $x^m \wedge 1$ conjugates x to an element in $\mathsf{R}(x)$. Hence we can conjugate the pair (x, x^m) by $x^m \wedge 1$, to apply ϕ to the second coordinate while keeping the first coordinate inside $\mathsf{R}(x)$. Iterating, we can assume that both x and x^m are recurrent elements.

Now, if x is either positive or negative, one checks immediately that $G_{\mathrm{Supp}(x)} = G_{\mathrm{Supp}(x^m)}$ is the parabolic closure of both x and x^m. If x is neither positive nor negative, one considers the Garside structure (G, G^+, Δ^N), with N big enough so that the left-numerator and the left-denominator of x are simple. In this case, the swap operation is equivalent to the twisted cycling operation, implying that x belongs to its ultra summit set. Then one can follow the arguments in [14, Theorem 8.2] to conclude that $G_{\mathrm{Supp}(x)} = G_{\mathrm{Supp}(x^m)}$ also in this case, hence the parabolic closures of x and x^m coincide. □

In the same way as it is done in [14] for Artin-Tits groups of spherical type, we can also conclude that all roots of an element in a parabolic subgroup belong to the parabolic subgroup:

Corollary 4.33. *Let (G, G^+, Δ) be a support-preserving LCM-Garside structure. If y belongs to a parabolic subgroup H, and $x \in G$ is such that $x^m = y$ for some nonzero integer m, then $x \in H$.*

Proof. This is the same proof as in [14, Corollary 8.3]. By Theorem 4.32, we have $\mathrm{PC}(x) = \mathrm{PC}(y)$. Since $y \in H$ and H is parabolic, it follows that $\mathrm{PC}(y) \subset H$. But then $x \in \mathrm{PC}(x) = \mathrm{PC}(y) \subset H$. □

Later we will show that other known Garside groups, apart from Artin-Tits groups of spherical type, also satisfy the hypotheses of Theorem 4.31 and Theorem 4.32. Before that, in the next section, we will provide the technical tools that will allow us to achieve this goal.

4.7. **Checking properties of a Garside structure.** In this section we will explain how one can check whether a Garside structure is LCM, and whether an LCM-Garside structure is support-preserving.

In order to show that a Garside structure is LCM, there are three properties to prove (recall Definition 4.7). The first two properties have a finite number of checkings (check whether $\Delta = \Delta_{\mathcal{A}}$ and, on the other hand, compute the elements of the form Δ_X for $X \subset \mathcal{A}$ and check whether their prefixes and their suffixes coincide). The third property can be verified thanks to a result from [30], which can be applied to the elements of the form $\delta = \Delta_X$ with $X \subset \mathcal{A}$.

Proposition 4.34. [30, Proposition 1.16] *Let (G, G^+, Δ) be a Garside structure, and let $\delta \in \mathrm{Div}(\Delta)$ be a balanced element. Then G_δ is a standard parabolic subgroup (hence $(G_\delta, G_\delta^+, \delta)$ is a Garside structure) if and only if, for every $x, y \in \mathrm{Div}(\delta)$, one has:*

$$xy \wedge \Delta \in \mathrm{Div}(\delta) \qquad \textit{and} \qquad xy \wedge^{\curvearrowleft} \Delta \in \mathrm{Div}(\delta).$$

Now let us assume that (G, G^+, Δ) is an LCM-Garside structure. We will explain how to determine whether the structure is support-preserving.

Given an element $x \in G$ which is conjugate to a positive element, one can compute the set $C^+(x)$ in two steps. First one finds one element $y \in C^+(x)$ by applying iterated swaps to x. Then, starting with y, one computes the directed graph $\mathcal{G}_{C^+(x)}$, defined as follows:

- The vertices of $\mathcal{G}_{C^+(x)}$ correspond to the elements of $C^+(x)$.
- There is an arrow labeled g with source u and target v if and only if g is non-trivial and positive, $u^g = v$ and $u^h \notin C^+(x)$ whenever $1 \prec h \prec g$.

The arrows in $\mathcal{G}_{C^+(x)}$ are called minimal positive conjugators. If the arrow starts at a vertex u, it is called a minimal positive conjugator for u. Since the graph $\mathcal{G}_{C^+(x)}$ is connected, one can compute the whole graph, starting with a single element $y \in C^+(x)$, provided that one knows how to compute the minimal positive conjugators for any given element.

A crucial property of these conjugating elements is that we just need to check that they are support-preserving, in order to show that an LCM-Garside structure is support-preserving:

Proposition 4.35. *An LCM-Garside structure (G, G^+, Δ) is support-preserving if and only if for every positive element $x \in G^+$, and every minimal positive conjugator c for x in $\mathcal{G}_{C^+(x)}$, one has:*

$$\left(G_{\mathrm{Supp}(x)}\right)^c = G_{\mathrm{Supp}(x^c)}.$$

Proof. The only if statement is trivial, so let us assume that the support in G is preserved under conjugation by minimal positive conjugators. Let x, y be a pair of conjugate positive elements and let $\alpha \in G$ such that $x^\alpha = y$.

If α is not positive, we know from Proposition 4.3 that $\beta\alpha$ is positive for some central element $\beta \in G$. Then

$$\left(G_{\mathrm{Supp}(x)}\right)^\alpha = \left(\left(G_{\mathrm{Supp}(x)}\right)^{\beta^{-1}}\right)^{\beta\alpha} = \left(G_{\mathrm{Supp}(x)}\right)^{\beta\alpha},$$

and $x^{\beta\alpha} = x^\alpha = y$. Hence, replacing α with $\beta\alpha$ if necessary, we can assume that α is positive.

Now we claim that every positive element α such that x^α is positive can be decomposed as a product of minimal positive conjugators, meaning that there is a path in $\mathcal{G}_{C^+(x)}$, starting at x, corresponding to a sequence of minimal positive conjugators whose product is α.

If α is either trivial or a minimal positive conjugator, the claim clearly holds. Otherwise, since α is not minimal, it can be decomposed as $\alpha = a_1 b_1$, where a_1 and b_1 are positive (so $1 \prec a_1 \prec \alpha$) and $x^{a_1} \in C^+(x)$. If a_1 is not a minimal positive conjugator, we keep going and find a_2 such that $1 \prec a_2 \prec a_1 \prec \alpha$ and $x^{a_2} \in C^+(x)$. Since we cannot have an infinite descending chain of positive elements in a Garside group, this process must stop, and we will obtain a minimal positive conjugator α_1 for x, with $\alpha = \alpha_1\beta_1$ and $\beta_1 \in G^+$. If β_1 is not trivial, we apply the same reasoning to β_1 and find a minimal positive conjugator α_2 for x^{α_1}, such that $\alpha = \alpha_1\alpha_2\beta_2$ and $\beta_2 \in G^+$. This process must also terminate, since a positive element cannot be decomposed as an arbitrarily large product of positive elements. So some β_r will be trivial, and $\alpha = \alpha_1 \cdots \alpha_r$ will be a product of minimal positive conjugators, starting at x. This shows the claim.

Finally, since we are assuming that the support is preserved under conjugation by minimal positive conjugators, it follows that the support will be preserved by α:

$$\left(G_{\mathrm{Supp}(x)}\right)^{\alpha} = \left(G_{\mathrm{Supp}(x)}\right)^{\alpha_1\alpha_2\cdots\alpha_r} = \left(G_{\mathrm{Supp}(x^{\alpha_1})}\right)^{\alpha_2\cdots\alpha_r} = \cdots = G_{\mathrm{Supp}(x^{\alpha_1\cdots\alpha_r})} = G_{\mathrm{Supp}(y)}$$

□

We will therefore be interested in describing minimal simple elements in detail. Given $x \in G^+$, we want to compute the arrows of the graph $\mathcal{G}_{C^+(x)}$ starting at the vertex x. This is explained in [23], and we will also describe here how to do it, as we will perform this computation in the following subsections.

Let us start by recalling the following result, which is a particular case of Proposition 4.22, and also holds for other sets like super summit sets, ultra summit sets, sliding circuits set and the like, although we will only use it in this case.

Lemma 4.36. [23, Proposition 4.8] *Let G be a Garside group, let $x \in G^+$. For every $u, v \in G$, if $x^u, x^v \in C^+(x)$ then $x^{u\wedge v} \in C^+(x)$.*

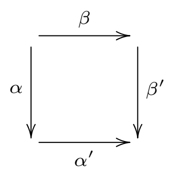
Now recall that $x \in G^+$, and that we want to compute the minimal positive conjugators for x. Take an atom $a \in G$, that is, a positive element which admits no nontrivial positive prefix. The set

$$\mathcal{C}_a(x) = \{\alpha \in G;\ a \preccurlyeq \alpha \text{ and } x^\alpha \in C^+(x)\} = \{\alpha \in G;\ a \preccurlyeq \alpha \text{ and } x^\alpha \in G^+\}$$

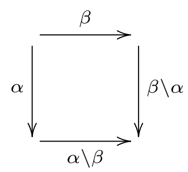
is nonempty ($\Delta \in \mathcal{C}_a(x)$), and is closed under $\wedge$ by Lemma 4.36. Hence, as it is formed by positive elements, and every $\preccurlyeq$-chain of positive elements must have a minimal element, it follows that $\mathcal{C}_a(x)$ has a unique $\preccurlyeq$-minimal element, that we denote $\rho_a(x)$.

Every minimal positive conjugator starting at x must have an atom as a prefix, so it must be equal to $\rho_a(x)$ for some atom a. Actually, the set of minimal positive conjugators starting at x is precisely the set of $\preccurlyeq$-minimal elements in the set $\mathcal{M}(x) = \{\rho_a(x);\ a \in \mathcal{A}\}$, where $\mathcal{A}$ is the set of atoms of G. Since the set $\mathcal{M}(x)$ is finite, we can compute the arrows starting at x if we are able to compute $\rho_a(x)$ for every atom a. Let us see how we can do it.

We will make extensive use of diagrams of the following kind, that we will call LCM-diagrams:



In this diagram, α and β are positive elements, $\alpha\alpha' = \beta\beta'$, and this is precisely the least common multiple $\alpha \vee \beta$. Concatenating several LCM-diagrams produce a new LCM-diagram, if one only reads the product of the arrows in the perimeter of the diagram (we will see many examples below). The element α' is usually denoted $\alpha\backslash\beta$, and it is called the right-complement of α in β. So we have $\alpha\backslash\beta = \alpha^{-1}(\alpha \vee \beta)$ and, similarly, $\beta\backslash\alpha = \beta^{-1}(\beta \vee \alpha)$. We can then express every LCM-diagram as follows:



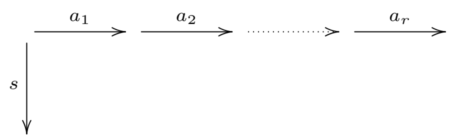
Let us then explain how to compute the element $\rho_a(x)$ for some atom a and some positive element x. The element $\rho_a(x)$ is a positive element c such that x^c is positive, that is, $c^{-1}xc = y \in G^+$ or, equivalently, $xc = cy$ for some $y \in G^+$. Since $a \preccurlyeq c \preccurlyeq cy$, this implies that $a \preccurlyeq xc$. At the same time, $xa \preccurlyeq xc$ since $a \preccurlyeq c$. Therefore $a \vee xa \preccurlyeq xc$. Let us compute $a \vee xa$ and write it as xc_1 for some positive element c_1. We have $xc_1 = xa \vee a = xa \vee x \vee a$. Left-multiplying by x^{-1} we obtain $c_1 = a \vee x^{-1}(x \vee a)$, that is, $c_1 = a \vee x\backslash a$. Notice that we have $xa \preccurlyeq xc_1 \preccurlyeq xc$, hence $a \preccurlyeq c_1 \preccurlyeq c$. Then we can apply the same reasoning, computing $c_2 = c_1 \vee x\backslash c_1$ and obtaining $a \preccurlyeq c_1 \preccurlyeq c_2 \preccurlyeq c$. And so on.

In this way we get a sequence

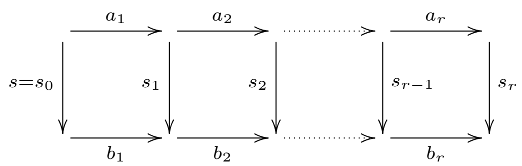
$$a = c_0 \preccurlyeq c_1 \preccurlyeq c_2 \preccurlyeq \cdots$$

of positive prefixes of $c = \rho_a(x)$, where $c_{i+1} = c_i \vee x\backslash c_i$ for every $i \geq 0$. We will call them the converging prefixes of $\rho_a(x)$. The sequence of converging prefixes must stabilize, and it is shown in [23] that $\rho_a(x) = c_m$ for the smallest m such that $c_m = c_{m+1}$, that is, we obtain $\rho_a(x)$ at the place in which the chain stabilizes.

Notice that all elements in the above chain are simple, since $\rho_a(x) \preccurlyeq \Delta$ (as $\Delta \in \mathcal{C}_a(x)$). Hence, in order to compute $\rho_a(x)$ we just need to know how to compute $s \vee \alpha$ for some simple element s and some positive element α. If we write $\alpha = a_1 \cdots a_r$ as a product of atoms, this computation is performed by starting with the following diagram:



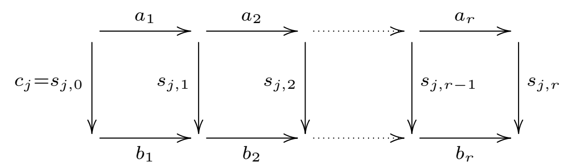
and filling the squares, from left to right, to obtain a concatenation of LCM-diagrams:



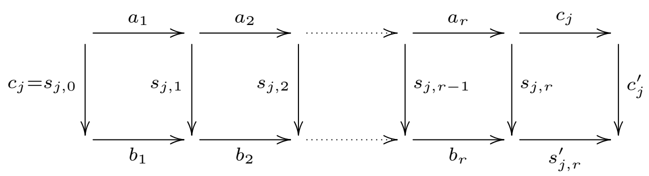
The top row represents the element $\alpha = a_1 \cdots a_r$. Let us denote $\beta = b_1 \cdots b_r$. Each square is an LCM-diagram, so $s_{i-1} \vee a_i = s_{i-1}b_i = a_i s_i$ for $i = 1, \ldots, r$. Since the concatenation of LCM-diagrams is an LCM-diagram, we obtain that $s \vee \alpha = s\beta = \alpha s_r$. In other words, $s_r = \alpha\backslash s$.

This means that in order to compute $s \vee \alpha$, we just need to know how to compute $t \vee a$ for any simple element t and any atom a. This is something that we should know how to compute, when working with a Garside group G. Since the set of simple elements and the set of atoms are finite, we may even have that information precomputed and stored.

Therefore, in order to compute the element $\rho_a(x)$ as explained above, and assuming that $x = a_1 \cdots a_r$ as a product of atoms, one starts with the initial converging prefix $c_0 = a$ and, for every $j \geq 0$, one computes a sequence of right-complements, from c_j to $x\backslash c_j$, by filling the following LCM-diagram:



By construction, for $i = 1, \dots, r$ we have $s_{j,i} = (a_1 \cdots a_i)\backslash c_j$, so $s_{j,r} = x\backslash c_j$. Hence, since $c_{j+1} = c_j \vee x\backslash c_j = c_j \vee s_{j,r}$, one can add one more square to the above LCM-diagram, as follows:



Then we have that $c_{j+1} = c_j c'_j$, which is the product of the top and right arrows in the last squared diagram.

This is repeated until one finds $c_m = c_{m+1}$ and then $\rho_a(x) = c_m$. For every $j = 0, \dots, m-1$ and every $i = 0, \dots, r$, the elements $s_{j,i} = (a_1 \cdots a_i)\backslash c_j$ will be called the pre-minimal conjugators for a and x. Notice that the pre-minimal conjugators are not necessarily prefixes of $\rho_a(x)$, but the converging prefixes $c_0, c_1, \dots, c_m$ are.

Therefore, by computing the pre-minimal conjugators and the converging prefixes, one can compute the element $\rho_a(x)$ for every atom a and every positive element x.

In the forthcoming subsections we will be able to describe the elements $\rho_a(x)$, in some cases, by performing a detailed study of the pre-minimal conjugators and the converging prefixes which are computed during the process. In other cases we will show that the element $\rho_a(x)$ that one would obtain is not a minimal positive conjugator, so there is no need to compute it.

There is a situation which occurs with all the monoids under study. Let us see that, in order to show that an LCM-Garside structure is support-preserving, we just need to care about the elements $\rho_a(x)$, where $a \notin \mathrm{Supp}(x)$.

Proposition 4.37. *Let (G, G^+, Δ) be an LCM-Garside structure. Suppose that for every $u \in G^+$ and every $b \notin \mathrm{Supp}(u)$ such that $\rho_b(u)$ is a minimal positive conjugator, $\#(\mathrm{Supp}(u^{\rho_b(u)})) \leq \#(\mathrm{Supp}(u))$. Then, for every $x \in G^+$ and every $a \in \mathrm{Supp}(x)$, one has $\mathrm{Supp}(x^{\rho_a(x)}) = \mathrm{Supp}(x)$ and*

$$\left(G_{\mathrm{Supp}(x)}\right)^{\rho_a(x)} = G_{\mathrm{Supp}(x)} = G_{\mathrm{Supp}(x^{\rho_a(x)})}.$$

Proof. Let $X = \mathrm{Supp}(x)$ and let $y = x^{\rho_a(x)}$ and $Y = \mathrm{Supp}(y)$. We want to prove that $X = Y$ and that $(G_X)^{\rho_a(x)} = G_X = G_Y$. We first notice that Δ_X is a positive element which admits a as a prefix, and which conjugates x to a positive element. Hence $\Delta_X \in \mathcal{C}_a(x)$. Since $\rho_a(x)$ is the minimal element in this set, it follows that $\rho_a(x) \preccurlyeq \Delta_X$. Hence $\rho_a(x) \in G_X^+$. This implies that $y = x^{\rho_a(x)} \in G_X^+$, hence $Y = \mathrm{Supp}(y) \subset X$.

From the above paragraph and the hypothesis, it follows that for every $u \in G^+$ and every atom $b \in \mathcal{A}$ such that $\rho_b(u)$ is a minimal positive conjugator, $\#(\mathrm{Supp}(u^{\rho_b(u)})) \leq \#(\mathrm{Supp}(u))$.

Let us go back to x and y, such that $x^{\rho_a(x)} = y$ with $a \in X$. We know that there is some positive power $(\Delta_X)^e$ which is central in G_X. If we denote $\alpha = \rho_a(x)^{-1}(\Delta_X)^e$, we see that α is a positive element such that $y^\alpha = x$. Now α can be decomposed as a product of minimal positive conjugators, and the conjugation by each one cannot increase the number of elements in the corresponding support. This implies that $\#(X) \leq \#(Y)$. Since X and Y are finite sets, and $Y \subset X$, it follows that $X = Y$.

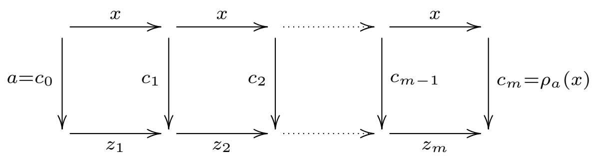
Finally, since $\rho_a(x) \in G_X^+$, conjugation by $\rho_a(x)$ is an inner automorphism of G_X, so it conjugates G_X to itself. In other words, $\rho_a(x)$ preserves the support. □

We finish this subsection with some useful results for computing the set of minimal positive conjugators for some $y \in G^+$.

Lemma 4.38. *Let $x \in G^+$ and let a be an atom. Write $x = a_1 \cdots a_r$ as a product of atoms. If for every $j = 0, \dots, m-1$ and every $i = 0, \dots, r-1$ we have $s_{j,i} \preccurlyeq s_{j,i+1}$, that is, if every pre-minimal conjugator is a prefix of the next one, then $c_{j+1} = s_{j,r}$ for every $j = 0, \dots, m-1$, and $\rho_a(x) = x^m\backslash a$.*

Proof. By hypothesis, for every $j = 0, \dots, m-1$ we have $c_j = s_{j,0} \preccurlyeq s_{j,1} \preccurlyeq \cdots \preccurlyeq s_{j,r} = x\backslash c_j$. Hence, as $c_{j+1} = c_j \vee x\backslash c_j$, it follows that $c_{j+1} = x\backslash c_j = s_{j,r}$.

The above property implies that, in order to compute new pre-minimal conjugators, one does not need to compute $c_j \vee x \backslash c_j$. Hence one can concatenate LCM-diagrams as follows, to obtain all converging prefixes for $\rho_a(y)$:



Hence $\rho_a(x) = x^m \backslash a$, as we wanted to show. □

Lemma 4.39. *Let $x \in G^+$ and let a and b be atoms of G. Suppose that $a \not\preccurlyeq \rho_b(x)$. If there is some converging prefix c_i for $\rho_a(x)$ such that $b \preccurlyeq c_i$, then $\rho_a(x)$ is not a minimal positive conjugator for x.*

Proof. We know from [23] that $\rho_a(x) = c_m$, where m is the place in which the chain of converging prefixes stabilizes. This means that $c_i \preccurlyeq c_m = \rho_a(x)$, and hence $b \preccurlyeq \rho_a(x)$.

Since $x^{\rho_a(x)}$ is positive by definition of $\rho_a(x)$, it follows that $\rho_a(x) \in \mathcal{C}_b(x)$. But $\rho_b(x)$ is the $\preccurlyeq$-minimal element in this set, so $\rho_b(x) \preccurlyeq \rho_a(x)$.

Now notice that $\rho_b(x) \neq \rho_a(x)$, since $a \preccurlyeq \rho_a(x)$ by definition and $a \not\preccurlyeq \rho_b(x)$ by hypothesis. Therefore $1 \prec \rho_b(x) \prec \rho_a(x)$. Since $x^{\rho_b(x)}$ is positive, this implies that $\rho_a(x)$ is not a minimal positive conjugator for x. □

Lemma 4.40. *Let $x \in G^+$ and let a and b be atoms of G. Suppose that $a \not\preccurlyeq \rho_b(x)$. Write $x = a_1 \cdots a_r$ as a product of atoms, and suppose that for every $j = 0, \dots, m-1$ and every $i = 0, \dots, r-1$ we have $s_{j,i} \preccurlyeq s_{j,i+1}$. If there is some pre-minimal conjugator $s_{j,i}$ for a and x such that $b \preccurlyeq s_{j,i}$, then $\rho_a(x)$ is not a minimal positive conjugator for x.*

Proof. If $b \preccurlyeq s_{j,i}$ for some pre-minimal conjugator $s_{j,i}$, by hypothesis we know that $s_{j,i} \preccurlyeq s_{j,i+1} \preccurlyeq \cdots \preccurlyeq s_{j,r}$, so $b \preccurlyeq s_{j,r}$. Also, since every pre-minimal conjugator is a prefix of the next one, from Lemma 4.38 we know that $s_{j,r} = c_{j+1}$. Hence $b \preccurlyeq c_{j+1}$. Therefore, by Lemma 4.39, $\rho_a(x)$ is not a minimal positive conjugator for x. □

5. Parabolic closures for complex braid groups

5.1. **The classical braid group.** We will start our study of particular complex braid groups by the case of the classical braid group $\mathcal{B}_{n+1}$ on $n+1$ strands. Since it is the same as the Artin group of type A_n, the results are already known. We nevertheless expose our new methods in detail for this case, as it will serve as a guide for the other groups, in which the same strategies will be used.

Let $G = \mathcal{B}_{n+1}$ endowed with classical Garside structure, for which the monoid of positive elements is given by the presentation with generators $\mathcal{A} = \{s_1, \dots, s_n\}$ (which are the atoms of this structure) and relations

- $s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$,
- $s_i s_j = s_j s_i$ if $|i-j| > 1$.

As usual, one can represent the braids in G as a collection of $n+1$ disjoint strands, up to homotopy fixing the endpoints, and the generator s_i corresponds to a braid in which the strands i and $i+1$ cross once (positively), and there is no other crossing. The simple elements in this structure corresponds to braids in which every two strands cross (positively) at most once, and the Garside element $\Delta = \Delta_{\mathcal{A}}$ is the braid in which every two strands cross (possitively) exactly once.

For every $X \subset \mathcal{A}$, the subgroup generated by X is a direct product of classical braid groups. The element Δ_X can be seen geometrically as well. Namely, if all elements in X are consecutive generators, $X = \{s_i, s_{i+1}, \dots, s_j\}$, then Δ_X is the braid which every two strands in $\{i, i+1, \dots, j+1\}$ cross (positively) exactly once:

$$\Delta_X = s_j (s_{j-1} s_j) \cdots (s_i s_{i+1} \cdots s_j).$$

Otherwise, if one considers that consecutive generators in $\{s_1,\dots,s_n\}$ are connected, and one decomposes X in connected components $X = X_1 \cup \cdots \cup X_r$, where $X_k = \{s_{i_k}, s_{i_k+1},\dots,s_{j_k}\}$ for some $i_k \leq j_k$, then $\Delta_X = \Delta_{X_1}\cdots\Delta_{X_r}$, where the factors are pairwise commuting. It is well known that in every case $X \subset \mathcal{A}$ is saturated, as the set of atoms which are prefixes of Δ_X is precisely X. Moreover, G_X is a standard parabolic subgroup of G. Therefore, the classical Garside structure of G is an LCM-Garside structure.

Now we need to introduce some special elements, which are analogous to the elementary ribbons defined by Godelle in [29]. We will do it for an arbitrary Garside group, as in [14].

Definition 5.1. Let $X \subset \mathcal{A}$ be a subset of atoms in a Garside group, and let $u \in \mathcal{A}$. Then we define $r_{X,u} = \Delta_X^{-1}\Delta_{X\cup\{u\}}$.

Lemma 5.2. *Let $X \subset \mathcal{A}$ be a subset of atoms in a Garside group, and let $u \in \mathcal{A}$. If $u \in \overline{X}$ then $r_{X,u} = 1$. Otherwise $r_{X,u}$ is a nontrivial simple element.*

Proof. If $u \in \overline{X}$ then $\overline{X} = \overline{X \cup \{u\}}$. Hence $\Delta_{X\cup\{u\}} = \Delta_{\overline{X\cup\{u\}}} = \Delta_{\overline{X}} = \Delta_X$, and then $r_{X,u} = 1$.

Let us suppose that $u \notin \overline{X}$. Recall that $\Delta_{X\cup\{u\}}$ is the join of all elements in $X \cup \{u\}$, hence all elements in X are prefixes of $\Delta_{X\cup\{u\}}$, which implies that $\Delta_X \preccurlyeq \Delta_{X\cup\{u\}}$. Hence $r_{X,u} = \Delta_X^{-1}\Delta_{X\cup\{u\}}$ is a positive element. Since $r_{X,u}$ is a suffix of $\Delta_{X\cup\{u\}}$, it is a suffix of the Garside element Δ, and hence it is a simple element. Now, since $\Delta_X = \Delta_{\overline{X}}$, an atom is a prefix of Δ_X if and only if it belongs to $\overline{X}$. Hence $u \not\preccurlyeq \Delta_X$. On the other hand $u \preccurlyeq \Delta_{X\cup\{u\}}$ by definition. Therefore $\Delta_X \neq \Delta_{X\cup\{u\}}$ and $r_{X,u}$ is nontrivial. □

We will show that if $x, y \in G^+$ are two positive braids such that $\mathrm{Supp}(x) = X \neq Y = \mathrm{Supp}(y)$, and that $x^\alpha = y$ for some minimal positive conjugator α, then α is a ribbon, namely $\alpha = r_{X,a}$ for some atom a. This is actually shown in [14, Proposition 6.3] for every Artin group of spherical type, but we will show it here in a different way, which will be helpful in the following sections, when we will study other Garside groups.

Given $1 \leq i \leq k \leq n$, we denote:

$$S_{i,k} = s_i s_{i+1}\cdots s_k, \qquad S_{k,i} = s_k s_{k-1}\cdots s_i.$$

The former braid can be drawn as the i-th strand crossing positively the strands $i+1, i+2, \dots, k+1$ in that precise order. The latter, as the $(k+1)$−st strand crossing positively the strands $k, k-1,\dots,i$.

Given $1 \leq i \leq j \leq k \leq n$, we can define the braid $\sigma_{i,j,k}$ in which every strand from $\{i,\dots,j\}$ crosses positively every strand in $\{j+1,\dots,k+1\}$ exactly once, and there is no other crossing. Hence, it is a simple braid, and it is easy to see that:

$$\sigma_{i,j,k} = S_{j,k}S_{j-1,k-1}\cdots S_{i,k-j+i} = S_{j,i}S_{j+1,i+1}\cdots S_{k,i+k-j}.$$

We will need the following:

Lemma 5.3. *Given $1 \leq i_1 \leq i_0 \leq j \leq k_0 \leq k_1 \leq n$, one has $\sigma_{i_0,j,k_0} \preccurlyeq \sigma_{i_1,j,k_1}$.*

Proof. It is well known that a simple braid s_0 is a prefix of a simple braid s_1 if and only if the pairs of strands that cross in s_0 also cross in s_1 (see [22]). The result follows immediately, as every $\sigma_{i,j,k}$ is a simple braid, and we know exactly the pairs of strands that cross in such an element. □

Recall also that one has:

$$\Delta_{\{s_i,\dots,s_{j-1}\}\cup\{s_{j+1},\dots,s_k\}} = \Delta_{\{s_i,\dots,s_{j-1}\}}\Delta_{\{s_{j+1},\dots,s_k\}}.$$

This also holds when either $i = j$ or $j = k$, that is, when one of the above factors is trivial.

Now, by checking the strands that cross in the corresponding braids, one can easily see that:

$$\Delta_{\{s_i,\dots,s_{j-1}\}\cup\{s_{j+1},\dots,s_k\}}\sigma_{i,j,k} = \Delta_{\{s_i,\dots,s_k\}}.$$

Therefore, we have shown the following:

Lemma 5.4. *Let $1 \leq i \leq j \leq k \leq n$, and let $X = \{s_i,\cdots s_{j-1}\} \cup \{s_{j+1},\dots,s_k\}$. Then one has:*

$$r_{X,s_j} = \sigma_{i,j,k}.$$

From this, we have a complete description of nontrivial ribbon elements:

Lemma 5.5. *Let $X \subset \{s_1, \dots, s_n\}$ and $s_j \notin X$. Decompose $X = X_0 \sqcup X_1$, where $X_0 \cup \{s_j\} = \{s_{i_0}, \dots, s_{k_0}\}$ is the connected component of $X \cup \{s_j\}$ containing s_j. Then*

$$r_{X,s_j} = r_{X_0,s_j} = \sigma_{i_0,j,k_0}.$$

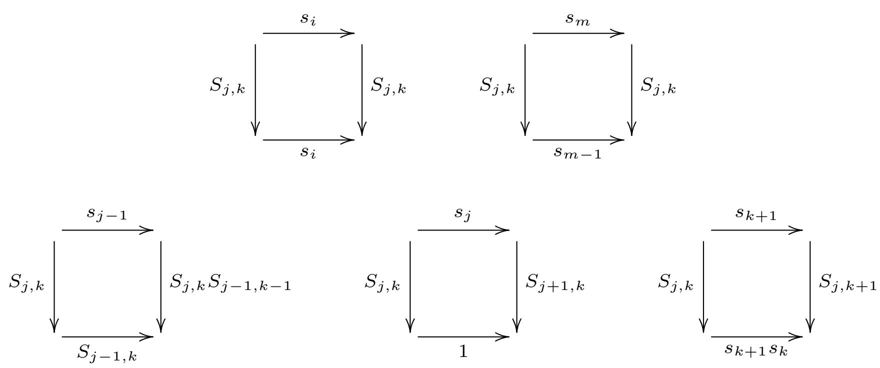
Proof. Since every atom of $X_0 \cup \{s_j\}$ commutes with every atom in X_1, we have $\Delta_X = \Delta_{X_0}\Delta_{X_1} = \Delta_{X_1}\Delta_{X_0}$ and $\Delta_{X\cup\{s_j\}} = \Delta_{X_0\cup\{s_j\}}\Delta_{X_1} = \Delta_{X_1}\Delta_{X_0\cup\{s_j\}}$. Hence

$$r_{X,s_j} = \Delta_X^{-1}\Delta_{X\cup\{s_j\}} = \Delta_{X_0}^{-1}\Delta_{X_1}^{-1}\Delta_{X_1}\Delta_{X_0\cup\{s_j\}} = \Delta_{X_0}^{-1}\Delta_{X_0\cup\{s_j\}} = r_{X_0,s_j}.$$

Now we can apply Lemma 5.4 to X_0 and we get $r_{X_0,s_j} = \sigma_{i_0,j,k_0}$. □

We have then obtained an explicit description of the ribbon elements in $G = \mathcal{B}_{n+1}$ as products of atoms. Let us show some technical but important results, which are well-known for specialists in braid groups.

Lemma 5.6. *Let $1 \le j \le k \le n$ and suppose that $i \notin \{j-1, \dots, k+1\}$ and that $m \in \{j+1, \dots, k\}$. The following are LCM-diagrams in the classical monoid for $\mathcal{B}_{n+1}$:*



Proof. The first diagram is clear, as $S_{j,k}$ cannot start with s_i, so the length of $S_{j,k} \vee s_i$ must be at least one letter bigger than $S_{j,k}$. Since $S_{j,k}s_i = s_iS_{j,k}$, this element admits $S_{j,k}$ and s_i as prefixes and has the minimal possible length, so $s_i \vee S_{j,k} = S_{j,k}s_i = s_iS_{j,k}$, as stated in the diagram.

The same argument holds for the second diagram, taking into account that $s_m \not\preccurlyeq S_{j,k}$ (as the only possible initial letter of $S_{j,k}$ is s_j, since $\{j, j+1\}$ is the only pair of consecutive strands that cross in $S_{j,k}$). This time we have:

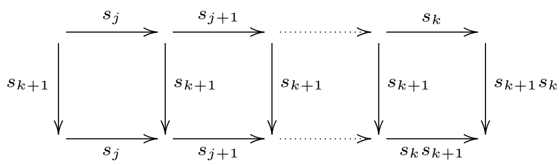
$$s_mS_{j,k} = s_mS_{j,m-2}s_{m-1}s_mS_{m+1,k} = S_{j,m-2}s_ms_{m-1}s_mS_{m+1,k}$$

$$= S_{j,m-2}s_{m-1}s_ms_{m-1}S_{m+1,k} = S_{j,m-2}s_{m-1}s_mS_{m+1,k}s_{m-1} = S_{j,k}s_{m-1}.$$

So this element is $s_m \vee S_{j,k}$, as we wanted to show.

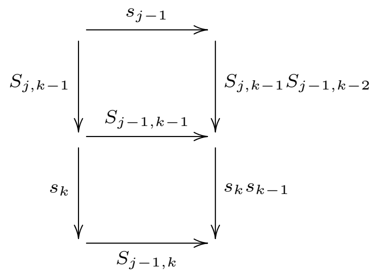
The fourth diagram is evident, as $S_{j,k} = s_jS_{j+1,k}$.

The fifth diagram can be shown by concatenating several known LCM-diagrams as follows:



The outermost paths in this diagram coincide with the elements of the fifth diagram in the statement (transposed).

It only remains to show that the third diagram is correct. If $j = k$ the result is clear, as $s_{j-1} \vee s_j = s_{j-1}s_js_{j-1} = s_js_{j-1}s_j$. So we assume that $j < k$ and that the claim holds for smaller values of k. Then we have, as $S_{j,k} = S_{j,k-1}s_k$, the following LCM-diagram:

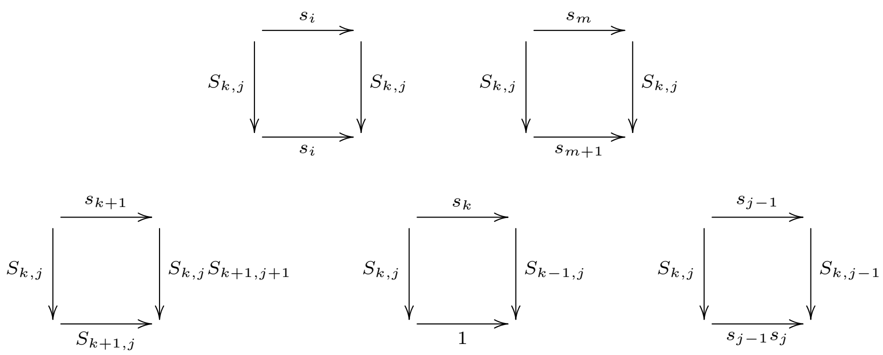


The top square holds by induction hypothesis, and the bottom one comes from the fifth diagram in the statement. This finishes the proof, as the vertical path on the right is:

$$S_{j,k-1}S_{j-1,k-2}s_k s_{k-1} = S_{j,k-1}s_k S_{j-1,k-2}s_{k-1} = S_{j,k}S_{j-1,k-1}.$$

□

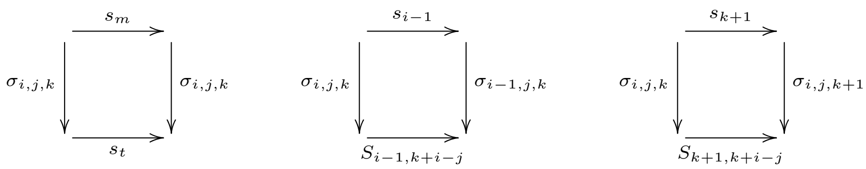
Lemma 5.7. *Let $1 \leq j \leq k \leq n$ and suppose that $i \notin \{j-1, \dots, k+1\}$ and that $m \in \{j, \dots, k-1\}$. The following are LCM-diagrams in the classical monoid for $\mathcal{B}_{n+1}$:*



Proof. These diagrams are obtained by conjugating those in Lemma 5.6 by Δ, an operation that preserves least common multiples, and sends s_i to s_{n-i} for every i. □

Using the above two results, we can derive the least common multiples of a ribbon element and a suitable atom:

Lemma 5.8. *Given $1 \leq i \leq j \leq k \leq n$ and $m \notin \{i-1, j, k+1\}$, the following are LCM-diagrams in the classical monoid for $\mathcal{B}_{n+1}$, whenever the elements involved are defined:*



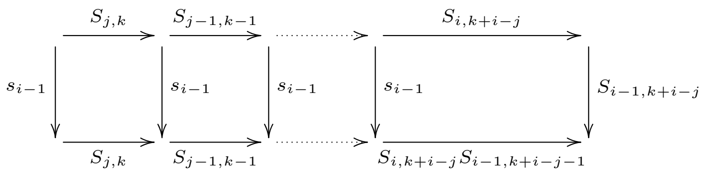
In the first diagram, s_t is an atom (conjugate of s_m by $\sigma_{i,j,k}$).

Proof. The first diagram holds since the only possible initial letter of $\sigma_{i,j,k}$ is σ_j (as j and $j+1$ are the only consecutive strands which cross in the simple element $\sigma_{i,j,k}$), hence the least common multiple of s_m and $\sigma_{i,j,k}$ must be bigger than $\sigma_{i,j,k}$. There is a common multiple one letter bigger, namely:

$$s_m\sigma_{i,j,k} = s_m\Delta^{-1}_{\{i,\dots,k\}\setminus\{j\}}\Delta_{\{i,\dots,k\}} = \Delta^{-1}_{\{i,\dots,k\}\setminus\{j\}}\Delta_{\{i,\dots,k\}}s_t = \sigma_{i,j,k}s_t.$$

Notice that Δ_X conjugates every atom in X to another atom in X, and commutes with every atom not adjacent to X, so the central equality above holds for some atom s_t. This element is thus the searched least common multiple, and the first diagram holds.

The second diagram is obtained from the following concatenation of LCM-diagrams, coming from Lemma 5.6:



The product of the arrows in the top row is $\sigma_{i,j,k}$, while the product of the arrows in the bottom row is $\sigma_{i-1,j,k}$. Hence the second diagram holds.

Finally, the third one is obtained from the second one after conjugation by Δ. $\square$

We can now determine the minimal positive conjugators in some suitable cases:

Proposition 5.9. *(see [14]) Let $G = \mathcal{B}_{n+1}$ be the braid group on $n+1$ strands. Let $x \in G^+$ and let $X = \operatorname{supp}(x)$. For every $s_j \in \mathcal{A} \setminus X$ one has $\rho_{s_j}(x) = r_{X,s_j}$.*

Proof. Recall that in order to find $\rho_{s_j}(x)$ one just needs to compute the pre-minimal conjugators and converging prefixes for s_j and x. The initial one is $c_0 = s_j = \sigma_{j,j,j}$. Then, by Lemma 5.8, if some pre-minimal conjugator is $\sigma_{i,j,k}$, the next one will be either $\sigma_{i,j,k}$, or $\sigma_{i-1,j,k}$, or $\sigma_{i,j,k+1}$ (here we are using that $s_j \notin X$). From Lemma 5.3, it follows that each pre-minimal conjugator is a prefix of the next one. Therefore, by Lemma 4.38, $\rho_{s_j}(x) = x^m \backslash s_j$ for some $m \geq 0$. That is, in order to obtain $\rho_{s_j}(x)$, we just need to compute pre-minimal conjugators until the sequence stabilizes.

Notice also that a pre-minimal conjugator $\sigma_{i,j,k}$ is different to the next one if and only if the corresponding atom in the word representing x is adjacent to $\{i, \dots, k\}$. This means that the sequence of pre-minimal conjugators will stabilize exactly when it reaches σ_{i_0,j,k_0}, where $\{i_0, \dots, k_0\}$ is the connected component of $X \cup \{s_j\}$ which contains s_j. Hence $\rho_{s_j}(x) = \sigma_{i_0,j,k_0}$ and, by Lemma 5.5, $\sigma_{i_0,j,k_0} = r_{X,\sigma_j}$, as we wanted to show. $\square$

The above result yields immediately that the classical Garside structure of G is support-preserving. This is shown in [14, Corollary 6.5] for every Artin group of spherical type, but we state it here for completeness.

Proposition 5.10. *(see [14, Corollary 6.5]) The classical Garside structure for the classical braid group is support-preserving.*

Proof. Let x, y be positive braids which are connected by a minimal positive conjugator, that is, $x^\rho = y$ where $\rho = \rho_{s_j}(x)$ for some atom s_j. Let $X = \operatorname{supp}(x)$ and $Y = \operatorname{supp}(y)$. We need to show that ρ conjugates G_X to G_Y.

If $s_j \notin X$ then, by Proposition 5.9, $\rho = r_{X,s_j}$. But conjugation by r_{X,s_j} sends the set of atoms X to another set of atoms Z of the same size. So the conjugation of x by ρ acts letter by letter, sending each letter of x to the corresponding letter of Z. It follows that Z is precisely the support of the obtained element y, so $Z = Y$ and conjugation by ρ sends the atoms of X to the atoms of Y, hence it sends G_X to G_Y, as we wanted to show.

Since conjugation by a minimal positive conjugator $\rho_{s_j}(x)$ where $s_j \notin X$ preserves the sizes of the supports, we can apply Proposition 4.37 to conclude that every $\rho_{s_i}(x)$ also preserves the support when $s_i \in X$. Therefore, the classical Garside structure if B is support-preserving. $\square$

In [14, Proposition 7.2], the above result, together with some technical arguments involving a set denoted $RSSS_\infty(x)$, was used to show the existence of the parabolic closure of an element in an Artin group of spherical type. From the definition and properties of recurrent elements shown in this paper, we already have this result immediately from Theorem 4.31.

Corollary 5.11. *Every element of the classical braid group admits a parabolic closure.*

Notice that, after [14, Proposition 7.2], we can also deduce the following from Theorem 4.31:

Corollary 5.12. *Every element in an Artin group of spherical type admits a parabolic closure.*

As we said, this result was already shown in [14], but we gave a complete proof here, using recurrent elements.

5.2. **The group** $G(e, e, n)$**.** In this section we will study the complex braid group B of type $G(e, e, n)$, endowed with the standard monoid structure of subsection 3.4 with set of atoms $\mathcal{A} = \{t_0, t_1, \ldots, t_{e-1}, s_3, s_4, \ldots, s_n\}$, with $e \geq 1$ and $n \geq 2$. We will show that the standard parabolic subgroups associated to the standard Garside structure are the 'parabolic subgroups' introduced in [9] §6.3, and we will show that it is a support-preserving LCM-Garside structure, so that every element admits a parabolic closure.

We remark that we modified the order of the elements t_i with respect to [12], using the permutation $t_i \mapsto t_{e-1-i}$. We now provide some important information about this Garside structure, taken from [12]:

If we denote $\tau = t_i t_{i+1}$ (which is the same element for any i), we have $\tau = t_0 \vee \cdots \vee t_{e-1}$, and also $\tau = t_i \vee t_j$ for every $i \neq j$.

For $k = 2, \ldots, n$, let $\Lambda_k = s_k s_{k-1} \cdots s_3 \tau s_3 \cdots s_{k-1} s_k$ (in particular, $\Lambda_2 = \tau$). Then the Garside element Δ is the LCM of the atoms, $\Delta = \Delta_{\mathcal{A}}$, which we can write in four different ways as follows:

$$\Delta = \Lambda_2 \Lambda_3 \cdots \Lambda_n = \Lambda_n \Lambda_{n-1} \cdots \Lambda_2 = (\tau s_3 \cdots s_n)^{n-1} = (s_n \cdots s_3 \tau)^{n-1}$$

One has, for every i:

$$\Delta \tau = \tau \Delta, \qquad \Delta s_i = s_i \Delta, \qquad \Delta t_i = t_{i-n} \Delta,$$

where the subindices of the t_i are taken modulo e. This shows the permutation of the atoms induced by conjugation by Δ. We can also see how conjugation by an element Λ_k affects some atoms:

$$\Lambda_k s_i = s_i \Lambda_k \ (i \neq k, k+1), \qquad \Lambda_2 t_i = t_{i-2} \Lambda_2, \qquad \Lambda_k t_i = t_{i-1} \Lambda_k \ (k > 2).$$

Finally, the **simple elements** in this structure are the elements of the form $p_2 \cdots p_r$, where each p_k is a prefix of Λ_k, taking into account that each Λ_k only admits the evident prefixes: those which correspond to a prefix of the word $s_k s_{k-1} \cdots s_3 t_i t_{i+1} s_3 \cdots s_{k-1} s_k$ for some i [12, Theorem 3.7].

With the above information, we will show that (B, B^+, Δ) is a support-preserving LCM-Garside structure. Let us denote $T_e = \{t_0, \ldots, t_{e-1}\}$ and $S_n = \{s_3, \ldots, s_n\}$, so the set of atoms of B is $\mathcal{A} = T_e \cup S_n$. We have:

Proposition 5.13. *The standard Garside structure* (B, B^+, Δ) *for* $G(e, e, n)$ *is an LCM-Garside structure. Moreover, if* $X \subset \mathcal{A}$ *is a set of atoms, then* X *is saturated if and only if* $\#(X \cap T_e) \in \{0, 1, e\}$*.*

Proof. We already know that $\Delta = \Delta_{\mathcal{A}}$, the least common multiple of all the atoms. We need to study the least common multiple Δ_X of subsets $X \subset \mathcal{A}$. The fact that $\#(X \cap T_e) \in \{0, 1, e\}$ implies that X is saturated is an immediate consequence of [9] §6.3. If $1 < \#(X \cap T_e) < e$, let $t_i, t_j \in X$ and let $t_k \notin X$. We have $t_i \vee t_j = \tau$, hence $\tau \preccurlyeq \Delta_X = x_1 \vee \cdots \vee x_m$, where $X = \{x_1, \ldots, x_m\}$. But $t_k \preccurlyeq \tau$, hence $t_k \preccurlyeq \Delta_X$, implying that X is not saturated. Hence, we only need to consider the elements Δ_X with $\#(X \cap T_e) \in \{0, 1, e\}$, and it follows from [9] that such an element is balanced and that G_{Δ_X} is a standard parabolic subgroup.

□

We have then shown that the standard parabolic subgroups of B are those generated by some set $X \subset \mathcal{A}$ which either contains no t_i, or contains exactly one, or contains them all. From the proof of the above result it follows that if X is not saturated (that is, if $1 < \#(X \cap T_e) < e$), then $\overline{X} = X \cup T_e$. Recall that we always have $\Delta_X = \Delta_{\overline{X}}$.

The rest of this section is devoted to show that (B, B^+, Δ) is a support-preserving Garside structure for B. By Proposition 4.35, we just need to show that for every positive element $x \in B^+$ and every minimal positive conjugator ρ for x such that $x^\rho = y$, then $(G_X)^\rho = G_Y$, where $X = \mathrm{Supp}(x)$ and $Y = \mathrm{Supp}(y)$.

We will then need to study in detail the minimal positive conjugators for an element in B^+, and to check that conjugation by such an element 'preserves the support' in the natural way.

Recall the concept of ribbon element given in Definition 5.1. As we did with braid groups in the previous section, we will show that the minimal conjugating elements that connect positive conjugates with different supports, are all ribbons of the form $r_{X,u}$, for some X and u.

When $\#(X \cap T_e) = 0$ and u is an atom, we already know that $X \cup \{u\}$ is contained into a submonoid of B^+ isomorphic to a braid monoid (with the same Garside structure), so we know exactly how the ribbon $r_{X,u}$ looks like. The same happens when $\#(X \cap T_e) = 1$ and $u \in S_n$. Hence we only need to study ribbons in the remaining cases.

Given $i, k \in \{2, \dots, n\}$, we denote:

$$\Lambda_{i,k} = s_i s_{i-1} \cdots s_3 \, \tau \, s_3 \cdots s_{k-1} s_k.$$

We consider that $\Lambda_{2,k} = \tau s_3 \cdots s_{k-1} s_k$, $\Lambda_{i,2} = s_i s_{i-1} \cdots s_3 \tau$, and $\Lambda_{2,2} = \tau$. Notice that $\Lambda_{k,k} = \Lambda_k$ for every $k = 2, \dots, n$.

We will also denote $S_{i,k} = s_i s_{i+1} \cdots s_k$ and $S_{k,i} = s_k s_{k-1} \cdots s_i$ whenever $3 \leq i \leq k \leq n$. We will extend this notation to $i = 2$, by denoting $s_2 = t_a$ for some given $a \in \{0, \dots, e-1\}$ which must be specified. Hence, we will denote $S^{(a)}_{2,k} = t_a s_3 \cdots s_k$ and $S^{(a)}_{k,2} = s_k \cdots s_3 t_a$. We will sometimes use the superscript even if $i \neq 2$: in that case $S^{(a)}_{i,k} = S_{i,k}$ and $S^{(a)}_{k,i} = S_{k,i}$.

Let $X \subset \mathcal{A}$ such that $X \cap T_e$ has more than one element. We will use the concept of adjacent generators coming from Artin groups, as follows. We will say that s_i and s_j are adjacent if and only if i and j are consecutive numbers, that s_3 is adjacent to all atoms in T_e, and that all atoms in T_e are adjacent to each other. We can imagine a graph whose vertices are the atoms in $\mathcal{A}$, where adjacent atoms are connected by an edge. In this way we can talk about connected subsets of $\mathcal{A}$, or about the connected components of a subset of $\mathcal{A}$.

We just need to determine the ribbon elements in two cases, given by the following two Lemmas.

Lemma 5.14. *Let $3 \leq j \leq k \leq n$. Denote $X = \{t_{i_1}, \dots, t_{i_p}\} \cup \{s_3, \dots, s_{j-1}\} \cup \{s_{j+1}, \dots, s_k\}$ for some $p > 1$. Then one has:*

$$r_{X,s_j} = \Lambda_{j,k} \Lambda_{j,k-1} \cdots \Lambda_{j,j} = \Lambda_{j,j} \Lambda_{j+1,j} \cdots \Lambda_{k,j}$$

Proof. Let $Y = X \cup \{s_j\} = \{t_{i_1}, \dots, t_{i_p}\} \cup \{s_3, \dots, s_k\}$. Notice that Y is not necessarily saturated but, as it contains more than one element from T_e, we have that $\overline{Y} = T_e \cup \{s_3, \dots, s_k\}$.

Suppose that $k - j = 0$. Then $\overline{X} = T_e \cup \{s_3, \dots, s_{j-1}\}$ and $\overline{X \cup \{s_j\}} = \overline{Y} = T_e \cup \{s_3, \dots, s_j\}$. This implies that $\Delta_X = \Lambda_2 \cdots \Lambda_{j-1}$ and that $\Delta_{X \cup \{s_j\}} = \Lambda_2 \cdots \Lambda_{j-1} \Lambda_j$. Hence $r_{X,s_j} = \Lambda_j = \Lambda_{j,j}$, as we wanted to show.

Now suppose that $k - j > 0$ and that the result holds for smaller values. Let $Z = X \setminus \{s_k\}$. Since $s_j \neq s_k$, we can apply the induction hypothesis to Z and s_j, to obtain $r_{Z,s_j} = \Lambda_{j,k-1} \Lambda_{j,k-2} \cdots \Lambda_{j,j} = \Lambda_{j,j} \Lambda_{j+1,j} \cdots \Lambda_{k-1,j}$.

We need to find r_{X,s_j} such that $\Delta_X r_{X,s_j} = \Delta_{X \cup \{s_j\}}$. We have:

$$\Delta_{X \cup \{s_j\}} = \Delta_{Z \cup \{s_j\}} \Lambda_k = \Delta_Z r_{Z,s_j} \Lambda_k = \Delta_Z \left(\prod_{m=1}^{k-j} \Lambda_{j,k-m} \right) \Lambda_k$$

Now let $X_2 = \{s_{j+1}, \dots, s_k\}$ and $X_1 = X \setminus X_2$ be the two connected components of X, and let $Z_2 = \{s_{j+1}, \dots, s_{k-1}\}$ and $Z_1 = Z \setminus Z_2$ be the two connected components of Z. We have $X_1 = Z_1$ and

$$\Delta_X = \Delta_{X_1} \Delta_{X_2} = \Delta_{Z_1} \Delta_{Z_2} r_{Z_2,s_k} = \Delta_Z r_{Z_2,s_k} = \Delta_Z (s_k s_{k-1} \cdots s_{j+1}),$$

where the last equality comes from Lemma 5.4, since $Z_2 \cup \{s_j\} \subset S_n$ and it is a connected set.

Now recall how conjugation by Λ_k affects the atoms. It follows that $s_i \Lambda_k = \Lambda_k s_i$ for $i \neq k, k+1$, and that $\tau \Lambda_k = \Lambda_k \tau$ for $k \geq 2$. Hence $\Lambda_{j,p} \Lambda_k = \Lambda_k \Lambda_{j,p}$ whenever $j \leq p < k$. Therefore:

$$\Delta_{X \cup \{s_j\}} = \Delta_Z \left(\prod_{m=1}^{k-j} \Lambda_{j,k-m} \right) \Lambda_k = \Delta_Z \Lambda_k \left(\prod_{m=1}^{k-j} \Lambda_{j,k-m} \right)$$

$$= \Delta_Z (s_k s_{k-1} \cdots s_{j+1}) \Lambda_{j,k} \left(\prod_{m=1}^{k-j} \Lambda_{j,k-m} \right) = \Delta_X \left(\prod_{m=0}^{k-j} \Lambda_{j,k-m} \right),$$

showing that $r_{X,s_j} = \Lambda_{j,k}\Lambda_{j,k-1}\cdots\Lambda_{j,j}$.

Finally, we have:

$$r_{X,s_j} = \Lambda_{j,k}(\Lambda_{j,k-1}\cdots\Lambda_{j,j}) = \Lambda_{j,k}(\Lambda_{j,j}\cdots\Lambda_{k-1,j}) =$$

$$= \Lambda_{j,j}\sigma_{j+1}\cdots\sigma_k(\Lambda_{j,j}\cdots\Lambda_{k-1,j}) = \Lambda_{j,j}(\sigma_{j+1}\Lambda_{j,j})\cdots(\sigma_k\Lambda_{k-1,j}) = \Lambda_{j,j}\Lambda_{j+1,j}\cdots\Lambda_{k,j},$$

where we have used the induction hypothesis and commutation relations only. □

Lemma 5.15. *For every $k \in \{2,\dots,n\}$ and every $b \in \{0,\dots,e-1\}$, let $X = \{t_{b-1}, s_3,\dots,s_k\}$. Then one has:*

$$r_{X,t_b} = (t_b s_3\cdots s_k)(t_{b+1}s_3\cdots s_{k-1})\cdots(t_{b+k-3}s_3)t_{b+k-2}.$$

Proof. For every $j \in \{2,\dots,n\}$ and every $a \in \{0,\dots,e-1\}$, let us denote $X_{2,j}^{(a)} = \{t_a, s_3,\dots,s_j\}$.

We will show the result by induction in k. When $k = 2$, we need to show that $r_{X,t_b} = t_b$. But this is clear, since in this case $X = \{t_{b-1}\}$ and $X\cup\{t_b\} = \{t_{b-1},t_b\}$, hence

$$\Delta_{\{t_{b-1}\}}t_b = t_{b-1}t_b = \tau = \Delta_{\{t_{b-1},t_b\}}.$$

Let us then assume that $k > 2$, and that the result is true for smaller values of k. We know that

$$\Delta_{X_{2,k}^{(b-1)}} = \Delta_{X_{2,k-1}^{(b-1)}}(s_k\cdots s_3t_{b-1})$$

by Lemma 5.4, since this is a relation in a braid monoid and $X = X_{2,k}^{(b-1)}$ is connected. Also, by induction hypothesis we have:

$$(t_b s_3\cdots s_k)(t_{b+1}s_3\cdots s_{k-1})\cdots(t_{b+k-3}s_3)t_{b+k-2} = (t_b s_3\cdots s_k)r_{X_{2,k-1}^{(b)},t_{b+1}}.$$

Finally, recalling how the element Λ_k conjugates the atoms, we see that

$$\Lambda_k r_{X_{2,k-1}^{(b)},t_{b+1}} = r_{X_{2,k-1}^{(b-1)},t_b}\Lambda_k.$$

Therefore:

$$\begin{aligned}
& \Delta_{X_{2,k}^{(b-1)}}(t_b s_3\cdots s_k)(t_{b+1}s_3\cdots s_{k-1})\cdots(t_{b+k-3}s_3)t_{b+k-2}\\
= {} & \Delta_{X_{2,k-1}^{(b-1)}}(s_k\cdots s_3t_{b-1})(t_b s_3\cdots s_k)r_{X_{2,k-1}^{(b)},t_{b+1}}\\
= {} & \Delta_{X_{2,k-1}^{(b-1)}}\Lambda_k r_{X_{2,k-1}^{(b)},t_{b+1}}\\
= {} & \Delta_{X_{2,k-1}^{(b-1)}} r_{X_{2,k-1}^{(b-1)},t_b}\Lambda_k\\
= {} & \Delta_{X_{2,k-1}^{(b-1)}\cup\{t_b\}}\Lambda_k\\
= {} & \Lambda_2\cdots\Lambda_{k-1}\Lambda_k\\
= {} & \Delta_{X_{2,k}^{(b-1)}\cup\{t_b\}}
\end{aligned}$$

This shows the result. □

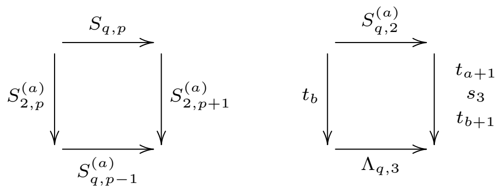
Now that we have the explicit description of the elements $r_{X,u}$ which will be relevant for our purposes, we can describe the minimal conjugators joining positive elements. There is one case which is already clear, as it follows from the results in the previous section.

Proposition 5.16. *Let $x \in B^+$ and let $X = \mathrm{supp}(x)$. Suppose that $X\cap T_e$ has at most one element. For every $u \in \mathcal{A}\setminus X$ such that $\#((X\cup\{u\})\cap T_e) \le 1$ one has $\rho_u(x) = r_{X,u}$.*

Proof. This follows immediately from Proposition 5.9, since in this case $X\cup\{u\}$ is contained in a monoid isomorphic to a braid monoid, with the same Garside structure. □

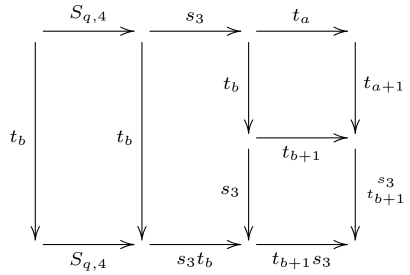
Now we need to see what happens when the atoms involved in the computations are not included into a braid monoid. We will need the following results:

Lemma 5.17. *Given $3 \leq p \leq q \leq n$ and $a, b \in \{0, \ldots, e-1\}$ with $a \neq b$, the following are LCM-diagrams:*



Proof. The first calculation is done in the monoid $\langle t_a, s_3, \ldots, s_q \rangle$, which is isomorphic to a braid monoid on q strands $\langle \sigma_1, \ldots, \sigma_{q-1} \rangle$. The element $S^{(a)}_{2,p}$ corresponds to a braid in which the first strand crosses once the strands $2, 3, \ldots, p$, and the element $S_{q,p}$ corresponds to the braid in which the q-th strand crosses once the strands $q-1, q-2, \ldots, p-1$. Both are simple braids, and their least common multiple will also be simple, thus determined by the crossings of its strands. The least common multiple must contain all the mentioned crossings (which are all different), but one readily sees that no simple element can contain exactly that set of crossings. Adding the crossing of strands 1 and q will produce a simple braid, which will then be the least common multiple. It corresponds to the element $S^{(a)}_{2,p} S^{(a)}_{q,p-1} = S_{q,p} S^{(a)}_{2,p+1}$.

For the second claim, we write $S^{(a)}_{q,2}$ as $S_{q,4} s_3 t_a$ (where the first factor would be trivial if $q = 3$). Then we have the following LCM-diagram, formed joining known LCM-diagrams:



The product of the arrows in the bottom row is precisely $\Lambda_{q,3}$ (as $t_b t_{b+1} = \tau$), hence the result is shown. $\square$

Now let us define the elements which will correspond to pre-minimal conjugators in one of the cases we need to study. Recall the elements $\sigma_{i,j,k}$ defined in the previous section. In the case of standard Garside structure of $G(e, e, n)$, we can define the same elements for $2 \leq i \leq j \leq k$, taking into account that if $i = 2$, a superscript (a) is needed, meaning that $s_2 = t_a$. Hence:

$$\sigma^{(a)}_{2,j,k} = S_{j,k} S_{j-1,k-1} \cdots S^{(a)}_{2,k-j+2} = S^{(a)}_{j,2} S_{j+1,3} \cdots S_{k,k-j+2}.$$

Also, given $3 \leq i \leq j \leq k \leq n$, let

$$\overline{\Lambda}_{i,j,k} = \Lambda_{j,i} \Lambda_{j+1,i} \cdots \Lambda_{k,i}.$$

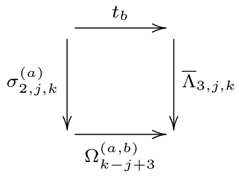
And, for any $p \in \{2, \ldots, n\}$ and every $a, b \in \{0, \ldots, e-1\}$ with $a \neq b$, we define:

$$\Omega^{(a,b)}_p = S^{(a+1)}_{2,p} S^{(a+2)}_{2,p-1} \cdots S^{(a+p-2)}_{2,3} S^{(b+p-2)}_{2,2}.$$

To avoid confusion, we notice that $\Omega^{(a,b)}_2 = t_b$, that $\Omega^{(a,b)}_3 = t_{a+1} s_3 t_{b+1}$, and that $\Omega^{(a,b)}_4 = t_{a+1} s_3 s_4 t_{a+2} s_3 t_{b+2}$.

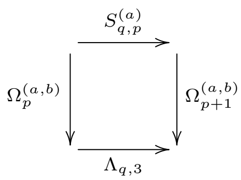
We already know, from Lemma 5.8, which is the least common multiple of $\sigma_{i,j,k}$ and an atom in a braid monoid. We need to show the only missing case:

Lemma 5.18. *Let $3 \le j \le k \le n$ and $a, b \in \{0, \dots, e-1\}$ with $a \neq b$. The following is an LCM-diagram:*

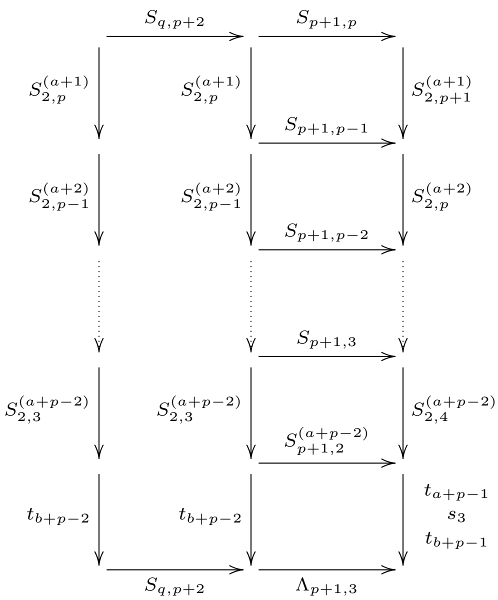


Moreover, $\sigma^{(a)}_{2,j,k} \preccurlyeq \overline{\Lambda}_{3,j,k}$.

Proof. First, we claim that the following is an LCM-diagram, where $2 \le p < q \le n$:

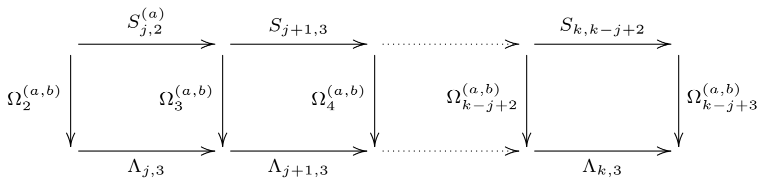


The claim is shown by the following concatenation of LCM-diagrams:



The first column holds by commutativity (since the involved atoms are not consecutive), and it only appears if $q \ge p+2$. The squares on the second column hold from Lemma 5.17. Then one just needs to notice that the products of the arrows at each edge of this diagram are precisely the elements of our claim, so the claim holds.

Now we finish the proof thanks to the following LCM-diagram:



Each square is an instance of the above claim. Since product of the arrows in the top row is $\sigma^{(a)}_{2,j,k}$, and the leftmost vertical arrow is $\Omega^{(a,b)}_2 = t_b$, this corresponds to the LCM-diagram of the statement.

Now let us show that $\sigma^{(a)}_{2,j,k} \preccurlyeq \overline{\Lambda}_{3,j,k}$. We just proved that the LCM-diagram of the statement holds, hence $\sigma^{(a)}_{2,j,k}\Omega^{(a,b)}_{k-j+3} = t_b\overline{\Lambda}_{3,j,k}$. Now, for every $m \in \{j,\dots,k\}$, we know that $t_b\Lambda_{m,3} = \Lambda_{m,3}t_{b+1}$. Hence:

$$t_b\overline{\Lambda}_{3,j,k} = t_b(\Lambda_{j,3}\Lambda_{j+1,3}\cdots\Lambda_{k,3}) = (\Lambda_{j,3}\Lambda_{j+1,3}\cdots\Lambda_{k,3})t_{b+k-j+1} = \overline{\Lambda}_{3,j,k}t_{b+k-j+1}.$$

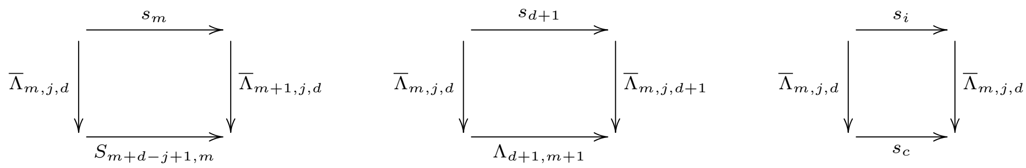
On the other hand, by definition: $\Omega^{(a,b)}_{k-j+3} \succcurlyeq t_{b+k-j+1}$, so $\Omega^{(a,b)}_{k-j+3} = Pt_{b+k-j+1}$ for some positive element P. Then

$$\sigma^{(a)}_{2,j,k}Pt_{b+k-j+1} = \sigma^{(a)}_{2,j,k}\Omega^{(a,b)}_{k-j+3} = t_b\overline{\Lambda}_{3,j,k} = \overline{\Lambda}_{3,j,k}t_{b+k-j+1}.$$

Cancelling $t_{b+k-j+1}$, we obtain that $\sigma^{(a)}_{2,j,k} \preccurlyeq \overline{\Lambda}_{3,j,k}$, as we wanted to show. □

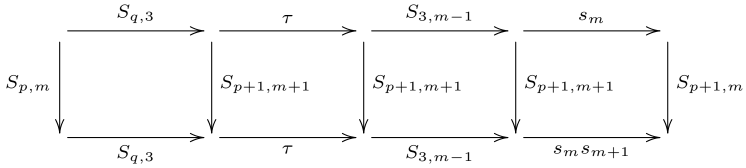
The elements $\sigma_{i,j,k}$ and $\overline{\Lambda}_{i,j,k}$ will be the pre-minimal conjugators in the case we are about to study, so we need to know the least common multiples of the latter with suitable atoms:

Lemma 5.19. *Let $3 \le m \le j \le d \le n$. The following are LCM-diagrams, where in the first one $m \neq j$, in the second one $d+1 \neq j$, and in the third one $i \neq m, j, d+1$.*

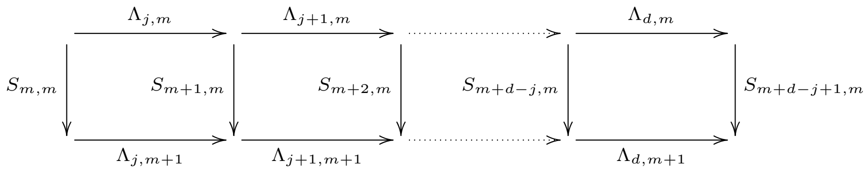


In the last diagram, s_c is some atom in S_n. Moreover, in each diagram the vertical arrow on the left is a prefix of the vertical arrow on the right.

Proof. First we observe that the following is an LCM-diagram whenever $3 \le m \le p < q$:

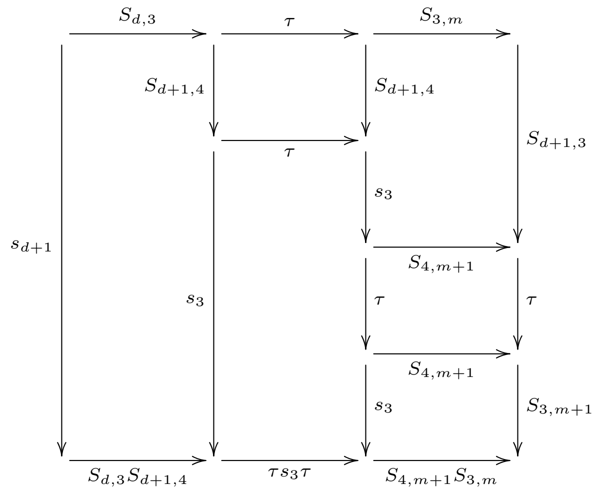


The top row represents $\Lambda_{p,m}$ and the bottom row represents $\Lambda_{p,m+1}$. Now we can concatenate the above diagram for different values of p and q (taking into account that $m < j$), to obtain:



This shows the first diagram in the statement. Let us see that the vertical arrow on the left (of the first diagram in the statement) is a prefix of the vertical arrow on the right, that is, $\overline{\Lambda}_{m,j,d} \preccurlyeq \overline{\Lambda}_{m+1,j,d}$. From the diagram, we have that $\overline{\Lambda}_{m,j,d}S_{m+d-j+1,m} = s_m\overline{\Lambda}_{m+1,j,d}$. Since $m < j$ in this case, we know that $s_m\Lambda_{k,m+1} = \Lambda_{k,m+1}s_m$ for every $k \in \{j,\dots,d\}$. Then $s_m\overline{\Lambda}_{m+1,j,d} = \overline{\Lambda}_{m+1,j,d}s_m$. On the other hand, $S_{m+d-j+1,m} = S_{m+d-j+1,m+1}s_m$. Therefore $\overline{\Lambda}_{m,j,d}S_{m+d-j+1,m+1}s_m = \overline{\Lambda}_{m+1,j,d}s_m$ and then, cancelling s_m, we obtain that $\overline{\Lambda}_{m,j,d} \preccurlyeq \overline{\Lambda}_{m+1,j,d}$.

Now suppose that $d+1 \neq j$. It is clear that for $p = j,\dots,d-1$ one has $s_{d+1}\vee\Lambda_{p,m} = s_{d+1}\Lambda_{p,m} = \Lambda_{p,m}s_{d+1}$, since s_{d+1} commutes with all atoms involving $\Lambda_{p,m}$. We then need to calculate the least common multiple of s_{d+1} and $\Lambda_{d,m}$. This is done in the following LCM-diagram:

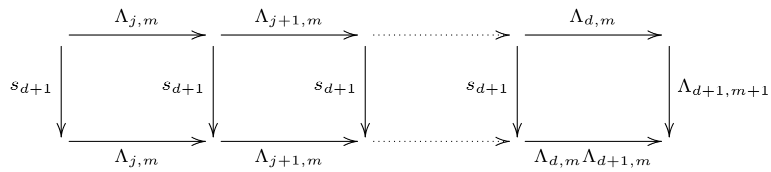


The squares in the above diagram either have already been considered, or they correspond to commuting atoms, or can be easily verified in the braid monoid, with the exception of the lower central one: $s_3 \vee \tau = s_3\tau s_3\tau = \tau s_3 \tau s_3$. These equalities can also be easily checked by decomposing them into suitable LCM-diagrams.

Notice that the lowest row corresponds to the following element:

$$\begin{aligned} S_{d,3}\underline{S_{d+1,4}}\tau s_3 \underline{\tau S_{4,m+1}}S_{3,m} &= S_{d,3}\tau \underline{S_{d+1,3}S_{4,m+1}}\tau S_{3,m} \\ &= S_{d,3}\tau S_{3,m}S_{d+1,3}\tau S_{3,m} \\ &= \Lambda_{d,m}\Lambda_{d+1,m}. \end{aligned}$$

Therefore, we have:



This shows the second LCM-diagram in the statement. In this case it is trivial that the vertical arrow on the left (of the second diagram in the statement) is a prefix of the vertical arrow on the right, since $\overline{\Lambda}_{m,j,d}\Lambda_{d+1,m} = \overline{\Lambda}_{m,j,d+1}$ by definition.

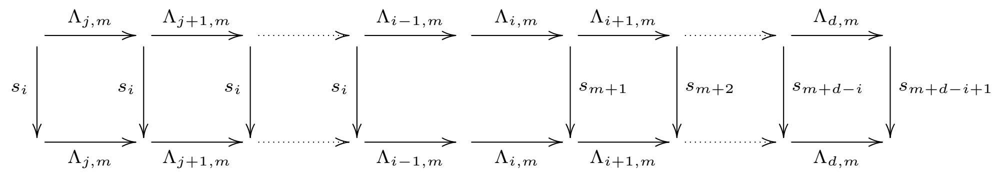
Notice that we also have shown that $s_{d+1} \vee \Lambda_{d,m} = \Lambda_{d,m}\Lambda_{d+1,m}s_{m+1}$. Since this element admits $\Lambda_{d,m}\Lambda_{d+1,m}$ as a prefix, it follows that:

$$s_{d+1} \vee \Lambda_{d,m}\Lambda_{d+1,m} = s_{d+1}\Lambda_{d,m}\Lambda_{d+1,m} = \Lambda_{d,m}\Lambda_{d+1,m}s_{m+1} \tag{5.1}$$

We will now suppose that either $i < m$ or $i > d+1$. It is easily seen that, in this case, $s_i \vee \Lambda_{p,m} = s_i\Lambda_{p,m} = \Lambda_{p,m}s_i$ for every $p \in \{j,\dots,d\}$. Therefore $s_i \vee \overline{\Lambda}_{m,j,d} = s_i\overline{\Lambda}_{m,j,d} = \overline{\Lambda}_{m,j,d}s_i$. Hence, the third diagram in the statement holds in this case, with $c = i$

If $m < i < j$, one has that $s_i \vee \Lambda_{p,m} = s_i\Lambda_{p,m} = \Lambda_{p,m}s_{i+1}$ for every $p \in \{j,\dots,d\}$. This implies that $s_i \vee \overline{\Lambda}_{m,j,d} = s_i\overline{\Lambda}_{m,j,d} = \overline{\Lambda}_{m,j,d}s_{i+j-d+1}$. So this case also holds, with $c = i+j-d+1$.

Finally, we suppose that $m \le j < i < d+1$. The statement in this case is shown by the following LCM-diagram:



The central rectangle comes from Equation (5.1), and all other squares have already been considered. This shows the final case, with $c = m + d - i + 1$.

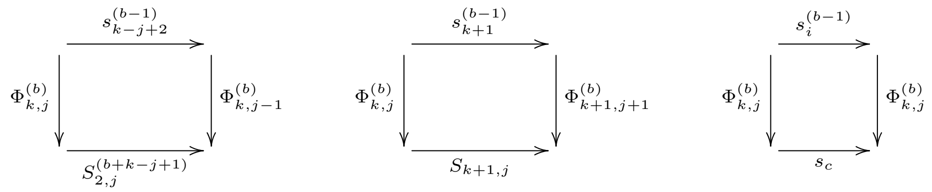
Since the vertical arrows on the left and on the right of the third diagram in the statement coincide, the former is a prefix of the latter. $\square$

The last technical result that we will need concerns the following elements, for $3 \leq j \leq k \leq n$ and $b \in \{0, \dots, e-1\}$:

$$\Phi_{k,j}^{(b)} = S_{2,k}^{(b)} S_{2,k-1}^{(b+1)} \cdots S_{2,j}^{(b+k-j)}.$$

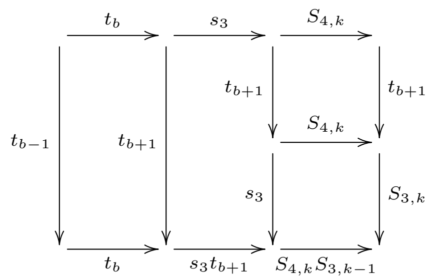
These elements will be the pre-minimal conjugators in the final case we will consider. Hence, we need to study their least common multiples with suitable atoms.

Lemma 5.20. *For $3 \leq j \leq k \leq n$ and $b \in \{0, \dots, e-1\}$, if we denote $s_2^{(b-1)} = t_{b-1}$ and $s_c^{(b-1)} = s_c$ for $c \geq 3$, the following are LCM-diagrams, where $i \neq k - j + 2, k + 1$:*



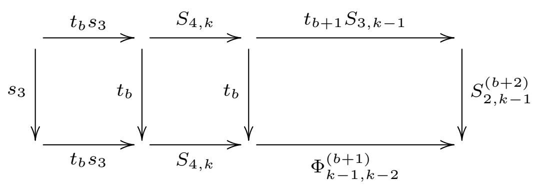
In the last diagram, s_c is some atom in S_n. Moreover, in each diagram the vertical arrow on the left is a prefix of the vertical arrow on the right.

Proof. Let us suppose first that $k - j + 2 = 2$. In this case $k = j$, so $s_{k-j+2}^{(b-1)} = t_{b-1}$ and, if $j > 3$, $\Phi_{k,j}^{(b)} = \Phi_{k,k}^{(b)} = S_{2,k}^{(b)} = t_b s_3 S_{4,k}$. Hence we have:



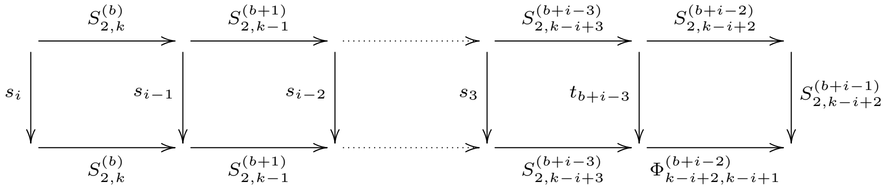
We see that the bottom row is: $t_b s_3 t_{b+1} S_{4,k} S_{3,k-1} = t_b s_3 S_{4,k} t_{b+1} S_{3,k-1} = S_{2,k}^{(b)} S_{2,k-1}^{(b+1)} = \Phi_{k,k-1}^{(b)} = \Phi_{k,j-1}^{(b)}$. And the rightmost column is $t_{b+1} S_{3,k} = S_{2,k}^{(b+1)}$. Hence, the first diagram in the statement holds in this case. If $j = k = 3$ we only have the two leftmost rectangles, but the result also holds, as $t_b s_3 t_{b+1} = S_{2,3}^{(b)} S_{2,2}^{(b-1)} = \Phi_{3,2}^{(b)} = \Phi_{k,j-1}^{(b)}$. And the rightmost column would be $t_{b+1} s_3 = S_{2,3}^{(b+1)}$, as desired.

Now suppose that $k - j + 2 = 3$. In this case we have



The square on the right hand side comes from the previous case, and the bottom row equals $\Phi_{k,k-2}^{(b)}$, so in this case the first diagram in the statement also holds.

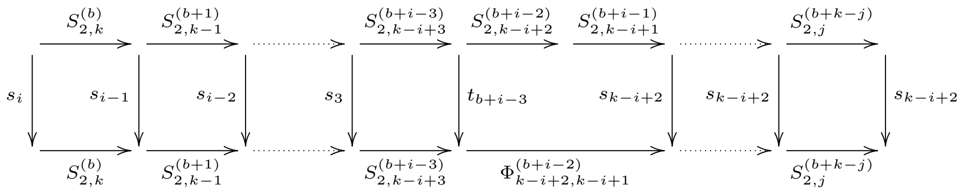
Now, if $i = k - j + 2 > 3$, we have:



Notice that the top row is $\Phi_{k,j}^{(b)}$ since $k - i + 2 = j$. The bottom row is then $\Phi_{k,j-1}^{(b)}$, and the rightmost column is $S_{2,j}^{(b+k-j+1)}$. Hence, the first diagram in the statement holds in every case.

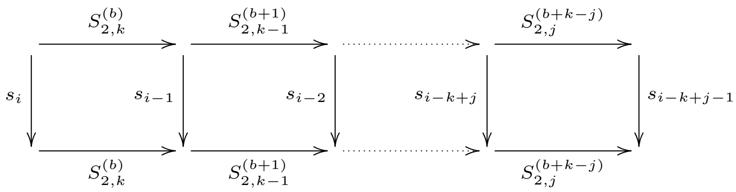
We see that the vertical arrow on the left hand side of the first diagram is a prefix of the vertical arrow on the right since, by definition, $\Phi_{k,j}^{(b)} S_{2,j-1}^{(b+k-j+1)} = \Phi_{k,j-1}^{(b)}$.

When $i < k - j + 2$, we have a diagram similar to the previous one, but where the vertical arrow becomes an element of T_e before arriving to the last square:



In this case, the top and the bottom rows are both equal to $\Phi_{k,j}^{(b)}$, so the third diagram in the statement holds, with $c = k - i + 2$.

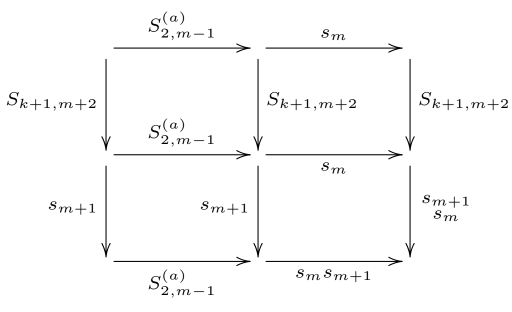
If $k - j + 2 < i < k + 1$, the diagram is easier:



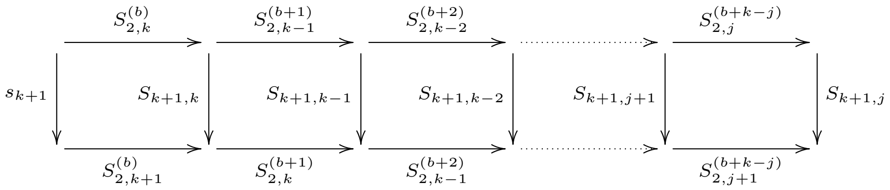
Hence, the third diagram in the statement holds for $c = i - k + j - 1$, in this case.

If $i > k+1$ the element s_i commutes with every atom in $\Phi_{k,j}^{(b)}$, hence $s_i \vee \Phi_{k,j}^{(b)} = s_i \Phi_{k,j}^{(b)} = \Phi_{k,j}^{(b)} s_i$, and the result is true also in this case, with $c = 1$. Hence, the third diagram of the statement holds in every case, and it is trivial that the vertical arrow on the left is a prefix of the vertical arrow on the right, as they coincide.

Finally, suppose that $i = k + 1$. We will first see that, for every $m = 3, \ldots, k$ and every $a \in \{1, \ldots, e-1\}$, we have that $S_{k+1,m+1} \vee S_{2,m}^{(a)} = S_{k+1,m+1} S_{2,m+1}^{(a)} = S_{2,m}^{(a)} S_{k+1,m}$. This is shown in the following diagram:



Using this property, we finish the proof with the following:



The top row is $\Phi_{k,j}^{(b)}$, and the bottom row is $\Phi_{k+1,j+1}^{(b)}$, so the second diagram in the statement holds, as we wanted to show.

We finish by showing that the vertical arrow on the left of the second diagram in the statement is a prefix of the vertical arrow on the right. We have already shown that $\Phi_{k,j}^{(b)} S_{k+1,j} = s_{k+1}\Phi_{k+1,j+1}^{(b)}$. Now we notice that $s_t S_{2,t}^{(a)} = S_{2,t}^{(a)} s_{t+1}$ for every $t \in \{j+1,\dots,k\}$ and every $a \in \{0,\dots,e-1\}$. Hence

$$s_{k+1}\Phi_{k+1,j+1}^{(b)} = s_{k+1}(S_{2,k+1}^{(b)} S_{2,k}^{(b+1)} \cdots S_{2,j+1}^{b+k-j}) = (S_{2,k+1}^{(b)} S_{2,k}^{(b+1)} \cdots S_{2,j+1}^{b+k-j}) s_j = \Phi_{k+1,j+1}^{(b)} s_j.$$

On the other hand, $S_{k+1,j} = S_{k+1,j+1} s_j$. Therefore

$$\Phi_{k,j}^{(b)} S_{k+1,j+1} s_j = \Phi_{k,j}^{(b)} S_{k+1,j} = s_{k+1}\Phi_{k+1,j+1}^{(b)} = \Phi_{k+1,j+1}^{(b)} s_j.$$

Cancelling s_j we obtain that $\Phi_{k,j}^{(b)} \preccurlyeq \Phi_{k+1,j+1}^{(b)}$, as we wanted to show.

□

We have already shown all technical lemmas, so we can finally prove the following:

Proposition 5.21. *The standard Garside structure of B for $W = G(e,e,n)$ is support-preserving.*

Proof. Let $x, y \in B^+$ be connected by a minimal positive conjugator, that is, $x^\rho = y$ where $\rho = \rho_u(x)$ for some atom u. Let $X = \mathrm{Supp}(x)$ and $Y = \mathrm{Supp}(y)$. We need to show that ρ conjugates G_X to G_Y.

Let us first suppose that $u \notin X$. The set $X \cup \{u\}$ may have several connected components, and we denote X_1 the subset of X such that $X_1 \cup \{u\}$ is the connected component containing u, and $X_2 = X \setminus X_1$.

Notice that $\Delta_{X\cup\{u\}} = \Delta_{X_1\cup\{u\}}\Delta_{X_2}$, where the two factors commute. And that $\Delta_X = \Delta_{X_1}\Delta_{X_2}$, where the two factors also commute. Hence $r_{X,u} = \Delta_X^{-1}\Delta_{X\cup\{u\}} = \Delta_{X_1}^{-1}\Delta_{X_1\cup\{u\}} = r_{X_1,u}$. We will see that $\rho = r_{X_1,u} = r_{X,u}$ in all cases in which ρ is a minimal simple conjugator.

Let us first suppose that $\#((X_1 \cup \{u\}) \cap T_e) \le 1$. In this case we can apply Proposition 5.16 to conclude that $\rho = r_{X_1,u} = r_{X,u}$. Hence, the same argument as in the proof of Proposition 5.10 applies, and it follows that $\rho = r_{X,u}$ conjugates X to Y, and then it conjugates G_X to G_Y, as claimed.

Suppose now that $\#((X_1 \cup \{u\}) \cap T_e) > 1$.

We will first assume that $u = s_j \in S_n$, hence $\#(X_1 \cap T_e) = \#(X \cap T_e) > 1$. Let $w = a_1 a_2 \cdots a_r$ be a word in X representing x. Let $\{t_{i_1},\dots,t_{i_p}\} = \{a_1,\dots,a_r\} \cap T_e$. Notice that $p > 1$, and that all atoms in $X \cap S_n$ must be present in w.

In order to compute $\rho_u(x)$, that is $\rho_{s_j}(x)$, one must compute the pre-minimal conjugators and the converging prefixes for s_j and x. We will see that each pre-minimal conjugator is a prefix of the following one, hence the converging prefix c_{i+1} will be equal to the pre-minimal conjugator $s_{i,r}$, for every $i \ge 0$, and we just need to care about the pre-minimal conjugators.

The first pre-minimal conjugator is $s_j = \sigma_{j,j,j}$. Using the relations in Lemma 5.8, we see that all pre-minimal conjugators have the form $\sigma_{i,j,k}$, and that each one is a prefix of the next one, until a pre-minimal conjugator is $\sigma_{3,j,k}$ and the next letter from w is t_a for some $a \in \{1,\dots,e-1\}$. In that case, the following pre-minimal conjugator will be $\sigma_{2,j,k}^{(a)}$.

The following steps either leave the pre-minimal conjugator untouched, or increase the third index (k), or, when the pre-minimal conjugator has the form $\sigma_{2,j,k}^{(a)}$ and the next letter from w is

t_b for some $b \neq a$, the following pre-minimal conjugator is $\overline{\Lambda}_{3,j,k}$ (by Lemma 5.18). Notice that, in all steps up to now, every pre-minimal conjugator is a prefix of the following one.

From this moment, we can apply Lemma 5.19, to find that every new pre-minimal conjugator will have the form $\overline{\Lambda}_{i,j,k}$ for some $i \leq j \leq k$, each one being a prefix of the following one. Hence, all elements in the sequence $c_0 \prec c_1 \prec \cdots \prec c_m$ will be either $\sigma_{i,j,k}$ or $\overline{\Lambda}_{i,j,k}$ for some $i \leq j \leq k$.

It remains to notice that the pre-minimal conjugators will be modified from c_i to c_{i+1}, unless $c_i = \overline{\Lambda}_{j,j,k}$ where $X_1 \cup \{s_j\} = \{t_{i_1}, \dots, t_{i_p}\} \cup \{s_3, \dots, s_k\}$, in which case all letters of w will leave the pre-minimal conjugator untouched. Therefore $c_m = \overline{\Lambda}_{j,j,k}$, that is, $\rho_{s_j}(x) = \overline{\Lambda}_{j,j,k}$. Since $\overline{\Lambda}_{j,j,k} = r_{X_1,s_j}$ by Lemma 5.14, we have that $\rho = r_{X_1,s_j} = r_{X,s_j}$, and then ρ conjugates X to Y, so G_X to G_Y, as we wanted to show.

Now suppose that $\#((X_1 \cup \{u\}) \cap T_e) > 1$ and $u = t_a \in T_e$. Notice that $X_1 \cap T_e$ must have at least one element, but cannot have more than one, as in this case it has e elements (as X is saturated being the support of x), and we would have $u \in X$, a contradiction with our initial assumption. Therefore $X_1 \cap T_e = \{t_{b-1}\}$ for some $b-1 \in \{0, \dots, e-1\}$. Since $X_1 \cup \{u\} = X_1 \cup \{t_a\}$ is connected, we must have $X_1 = \{t_{b-1}, s_3, \dots, s_k\}$ for some $k \in \{2, \dots, n\}$ (the case $k = 2$ means $X_1 = \{t_{b-1}\}$).

The interesting case happens when $t_a = t_b$. In this case we claim that $\rho = \rho_{t_b}(x) = r_{X_1,t_b}$. Hence we will have that $\rho = r_{X,t_b}$ conjugates X to Y.

To show this claim, denote $X_j = \{s_3, \dots, s_j\}$ and $X_j^{(b-1)} = \{t_{b-1}, s_3, \cdots, s_j\}$ for $j = 2, \dots, k$. We will show that all pre-minimal conjugators for t_b and x have the form $\Phi_{j,i}^{(b)}$ for some $2 \leq i \leq j \leq k$.

Indeed, the first pre-minimal conjugator is $t_b = \Phi_{2,2}^{(b)}$. The atoms which appear in any word representing x are precisely those in X, which means that we can apply Lemma 5.20 at every step, so if some pre-minimal conjugator is $\Phi_{j,i}^{(b)}$, the following one will be either $\Phi_{j,i}^{(b)}$, or $\Phi_{j,i-1}^{(b)}$, or $\Phi_{j+1,i+1}^{(b)}$. Moreover, the pre-minimal conjugator will be modified after reading all letters from x, unless it is equal to $\Phi_{k,2}^{(b)}$. Since each pre-minimal conjugator is a prefix of the following one, the elements $c_0 \prec c_1 \prec \cdots \prec c_m$ will also have this form, and the last of these elements will be $\rho = c_m = \Phi_{k,2}^{(b)}$. Since $\Phi_{k,2}^{(b)} = r_{X_1,t_b}$ by Lemma 5.15, it follows that $\rho = r_{X_1,t_b} = r_{X,t_b}$, so it conjugates X to Y, as we wanted to show.

Still considering $u \notin X$, it only remains the case $X_1 = \{t_{b-1}, s_3, \dots, s_k\}$ and $u = t_a$, with $a \neq b-1, b$. In this case we see that, when computing the pre-minimal conjugators as in the previous case, one obtains some pre-minimal conjugator of the form $\Phi_{2,p}^{(b)}$, and then some c_i will admit $\Phi_{2,p}^{(b)}$ as a prefix. This implies that c_i admits t_b as a prefix. On the other hand, from the previous case we know that $\rho_{t_b}(x) = \Phi_{2,k}^{(b)}$, which does not admit t_a as a prefix. Hence, it follows that $\rho_{t_a}(x)$ is not a minimal conjugating element by Lemma 4.39.

We have then shown that, if $u \notin X$, the associated minimal positive conjugator sends the set X to the set Y, so $\#(X) = \#(Y)$ in this case. By Proposition 4.37, when $u \in X$, the element $\rho_u(x)$ also sends G_X to G_Y (which in this case equals G_X). Therefore, the Garside structure is support-preserving. □

5.3. **Dual monoids for exceptional groups.** We fix a well-generated irreducible complex reflection group W having all its reflections of order 2, and a Coxeter element $c \in W$. As in subsection 3.5 this defines a dual braid monoid $M(c)$ endowed with a Garside structure with $\Delta = \mathbf{c}$. The first results we prove are valid for all these groups.

Proposition 5.22. *The dual braid monoid $M(c)$ is LCM-Garside.*

Proof. Recall that the dual monoids $M(c)$ are known to satisfy condition (3.1) by Proposition 3.12. In order to prove that they are LCM-Garside, we need to check the conditions of Definition 4.7. Conditions (1) and (2) are immediate, so we need to prove (3). Let $X \subset \mathcal{A}$. Then $\delta = \Delta_X$ is a simple element, so that $\delta = \mathbf{c}_0$ for $c_0 \in [1, c]$. By Proposition 3.11 (3) this is a Coxeter element, attached to some parabolic subgroup W_0 of W. Then G_δ is the subgroup of B generated by the atoms dividing δ, that is the elements $\mathbf{s}$ for $s \in W$ a reflection such that $\ell(s) + \ell(s^{-1}c_0) = \ell(c_0)$. By Proposition 3.11 (3) such reflections necessarily belong to W_0.

On the other hand, we have an injective morphism $M(c_0) \to M(c)$ which identifies $M(c_0)$ to a submonoid of $M(c)$ and induces an injective group homomorphism $B_0 \to B$, where B_0 is the group of fractions of $M(c_0)$. Thus B_0 is identified with G_δ. Moreover, we have $G_\delta^+ = G_\delta \cap G^+ = B_0 \cap M(c) = M(c_0) \subset M(c)$ by Proposition 3.13 (3).

Therefore we need to prove that, for any $c_0 \in [1, c]$, $\mathrm{Div}_{M(c_0)}(\mathbf{c_0}) = M(c_0) \cap \mathrm{Div}_{M(c)}(\mathbf{c})$. Clearly $\mathrm{Div}_{M(c_0)}(\mathbf{c_0}) \subset M(c_0) \cap \mathrm{Div}_{M(c)}(\mathbf{c})$. Let $\mathbf{m} \in M(c_0) \cap \mathrm{Div}_{M(c)}(\mathbf{c})$. We need to prove that $\mathbf{m}$ divides $\mathbf{c}_0$. For this we introduce the natural monoid morphism $\pi : M(c) \to W$, which maps $M(c_0)$ onto W_0, and $\mathbf{w}$ to w for every $w \in [1, c]$. From this we get that $\pi(\mathbf{m}) = m \in W_0 \cap [1, c] = [1, c_0]$ by property (3.1). Now, clearly the length of $\mathbf{m}$ in $M(c_0)$ is the same as the length of $\mathbf{m}$ in $M(c)$, that is the length of m with respect to set $\mathcal{R}$ of reflections of W (see Proposition 3.6 (3)). Since $m \in W_0$ this length is equal to the length with respect to set $\mathcal{R}_0 = \mathcal{R} \cap W_0$ of reflections of W_0 by Proposition 3.11 (2). So, we have $m \in [1, c_0]$ and $\ell(m) = \ell(\mathbf{m})$. By Lemma 3.10 this implies $\mathbf{m} \in \mathrm{Div}_{M(c_0)}(\mathbf{c}_0)$ and this concludes the proof. □

We want to check whether these Garside structures are support-preserving.

The following is an easy consequence of Proposition 3.9.

Lemma 5.23. *Let $\mathbf{a}, \mathbf{b}$ be two distinct atoms of $M(c)$. If $\mathbf{ba}$ is simple, then $\mathbf{a} \vee \mathbf{b} = \mathbf{ba} = \mathbf{as} = \mathbf{tb}$, for some atoms $\mathbf{s} \neq \mathbf{a}$ and $\mathbf{t} \neq \mathbf{b}$.*

Proof. Since $\mathbf{ba}$ is simple, it is balanced by Proposition 3.9 (2). It admits $\mathbf{a}$ as a suffix, so it also admits $\mathbf{a}$ as a prefix. Since $\ell(\mathbf{ba}) = 2$ by Proposition 3.9 (1), it follows that $\mathbf{ba} = \mathbf{as}$ for some atom $\mathbf{s}$. Recall that $\mathbf{a}$ and $\mathbf{b}$ are two distinct atoms, so $\mathbf{a} \vee \mathbf{b}$ cannot be a single atom, and its length must be greater than 1. Since we have a common multiple of $\mathbf{a}$ and $\mathbf{b}$ of length 2, it follows that $\mathbf{a} \vee \mathbf{b} = \mathbf{ba} = \mathbf{as}$. Moreover, $\mathbf{s} \neq \mathbf{a}$, since a simple element is square-free, by Proposition 3.9 (3).

In the same way, $\mathbf{b}$ is a prefix of $\mathbf{ba}$, hence it is also a suffix. By the above arguments, there exists an atom $\mathbf{t} \neq \mathbf{b}$ such that $\mathbf{ba} = \mathbf{tb}$. □

Lemma 5.24. *Let G be one of the 2-reflection groups G_{24}, G_{27}, G_{29}, G_{33}, G_{34}, equipped with the dual Garside structure $(G, M(c), \Delta)$, where $\pi(\Delta) = c$. Let $X \subsetneq \mathcal{A}$ be a saturated set of atoms. Let $A_0 = \{\mathbf{a} \in \mathcal{A};\ \Delta_X \mathbf{a} \preccurlyeq \Delta\}$. Then, there is a sequence $A_0 \subset A_1 \subset \cdots \subset A_r = \mathcal{A} \setminus X$ such that, for every $i > 0$ and every atom $\mathbf{a} \in A_i$, there is a (possibly trivial) saturated subset $X' \subsetneq X$ such that:*

(1) For every $\mathbf{b} \in X'$, one has $\mathbf{a} \preccurlyeq \mathbf{b} \backslash \mathbf{a}$.
(2) For every $\mathbf{b} \in X \setminus X'$, there exists $\mathbf{d} \in A_{i-1}$ such that $\mathbf{d} \preccurlyeq \mathbf{b} \backslash \mathbf{a}$.

Proof. For every such saturated set X, one applies by computer the following algorithm, with the notation that $a \leftarrow b$ indicates that the variable a receives the content of b, and where $\mathcal{P}_{sat}(\mathcal{A})$ denotes de saturated subsets of $\mathcal{A}$.

(1) Compute $\mathcal{A}_X = \{\mathbf{a} \in \mathcal{A} \mid \mathbf{a} \prec \Delta_X^{-1} \Delta\}$.
(2) Set $\mathcal{B} = \mathcal{A} \setminus (X \cup \mathcal{A}_X)$, and $\mathcal{D} = \{(\mathbf{a}, \mathbf{b}, V) \in \mathcal{B} \times X \times \mathcal{P}_{sat}(\mathcal{A}) \mid \Delta_V = \mathbf{b} \backslash \mathbf{a} = \mathbf{b}^{-1}(\mathbf{a} \vee \mathbf{b})\}$.
(3) Set $\mathcal{N} = \emptyset$.
(4) Repeat
  (a) $\mathcal{A}_X \leftarrow \mathcal{A}_X \cup \mathcal{N}$ `#` $\mathcal{A}_X$ `will be` A_i `for` $i = 0, 1, 2 \ldots$
  (b) $\mathcal{D} \leftarrow \{(\mathbf{a}, \mathbf{b}, V) \in \mathcal{D} \mid \mathcal{A}_X \cap V = \emptyset\}$
  `# Remove those satisfying` $\mathbf{d} \preccurlyeq \mathbf{b} \backslash \mathbf{a}$ `for some` $\mathbf{d} \in A_{i-1}$
  (c) For every $\mathbf{a} \in \mathcal{A}$, do :
    • If every $(\mathbf{a}, \mathbf{b}, V) \in \mathcal{D}$ satisfies $\mathbf{a} \in V$ (that is, $\mathbf{a} \preccurlyeq \mathbf{b} \backslash \mathbf{a}$) and there is some saturated $X' \subsetneq X$ such that every $(\mathbf{a}, \mathbf{b}, V)$ satisfies $\mathbf{b} \in X'$, then $\mathcal{D} \leftarrow \{(\mathbf{z}, \mathbf{b}, V) \in \mathcal{D} \mid \mathbf{z} \neq \mathbf{a}\}$
    `# Remove if all the remaining cases satisfy condition (1).`
  (d) $\mathcal{N} \leftarrow \{\mathbf{z} \in \mathcal{B} \setminus \mathcal{A}_X \mid \forall (\mathbf{a}, \mathbf{b}, V) \in \mathcal{D}\ \mathbf{z} \neq \mathbf{a}\}$
  `# The new atoms to add to` $\mathcal{A}_X$ `are those completely removed from` $\mathcal{D}$.
(5) until $\mathcal{N} = \emptyset$.

(6) if $\mathcal{D} = \emptyset$ then return true else return false.

The fact that this algorithm terminates in all these cases with the statement true proves the claim. □

Proposition 5.25. *Let G be one of the 2-reflection groups G_{24}, G_{27}, G_{29}, G_{33}, G_{34}. The dual Garside structure $(G, M(c), \Delta)$ is support-preserving.*

Proof. Let $\mathbf{x} \in M(c)$ be a positive element, and let $X = \mathrm{Supp}(\mathbf{x})$. If $X = \mathcal{A}$, the structure is support preserving by Proposition 4.37. Hence we can assume that $X \subsetneq \mathcal{A}$.

Consider the chain of subsets $A_0 \subset A_1 \subset \cdots \subset A_r = \mathcal{A} \setminus X$ described in Lemma 5.24.

Let $\mathbf{a} \in A_0$. By definition, $\Delta_X \mathbf{a}$ is a simple element. Given $\mathbf{b} \in X$, we know that $\mathbf{b}$ is a suffix of Δ_X, hence $\mathbf{ba}$ is a suffix of the simple element $\Delta_X \mathbf{a}$, so $\mathbf{ba}$ is simple. By Lemma 5.23, $\mathbf{a} \vee \mathbf{b} = \mathbf{ba} = \mathbf{as}$ for some atom $\mathbf{s}$. This holds for every $\mathbf{b} \in X$, therefore conjugation by $\mathbf{a}$ sends X to some subset $Y \subset \mathcal{A}$. It follows that $\mathbf{a}^{-1}\mathbf{xa}$ is a positive element $\mathbf{y}$, and this implies that $\rho_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}$. So, in order to see that this minimal positive conjugator preserves the support, we need to show that $\mathrm{Supp}(\mathbf{y}) = Y$.

Consider a subset $X' = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset X$ such that $\mathbf{x}_1 \vee \cdots \vee \mathbf{x}_m = \Delta_X$. Let us see that $\mathbf{x}_1\mathbf{a} \vee \cdots \vee \mathbf{x}_m\mathbf{a} = \Delta_X \mathbf{a}$. Since $\Delta_X \mathbf{a}$ is simple and each $\mathbf{x}_i$ is a suffix of Δ_X, it follows that $\mathbf{x}_i\mathbf{a}$ is a suffix, hence a prefix of $\Delta_X \mathbf{a}$, for $i = 1, \ldots, m$. Therefore $\mathbf{x}_1\mathbf{a} \vee \cdots \vee \mathbf{x}_m\mathbf{a} \preccurlyeq \Delta_X \mathbf{a}$. But then we have that $\Delta_X = \mathbf{x}_1 \vee \cdots \vee \mathbf{x}_m \preccurlyeq \mathbf{x}_1\mathbf{a} \vee \cdots \vee \mathbf{x}_m\mathbf{a} \preccurlyeq \Delta_X \mathbf{a}$. Since $\ell(\Delta_X \mathbf{a}) = \ell(\Delta_X) + 1$ (because the Garside structure is homogeneous), if an element α satisfies $\Delta_X \preccurlyeq \alpha \preccurlyeq \Delta_X \mathbf{a}$, we must have either $\ell(\alpha) = \ell(\Delta_X)$ or $\ell(\alpha) = \ell(\Delta_X) + 1$. In the former case $\alpha = \Delta_X$, and in the latter $\alpha = \Delta_X \mathbf{a}$, since two positive elements of the same length, one being prefix of the other, must be equal. But in our case $\mathbf{x}_1\mathbf{a}$ cannot be a prefix of Δ_X, since $\mathbf{a} \notin X$ and X is saturated. Therefore $\mathbf{x}_1\mathbf{a} \vee \cdots \vee \mathbf{x}_m\mathbf{a} \neq \Delta_X$, and then $\mathbf{x}_1\mathbf{a} \vee \cdots \vee \mathbf{x}_m\mathbf{a} = \Delta_X \mathbf{a}$, as claimed.

We keep considering $X' = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset X$ such that $\mathbf{x}_1 \vee \cdots \vee \mathbf{x}_m = \Delta_X$. For $i = 1, \ldots, m$, let $\mathbf{y}_i = \mathbf{a}^{-1}\mathbf{x}_i\mathbf{a} \in Y$, and denote $Y' = \{\mathbf{y}_1, \ldots, \mathbf{y}_m\} \subset Y$. Then we have:

$$\mathbf{y}_1 \vee \cdots \vee \mathbf{y}_m = \mathbf{a}^{-1}\mathbf{x}_1\mathbf{a} \vee \cdots \vee \mathbf{a}^{-1}\mathbf{x}_m\mathbf{a} = \mathbf{a}^{-1}(\mathbf{x}_1\mathbf{a} \vee \cdots \vee \mathbf{x}_m\mathbf{a}) = \mathbf{a}^{-1}\Delta_X \mathbf{a}.$$

Notice that the final result of the equality does not depend on X', but on X. If $X' = X$ we have $Y' = Y$, by construction. Hence the above equality shows that $\Delta_Y = \mathbf{a}^{-1}\Delta_X\mathbf{a}$. But then, for every other subset X' we obtain $\mathbf{y}_1 \vee \cdots \vee \mathbf{y}_m = \Delta_Y$ as well.

Let us see that Y is a saturated set of atoms. Let $\mathbf{s}$ be an atom such that $\mathbf{s} \preccurlyeq \Delta_Y$. We need to show that $\mathbf{s} \in Y$. From $\mathbf{s} \preccurlyeq \Delta_Y$ it follows that $\mathbf{as} \preccurlyeq \mathbf{a}\Delta_Y = \Delta_X\mathbf{a}$. Since the latter is a simple element, it follows that $\mathbf{as}$ is simple. Hence, by Lemma 5.23, there exists an atom $\mathbf{t}$ such that $\mathbf{as} = \mathbf{ta}$. Now $\mathbf{ta}$ is a prefix, hence a suffix of $\Delta_X\mathbf{a}$, and this implies that $\mathbf{t}$ is a suffix of Δ_X. Since X is saturated, $\mathbf{t} \in X$. Therefore $\mathbf{s} = \mathbf{a}^{-1}\mathbf{ta} \in \mathbf{a}^{-1}X\mathbf{a} = Y$. This shows that Y is saturated.

Now let $\mathbf{u}_1 \cdots \mathbf{u}_k$ be a decomposition of $\mathbf{x}$ as a product of atoms (notice that $\mathbf{u}_i \in X$ for every i). For $i = 1, \ldots, k$, let $\mathbf{v}_i = \mathbf{a}^{-1}\mathbf{u}_i\mathbf{a}$. Then $\mathbf{y} = \mathbf{v}_1 \cdots \mathbf{v}_k$. We know that $\mathrm{Supp}(\mathbf{x}) = X$, that is, $\Delta_X = \mathbf{u}_1 \vee \cdots \vee \mathbf{u}_k$. From the above arguments, taking $X' = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ and $Y' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ (it does not matter if some atoms are repeated), we have $\mathbf{v}_1 \vee \cdots \vee \mathbf{v}_k = \Delta_Y$, where Y is saturated. Hence $\mathrm{Supp}(\mathbf{y}) = Y$.

We know that $\rho_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}$ conjugates X to Y, so it conjugates G_X to G_Y. Hence $\rho_{\mathbf{a}}(\mathbf{x})$ preserves the support if $\mathbf{a} \in A_0$.

Now suppose that $\mathbf{a} \in A_i \setminus A_0$ for some $i > 0$. We claim that $\rho_{\mathbf{a}}(\mathbf{x})$ is not a minimal positive conjugator. We show this claim by induction on i.

Suppose that $i = 1$, and let X' be the subset of X described in Lemma 5.24. Write $\mathbf{x} = \mathbf{u}_1 \cdots \mathbf{u}_k$ as a product of atoms. Notice that not all these atoms can belong to X', otherwise $\mathrm{Supp}(\mathbf{x}) \subset X' \subsetneq X$, which is not possible. Let j be the first index such that $\mathbf{u}_j \in X \setminus X'$. We proceed to compute the pre-minimal conjugators, starting with $\mathbf{s}_{0,0} = \mathbf{a}$. If $\mathbf{a} \preccurlyeq \mathbf{s}_{0,t}$ for some $t < j$, then $\mathbf{a} \preccurlyeq \mathbf{u}_t \backslash \mathbf{a} \preccurlyeq \mathbf{u}_t \backslash \mathbf{s}_{0,t} = \mathbf{s}_{0,t+1}$. This shows that $\mathbf{a} \preccurlyeq \mathbf{s}_{0,j-1}$. Then, as $\mathbf{u}_j \in X \setminus X'$, there exists $\mathbf{d} \in A_0$ such that $\mathbf{d} \preccurlyeq \mathbf{u}_j \backslash \mathbf{a} \preccurlyeq \mathbf{u}_j \backslash \mathbf{s}_{0,j-1} = \mathbf{s}_{0,j}$. So $\mathbf{d}$ is a prefix of a pre-minimal conjugator. But then we know that all subsequent pre-minimal conjugators will admit $\mathbf{d}$ as a prefix, since $\mathbf{u} \backslash \mathbf{d} = \mathbf{d}$ for every $\mathbf{u} \in X$ and every $\mathbf{d} \in A_0$. Therefore, we will have $\mathbf{d} \preccurlyeq \mathbf{s}_{0,k}$, which implies that $\mathbf{d}$ is a prefix of the converging prefix c_1. We already know from the previous case that $\rho_{\mathbf{d}}(\mathbf{x}) = \mathbf{d}$, so it does not admit $\mathbf{a}$ as a prefix. Therefore, by Lemma 4.39, $\rho_{\mathbf{a}}(\mathbf{x})$ is not a minimal positive conjugator for $\mathbf{x}$.

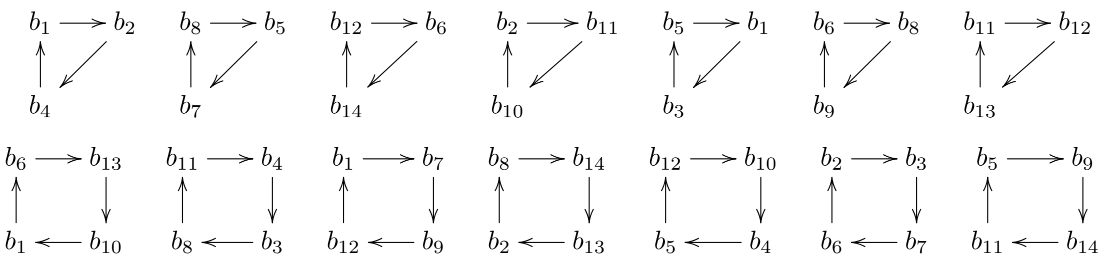
Now suppose that $i > 1$ and that the claim holds for smaller values of i. Let $\mathbf{a} \in A_i \setminus A_0$ and, as above, let X' be the subset of X described in Lemma 5.24. Using the above argument, we know that some pre-minimal conjugator for $\mathbf{a}$ will admit an atom $\mathbf{b} \in A_{i-1}$ as a prefix. Applying Lemma 5.24 again, all subsequent pre-minimal conjugators will admit an atom from A_{i-1} as a prefix. Hence, a converging prefix for $\mathbf{a}$ and $\mathbf{x}$ admits some atom $\mathbf{s} \in A_{i-1}$ as a prefix. It follows that $\mathbf{s} \preccurlyeq \rho_{\mathbf{a}}(\mathbf{x})$. If $\mathbf{s} \in A_0$, the argument in the previous paragraph shows that $\rho_{\mathbf{a}}(\mathbf{x})$ is not a minimal positive conjugator. If, on the contrary, $\mathbf{s} \in A_{i-1} \setminus A_0$, from $\mathbf{s} \preccurlyeq \rho_{\mathbf{a}}(\mathbf{x})$ we obtain $\rho_{\mathbf{s}}(\mathbf{x}) \preccurlyeq \rho_{\mathbf{a}}(\mathbf{x})$. But we know that $\rho_{\mathbf{s}}(\mathbf{x})$ is not a minimal positive conjugator, by induction hypothesis. Therefore $\rho_{\mathbf{a}}(\mathbf{x})$ is not a minimal positive conjugator, and the claim is shown.

Since we have $A_r = \mathcal{A} \setminus X$ by Lemma 5.24, we have shown that for every atom $\mathbf{a} \in \mathcal{A} \setminus (A_0 \cup X)$ the element $\rho_{\mathbf{a}}(\mathbf{x})$ is not a minimal positive conjugator, while for every $\mathbf{a} \in A_0$ the element $\rho_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}$ is a positive minimal conjugator and preserves the support, sending the set X to a set Y of the same size. By Proposition 4.37, the atoms in X also preserve the support, so the Garside structure is support-preserving. $\square$

In order to better understand the above result, we will use G_{24} as an example. See [40] for a detailed explanation of this Garside structure of G_{24}.

There are 14 atoms in the mentioned Garside structure of G_{24}, named $b_1, \dots, b_{14}$, and the Garside element is $c = b_1 b_2 b_3$. A presentation of the Garside monoid (and of the Garside group), using these atoms is given in Figure 1 (note that these are **not** LCM diagrams).

Figure 1. Defining relations of the dual braid monoid in type G_{24}.



The relations are defined as follows. In every diagram of Figure 1, all possible products of two consecutive elements (following the arrows) determine the same element in the monoid (and in the group). For instance, the first diagram produces the relations $b_1 b_2 = b_2 b_4 = b_4 b_1$.

Notice that all relations are homogeneous (with elements of length 2). Hence, as the Garside element $(b_1 b_2 b_3)$ has length 3, all proper simple elements have length at most 2. If two different atoms lie in the same diagram, their least common multiple has length 2 (for example, $b_6 \vee b_{10} = b_6 b_{13} = b_{10} b_1$). On the other hand, if there is no diagram containing two given atoms, their least common multiple is c (this happens in G_{24}, but it does not in other groups considered in this section).

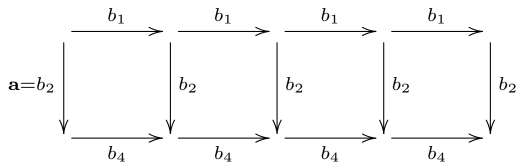
It is worth mentioning that we have ordered the diagrams in Figure 1 in such a way that, if we conjugate an element in Figure 1 by c, we obtain the corresponding element in the diagram on its right (if the element appears in the rightmost diagram, the corresponding element is in the leftmost one). So conjugation by c permutes the atoms as follows:

$$b_1 \to b_8 \to b_{12} \to b_2 \to b_5 \to b_6 \to b_{11} \to b_1,$$

$$b_4 \to b_7 \to b_{14} \to b_{10} \to b_3 \to b_9 \to b_{13} \to b_4.$$

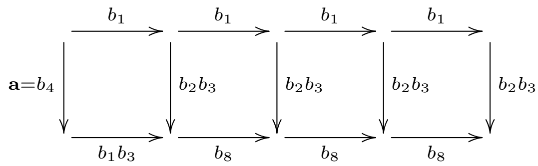
Now notice that the nontrivial standard parabolic subgroups of G_{24}, with respect to this structure, are the following ones: cyclic subgroups generated by one atom; subgroups generated by the atoms in a diagram of Figure 1; the whole group G_{24}.

Example 5.26. Let $\mathbf{x} = b_1^4$. We have $X = \mathrm{Supp}(x) = \{b_1\}$. Then $A_0 = \{b_2, b_3, b_6, b_7\}$. If $\mathbf{a} \in A_0$, for instance if $\mathbf{a} = b_2$, the computation of the pre-minimal conjugators for $\mathbf{a}$ and $\mathbf{x}$ is the following:



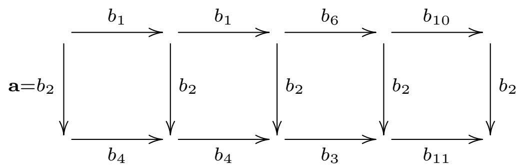
Hence, the only converging prefix is $\mathbf{a}$, and $\rho_{\mathbf{a}} = \mathbf{a} = b_2$. In this case, one has $\mathbf{x}^{\mathbf{a}} = (b_1^4)^{\mathbf{a}} = b_4^4 = \mathbf{y}$, and $\mathbf{a}$ conjugates $X = \{b_1\}$ to $Y = \{b_4\}$.

On the other hand, if $\mathbf{a} \notin X \cup A_0$, for instance if $\mathbf{a} = b_4$, we have the following:



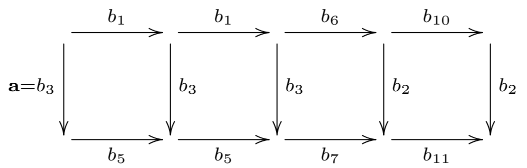
From this diagram we see that $b_2 \in A_0$ is a prefix of a converging prefix for $\mathbf{a}$ (the converging prefix $c_1 = b_4 \vee b_2 b_3 = \Delta$), hence $b_1 = \rho_{b_2}(\mathbf{x}) \preccurlyeq \rho_{\mathbf{a}}(\mathbf{x})$. Hence $\rho_{\mathbf{a}}(x)$ is not minimal.

Example 5.27. Let $\mathbf{x} = b_1 b_1 b_6 b_{10}$. In this case $X = \{b_1, b_6, b_{10}, b_{13}\}$, and $A_0 = \{b_2\}$. If $\mathbf{a} \in A_0$, that is if $\mathbf{a} = b_2$, the computation of the pre-minimal conjugators for $\mathbf{a}$ and $\mathbf{x}$ is:



We see that $\rho_{\mathbf{a}}(\mathbf{x}) = \mathbf{a} = b_2$, and that $\mathbf{a}$ conjugates $X = \{b_1, b_6, b_{10}, b_{13}\}$ to $Y = \{b_4, b_3, b_{11}, b_8\}$. Hence it conjugates $\mathbf{x}$ to $\mathbf{y} = b_4 b_4 b_3 b_{11}$, whose support is Y.

On the other hand, if $\mathbf{a} \notin X \cup A_0$, for instance if $\mathbf{a} = b_3$, we have the following:



In this case we see that b_3 appears as a pre-minimal conjugator as long as the letters of $\mathbf{x}$ equal b_1, but it changes to a pre-minimal conjugator starting with b_2 when one encounters the letter b_6, which does not belong to A_0. From that moment on, the forthcoming pre-minimal conjugators start with b_2, so the converging prefix $c_1 = b_3 \vee b_2$ admits b_2 as a prefix. Therefore, $\rho_{\mathbf{a}}(\mathbf{x})$ is not a minimal positive conjugator in this case.

We have then shown that the dual Garside structures of the groups G_{24}, G_{27}, G_{29}, G_{33} and G_{34} are support-preserving LCM-structures. Therefore, by Theorem 4.31, every element admits a parabolic closure.

6. Intersection of parabolic subgroups

In this section we will adapt the arguments in [14] to show that, in any irreducible complex braid group (except possibly in G_{31}), the intersection of parabolic subgroups is a parabolic subgroup.

6.1. **The main Garside-theoretic argument.** We will consider first the complex braid groups which admit a support-preserving LCM-Garside structure (B, B^+, Δ), as shown in the previous section. We also notice that, in all those Garside structures, we have the following two properties:

(1) Relations are homogeneous. This implies that all positive words representing an element $x \in B^+$ have the same length, and we can use that quantity as the length of x.
(2) Simple elements are square-free. This means that, for every atom $a \in \mathcal{A}$, the element aa is not simple. This implies that aa can never be a factor of a simple element. It also implies that, given two simple elements α and β, if the set $\mathrm{Suff}(\alpha)$ of atoms which are suffixes of α contains the set $\mathrm{Pref}(\beta)$ of atoms which are prefixes of β, then the product $\alpha\beta$ is in left normal form.

The first property is clear, because the presentations providing the Garside structures are homogeneous. The second one is, by Proposition 3.3, a consequence of the fact that all the Garside structures involved here can be defined as interval monoids associated to a generating set for W made of involutions. This covers the case of the monoids attached to real reflection groups, to the $G(e, e, n)$, and to the well-generated exceptional 2-reflection groups.

In this section, B will be one of the mentioned complex braid groups, and (B, B^+, Δ) will denote the studied Garside structure, which is a support-preserving LCM-Garside structure, and satisfies the above two properties. Moreover, the parabolic subgroups for these Garside structures correspond to the parabolic subgroups of B which were defined independently on the choice of a Garside structure, so there is no ambiguity here when using this term.

Given such a complex braid group B endowed with the corresponding Garside structure, we start by defining, for every parabolic subgroup H, a special element $z_H \in H$. In the case of an irreducible parabolic subgroup, we shall see that it coincides that our previous definition. We start by defining it for standard parabolic subgroups. Notice that this definition coincides with the ones in [29] and [14] for Artin–Tits groups of spherical type.

Definition 6.1. Let $B_0 = G_X$ be a standard parabolic subgroup of B. We define $z_{B_0} = (\Delta_X)^e$, where e is the smallest positive integer such that $(\Delta_X)^e$ is central in B_0.

Proposition 6.2. *Let $B_1 = G_X$ and $B_2 = G_Y$ be two standard parabolic subgroups of B which are conjugate. Then Δ_X is conjugate to Δ_Y. Moreover, if $z_{B_1} = (\Delta_X)^e$ then $z_{B_2} = (\Delta_Y)^e$, so z_{B_1} is conjugate to z_{B_2}.*

Proof. Let $c_1 \in B$ be such that $(B_1)^{c_1} = B_2$, and consider $(\Delta_X)^{c_1} \in B_2$. Since B_2 is standard, we can conjugate $(\Delta_X)^{c_1}$ by an element $c_2 \in B_2$ (a conjugating element for iterated swaps), so that $(\Delta_X)^{c_1c_2}$ is recurrent. But $\mathsf{R}(\Delta_X) = C_+(\Delta_X)$, hence $(\Delta_X)^{c_1c_2}$ is positive.

We can now assume that B_1 is not equal to B, otherwise $B_1 = B_2 = B$ and the result is trivial. It follows that $\inf(\Delta_X) = 0$ and $\sup(\Delta_X) = 1$, and that one cannot conjugate Δ_X to an element of bigger infimum. Hence $\inf((\Delta_X)^{c_1c_2}) = 0$. We can now apply iterated *decycling*, to decrease the supremum of $(\Delta_X)^{c_1c_2}$ to its minimal possible value (which is 1). All the conjugating elements will belong to B_2, hence there is $c_3 \in B_2$ such that $(\Delta_X)^{c_1c_2c_3}$ is a simple element in $B_2 = G_Y$. Therefore $1 \preccurlyeq (\Delta_X)^{c_1c_2c_3} \preccurlyeq \Delta_Y$.

Since relations are homogeneous in B, it follows that $|\Delta_X| = |(\Delta_X)^{c_1c_2c_3}| \leq |\Delta_Y|$. Now, inverting the roles of B_1 and B_2, the same proof yields $|\Delta_Y| \leq |\Delta_X|$. Therefore $|\Delta_X| = |\Delta_Y|$.

Recall that $(\Delta_X)^{c_1c_2c_3} \preccurlyeq \Delta_Y$. Since both positive elements have the same length, they are equal. This shows that Δ_X is conjugate to Δ_Y.

Now notice that $(G_X)^{c_1c_2c_3} = (G_Y)^{c_2c_3} = G_Y$. Hence the center of G_X is conjugated by $c_1c_2c_3$ to the center of G_Y. Since $((\Delta_X)^k)^{c_1c_2c_3} = (\Delta_Y)^k$ for every $k > 0$, it follows that if $z_{B_1} = (\Delta_X)^e$ then $z_{B_2} = (\Delta_Y)^e$. Clearly this implies that $z_{B_1}$is conjugate (by $c_1c_2c_3$) to z_{B_2}. □

Now we can define z_{B_0} for an arbitrary parabolic subgroup B_0:

Definition 6.3. Let B_0 be a parabolic subgroup of B. Let $c \in B$ be such that B_0^c is standard, say $B_0^c = G_X$. Then we define $z_{B_0} = (z_{G_X})^{c^{-1}}$.

Proposition 6.4. *Under the above conditions, the element z_{B_0} is well defined.*

Proof. Suppose that $c_1, c_2 \in B$ are such that $B_0^{c_1} = G_X$ and $B_0^{c_2} = G_Y$. We need to show that $(z_{G_X})^{c_1^{-1}} = (z_{G_Y})^{c_2^{-1}}$. But this follows from Proposition 6.2, since we have $(G_X)^{c_1^{-1}c_2} = G_Y$, and this implies that $(z_{G_X})^{c_1^{-1}c_2} = z_{G_Y}$. □

Proposition 6.5. *If B_0 is an irreducible parabolic subgroup of B, then z_{B_0} is indeed the canonical element defined in the introduction. In particular, it is independent on the choice of a Garside structure.*

Proof. We only need to check this for a standard parabolic subgroup. The proof is then easy in all cases, the expression of the canonical element z_{B_0} being known in all cases. □

Notice that the above statement is specific to the case where B_0 is irreducible. For instance, if $B = \mathcal{B}_5$ with generators $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, and $B_0 = \langle \sigma_1 \rangle \times \langle \sigma_3, \sigma_4 \rangle$, then the element '$z_{B_0}$' considered in this section would be $\sigma_1^2(\sigma_3\sigma_4\sigma_3)^2$ when considering the classical Garside structure, while the one associated to the dual braid monoid would yield $z_{B_0} = \sigma_1^3(\sigma_3\sigma_4)^3 = \sigma_1^3(\sigma_3\sigma_4\sigma_3)^2$.

The following are three important properties of z_{B_0}:

Proposition 6.6. *Let B_0 be a parabolic subgroup of B. Then* $\mathrm{PC}(z_{B_0}) = B_0$.

Proof. If B_0 is standard, say $B_0 = G_X$ (for a saturated X), then Δ_X is a positive element whose support is X. Hence $\mathrm{PC}(\Delta_X) = G_X$. By Theorem 4.32, $\mathrm{PC}(z_{B_0}) = \mathrm{PC}((\Delta_X)^e) = G_X = B_0$.

If B_0 is not standard, let c be such that B_0^c is standard, so $\mathrm{PC}(z_{B_0^c}) = B_0^c$. By definition, $z_{B_0} = (z_{B_0^c})^{c^{-1}}$. Then, by Lemma 4.30 and by the standard case, $\mathrm{PC}(z_{B_0}) = \mathrm{PC}((z_{B_0^c})^{c^{-1}}) = \mathrm{PC}(z_{B_0^c})^{c^{-1}} = (B_0^c)^{c^{-1}} = B_0$. □

Proposition 6.7. *Let B_1 and B_2 be two parabolic subgroups of B. For every $c \in B$, one has $(B_1)^c = B_2$ if and only if $(z_{B_1})^c = z_{B_2}$.*

Proof. Suppose that $(B_1)^c = B_2$. Let $d \in B$ be such that $B_1^d = G_X$. Then $c^{-1}d$ is such that $B_2^{c^{-1}d} = G_X$ and, by definition, $z_{B_1} = (z_{G_X})^{d^{-1}}$ and $z_{B_2} = (z_{G_X})^{d^{-1}c}$. Therefore $(z_{B_1})^c = z_{B_2}$.

Conversely, suppose that $(z_{B_1})^c = z_{B_2}$. Then, by Lemma 4.30, $\mathrm{PC}(z_{B_1})^c = \mathrm{PC}(z_{B_2})$ and, by Proposition 6.6, $(B_1)^c = B_2$. □

Proposition 6.8. *Let B_0 be a parabolic subgroup of B. Then B_0 is standard if and only if z_{B_0} is positive.*

Proof. If B_0 is standard, say $B_0 = G_X$, we know that z_{B_0} is a positive power of Δ_X, so it is positive.

Conversely, suppose that z_{B_0} is positive. Then it is recurrent and, by Theorem 4.29, $PC(z_{B_0}) = G_X$, where $X = \mathrm{Supp}(z_{B_0})$. On the other hand, by Proposition 6.6 we know that $PC(z_{B_0}) = B_0$. Therefore $B_0 = G_X$ is standard. □

Let us then prove that the intersection of parabolic subgroups is a parabolic subgroup. We need the following definition, taken from [14]:

Definition 6.9. (See [14, Definition 9.3]) For every element $\gamma \in B$ we define an integer $\varphi(\gamma)$ as follows: Conjugate γ to $\gamma' \in \mathsf{R}(\gamma)$. Let $U = \mathrm{Supp}(\gamma')$. Then let $\varphi(\gamma) = |\Delta_U|$, the length of the positive element Δ_U as a word in the atoms.

Notice that we used $\mathsf{R}(\gamma)$, instead of the set $RSSS_\infty(\gamma)$ which is used in [14]. This is because it is theoretically much simpler, and satisfies all the needed properties.

Proposition 6.10. [14, Proposition 9.4] *The integer $\varphi(\gamma)$ is well defined. Moreover, if γ is conjugate to a positive element, then $\varphi(\gamma) = |\Delta_X|$, where $X = \mathrm{Supp}(\beta)$ for any positive element β conjugate to γ.*

Proof. We adapt the proof in [14]. Suppose that $\gamma', \gamma'' \in \mathsf{R}(\gamma)$, and let $U = \mathrm{Supp}(\gamma')$ and $V = \mathrm{Supp}(\gamma'')$. Then $\mathrm{PC}(\gamma') = G_U$ and $\mathrm{PC}(\gamma'') = G_V$. Since γ' and γ'' are conjugate, so are its parabolic closures G_U and G_V. Hence, by Proposition 6.2, the elements Δ_U and Δ_V are also conjugate. The homogeneous relations of B imply that $|\Delta_U| = |\Delta_V|$. This shows that $\varphi(\gamma)$ is well defined.

If γ is conjugate to a positive element, the result follows from Proposition 4.17, as in that case $\mathsf{R}(\gamma) = C^+(\gamma)$. $\square$

Finally, we can adapt the main result in [14] to our case, but simplifying considerably the proof. We will not need results analogous to Lemma 9.1 and Lemma 9.2 in [14], and the final argument is much simpler.

Theorem 6.11. *Let B_1 and B_2 be two parabolic subgroups of B with Garside structure (B, B^+, Δ) as above. Then $B_1 \cap B_2$ is also a parabolic subgroup.*

Proof. First, we can assume that $B_1 \cap B_2 \neq \{1\}$, otherwise the result is trivial. Then we consider a nontrivial element $\alpha \in B_1 \cap B_2$ such that $\varphi(\alpha)$ is maximal, and we will show that $B_1 \cap B_2 = \mathrm{PC}(\alpha)$, so $B_1 \cap B_2$ is a parabolic subgroup.

Up to conjugation of α, B_1 and B_2 by the same element, we can assume that α is a recurrent element. Hence, if $X = \mathrm{Supp}(\alpha)$, we have that $\mathrm{PC}(\alpha) = G_X$, and we need to show that $G_X = B_1 \cap B_2$.

Since α belongs to the parabolic subgroup B_i, for $i = 1, 2$, it follows that $\mathrm{PC}(\alpha) \subset B_i$ for $i = 1, 2$. Hence $G_X = \mathrm{PC}(\alpha) \subset B_1 \cap B_2$, so we have one inclusion. Notice that this also yields $\Delta_X \in B_1 \cap B_2$.

Let us then show the inclusion $B_1 \cap B_2 \subset G_X$. Let $w \in B_1 \cap B_2$. In order to show that $w \in G_X$, we will consider the sequence of elements $\beta_m = w(\Delta_X)^m \in B_1 \cap B_2$, for $m > 0$. We will first show, using an argument taken from [14], that β_m is conjugate to a positive element, for m big enough.

Consider the reduced left-fraction decomposition $w = a^{-1}b$, where a is a product of r simple elements and b is a product of s simple elements. We have that $\beta_m = a^{-1}b(\Delta_X)^m$, so the denominator $D_L(\beta_m)$ is the product of at most r simple elements, and the numerator $N_L(\beta_m)$ is the product of at most $s + m$ simple elements. Notice that these numbers could decrease, but never increase, if one applies iterated swaps to β_m.

We can now conjugate $\beta_m \in B_1 \cap B_2$ to a recurrent element $\widetilde{\beta}_m$ by iterated swaps. Let $U_m = \mathrm{Supp}(\widetilde{\beta}_m)$, and denote $n = |\Delta_X| = \varphi(\alpha)$. By definition, $|\Delta_{U_m}| = \varphi(\beta_m) \leq \varphi(\alpha) = n$.

As we pointed out, if we consider the reduced left-fraction decomposition $\widetilde{\beta}_m = x_m^{-1}y_m$, then x_m is the product of at most r simple elements, and y_m is the product of at most $s+m$ simple elements. Since $\widetilde{\beta}_m$ is recurrent, $x_m, y_m \in G_{U_m}^+$. It follows that the simple factors in the normal forms of x_m and y_m have length at most n. Let N_m be the number of simple factors, in y_m, whose length is smaller than n. Since y_m has at most $s + m$ simple elements, we have that $|y_m| \leq (s + m)n - N_m$. It follows that the exponent sum of $\widetilde{\beta}_m$ written as a product of atoms and their inverses, that is $\ell(\widetilde{\beta}_m)$ for $\ell : B \twoheadrightarrow \mathbb{Z}$ the natural homomorphism, satisfies $\ell(\widetilde{\beta}_m) \leq (s + m)n - N_m$. But since $\widetilde{\beta}_m$ is a conjugate of β_m we have $\ell(\widetilde{\beta}_m) = \ell(\beta_m) = \ell(w) + \ell((\Delta_X)^m) = \ell(w) + nm$. We then have $\ell(w) + nm \leq (s+m)n - N_m$, so $N_m \leq sn - \ell(w)$, where the right-hand side of the inequality does not depend on m. Therefore, the number of 'small' factors in y_m is bounded by a number which is independent of m. This implies, in particular, that there exists $M > 0$ such that, for every $m > M$, some factor in the left normal form of y_m has length n. Since the simple elements in G_{U_m} have length at most n, it follows that some factor in the left normal form of y_m equals Δ_{U_m}, and that $|\Delta_{U_m}| = n$. But $x_m \in G_{U_m}^+$, which implies that $x_m = 1$ since otherwise there would be cancellation between x_m and y_m. Hence, for m big enough $(m > M)$, we have $\widetilde{\beta}_m = y_m \in B^+$.

Notice that, for every $m > M$, the left normal form of $\widetilde{\beta}_m$ is $\Delta_{U_m}^{m-N_m} s_1 \cdots s_{N_m}$. Let us denote $R_m = s_1 \cdots s_{N_m}$ the non-Delta part of the left normal form of $\widetilde{\beta}_m$. Since N_m is bounded above by a number independent of m, it follows that the sequence $\{R_m\}_{m \geq 1}$ can take finitely many possible values.

Also, for every $m > M$, let c_m be the minimal positive element such that $c_m\beta_m c_m^{-1}$ is positive. We know from 4.16 that c_m is precisely the conjugating element for iterated swaps, hence $c_m\beta_m c_m^{-1} = \widetilde{\beta}_m$. It is well known, from Remark [4], that there exists a conjugating element from β_m to a positive element, whose length is bounded above by $\inf(\beta_m) \cdot |\Delta_X|$. Since $\inf(\beta_m) \leq r$ for every m, it follows that the length of the positive element c_m is bounded above by an integer not depending on m. That is, the sequence $\{c_m\}_{m\geq 1}$ can take a finite number of possible values.

Let $e > 0$ be such that $(\Delta_V)^e$ is central for every subset V of atoms. Since the sequences $\{R_{em}\}_{m\geq 1}$ and $\{c_{em}\}_{m\geq 1}$ can take finitely many possible values, and there are only a finite number of subsets of atoms, there exist integers m_1 and m_2, with $M < m_1 < m_2$, such that $c_{em_1} = c_{em_2}$, $R_{em_1} = R_{em_2}$ and $U_{em_1} = U_{em_2}$. Let us denote $c = c_{em_1} = c_{em_2}$, $R = R_{em_1} = R_{em_2}$, $U = U_{em_1} = U_{em_2}$ and $t = m_2 - m_1$. We have $\widetilde{\beta}_{em_1} = c\beta_{em_1}c^{-1}$ and:

$$\widetilde{\beta}_{em_2} = c\beta_{em_2}c^{-1} = c\beta_{em_1}\Delta_X^{et}c^{-1} = (c\beta_{em_1}c^{-1})(c\Delta_X^{et}c^{-1}) = \widetilde{\beta}_{em_1}(c\Delta_X^{et}c^{-1}).$$

On the other hand, since $R = R_{em_1} = R_{em_2}$, it follows that $N := N_{em_1} = N_{em_2}$. Hence:

$$\widetilde{\beta}_{em_2} = \Delta_U^{em_2-N}R = \Delta_U^{em_2-em_1}\Delta_U^{em_1-N}R = \Delta_U^{et}\widetilde{\beta}_{em_1} = \widetilde{\beta}_{em_1}\Delta_U^{et}.$$

Therefore, $c\Delta_X^{et}c^{-1} = \Delta_U^{et}$. That is, $(\Delta_U^{et})^c = \Delta_X^{et}$.

Now notice that $\mathrm{PC}(\Delta_U^{et}) = G_U$ and $\mathrm{PC}(\Delta_X^{et}) = G_X$. Hence, by Lemma 4.30, $(G_U)^c = \mathrm{PC}(\Delta_U^{et})^c = \mathrm{PC}((\Delta_U^{et})^c) = \mathrm{PC}(\Delta_X^{et}) = G_X$. On the other hand, we know that $\mathrm{PC}(\widetilde{\beta}_{em_1}) = G_U$. Hence, again by Lemma 4.30, $\mathrm{PC}(\beta_{em_1}) = \mathrm{PC}((\widetilde{\beta}_{em_1})^c) = \mathrm{PC}(\widetilde{\beta}_{em_1})^c = (G_U)^c = G_X$. This implies that $\beta_{em_1} \in G_X$. Hence, as $w = \beta_{em_1}\Delta_X^{-em_1}$, we finally obtain that $w \in G_X$, as we wanted to show. □

6.2. **Characterization of adjacency for the Garside groups.** In the introduction we defined the curve graph Γ, as the graph whose vertices are irreducible parabolic subgroups, and where two such subgroups B_1 and B_2 are adjacent if either $B_1 \subset B_2$, or $B_2 \subset B_1$, or $B_1 \cap B_2 = [B_1, B_2] = \{1\}$.

We will see in this section that this notion of adjacency, which is very natural from the point of view of curves in a surface, can be characterized very easily in terms of the elements z_{B_1} and z_{B_2}.

As we did in subsection 6.1, we consider B to be one of the irreducible complex braid groups whose Garside structure (B, B^+, Δ) has been studied in this paper, satisfying the two conditions stated at the beginning of subsection 6.1 (relations are homogeneous and simple elements are square-free).

Proposition 6.12. *Two irreducible parabolic subgroups $B_1, B_2 \subset B$ with Garside structure (B, B^+, Δ) are adjacent in Γ if and only if z_{B_1} and z_{B_2} commute.*

Proof. Suppose that B_1 and B_2 are adjacent. If $B_1 \subset B_2$, since z_{B_2} is central in B_2 it follows that $[z_{B_1}, z_{B_2}] = 1$. If $B_2 \subset B_1$ the argument is the same. Finally, if $B_1 \cap B_2 = [B_1, B_2] = 1$, every element of B_1 commutes with every element of B_2, hence z_{B_1} commutes with z_{B_2} also in this case.

Conversely, suppose that z_{B_1} commutes with z_{B_2}. The first part of the proof follows the arguments of [14, Theorem 2.2]: we will simultaneously conjugate B_1 and B_2 to standard parabolic subgroups.

Since B_1 is a parabolic subgroup, it is conjugate to a standard parabolic subgroup G_T for some subset of atoms $T \subset \mathcal{A}$. We can simultaneously conjugate B_1 and B_2 to assume that $B_1 = G_T$ is already standard. Hence z_{B_1} is a positive element.

Now consider the reduced left-fraction decomposition of z_{B_2}, say $z_{B_2} = a^{-1}b$. By definition of reduced left-fraction decomposition, $a \wedge b = 1$. Left-multiplying this equality by a^{-1} we obtain $1 \wedge a^{-1}b = a^{-1}$, that is, $1 \wedge z_{B_2} = a^{-1}$.

It is clear that, as z_{B_1} is positive (hence recurrent), z_{B_1} conjugated by 1 is positive (hence recurrent). And, since z_{B_1} and z_{B_2} commute, z_{B_1} conjugated by z_{B_2} is also positive (hence recurrent). Therefore, by Proposition 4.22, $(z_{B_1})^{a^{-1}} = (z_{B_1})^{1\wedge z_{B_2}}$ is recurrent (hence positive, as the recurrent conjugates of z_{B_1} are precisely the positive conjugates of z_{B_1}).

It follows that, if we simultaneously conjugate z_{B_1} and z_{B_2} by a^{-1}, we replace z_{B_1} by a positive conjugate, and we replace z_{B_2} by $\phi(z_{B_2})$.

The interested reader may see that all the requirements of [13, Lemma 9 and Theorem 3] are fulfilled (interchanging the prefix and the suffix orders) in the groups we are working with, and this implies that $\phi(z_{B_2})$ is already a positive element (the theorem says that a is the shortest positive element such that $az_{B_2}a^{-1}$ is positive).

But we do not need to use the results in [13] to finish this proof. We have that the obtained conjugates of z_{B_1} and z_{B_2} satisfy the same initial hypotheses: they commute and the first one is positive. Hence we can simultaneously conjugate both elements by the left denominator of the second one, obtaining a new pair of elements satisfying the same hypotheses. Iterating the process, since we know that z_{B_2} is conjugate to a positive element and that such a conjugate can be obtained by iterated swaps, we finally obtain simultaneous conjugates of z_{B_1} and z_{B_2} which are both positive (and commute).

We denote z_1 the positive conjugate of z_{B_1} and z_2 the positive conjugate of z_{B_2}. Being conjugates of some z_{B_i}, we have that $z_i = z_{B'_i}$ for some parabolic subgroup B'_i, for $i = 1, 2$. Since z_1 and z_2 are positive, it follows from Proposition 6.8 that B'_1 and B'_2 are both standard parabolic subgroups, as we wanted to show.

We can then assume, up to a simultaneous conjugation, that $B_1 = G_X$ and $B_2 = G_Y$ are standard parabolic subgroups, where X and Y are saturated subsets of atoms.

Let us denote X' and Y' the images of X and Y, respectively, on the reflection group W. Let W_X and W_Y be the subgroups of W generated by X' and Y', respectively. These are irreducible parabolic subgroups of W, by definition.

Now, by Theorem 2.11, we have three possibilities. The first one is that $W_X \subset W_Y$, which implies that the set X' is contained in Y'. Since the map from B to W is injective on the atoms, this implies that $X \subset Y$, hence $G_X \subset G_Y$.

The second possibility is that $W_Y \subset W_X$. By the same argument, we obtain that $Y \subset X$, and then $G_Y \subset G_X$.

The third and final possibility is that $W_X \cap W_Y = 1$. In this case $X' \cap Y' = \varnothing$, hence $X \cap Y = \varnothing$. This implies that $G_X \cap G_Y = 1$, since for every element $x \in G_X \cap G_Y$, its reduced left-fraction decomposition $a^{-1}b$ is such that $a, b \in G_X^+ \cap G_Y^+$ and then, as X and Y are saturated, every representative of a (and also of b) is a word in X and also in Y. This is only possible if $a = b = 1$. Hence $x = 1$.

It remains to show that, in this third case, $[G_X, G_Y] = 1$. This is equivalent to say that every element of X commutes with every element of Y. Let $x \in X$ and $y \in Y$, and let x' and y' be their images in W. We know that W_X and W_Y commute, hence $x'y' = y'x'$. Since x' and y' are different (as $W_X \cap W_Y = 1$) and have order 2, the element $x'y' = y'x'$ has order 2, and then $xy = yx$ is a relation in the considered interval monoid (associated to either an Artin group, $G(e, e, n)$, or an exceptional group). Therefore G_X and G_Y commute. □

6.3. **General complex braid groups.** We can then prove in general Theorem 1.1, Theorem 1.2 and finally Theorem 1.3 of the introduction. We start with Theorem 1.1 and Theorem 1.2. If W is isodiscriminantal to some reflection group W' to which the methods of the previous sections can be applied, we have already proved Theorem 1.1 and Theorem 1.2 in this case. Indeed, the case of an arbitrary collection of parabolics $(B_i)_{i \in I}$ is easily deduced from the case $|I| = 2$, first for the case where I is finite, and then by noticing that, the $B_1 \cap B_2 \subsetneq B_1$ is possible only if the rank of $B_1 \cap B_2$ is smaller than the rank of B_1, so that $\bigcap_{i \in I} B_i = \bigcap_{i \in J} B_i$ for some finite $J \subset I$.

This covers all well-generated reflection groups, so we now consider the other ones which are not G_{31}.

If W belongs to the general series $G(de, e, n)$ for $d > 1$ (or is isodiscriminantal to one of these groups), by the description of the parabolics in Proposition 3.2, Theorem 1.2 follows immediately from the case of $G(de, 1, n)$. But then, the existence of a parabolic closure in Theorem 1.1 is an immediate consequence of Theorem 1.2 (consider the intersection of all parabolic subgroups containing the given element). Moreover, if $x \in B$ is such that x^m belongs to some parabolic subgroup B_0, writing $B_0 = B \cap \hat{B}_0$ with $\hat{B}_0$ some parabolic subgroup of the braid group $\hat{B}$ of type $G(de, 1, n)$, from $x^m \in \hat{B}_0$ we get readily $x \in \hat{B}_0$ hence $x \in \hat{B}_0 \cap B = B_0$, and we get Theorem 1.1.

Therefore both theorems are proved for the infinite series. We now prove the third one. Let B_0 be an irreducible parabolic subgroup of B, that we write as $\hat{B}_0 \cap B$ for $\hat{B}_0$ an irreducible parabolic subgroup of $\hat{B}$ by Proposition 3.2.

First notice that such a parabolic $\hat{B}_0$ is unique. Indeed, if we had another one $\hat{B}'_0$, then $\hat{B}_0 \cap \hat{B}'_0$ would be another parabolic of $\hat{B}$ containing B_0, and therefore having the same rank as $\hat{B}_0$. By Corollary 2.9 this proves $\hat{B}_0 = \hat{B}_0 \cap \hat{B}'_0 = \hat{B}'_0$ and the uniqueness of such a $\hat{B}_0$.

Setting $m_0 = |Z(\hat{W}_0)|/|Z(W_0)|$ we have $z_{B_0} = z_{\hat{B}_0}^{m_0}$ by Proposition 3.2. Denoting PC and $\widehat{\mathrm{PC}}$ the parabolic closures taken inside B and $\hat{B}$, respectively, we have

$$\mathrm{PC}(z_{B_0}) = B \cap \widehat{\mathrm{PC}}(z_{B_0}) = B \cap \widehat{\mathrm{PC}}(z_{\hat{B}_0}^{m_0}) = B \cap \widehat{\mathrm{PC}}(z_{\hat{B}_0}) = B \cap \hat{B}_0 = B_0.$$

Finally, if B_1, B_2 are two irreducible parabolic subgroups of B, for each $i = 1, 2$ we have $B_i = B \cap \hat{B}_i$ for some uniquely defined irreducible parabolic subgroup $\hat{B}_i$, and we have $z_{B_i} = z_{\hat{B}_i}^{m_i}$, for some $m_i \geq 1$. First assume we have $g \in B$ such that $B_1^g = B_2$. Then $\hat{B}_1^g$ is a parabolic subgroup of $\hat{B}$ such that $\hat{B}_1^g \cap B = (\hat{B}_1 \cap B)^g = B_1^g = B_2 = \hat{B}_2 \cap B$, whence $\hat{B}_1^g = \hat{B}_2$ by uniqueness of the parabolic subgroup of $\hat{B}$ corresponding to B_2. By Theorem 1.3 for $\hat{B}$ it follows that $z_{\hat{B}_1}^g = z_{\hat{B}_2}$. Now notice that $m_1 = m_2$, as the images of $\hat{B}_i$ and of the B_i inside W are also conjugates under the image of g. Therefore $z_{B_1}^g = z_{B_2}$.

Assume now that $[z_{B_1}, z_{B_2}] = 1$. This means $[z_{\hat{B}_1}^{m_1}, z_{\hat{B}_2}^{m_2}] = 1$. But then,

$$\widehat{\mathrm{PC}}(z_{\hat{B}_1})^{z_{\hat{B}_2}^{m_2}} = \widehat{\mathrm{PC}}(z_{\hat{B}_1}^{m_1})^{z_{\hat{B}_2}^{m_2}} = \widehat{\mathrm{PC}}((z_{\hat{B}_1}^{m_1})^{z_{\hat{B}_2}^{m_2}}) = \widehat{\mathrm{PC}}(z_{\hat{B}_1}^{m_1}) = \widehat{\mathrm{PC}}(z_{\hat{B}_1})$$

hence, by Proposition 6.7, we get $[z_{\hat{B}_1}, z_{\hat{B}_2}^{m_2}] = 1$. Then,

$$\widehat{\mathrm{PC}}(z_{\hat{B}_2})^{z_{\hat{B}_1}} = \widehat{\mathrm{PC}}(z_{\hat{B}_2}^{m_2})^{z_{\hat{B}_1}} = \widehat{\mathrm{PC}}((z_{\hat{B}_2}^{m_2})^{z_{\hat{B}_1}}) = \widehat{\mathrm{PC}}(z_{\hat{B}_2}^{m_2}) = \widehat{\mathrm{PC}}(z_{\hat{B}_2})$$

and applying again Proposition 6.7, we get $[z_{\hat{B}_1}, z_{\hat{B}_2}] = 1$. Then Theorem 1.3 applied to $\hat{B}$ implies that, either $\hat{B}_1 \subset \hat{B}_2$, or $\hat{B}_2 \subset \hat{B}_1$, or $\hat{B}_1 \cap \hat{B}_2 = [\hat{B}_1, \hat{B}_2] = \{1\}$. In the first two cases, $\hat{B}_i \subset \hat{B}_j$ implies immediately $B_i = \hat{B}_i \cap B \subset \hat{B}_j \cap B = B_j$, while in the third case we get $B_1 \cap B_2 \subset \hat{B}_1 \cap \hat{B}_2 = \{1\}$ and $[B_1, B_2] \subset [\hat{B}_1, \hat{B}_2] = \{1\}$. This completes the proof of Theorem 1.3 for the infinite series.

6.4. **Special cases in rank 2.** It remains to consider the exceptional groups G_{12}, G_{13} and G_{22}. Since they have rank 2, we only need to prove Theorem 1.1. Indeed, assuming it to hold, let us consider two parabolic subgroups B_1, B_2. Then, either one of them is $\{1\}$ or B, in which case $B_1 \cap B_2 \in \{B_1, B_2\}$ is obviously a parabolic subgroup, or they have both rank 1. Therefore, either $B_1 \cap B_2 = \{1\}$, or $B_1 \cap B_2$ contains a nontrivial element x, and we have $\{1\} \subsetneq \mathrm{PC}(x) \subset B_i$ for $i = 1, 2$. But since $\mathrm{PC}(x)$ has rank 1, by Corollary 2.9 this implies $B_1 = \mathrm{PC}(x) = B_2$ and thus $B_1 \cap B_2 = B_1 = B_2$ is a parabolic subgroup. From this one gets Theorem 1.2, so we only need to prove Theorem 1.1 in order to get both theorems.

For G_{12} and G_{22} we use the description of subsection 3.6, and the fact that the presentations given there provide a Garside structure, with $\Delta = stus$ and $\Delta = stust$, respectively. It is obvious that these structures are LCM-Garside and it can be checked that they are support-preserving, in order to apply Theorem 4.31. However, it is quicker (and somewhat simpler) to apply the following property, actually shared by all the Garside monoids used in this paper.

Proposition 6.13. *([19], Proposition 2.2) Such monoids M satisfy that, for any atom r and $x, y \in M$, if $r^n x = xy$ for some $n > 0$, then $y = t^n$ for some atom t such that $rx = xt$.*

This property has the immediate consequence that, if r is an atom of M, and $x \in M$ centralizes some r^n for $n > 0$, then $r^n = q^n$ for some atom q. But here $\mathrm{lcm}(r, q) = \Delta$ if $r \neq q$, hence $r^n = q^n$ implies $r = q$, and then $rx = xr$. This proves that the centralizer of r^n is equal to the centralizer of r.

Let $x \in B$ being nontrivial. We want to prove that it is contained inside a unique minimal parabolic. Since we are in rank 2, the only proper parabolics containing x are cyclic subgroups generated by braided reflections, and since all such braided reflections are conjugates, we can assume that x is a conjugate of some nontrivial power s^k of s. Up to exchanging x with x^{-1} we can assume $k > 0$. Therefore we can assume $x = s^k$ for some k.

Then, if $x = s^k$ were also contained in some other parabolic $\{cs^nc^{-1}; n \in \mathbb{Z}\}$ we would have $x = cs^nc^{-1}$ for some n. But since n is equal to $\ell(x) = k$ we get $n = k$. Moreover, up to multiplying c with some central power of Δ we can assume that $c \in B^+$. Hence c centralizes s^n inside B^+, which implies that c centralizes s. This proves that the parabolic closure of x is well-defined, and equal to $\langle s \rangle$. Since this is independent of $k > 0$, this is also the parabolic closure of every x^m for $m \geq 1$, whence $\mathrm{PC}(x^m) = \mathrm{PC}(x)$.

For G_{13}, we proved in subsection 3.6 that, inside its braid group B, which can be identified with the Artin group of type $I_2(6)$, $B = \langle a, b \mid (ab)^3 = (ba)^3 \rangle$, with Garside structure $B^+ = \langle a, b \mid (ab)^3 = (ba)^3 \rangle^+$, $\Delta = (ab)^3$, the proper parabolic subgroups of B are conjugates of either $\langle b^{-1} \rangle$ or to $\langle \Delta a^{-2} \rangle$. Obviously, the second one is not a parabolic subgroup for the Garside structure, but we shall nevertheless be able to use this structure in order to deal with this case.

Let $x \in B$ being non trivial. We want to prove that it is contained inside a unique minimal parabolic. Since we are in rank 2, we can assume that x is conjugate to a power of either b or $\Delta.a^{-2}$.

We first prove that it cannot belong to two such parabolics of different type, that is one conjugate of $\langle b \rangle$ and one conjugate of $\langle \Delta.a^{-2} \rangle$. The reason for this is that the abelianization map $B \to \mathbb{Z}^2$ mapping a to $(1,0)$ and b to $(0,1)$ is injective on such cyclic subgroups and maps subgroups of the first kind to $\mathbb{Z}(1,0)$ and subgroups of the second kind to $\mathbb{Z}(1,3)$. Since these two subgroups of $\mathbb{Z}^2$ have trivial intersection this proves this claim.

Now assume that x belongs to two subgroups conjugates of $\langle b \rangle$. Without loss of generality we can assume $x = b^k$ for some $k > 0$ and $x = cb^kc^{-1}$ for some $c \in B$. Up to multiplying c with some central power of Δ we can assume that $c \in B^+$. Then $cb^k = b^kc$ hence by Proposition 6.13 we have $cb = bc$ and the two subgroups are the same. Also, $\mathrm{PC}(x^m) = \mathrm{PC}(x)$ for every $m \geq 1$.

Finally assume that x belongs to two subgroups conjugates of $\langle \Delta a^{-2} \rangle$. Without loss of generality we can assume $x = (\Delta.a^{-2})^k = \Delta^k.a^{-2k}$, and $x = c\Delta^k.a^{-2k}c^{-1}$ for some $c \in B$, with $k > 0$. Dividing by $\Delta^k \in Z(B)$ and taking the inverse we get $y = (\Delta^{-k}x)^{-1} = a^{2k}$ and $y = ca^{2k}c^{-1}$, so as before we get that c commutes with a and we get $\mathrm{PC}(x) = \langle \Delta a^{-2} \rangle$. Since we also have $\mathrm{PC}(x^m) = \langle \Delta a^{-2} \rangle$ for every $m \geq 1$ this yields the conclusion in that case, too.

We then prove Theorem 1.3 in the case of the groups of rank 2. Let B_0 be an irreducible parabolic subgroup. We have $\mathrm{PC}(z_B) = B$, for otherwise z_B would be the power of some braided reflection σ, so that z_B and σ would have some nontrivial common power inside the pure braid group P, and considering the image inside $P^{ab} = H_1(X, \mathbb{Z}) \simeq \mathbb{Z}\mathcal{A}$ immediately yields a contradiction, as the hyperplane arrangement $\mathcal{A}$ contains at least 2 hyperplanes. If B_0 has rank 1, clearly $B_0 = \langle z_{B_0} \rangle$, so that $\mathrm{PC}(z_{B_0}) = B_0$. This proves the first part of the statement. Let B_1, B_2 be two irreducible parabolic subgroups. Without loss of generality we can assume that they have rank 1, for otherwise the statements are trivially true. If $B_1^g = B_2$ for some $g \in B$, since $B_i = \langle z_{B_i} \rangle$ and z_{B_i} is the unique positive generator of B_i (in the sense of subsection 2.5) we have necessarily $z_{B_1}^g = z_{B_2}$.

Then, $z_{B_1}z_{B_2} = z_{B_2}z_{B_1}$ trivially implies $[B_1, B_2] = 1$, as $B_i = \langle z_{B_i} \rangle$. Moreover, if $B_1 \cap B_2$ contains some nontrivial element, there exists $a, b \neq 0$ such that $z_{B_1}^a = z_{B_2}^b$ whence

$$B_1 = \mathrm{PC}(z_{B_1}) = \mathrm{PC}(z_{B_1}^a) = \mathrm{PC}(z_{B_2}^b) = \mathrm{PC}(z_{B_2}) = B_2$$

and $B_1 = B_2$, which completes the proof of Theorem 1.3.

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